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Novel exact surface wave solutions for layered structures

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Abstract

Novel exact solutions describing surface acoustic waves on general layered structures have been found by the method of variable separation. First, solutions have been constructed with plane wavefronts and involving polynomial dependence on lateral variables. Second, their inhomogeneous plane-wave analogues have been found. At last, beam-like solutions highly localized at large lateral distances in a given sector have also been considered. ***To cite this article: A.P. Kiselev et al., C. R. Mecanique 335 (2007).***

Résumé

Des solutions exactes inédites de la propagation d'ondes acoustiques de surface dans un milieu stratifié. De nouvelles solutions exactes, caractéristiques d'ondes acoustiques de surface dans un milieu stratifié quelconque, sont obtenues par séparation des variables. Nous nous attachons tout d'abord aux solutions présentant des fronts d'ondes plan mais dont l'amplitude varie de façon polynomiale selon les directions transversales, ensuite aux ondes planes hétérogènes qui leur sont analogues, et enfin aux solutions de type « rayon » qui sont fortement localisées dans un secteur angulaire donné et comportent une focalisation. ***Pour citer cet article : A.P. Kiselev et al., C. R. Mecanique 335 (2007).***

Keywords: Waves; Surface wave; Exact solution; Beam

Mots-clés : Ondes ; Onde de surface ; Solution exacte ; Rayon

1. Introduction and statement of the problem

Known explicit solutions describing surface acoustic waves in layered structures have the form of plane waves see, e.g., [1,2]. In the present Note we extend the class of explicit solutions for surface waves using a straightforward separation of variables. We give a special attention to modes with plane wavefronts but involving polynomial dependencies

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on lateral variables and to their inhomogeneous plane-wave analogues. In addition, we discuss beam-like solutions becoming highly localized in a given sector at large lateral distances.

Let a layered half-space be characterized by the acoustic stiffness $\mu(z) > 0$ and the density $\rho(z) > 0$. We consider time-harmonic surface waves described by the scalar equation

$$\mu(z)(u_{xx} + u_{yy}) + \bigl(\mu(z)u_z\bigr)_z + \omega^2\rho(z)u = 0, \quad z > 0 \tag{1}$$

where $\omega > 0$ is the circular frequency. When $\mu(z)$ has finite jumps at $z = z_1, \ldots, z_N$, the classical matching conditions

$$[u]|_{z=z_m} = 0, \quad [\mu u_z]|_{z=z_m} = 0, \quad m = 1, \ldots, N \tag{2}$$

hold, where $[\ ]|_{z=z_m}$ stands for the jumps at $z = z_m$. We assume the classical free-surface boundary condition

$$u_z|_{z=0} = 0 \tag{3}$$

and the damping condition at infinity

$$u \to 0 \quad \text{as } z \to \infty \tag{4}$$

In what follows the time-dependence $\mathrm{e}^{-\mathrm{i}\omega t}$ is understood.

2. Standard plane-wave solution

The commonly known solution for a plane surface wave propagating along the axis x has the form [1,2]

$$u^0 = \mathrm{e}^{\mathrm{i}kx} v(z; k) \tag{5}$$

where $k > 0$ is a constant wave number. The quantity k^2 is the eigennumber corresponding to an eigenfunction $v(z; k)$ of the spectral problem for the ordinary differential operator defined by

$$(\mu v')' + \bigl(\omega^2\rho - \mu k^2\bigr)v = 0 \tag{6}$$

with the conditions (2), (3) and (4). The symbol 'prime' denotes the derivative $\mathrm{d}/\mathrm{d}z$. We assume that, given ω, such an eigenvalue exists and fix it. The eigenfunction $v(z; k)$ is then unique up to a constant factor. Solution (5) describes a surface wave running along the axis x with the phase velocity $c = \omega/k$. Its wave front is given by $x = ct$ and amplitude is independent of lateral variables x and y.

3. Generalisation of plane-wave solutions

3.1. Plane waves with polynomial amplitudes

First, we construct solutions of (1)–(4) describing surface waves running along the axis x and whose amplitudes depend polynomially on y and x. We introduce polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$, $r \geqslant 0$, $0 \leqslant \varphi < 2\pi$. Apparently, problem (1)–(4) is invariant with respect to φ. Therefore, the φ-derivatives of (5),

$$u = (\partial/\partial\varphi)^m u^0 = P_m(x, y)u^0, \quad m = 1, 2, \ldots \tag{7}$$

are also solutions of (1), (3), and (4). Here $P_m = \mathrm{e}^{-\mathrm{i}kr\cos\varphi}(\partial/\partial\varphi)^m \mathrm{e}^{\mathrm{i}kr\cos\varphi}$ are polynomials of x and y,

$$P_1 = -\mathrm{i}ky \tag{8}$$

$$P_2 = (\mathrm{i}ky)^2 - \mathrm{i}kx, \qquad P_3 = -(\mathrm{i}ky)^3 + 3\mathrm{i}kxy + \mathrm{i}ky, \qquad \ldots \tag{9}$$

$$P_{m+1} = (\partial/\partial\varphi - \mathrm{i}kr\sin\varphi)P_m = (x\partial/\partial y - y\partial/\partial x - \mathrm{i}ky)P_m \tag{10}$$

Solutions (7) describe surface waves with plane wave front. They propagate along the axis x with amplitudes dependent on the transverse lateral variable y. When $m > 1$, the amplitudes also necessarily depend on the longitudinal variable x. Note that the elastic Rayleigh wave traveling on a homogeneous isotropic half-space with a linear dependence on the transverse lateral variable y—an analogue of (8)—has recently been described [3].

3.2. General solutions of moderate growth

The structure of solutions (7) prompts us to seek solutions of (1)–(4) in a general form

$$u = P(x, y)v(z; k) \tag{11}$$

Substitution of (11) into (1) immediately gives the Helmholtz equation,

$$P_{xx} + P_{yy} + k^2 P = 0 \tag{12}$$

As it is well known, any solution of (12) of moderate growth (that is, growing at infinity not faster than a polynomial) can be represented by

$$P(x, y) = \int_0^{2\pi} \mathrm{e}^{\mathrm{i}kr\cos(\varphi-\phi)} A(\phi)\,\mathrm{d}\phi \tag{13}$$

where the density $A(\phi)$ is a distribution on the unit circle, see, e.g. [4]. The solutions (7), (8), (9), ..., can be obtained by putting $A(\phi) = \delta(\phi)$, $A(\phi) = \delta'(\phi)$, ..., where δ, δ', ..., is the Dirac delta function on unit circle, its derivative, etc.

3.3. Beam-like solutions

We assume now that $A(\phi)$ is infinitely differentiable. Consider the asymptotic behavior of (13) at large values of kr. The phase function $kr\cos(\varphi - \phi)$ has two stationary points $\phi = \varphi$ and $\phi = \pi + \varphi$. The standard stationary phase method [5] yields the asymptotic expansion

$$P \sim P^+ + P^- \tag{14}$$

where

$$P^\pm \sim \frac{\mathrm{e}^{\pm\mathrm{i}kr}}{\sqrt{kr}}\left(A^\pm(\varphi) + \sum_{m=1}^{\infty} \frac{A_m^\pm(\varphi)}{(kr)^m}\right), \quad kr \to \infty \tag{15}$$

with $A^+(\varphi) = \sqrt{2\pi}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}} A(\varphi)$, $A^-(\varphi) = \sqrt{2\pi}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}} A(\pi + \varphi)$. Here, P^- and P^+ describe incoming and outgoing waves, respectively. Higher-order terms $A_m^\pm(\varphi)$ can be found by applying certain differential operators with constant coefficients to the corresponding directivity patterns $A^\pm(\varphi)$.

Let us confine ourselves to functions $A(\varphi)$ vanishing outside a sector given. We choose the sector $\Phi^+ = \{-\alpha < \varphi < \alpha\}$, where $0 < \alpha < \pi/2$ (here we identify values of ϕ which differ by $2\pi n$, $n = \pm 1, \pm 2, \ldots$). Then the incoming wave P^- will be, up to any power of $(kr)^{-1}$, non-zero at large distances only inside $\Phi^- = \{\pi - \alpha < \varphi < \pi + \alpha\}$, and P^+ will vanish with the same accuracy outside the opposite sector Φ^+.

As an example, let us put

$$A(\phi) = \exp\left(a\frac{\phi^2}{\phi^2 - \alpha^2}\right) \quad \text{for } |\phi| < \alpha^2, \qquad A(\phi) = 0 \quad \text{otherwise} \tag{16}$$

where a is a free positive constant. Fig. 1 shows the result of numerical simulations of the solution (11), (13), (16).

3.4. Inhomogeneous plane waves with polynomial amplitudes

Eq. (11) also allows solutions growing faster than a polynomial as $kr \to \infty$. For instance,

$$P(x, y) = \mathrm{e}^{\mathrm{i}kr\cos(\varphi-\vartheta)} \tag{17}$$

with a complex ϑ. Rayleigh waves of this type on isotropic homogeneous half-space have earlier been considered in [6]. Differentiating (17) with respect to φ yields solutions with polynomial amplitudes, analogous to (8)–(10).

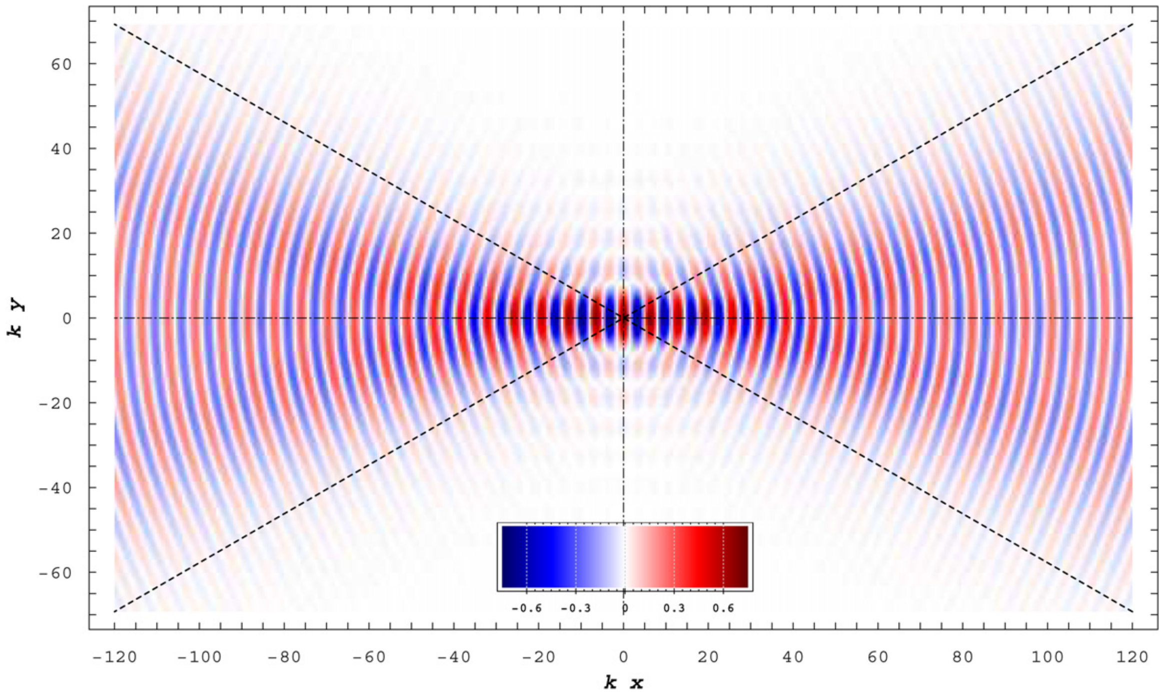

Fig. 1. Real part of solution (11), (13), (16) at $\alpha = \pi/6$ and $a = 0.5$ on surface $z = 0$. It is assumed that $v(0; k) = 1$.

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