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Bloch wave homogenization in a medium perforated by critical holes

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Abstract

In this Note, we use the Bloch wave method to study the asymptotic behavior of the solution of the Laplace equation in a periodically perforated domain, under a non-homogeneous Neumann condition on the boundary of the holes, as the hole size goes to zero more rapidly than the domain period. We prove that for a critical size, the non-homogeneous boundary condition generates an additional term in the homogenized problem, commonly referred to as 'the strange term' in the literature. ***To cite this article: J. Ortega et al., C. R. Mecanique 335 (2007).***

Résumé

Homogénéisation par ondes de Bloch dans un milieux perforé par trous critiques. Dans cette Note, nous utilisons la méthode des ondes de Bloch dans l'étude du comportement asymptotique de la solution de l'équation de Laplace dans un domaine périodiquement perforé sous une condition Neumann non homogène sur la frontière des perforations quand la taille des trous converge vers zéro plus rapidement que la période du domaine. On prouve que pour une taille critique, la condition non homogène génère un terme additionnel dans le problème homogénéisé, lequel est connue dans la littérature comme « le terme étrange ». ***Pour citer cet article : J. Ortega et al., C. R. Mecanique 335 (2007).***

Keywords: Computational solid mechanics; Bloch wave method

Mots-clés : Mécanique des solides numérique ; Méthode des ondes de Bloch

Version française abrégée

Soit $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, un ouvert borné et soit ε un réel positive. On dénote par Ω^ε le domaine Ω perforé par un réseaux périodique des perforations de forme $r(\varepsilon)T$ et période $2\pi\varepsilon$.

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On s'intéresse au comportement asymptotique de la solution v^ε du problème (2) quand ε tend vers zéro et quand la taille $r(\varepsilon)$ est telle que (5).

Le théorème principal de ce travail est le suivant :

Théorème 0.1. *On suppose que $r(\varepsilon)$ vérifies la condition* (5) *et que $\gamma \neq 0$. Soit v^ε dans V^ε la suite des solutions du problème* (2). *Donc, pour une famille quelconque d'opérateurs linéaires $\{P^\varepsilon\}$ vérifiant* (4) *on a*

$$r(\varepsilon)^{-(N-1)}\varepsilon^N P^\varepsilon v^\varepsilon \rightharpoonup v \quad \text{faible dans } H_0^1(\Omega), \text{ quand } \varepsilon \to 0$$

où v est l'unique solution du problème homogénéise suivant :

$$\begin{cases} -\Delta v = \dfrac{|\partial T|\gamma}{(2\pi)^N} & \text{dans } \Omega \\ v = 0 & \text{sur } \partial\Omega \end{cases}$$

Ce problème a été déjà étudie dans [1] en utilisant la méthode classique des fonctions test de Tartar. La principale nouveauté dans cette Note est que la preuve de ce théorème est faite avec la méthode spectrale des ondes de Bloch.

On commence la preuve en utilisant une fonction de troncature $\varphi \in \mathscr{D}(\Omega)$ qui transforme le système (2) en le nouveau système (9), écrit dans $\mathbb{R}^N$ périodiquement perforé.

On prend la première transformée de Bloch pour obtenir l'identité (11). Pour passer à la limite quand $\varepsilon \to 0$, on utilise les Lemmes 3.1–3.3 et on obtient l'identité (12).

La principale difficulté dans le passage à la limite est concentrée dans le terme de bord. Ce terme est décomposé en quatre parties notés par $I_1, \ldots, I_4$. La première intégrale converge vers la transformé de Fourier du terme étrange dans l'équation homogénéisé (voir (17)). Les autres intégrales convergent vers zéro dans $L^\infty_{\text{loc}}(\mathbb{R}^N)$. Après le passage à la limite on trouve l'équation homogénéisé dans l'espace de Fourier.

1. Introduction

The aim of this Note is to study the asymptotic behavior of the solution of a problem associated with the Laplace operator in a periodically perforated domain using the spectral method of Bloch waves. On the exterior boundary we consider a homogeneous Dirichlet condition and on the boundary of the holes the condition is non-homogeneous Neumann-type. We are interested in the case when the hole size goes to zero faster than the domain period. We recall that the homogenization process in this same problem, using the classical method of Tartar's test functions, was studied in Conca and Donato [1]. There exists a critical hole size when the non-homogeneous boundary condition generates an additional term in the homogenized problem, which is referred to as 'the strange term' in the literature. This terminology has been introduced by Cioranescu and Murat [2] in the study of a similar homogenization problem, with a Dirichlet boundary condition on the holes.

The main novelty of this Note is to show that the strange term in the homogenized equation is amenable to analysis using the Bloch wave decomposition. This strange term is a consequence of the mean value of the non-homogeneous boundary condition on the holes.

The literature on homogenization theory contains a large number of works (see, for instance, [3–7]). The Bloch waves applied in homogenization problems appear for example, in [8–10]. In terms of using the Bloch wave decomposition to solve problems on periodically perforated domains, we indicate the works [11] and [12].

2. Setting of the problem and main result

Let $\Omega \subset \mathbb{R}^N,\ N \geqslant 2$, be a smooth open bounded set and let ε be a positive real number. Assume that $r : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous map verifying $r(\varepsilon) < \pi\varepsilon$ and

$$\lim_{\varepsilon \to 0} \frac{r(\varepsilon)}{\varepsilon} = 0 \tag{1}$$

Let $T \subset \mathbb{R}^N$ be a non-empty open bounded set with a smooth boundary. We suppose that T is star-shaped with respect to 0, $d(0, \partial T) > 0$ and $\max_{s\in\partial T} |s| = 1$. For any integer vector $p \in \mathbb{Z}^N$, we denote by $T_p^\varepsilon = 2\pi\varepsilon p + r(\varepsilon)T$ and by T^ε the set of all holes contained in Ω, i.e., $T^\varepsilon = \bigcup\{T_p^\varepsilon \mid \overline{T_p^\varepsilon} \subset \Omega,\ p \in \mathbb{Z}^N\}$. We set $\Omega^\varepsilon = \Omega \setminus \overline{T^\varepsilon}$, the periodically perforated domain with holes of size $r(\varepsilon)$.

We are interested in studying the asymptotic behavior, as ε goes to zero, of the solution v^ε of the following non-homogeneous Neumann boundary-value problem:

$$\begin{cases} -\Delta v^\varepsilon = 0 & \text{in } \Omega^\varepsilon \\ \dfrac{\partial v^\varepsilon}{\partial n} = \gamma & \text{on } \partial T^\varepsilon \\ v^\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

where γ is a non-zero constant. The variational formulation of problem (2) is: Find $v^\varepsilon \in V^\varepsilon$ such that

$$\int_{\Omega^\varepsilon} \nabla v^\varepsilon \cdot \nabla\varphi \,\mathrm{d}x = \gamma \int_{\partial T^\varepsilon} \varphi \,\mathrm{d}s \quad \forall \varphi \in V^\varepsilon \tag{3}$$

where $V^\varepsilon = \{\varphi \in H^1(\Omega^\varepsilon) \mid \varphi = 0 \text{ on } \partial\Omega\}$ is a Hilbert space endowed with the inner product in $H^1(\Omega^\varepsilon)$.

Since the functions v^ε are only defined in Ω^ε, we shall introduce a family of linear continuous extension operators $P^\varepsilon \in \mathscr{L}(V^\varepsilon, H_0^1(\Omega))$ such that

$$P^\varepsilon\varphi|_{\Omega^\varepsilon} = \varphi \quad \text{and} \quad \|\nabla P^\varepsilon\varphi\|_{L^2(\Omega)} \leqslant C\|\nabla\varphi\|_{L^2(\Omega^\varepsilon)} \quad \forall\varphi \in V^\varepsilon \tag{4}$$

where C is a real positive constant independent of ε. The proof of the existence of one such family can be found in [4] or [1].

Let us now state the homogenization result that we will prove in this Note, by using the Bloch wave decomposition, concerning with the asymptotic behavior of the solution v^ε, as $\varepsilon \to 0$. Depending on the way as $\frac{r(\varepsilon)}{\varepsilon}$ goes to zero, the solution v^ε has different asymptotic behaviors. In this Note, we are interested in the case where the hole size is equal to the so-called critical size, that is,

$$\lim_{\varepsilon\to 0} \varepsilon^{-2} F\left(\frac{r(\varepsilon)}{\varepsilon}\right) = \ell \in (0, \infty) \tag{5}$$

with F a real function given by:

$$F(r) = \begin{cases} -(\ln r)^{-1} & \text{if } N = 2 \\ r^{N-2} & \text{if } N \geqslant 3 \end{cases} \tag{6}$$

Theorem 2.1. *Assume that $r(\varepsilon)$ verifies (5) and $\gamma \neq 0$. Let v^ε in V^ε be the sequence of the unique solutions of the problem (2). Then, for any family of linear continuous extension operators $\{P^\varepsilon\}$ verifying (4), we have that*

$$r(\varepsilon)^{-(N-1)}\varepsilon^N P^\varepsilon v^\varepsilon \rightharpoonup v \quad \textit{weakly in } H_0^1(\Omega),\ \textit{as } \varepsilon \to 0$$

where v is the unique solution of the following homogenized problem:

$$\begin{cases} -\Delta v = \dfrac{|\partial T|\gamma}{(2\pi)^N} & \textit{in } \Omega \\ v = 0 & \textit{on } \partial\Omega \end{cases} \tag{7}$$

The complete proof of this result is given in [13]. Moreover, there we also study the case where the hole size is not equal to the critical one.

3. Sketch of the proof

Due to Lemma 2.1 in [1] and the Poincaré inequality in Ω, we can extract a subsequence such that

$$r(\varepsilon)^{-(N-1)}\varepsilon^N P^\varepsilon v^\varepsilon \rightharpoonup v \quad \text{weakly in } H_0^1(\Omega),\ \text{as } \varepsilon \to 0 \tag{8}$$

Our goal is to prove, using the spectral method of Bloch waves, that the limit v is the unique solution of the problem (7). For this purpose, we consider a cut-off function $\varphi \in \mathscr{D}(\Omega)$. Since v^ε is solution of the problem (2), we deduce that φv^ε satisfies

$$\begin{cases} -\Delta(\varphi v^\varepsilon) = -2\nabla\varphi \cdot \nabla v^\varepsilon - \Delta\varphi\, v^\varepsilon & \text{in } \mathbb{R}^N \setminus \overline{S^\varepsilon} \\ \dfrac{\partial(\varphi v^\varepsilon)}{\partial n} = \gamma\varphi + v^\varepsilon \dfrac{\partial\varphi}{\partial n} & \text{on } \partial S^\varepsilon \end{cases} \tag{9}$$

where S^ε denotes the set of all holes in $\mathbb{R}^N$, that is, $S^\varepsilon = \bigcup_{p\in\mathbb{Z}^N} T_p^\varepsilon$.

We apply the first Bloch transform to the problem (9) in order to get

$$\begin{aligned} &\lambda_1^\varepsilon\big(r(\varepsilon);\xi\big)\big(B_1^\varepsilon(\varphi v^\varepsilon)\big)(\xi) - \int\limits_{\partial T^\varepsilon} \frac{\partial(\varphi v^\varepsilon)}{\partial n}(s)\mathrm{e}^{-\mathrm{i}\xi\cdot s}\overline{\phi_1^\varepsilon}\big(r(\varepsilon);s;\xi\big)\,\mathrm{d}s \\ &\quad = -2B_1^\varepsilon(\nabla\varphi\cdot\nabla v^\varepsilon)(\xi) - B_1^\varepsilon\big((\Delta\varphi)v^\varepsilon\big)(\xi) \end{aligned} \tag{10}$$

where $\lambda_1^\varepsilon(r(\varepsilon);\xi)$ and $\phi_1^\varepsilon(r(\varepsilon);\cdot;\xi)$ denote the Bloch waves at ε-scale. By homothecy, these waves are related to the Bloch waves at a reference-scale, λ_1 and ϕ_1, by the relations $\lambda_1^\varepsilon(r(\varepsilon);\xi) = \varepsilon^{-2}\lambda_1(\frac{r(\varepsilon)}{\varepsilon};\eta)$ and $\phi_1^\varepsilon(r(\varepsilon);x;\xi) = \phi_1(\frac{r(\varepsilon)}{\varepsilon};y;\eta)$. The variables $(x;\xi)$ and $(y;\eta)$ are related by $y = \frac{x}{\varepsilon}$, $\eta = \varepsilon\xi$.

In order to pass to the limit in the previous Bloch equation, we need the three following lemmas (for details, see [13]):

Lemma 3.1. *Let h^ε be a sequence in $L^2(\Omega^\varepsilon)$ and $h \in L^2(\mathbb{R}^N)$. If $\widetilde{h^\varepsilon} \rightharpoonup h$ weakly in $L^2(\mathbb{R}^N)$, then*

$$B_1^\varepsilon \widetilde{h^\varepsilon} \rightharpoonup \widehat{h} \quad \text{weakly in } L^2_{\mathrm{loc}}\big(\mathbb{R}^N\big)$$

where $\widetilde{(\cdot)}$ denotes the extension by zero on $\mathbb{R}^N \setminus \overline{\Omega^\varepsilon}$ and $\widehat{h}$ the usual Fourier transform of h.

Lemma 3.2. *If* (1) *holds, then for any $\psi \in \mathscr{D}(\Omega)$ we have that $\widetilde{\psi|_{\Omega^\varepsilon}} \to \psi$ strongly in $L^2(\mathbb{R}^N)$.*

Lemma 3.3. *If* (1) *holds, then* $\lim_{\varepsilon\to 0}\lambda_1^\varepsilon(r(\varepsilon);\xi) = \delta_{kl}\xi_k\xi_l$.

Before using the previous lemmas, let us write the identity (10) as follows

$$\begin{aligned} &\lambda_1^\varepsilon\big(r(\varepsilon);\xi\big)\big(B_1^\varepsilon\big(\widetilde{\varphi|_{\Omega^\varepsilon}} r(\varepsilon)^{-(N-1)}\varepsilon^N P^\varepsilon v^\varepsilon\big)\big)(\xi) - r(\varepsilon)^{-(N-1)}\varepsilon^N \int\limits_{\partial T^\varepsilon} \frac{\partial(\varphi v^\varepsilon)}{\partial n}(s)\mathrm{e}^{-\mathrm{i}\xi\cdot s}\overline{\phi_1^\varepsilon}\big(r(\varepsilon);s;\xi\big)\,\mathrm{d}s \\ &\quad = -2B_1^\varepsilon\big(\widetilde{(\nabla\varphi)|_{\Omega^\varepsilon}}\cdot r(\varepsilon)^{-(N-1)}\varepsilon^N\nabla(P^\varepsilon v^\varepsilon)\big)(\xi) - B_1^\varepsilon\big(\widetilde{(\Delta\varphi)|_{\Omega^\varepsilon}} r(\varepsilon)^{-(N-1)}\varepsilon^N P^\varepsilon v^\varepsilon\big)(\xi) \end{aligned} \tag{11}$$

Then, we get

$$-\widehat{(\varphi\Delta v)}(\xi) = \lim_{\varepsilon\to 0} r(\varepsilon)^{-(N-1)}\varepsilon^N \int\limits_{\partial T^\varepsilon} \frac{\partial(\varphi v^\varepsilon)}{\partial n}(s)\mathrm{e}^{-\mathrm{i}\xi\cdot s}\overline{\phi_1^\varepsilon}\big(r(\varepsilon);s;\xi\big)\,\mathrm{d}s \tag{12}$$

Now, we proceed to pass to the limit in the above boundary integral. To do this, we decompose it into the sum of the following four integrals:

$$I_1 = r(\varepsilon)^{-(N-1)}\varepsilon^N \int\limits_{\partial T^\varepsilon} \gamma\varphi(s)\mathrm{e}^{-\mathrm{i}\xi\cdot s}\overline{\phi_1^\varepsilon}\big(r(\varepsilon);s;0\big)\,\mathrm{d}s \tag{13}$$

$$I_2 = r(\varepsilon)^{-(N-1)}\varepsilon^N \int\limits_{\partial T^\varepsilon} \frac{\partial\varphi}{\partial n}(s)v^\varepsilon(s)\mathrm{e}^{-\mathrm{i}\xi\cdot s}\overline{\phi_1^\varepsilon}\big(r(\varepsilon);s;0\big)\,\mathrm{d}s \tag{14}$$

$$I_3 = r(\varepsilon)^{-(N-1)}\varepsilon^N \int\limits_{\partial T^\varepsilon} \gamma\varphi(s)\mathrm{e}^{-\mathrm{i}\xi\cdot s}\big[\overline{\phi_1^\varepsilon}\big(r(\varepsilon);s;\xi\big) - \overline{\phi_1^\varepsilon}\big(r(\varepsilon);s;0\big)\big]\,\mathrm{d}s \tag{15}$$

$$I_4 = r(\varepsilon)^{-(N-1)}\varepsilon^N \int\limits_{\partial T^\varepsilon} \frac{\partial\varphi}{\partial n}(s)v^\varepsilon(s)\mathrm{e}^{-\mathrm{i}\xi\cdot s}\big[\overline{\phi_1^\varepsilon}\big(r(\varepsilon);s;\xi\big) - \overline{\phi_1^\varepsilon}\big(r(\varepsilon);s;0\big)\big]\,\mathrm{d}s \tag{16}$$

In order to pass to the limit in the first integral I_1, let us first note that $\phi_1^\varepsilon(r(\varepsilon); x; 0) = |Y_\varepsilon^\star|^{-1/2}$, where $Y_\varepsilon^\star = [-\pi, \pi[^N \backslash (\frac{r(\varepsilon)}{\varepsilon})\overline{T}$ is the reference cell. Then, we use similar arguments as in [1] to obtain

$$\lim_{\varepsilon\to 0} I_1 = \lim_{\varepsilon\to 0} \gamma |Y_\varepsilon^\star|^{-1/2}\left(\frac{|\partial T||\Omega|}{(2\pi)^N}\right)\cdot \frac{1}{|\partial T^\varepsilon|}\int_{\partial T^\varepsilon} \varphi(s)\mathrm{e}^{-\mathrm{i}\xi\cdot s}\,\mathrm{d}s$$
$$= \gamma \frac{1}{(2\pi)^{N/2}}\left(\frac{|\partial T||\Omega|}{(2\pi)^N}\right)\cdot \frac{1}{|\Omega|}\int_{\Omega} \varphi(x)\mathrm{e}^{-\mathrm{i}\xi\cdot x}\,\mathrm{d}x = \frac{\gamma|\partial T|}{(2\pi)^N}\hat{\varphi}(\xi) \tag{17}$$

For the integral I_2, we use Green's formula and the Cauchy–Schwarz inequality to get

$$|I_2| \leqslant C\big(1+|\xi|\big)|Y_\varepsilon^\star|^{-1/2} r(\varepsilon)^{-(N-1)}\varepsilon^N \|P^\varepsilon v^\varepsilon\|_{H^1(\Omega)}\left(\frac{r(\varepsilon)}{\varepsilon}\right)^{N/2} \leqslant C(1+M)\left(\frac{r(\varepsilon)}{\varepsilon}\right)^{N/2}$$

Then, the convergence to zero of I_2 in $L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ is a direct consequence of the condition (1).

In the sequel, we study the convergence of the boundary integral I_3, defined in (15). We begin by using the variational formulation (3) as follows:

$$I_3 = r(\varepsilon)^{-(N-1)}\varepsilon^N \gamma \int_{\Omega^\varepsilon} \nabla v^\varepsilon(x)\cdot \nabla\big(\varphi(x)\mathrm{e}^{-\mathrm{i}\xi\cdot x}\big[\,\overline{\phi_1^\varepsilon}\big(r(\varepsilon); x; \xi\big) - \overline{\phi_1^\varepsilon}\big(r(\varepsilon); x; 0\big)\big]\big)\,\mathrm{d}x$$

Since the application $\eta \mapsto \phi_1(\frac{r(\varepsilon)}{\varepsilon}; \cdot; \eta) \in H^1_\#(Y_\varepsilon^\star)$ is Lipschitz, we obtain $|I_3| \leqslant C(1+|\xi|)|\xi|\varepsilon$. Then, the integral I_3 converges to zero in $L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$.

In order to estimate the term I_4, we introduce the auxiliary functions $\hat{v}_j^\varepsilon$, $j = 1, \ldots, N$, as the solutions of the problem: Find $\hat{v}_j^\varepsilon \in V^\varepsilon$ such that

$$\int_{\Omega^\varepsilon} \nabla \hat{v}_j^\varepsilon \cdot \nabla v\,\mathrm{d}x = \int_{\partial T^\varepsilon} v^\varepsilon n_j v\,\mathrm{d}s \quad \forall v \in V^\varepsilon \tag{18}$$

where n_j denotes the jth component of the normal unit vector.

Using these auxiliary functions, the integral I_4 can be rewritten as follows:

$$I_4 = r(\varepsilon)^{-(N-1)}\varepsilon^N \int_{\Omega^\varepsilon} \nabla \hat{v}_j^\varepsilon(x)\cdot \nabla\left(\frac{\partial\varphi}{\partial x_j}(x)\mathrm{e}^{-\mathrm{i}\xi\cdot x}\big[\overline{\phi_1^\varepsilon}\big(r(\varepsilon); x; \xi\big) - \overline{\phi_1^\varepsilon}\big(r(\varepsilon); x; 0\big)\big]\right)\mathrm{d}x$$

then by similar arguments as in the integral I_3, we conclude that I_4 converges to zero.

Hence, from all the previous convergence of the integrals $I_1, \ldots, I_4$ it follows that (12) becomes

$$\big(\widehat{-\varphi\Delta v}\big)(\xi) = \left(\widehat{\frac{\gamma|\partial T|}{(2\pi)^N}\varphi}\right)(\xi) \tag{19}$$

Taking the inverse Fourier transform of (19), and since the above relation is valid for all φ in $\mathscr{D}(\Omega)$, we conclude.

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