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On the non-conservativeness of a class of anisotropic damage models with unilateral effects

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Abstract

The aim of this Note is to show that a class of anisotropic elastic-damage models including unilateral effects can be considered, for constant damage values, as non-linear and non-conservative elastic. The conservative character of corresponding constitutive models is related to the symmetry of the Hessian tensor. For the models under consideration, it is shown that the condition of conservativeness (existence of the elastic potential energy function) is obtained only when there is coaxiality of the strain and damage tensors. ***To cite this article: N. Challamel et al., C. R. Mecanique 334 (2006).***
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Résumé

Sur le caractère non-conservatif de quelques modèles d'endommagement anisotrope avec conditions unilatérales. L'objectif de cette note est de montrer qu'une classe de modèles élastique-endommageables anisotropes avec prise en compte spécifique des effets unilatéraux représente en réalité, à endommagement constant, une forme d'élasticité non-linéaire non-conservative. Le caractère conservatif (existence du potentiel thermodynamique) est équivalent à la symétrie du tenseur Hessien. On montre alors que, pour la classe de modèles considérés, la condition de conservation de l'énergie n'est assurée que lorsque les directions principales des tenseurs d'endommagement et de déformation coïncident (coaxialité au sens du dommage). ***Pour citer cet article : N. Challamel et al., C. R. Mecanique 334 (2006).***
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Mots-clés : Endommagement ; Elasticité non-linéaire ; Endommagement tensoriel ; Effet unilatéral ; Systeme non conservatif ; Hessien

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La modélisation des phénomènes de microfissuration par la théorie de l'endommagement prenant en compte l'anisotropie induite et les effets de refermeture de fissures reste un problème difficile sur lequel un consensus n'a pas

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encore été trouvé, malgré une vingtaine d'années de recherche sur le sujet [1–4]. L'objectif de cette note est de montrer que beaucoup de modèles couplés du type élasticité/endommagement avec prise en compte des effets unilatéraux sont en réalité, à endommagement constant, des modèles élastiques non-linéaires non-conservatifs. Le traitement des systèmes continus élastiques non-conservatifs a fait l'objet de nombreuses études depuis les années 1960 (voir par exemple [5]). Ces études se sont principalement focalisées sur des systèmes dont l'aspect conservatif ou non-conservatif provient du chargement. Une autre source de non-conservativité peut être le comportement élastique non-linéaire dont le Hessien est non symétrique [6]. Dans cette note, nous supposons que le chargement est conservatif et nous cherchons à caractériser la loi de comportement d'un matériau élastique-endommageable au travers de son caractère conservatif. Une classe de lois qui englobe plusieurs modèles de la littérature s'exprime sous la forme (1). La contrainte $\underset{\sim}{\sigma}$ est une fonction non-linéaire de $\underset{\sim}{\varepsilon}$, la non-linéarité étant liée à la présence du terme de déformation positive. $\Delta\underset{\approx}{\mathbf{C}}$ est un tenseur du quatrième ordre qui correspond à la chute de rigidité due aux fissures ouvertes. Ce tenseur est supposé dépendre d'une variable interne d'endommagement $\underset{\sim}{\mathbf{D}}$, dont l'évolution n'est pas considérée ici, ramenant l'étude du matériau endommageable à celle d'un matériau élastique non-linéaire. $\underset{\sim}{\varepsilon}^+$ est le tenseur dit de « déformation positive », défini en (2). Une condition nécessaire et suffisante pour que le tenseur des contraintes dérive d'un potentiel est que le tenseur Hessien $\underset{\approx}{\mathbf{H}} = \frac{\partial \underset{\sim}{\sigma}}{\partial \underset{\sim}{\varepsilon}}$ soit symétrique. Le deuxième terme de la relation (1) étant seule source d'une potentielle non-conservativité, le Hessien lui correspondant se calcule sous la forme (3) dans laquelle apparaît le tenseur de dérivation $\frac{\partial \underset{\sim}{\varepsilon}^+}{\partial \underset{\sim}{\varepsilon}}$ dont on peut trouver une expression formelle dans [9] (voir (4)). En introduisant les six tenseurs propres $\underset{\sim}{\mathbf{E}}_i(\underset{\sim}{\varepsilon});\ i \in \{1, 6\}$, de la décomposition de Kelvin de $\frac{\partial \underset{\sim}{\varepsilon}^+}{\partial \underset{\sim}{\varepsilon}}$, la proposition (6) peut être énoncée : le tenseur Hessien est symétrique si et seulement si les tenseurs $\underset{\sim}{\mathbf{E}}_i(\underset{\sim}{\varepsilon});\ i \in \{1, 6\}$, sont des tenseurs propres de $\Delta\underset{\approx}{\mathbf{C}}$. Ainsi, la propriété de conservation de l'énergie impose une condition sur la structure du tenseur de chute de rigidité. Cette condition supplémentaire exprime simplement le fait que seuls les termes diagonaux de la relation contrainte–déformation (lorsqu'elle s'exprime sous la forme (1)) dans la base principale du tenseur de contrainte (ou de déformation) sont affectés par la refermeture de fissures : on reconnaît là la condition de continuité de la relation contrainte–déformation de [2], justifiée cette fois-ci du point de vue énergétique.

L'analyse de quelques modèles significatifs de la littérature est alors proposée, sur la base du critère défini en (6). Dans certains cas, le paradoxe de Green et Naghdi [6] peut être observé, conduisant à une dissipation infinie pour des cycles fermés de déformations. L'aspect non-conservatif de quelques modèles d'endommagement avec conditions unilatérales avait déjà été souligné par Carol et Willam [17], à partir d'exemples de chargement cyclique, notamment pour le modèle de Ju [7] ou pour des modèles similaires. Le critère de symétrie de la matrice Hessienne paraît plus général. La perte de symétrie de cette matrice pour beaucoup de modèles construits à partir d'une correction de la relation contrainte–déformation pour prendre en compte les effets unilatéraux montre sans ambiguïté que ces modèles ne peuvent pas découler d'un potentiel.

1. Introduction

Modelling microcracking phenomena by Continuum Damage Mechanics including both the induced anisotropy and the damage activation/deactivation effect (for example due to crack opening/closure) is still a challenging problem, despite more than twenty years of studies devoted to this subject ([1–4], among many others). The aim of this Note is to show that a significant class of elastic-damage models taking into account unilateral effects has to be considered, for a given damage configuration, as non-linear and non-conservative elastic.

2. Framework of the study

A conservative system is defined by the condition that the external work and the internal work of the system in any admissible displacement of the system depends solely on the initial and final configurations of the system. Special attention has been paid to the study of non-conservative elastic continuous systems since the 1960s (see, for instance, the reference book [5]). Most of these works have been devoted to non-conservative loading such as circulatory loading. Another origin of non-conservativeness is the nonlinear elastic behaviour with a non-symmetric Hessian tensor [6]. In this Note, the loading is assumed to be conservative and the conservative aspect of the constitutive law is investigated. A broad class of stress-strain relationships regarding some reference models (see, among others, [2,7], ...) may be expressed as:

$$\underset{\sim}{\sigma} = \underset{\approx}{\mathbf{C}}^0 : \underset{\sim}{\varepsilon} + \Delta\underset{\approx}{\mathbf{C}}(\underset{\sim}{\mathbf{D}}) : \underset{\sim}{\varepsilon}^+(\underset{\sim}{\varepsilon}) \tag{1}$$

where $\underset{\sim}{\varepsilon}$ and $\underset{\sim}{\sigma}$ are the strain and the stress tensor, respectively. The positive part $\underset{\sim}{\varepsilon}^+$ of $\underset{\sim}{\varepsilon}$ is defined by:

$$\underset{\sim}{\varepsilon}^+(\underset{\sim}{\varepsilon}) = \sum_{i\in\{I,II,III\}} \varepsilon_i(\underset{\sim}{\varepsilon}) h\big[\varepsilon_i(\underset{\sim}{\varepsilon})\big]\vec{e}_i(\underset{\sim}{\varepsilon}) \otimes \vec{e}_i(\underset{\sim}{\varepsilon}) \tag{2}$$

with $h(x) = 1$ if $x > 0$ and $h(x) = 0$ if $x \leqslant 0$. $h(x)$ is the Heaviside function. ε_i is the ith eigenvalue of $\underset{\sim}{\varepsilon}$ and $\vec{e}_i$ is the normalized associated eigenvector. It can be remarked that, unlike the work by Carol and Willam [17], the introduction of any non-linear projection operator (sometimes called positive projection operator) is not useful in the following. $\underset{\approx}{\mathbf{C}}^0$ is the fourth-order elasticity tensor of the undamaged material. $\Delta\underset{\approx}{\mathbf{C}}$ is a fourth-order tensor which corresponds to the stiffness reduction associated to active damage (e.g., open microcracks). When the material is subjected to predominant compressive strain, the initial stiffness $\underset{\approx}{\mathbf{C}}^0$ may be restored. The tensor $\Delta\underset{\approx}{\mathbf{C}}$ is assumed to depend on an internal symmetric second-order tensorial variable $\underset{\sim}{\mathbf{D}}$, whose evolution is disregarded in the present study (a given damage state $\underset{\sim}{\mathbf{D}}$ is being considered), thus converting the inelastic law (when damage growth is considered) into a non-linear elastic one. It can be outlined that the conclusions below may also be addressed to dual formulations involving positive stress $\underset{\sim}{\sigma}^+$, i.e., $\underset{\sim}{\varepsilon} = \underset{\approx}{\mathbf{S}}^0 : \underset{\sim}{\sigma} + \Delta\underset{\approx}{\mathbf{S}}(\underset{\sim}{\mathbf{D}}) : \underset{\sim}{\sigma}^+(\underset{\sim}{\sigma})$, where $\underset{\approx}{\mathbf{S}}^0$ and $\Delta\underset{\approx}{\mathbf{S}}$ are the compliance of the undamaged material and the compliance modification due to the presence of microcracks respectively. Such expressions may be found, for example, in [8].

3. Stiffness degradation and symmetry condition of the Hessian

The stress tensor (1) derives from a potential if and only if the Hessian tensor $\underset{\approx}{\mathbf{H}} = \frac{\partial \underset{\sim}{\sigma}}{\partial \underset{\sim}{\varepsilon}}$ possesses the major symmetry ($H_{ijkl} = H_{klij}$). As only the second term of the relation (1) is potentially associated to the non-conservativeness, the Hessian tensor $\Delta\underset{\approx}{\mathbf{H}}$ corresponding to this term is calculated as:

$$\Delta\underset{\approx}{\mathbf{H}} = \frac{\partial}{\partial \underset{\sim}{\varepsilon}}\big[\Delta\underset{\approx}{\mathbf{C}}(\underset{\sim}{\mathbf{D}}) : \underset{\sim}{\varepsilon}^+(\underset{\sim}{\varepsilon})\big] = \Delta\underset{\approx}{\mathbf{C}}(\underset{\sim}{\mathbf{D}}) : \frac{\partial \underset{\sim}{\varepsilon}^+(\underset{\sim}{\varepsilon})}{\partial \underset{\sim}{\varepsilon}} \tag{3}$$

An expression for the tensor $\frac{\partial \underset{\sim}{\varepsilon}^+}{\partial \underset{\sim}{\varepsilon}}$ has been formally given by Ekh et al. [9] by using a smooth function instead of the discontinuous Heaviside function h. The tensor $\frac{\partial \underset{\sim}{\varepsilon}^+}{\partial \underset{\sim}{\varepsilon}}$ can be simplified in case of non-vanishing principal strains as follows:

$$\frac{\partial \underset{\sim}{\varepsilon}^+}{\partial \underset{\sim}{\varepsilon}} = \sum_{i\in\{I,II,III\}} h\big[\varepsilon_i(\underset{\sim}{\varepsilon})\big]\vec{e}_i(\underset{\sim}{\varepsilon}) \otimes \vec{e}_i(\underset{\sim}{\varepsilon}) \otimes \vec{e}_i(\underset{\sim}{\varepsilon}) \otimes \vec{e}_i(\underset{\sim}{\varepsilon}) + \sum_{i\in\{I,II,III\}} \varepsilon_i(\underset{\sim}{\varepsilon}) h\big[\varepsilon_i(\underset{\sim}{\varepsilon})\big]\frac{\partial[\vec{e}_i(\underset{\sim}{\varepsilon}) \otimes \vec{e}_i(\underset{\sim}{\varepsilon})]}{\partial \underset{\sim}{\varepsilon}} \tag{4}$$

Such an expression can be generalized to the specific case of vanishing principal strains, by using the concept of generalized gradient (see for instance [10]). The term $\frac{\partial[\vec{e}_i(\underset{\sim}{\varepsilon})\otimes\vec{e}_i(\underset{\sim}{\varepsilon})]}{\partial \underset{\sim}{\varepsilon}}$ in (4) can be found explicitly in [11]. As a consequence, the fourth-order derivation tensor $\frac{\partial \underset{\sim}{\varepsilon}^+}{\partial \underset{\sim}{\varepsilon}}$ can be decomposed into the six Kelvin's eigentensors (see for instance [12]) $\underset{\sim}{\mathbf{E}}_i(\underset{\sim}{\varepsilon})$; $i \in \{1, 6\}$ expressed by:

$$\vec{e}_1 \otimes \vec{e}_1,\ \vec{e}_2 \otimes \vec{e}_2,\ \vec{e}_3 \otimes \vec{e}_3,\ \frac{1}{\sqrt{2}}\big(\vec{e}_1 \otimes \vec{e}_2 + \vec{e}_2 \otimes \vec{e}_1\big),\ \frac{1}{\sqrt{2}}\big(\vec{e}_1 \otimes \vec{e}_3 + \vec{e}_3 \otimes \vec{e}_1\big),\ \frac{1}{\sqrt{2}}\big(\vec{e}_2 \otimes \vec{e}_3 + \vec{e}_3 \otimes \vec{e}_2\big) \tag{5}$$

The following proposition can then be shown:

$$\underset{\sim}{\mathbf{E}}_i(\underset{\sim}{\varepsilon});\ i \in \{1, 6\}, \text{eigentensors of } \Delta\underset{\approx}{\mathbf{C}} \quad \Leftrightarrow \quad \underset{\approx}{\mathbf{H}} \text{ symmetric} \tag{6}$$

It is easy to prove (6) from the fundamental properties of the product of two symmetric tensors. Thus, the energy conservativeness leads to a restriction on the structure of the reduced stiffness tensor. Condition (6) simply expresses the fact that only the diagonal terms of the stress-strain relation (1) in the principal basis of the strain tensor are affected by the closure of the pre-existing cracks, which is related to the postulate of Chaboche [2]. This paper provides arguments from an energetic viewpoint. When the condition (6) is not verified, the paradox of Green and Naghdi [6] can be observed, leading to an infinite dissipation for closed strain cycles.

4. Some particular damage models

This section aims at reviewing some damage models regarding the criterion (6) relative to existence of the elastic potential energy function.

4.1. Isotropic stiffness loss

One of the simplest models consists in considering a stiffness loss tensor proportional to the unit tensor:

$$\Delta C_{ijkl} = \frac{\alpha}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \quad \Rightarrow \quad \underset{\sim}{\sigma} = \underset{\approx}{\mathbf{C}}^0 : \underset{\sim}{\varepsilon} + \alpha \underset{\sim}{\varepsilon}^+ \tag{7}$$

Condition (6) is verified whatever $\underset{\sim}{\varepsilon}^+$ is, and $\underset{\sim}{\sigma}$ derives without any ambiguity from a potential (the model is then hyperelastic). The term of the energy potential corresponding to $\underset{\sim}{\varepsilon}^+$ can be explicitly given, as in Challamel et al. [13]:

$$\frac{\partial[\frac{1}{2}\operatorname{tr}(\underset{\sim}{\varepsilon}^+ . \underset{\sim}{\varepsilon}^+)]}{\partial \underset{\sim}{\varepsilon}} = \underset{\sim}{\varepsilon}^+ \tag{8}$$

4.2. Explicit tensorial damage models

Another model which falls within the framework (1) is based on:

$$\Delta C_{ijkl} = \frac{\alpha}{2}(D_{ik}\delta_{jl} + D_{il}\delta_{jk} + D_{jl}\delta_{ik} + D_{jk}\delta_{il}) \quad \Rightarrow \quad \underset{\sim}{\sigma} = \underset{\approx}{\mathbf{C}}^0 : \underset{\sim}{\varepsilon} + \alpha\left(\underset{\sim}{\varepsilon}^+ . \underset{\sim}{\mathbf{D}} + \underset{\sim}{\mathbf{D}} . \underset{\sim}{\varepsilon}^+\right) \tag{9}$$

where $\underset{\sim}{\mathbf{D}}$ stands for the damage variable. Eq. (9) is a particular version of the model by Halm and Dragon [3] (in this case, $\underset{\sim}{\varepsilon}^+$ is replaced by $\underset{\sim}{\varepsilon}^-$, built from the negative eigenvalues of $\underset{\sim}{\varepsilon}$), Murakami and Kamiya [14] or that by Challamel et al. [13] when the principal directions of the damage tensor coincide with the ones of the strain tensor. In this case, the condition (6) is verified, and the relation (9) derives from a potential (non-linear conservative elastic model). Nevertheless, when the principal directions of $\underset{\sim}{\mathbf{D}}$ and $\underset{\sim}{\varepsilon}$ do not coincide, the Hessian tensor corresponding to (9) is no longer symmetric and the model is not conservative.

4.3. Quadratic damage model

An analogous remark can be formulated for the following relation:

$$\Delta C_{ijkl} = \frac{\alpha}{2}(A_{ik}A_{jl} + A_{il}A_{jk}) \quad \Rightarrow \quad \underset{\sim}{\sigma} = \underset{\approx}{\mathbf{C}}^0 : \underset{\sim}{\varepsilon} + \alpha \underset{\sim}{\mathbf{A}} . \underset{\sim}{\varepsilon}^+ . \underset{\sim}{\mathbf{A}} \tag{10}$$

When the principal directions of the $\underset{\sim}{\mathbf{A}}$ tensor coincide with the ones of the strain tensor, the Hessian tensor is symmetric. In this case again, the relation (10) derives from a potential. Nevertheless, as in the previous case, when the principal directions of both tensors do not coincide, the model is no more conservative. Note that the second term of (10) is a variant of the model considered by Lemaitre et al. [15], when $\underset{\sim}{\mathbf{A}} = (\underset{\sim}{\mathbf{1}} - \underset{\sim}{\mathbf{D}})^{-1/2}$: in a dual formulation (permutation of the stress and strain tensors) and when the principal directions of $\underset{\sim}{\mathbf{D}}$ and $\underset{\sim}{\sigma}$ coincide, the definition of $\underset{\sim}{\sigma}^+$ used by Lemaitre et al. [15] and Eq. (2) applied to stress lead to the same expression.

5. Conclusions

This Note shows that several anisotropic elastic-damage models containing explicitly the entity $\underset{\sim}{\varepsilon}^+$ in the stress-strain relationship to deal with unilateral effects can be considered, for a given damage state, as non-linear and non-conservative elastic. The conservative aspect of a general class of constitutive models is checked by means of the symmetry of the Hessian tensor. The conservativeness criterion leads to the stress-strain continuity condition introduced by Chaboche [2]. When the principal directions of the strain and the damage tensors coincide, certain models considered in this note derive from a potential.

Nevertheless, the loss of symmetry of the Hessian tensor in the general case (when the principal directions of the strain and the damage tensors do not coincide) indicates that no thermodynamic potential exists. The corresponding models belong to the class of non-linear elastic materials, leading to the resolution of a non-conservative problem which is not self-adjoint [16]. The paradox of Green and Naghdi [6] may be observed and infinite dissipation may be induced for closed strain cycles. The non-conservativeness of some damage models including unilateral behavior (as, for example, for the model of Ju [7]) has been previously suggested by Carol and Willam [17], who gave specific examples of cyclic loading.

The Hessian tensor viewpoint confirms this spurious effect for the set of models considered. A simple way to avoid this consists in conceiving a conservative model, based on the explicit expression of the thermodynamic potential from the state variables. This note also illustrates the persistent difficulty to model rigorously anisotropic damage allowing for unilateral effect through the formalism (1) along complex non proportional loading paths, since the highly restrictive condition (6) has to be satisfied.

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