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Optimal control for a Timoshenko beam

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Abstract

We consider a linear model of a rotating Timoshenko beam, which is clamped at one end to a disk the other being free. The motion of the beam is controlled by the angular acceleration of the disk. We study the minimization problem of mean square deviation of the Timoshenko beam from a given position. For the minimization problem of the first mode we prove that optimal control is the chattering control, i.e., it has an infinite number of switches in a finite time interval. We construct a suboptimal control with a finite number of switches. ***To cite this article: M.I. Zelikin, L.A. Manita, C. R. Mecanique 334 (2006).***

Résumé

Contrôle optimal d'une barre de Timoshenko. On considére une barre de Timoshenko en rotation, dont une extrêmité est reliée à un disque et l'autre est libre. Son mouvement est controlé par l'accéleration angulaire du disque. Nous étudions le problème de la minimisation de la moyenne quadratique de la deviation. Pour le problème de la minimisation du premier mode, nous démontrons qu'un contrôle optimal a une infinité de points de discontinuité en temps fini. Nous proposons une procédure pour construire d'un contrôle sous-optimale qui a un nombre fini de points de discontinuité. ***Pour citer cet article : M.I. Zelikin, L.A. Manita, C. R. Mecanique 334 (2006).***

Keywords: Computational solid mechanics; Timoshenko beam; Singular regimes; Chattering solutions

Mots-clés : Mécanique des solides numérique ; Barre de Timoshenko ; Règimes singuliers ; Solutions avec un nombre infini de points de discontinuité

1. Introduction

We consider a Timoshenko beam, which is clamped at one end to a disk and free at the other [1]. We suppose that the beam motion is controlled by the angular acceleration of the disk. The equations of the Timoshenko beam can be transformed into a system of partial differential equations of second order with a single real parameter $\gamma > 0$ [2]

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$$\frac{\partial^2 w(x,t)}{\partial t^2} - \frac{1}{\gamma}\frac{\partial^2 w(x,t)}{\partial x^2} + \frac{1}{\gamma}\frac{\partial \xi(x,t)}{\partial x} = -\ddot{\theta}(t)(r+x)$$
$$\frac{\partial^2 \xi(x,t)}{\partial t^2} - \frac{\partial^2 \xi(x,t)}{\partial x^2} + \frac{1}{\gamma}\xi(x,t) - \frac{1}{\gamma}\frac{\partial w(x,t)}{\partial x} = -\ddot{\theta}(t) \tag{1}$$

Here the X-axis coincides with the beam when at rest, $w(x,t)$ is the displacement of the center line of the beam in the direction perpendicular to the X-axis and $\xi(x,t)$ is the rotation angle of the cross-section area at the location $x \in [0,l]$ and time t; $\theta(t)$ is the rotation angle of the disk, r is the radius of the disk, l is the beam length. In the following, we assume that $l=1$, $\gamma=1$. The boundary conditions are

$$w(0,t) = \xi(0,t) = 0, \quad w_x(l,t) - \xi(l,t) = 0, \quad \xi_x(l,t) = 0 \tag{2}$$

The initial state of the beam is

$$w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \quad \xi(x,0) = \xi_0(x), \quad \xi_t(x,0) = \xi_1(x), \quad x \in [0,l] \tag{3}$$

The control problems of slowly rotating beams were studied in [2–4]. The controllability of the beam from a position of rest into a position of rest under a given angle within a given time T (for sufficiently large T) was proved in [3], with a method of construction of a piecewise constant control solving the problem. The control allowing to stabilize the system (the beam plus the disk) in a position of rest was constructed in [4]. It was shown that there exists at most a countable set of singular values of the disk radius such that the system is not controllable. For non-singular r a control solving the problem of controllability was constructed. In this Note we consider an optimal control problem for a rotating uniform Timoshenko beam. We prove that optimal trajectories for the minimization problem of the first mode of the beam have an infinite number of control switches in a finite time interval. We construct a suboptimal control with a finite number of switches.

2. The optimal control problem of the Timoshenko beam

Consider the minimization problem of mean square deviation of the Timoshenko beam:

$$\int_0^\infty \left\| \begin{pmatrix} w(x,t) \\ \xi(x,t) \end{pmatrix} \right\|^2_{L_2([0,1],\mathbb{R}^2)} \mathrm{d}t \to \inf \tag{4}$$

Let $H = L^2([0,1],\mathbb{C}^2)$. Put

$$D = \left\{ \begin{pmatrix} y \\ z \end{pmatrix} \in H \;\middle|\; y(0) = z(0) = 0,\ y'(1) - z(1) = 0,\ z'(1) = 0 \right\}.$$

Define a linear operator $A : D \to H$ by $A\binom{y}{z} = \binom{-y''+z'}{-y'-z''+z}$. Let

$$\boldsymbol{\omega}(x,t) = \begin{pmatrix} w(x,t) \\ \xi(x,t) \end{pmatrix}, \qquad \boldsymbol{g}(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix} = \begin{pmatrix} -(r+x) \\ -1 \end{pmatrix}$$

Then the problem (1), (4) takes on the following form: minimize the functional

$$\int_0^\infty \|\boldsymbol{\omega}(x,t)\|^2_H \,\mathrm{d}t \tag{5}$$

subject to the control system

$$\frac{\partial^2}{\partial t^2}\boldsymbol{\omega}(\cdot,t) + A\boldsymbol{\omega}(\cdot,t) = u(t)\boldsymbol{g}(\cdot), \quad t > 0 \tag{6}$$

Here $u(t) = \ddot{\theta}(t)$ is a scalar control. We assume that $-1 \leqslant u \leqslant 1$.

The properties of the spectrum of A were studied in [2,3]. It was proved that the operator A is self-adjoint in H and positive. A has an orthonormal sequence of eigenfunctions $\boldsymbol{h}_j(x) = \binom{y_j(x)}{z_j(x)}$ ($j \in \mathbb{N}$) and a corresponding sequence of eigenvalues $\{\lambda_j \in \mathbb{R}\}$ such that $1 < \lambda_j \uparrow \infty$ as $j \to \infty$. The eigenvalues λ_j solve the equation [5]

$$1 + \cos\sqrt{\lambda - \sqrt{\lambda}}\cos\sqrt{\lambda + \sqrt{\lambda}} - \frac{\lambda}{\sqrt{(\lambda-1)\lambda}}\sin\sqrt{\lambda - \sqrt{\lambda}}\sin\sqrt{\lambda + \sqrt{\lambda}} = 0$$

It was proved that for any sufficiently large k there exist two close eigenvalues λ_k^- and λ_k^+ of the operator A such that $\lambda_k^- < (\pi(2k-1)/2)^2 < \lambda_k^+$ and $\lambda_k^- - \lambda_k^+ \to 2\cos(1/2)$ as $k \to \infty$.

Since the eigenfunctions $(\boldsymbol{h}_j(x))_{j=1}^{\infty}$ of the operator A form complete orthonormal system, let us expand the solution of Eq. (6) and the right-hand side of this equation as follows:

$$\boldsymbol{\omega}(x,t) = \sum_{j=1}^{\infty} s_j(t)\boldsymbol{h}_j(x), \qquad \boldsymbol{g}(x) = \sum_{i=1}^{\infty} C_j\boldsymbol{h}_j(x) \tag{7}$$

where $s_j(t) = \langle\boldsymbol{\omega}(x,t), \boldsymbol{h}_j(x)\rangle_H$, $C_j = \langle\boldsymbol{g}(x), \boldsymbol{h}_j(x)\rangle_H$. If $C_j = 0$ for some j, then the value r is called singular. It was proved [4] that there exists, at most, a countable set of singular values of the disk radius. In the sequel we assume that r is non-singular, i.e. $C_j \neq 0\ \forall j \in \mathbb{N}$. On substituting (7) into (6) we get $\sum_{j=1}^{\infty}[\ddot{s}_j(t) + \lambda_j s_j(t) - C_j u(t)]\boldsymbol{h}_j(x) = 0$. Hence we have a countable system of ordinary differential equations $\ddot{s}_j(t) + \lambda_j s_j(t) = C_j u(t)$, $j \in \mathbb{N}$. Using the Parseval identity, we see that the functional (5) can be written in the form $\int_0^{\infty}\sum_{j=1}^{\infty} s_j^2(t)\,\mathrm{d}t$. By expanding the initial values (3)

$$\begin{pmatrix} w_0(x) \\ \xi_0(x) \end{pmatrix} = \sum_{j=1}^{\infty} \tau_j^0 \boldsymbol{h}_j(x), \qquad \begin{pmatrix} w_1(x) \\ \xi_1(x) \end{pmatrix} = \sum_{i=1}^{\infty} \tau_j \boldsymbol{h}_j(x)$$

we obtain the initial conditions for the functions $s_j(t)$: $s_j(0) = \tau_j^0$, $s_j'(0) = \tau_j$. Thus, we reduce the problem (1)–(4) to the following problem:

$$\int_0^{\infty}\sum_{j=1}^{\infty} s_j^2(t)\,\mathrm{d}t \to \min, \quad \ddot{s}_j(t) + \lambda_j s_j(t) = C_j u(t), \quad s_j(0) = \tau_j^0, \quad s_j'(0) = \tau_j \quad (j \in \mathbb{N})$$

Since usually the main part of vibration energy is concentrated in the first mode of the beam, in the following section we consider the minimization problem of the first mode of the beam. We prove that the optimal trajectories in this problem are the so-called *chattering trajectories*.

3. Singular regimes and chattering trajectories

Consider the optimal control problem

$$\int_0^T \bigl(\varphi_0(x) + u\varphi_1(x)\bigr)\,\mathrm{d}t \to \min, \quad \dot{x} = f_0(x) + uf_1(x)$$

with $x(0) \in B_0 \subset \mathbb{R}^n$, $x(T) \in B_T \subset \mathbb{R}^n$. Here x is a state variable, u is a scalar control, $|u| \leqslant 1$. $\varphi_i : \mathbb{R}^n \to \mathbb{R}$, $f_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 0, 1$, the functions φ_i, f_i are smooth enough, B_0, B_T are smooth manifolds. The admissible controls $u(t)$ need to be measurable, the admissible trajectories $x(t)$ are assumed to be absolutely continuous. Let us consider Pontryagin's maximum principle with the Hamiltonian $H = H_0(x,\psi) + uH_1(x,\psi)$, where $H_0(x,\psi) = f_0(x)\psi - \frac{1}{2}\varphi_0(x)$, $H_1(x,\psi) = f_1(x)\psi - \frac{1}{2}\varphi_1(x)$. We have the following Hamiltonian system:

$$\dot{x} = \frac{\partial H}{\partial \psi}, \qquad \dot{\psi} = -\frac{\partial H}{\partial x} \tag{8}$$

The maximum condition yields: $u = +1$ for $H_1 > 0$, $u = -1$ for $H_1 < 0$. The surface $H_1 = 0$ is a discontinuity surface of the Hamiltonian system. An extremal $(x(t), \psi(t))$, $t \in (t_0, t_1)$ is called singular if $H_1(x(t), \psi(t)) = 0$ for $t \in (t_0, t_1)$. This means that singular extremals lie on the zero-level surface of the function H_1. To find the value of optimal control on a singular trajectory one has to differentiate the identity $H_1(x(t), \psi(t)) = 0$ along a solution of the system (8) with respect to t. It is known that the non-zero coefficient of u can arise for the first time only at even step of differentiation $2q$. The number q is called the order of the singular trajectory. More precisely, we say that a number q is the order of a singular trajectory (intrinsic order) iff

$$ad_{H_1} ad^i_{H_0} H_1 = 0, \quad i = 0, \dots, 2q-2, \quad ad_{H_1} ad^{2q-1}_{H_0} H_1 \neq 0 \tag{9}$$

in some open neighborhood of the singular trajectory $(x(t), \psi(t))$. If conditions (9) are valid only at points of the trajectory $(x(t), \psi(t)), t \in (t_0, t_1)$ we say that q is a local order of singular trajectory. The necessary condition for optimality of singular trajectory is the following Kelley's condition:

$$(-1)^q \frac{\partial}{\partial u} \frac{\mathrm{d}^{2q} H_1}{\mathrm{d}t^{2q}} \leqslant 0$$

It is known that the conjugation of a piecewise smooth trajectory with a singular arc of even order is non-optimal. Therefore, for optimal trajectories a singular arc is joined with a chattering trajectory. A chattering trajectory is a trajectory with infinite number of control switchings in a finite time interval. In [6] a complete theory of chattering trajectories of the second order was constructed.

4. Problem of controlling an oscillator

Consider the minimization problem for the first mode of the Timoshenko beam

$$\int_0^\infty s^2(t)\,\mathrm{d}t \to \min \tag{10}$$

$$\ddot{s}(t) + \lambda s(t) = Cu(t), \quad s(0) = \tau_1^0, \quad \dot{s}(0) = \tau_1 \tag{11}$$

Here s is a state variable, u is a scalar control, $-1 \leqslant u \leqslant 1$.

Theorem 4.1. *The origin $(s, \dot{s}) = (0, 0)$ is a singular trajectory of second order. There exists a neighbourhood U_ε of the origin such that the following statements hold*:

(a) *For any $(\tau_1^0, \tau_1) \in U_\varepsilon$ there exists an optimal trajectory $(s^*(t), \dot{s}^*(t))$, $(s^*(0) = \tau_1^0, \dot{s}^*(0) = \tau_1)$. The trajectory $(s^*(t), \dot{s}^*(t))$ reaches the origin in a finite time with an infinite number of control switchings.*
(b) *In U_ε the optimal switching curve has the form*

$$\Gamma = \begin{cases} s = \mu_1(\dot{s})\dot{s}^2, & \dot{s} > 0 \\ s = \mu_2(\dot{s})\dot{s}^2, & \dot{s} < 0 \end{cases}$$

where $\mu_i(\dot{s}) \in C^1$, $\mu_1(0) \in (-1/2C, 0)$, $\mu_2(0) \in (0, 1/2C)$. The optimal feedback control $\tilde{u}(s, \dot{s}) = -C$ on the right-hand side of the curve Γ, and $\tilde{u}(s, \dot{s}) = C$ on its left-hand side (*see Fig.* 1).

Proof. We show that the origin is a singular trajectory of order 2. We write the Pontryagin function and use Pontryagin's maximum principle:

$$H = \varphi v + \psi(-\lambda s + Cu) - \frac{1}{2}s^2 = H_0 + uH_1$$

where

$$H_0 = \varphi v - \lambda \psi s - \frac{1}{2}s^2, \quad H_1 = C\psi \quad \text{and} \quad u_{\text{opt}} = \begin{cases} -1, & H_1 > 0 \\ 1, & H_1 < 0 \end{cases}$$

The system of adjoint equations has the form $\dot{\varphi} = -\frac{\partial H}{\partial s} = s + \lambda\psi$, $\dot{\psi} = -\frac{\partial H}{\partial v} = -\varphi$. Let find singular trajectories, i.e., trajectories on which the coefficient of control u in the Pontryagin function equals zero. We have:

$$H_1 = \psi = 0 \Rightarrow \varphi = 0 \Rightarrow s = 0 \Rightarrow v = 0 \Rightarrow u = 0.$$

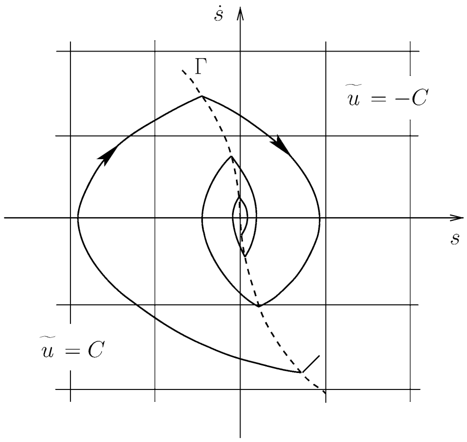

Fig. 1. Optimal feedback control.

Thus, the origin ($s = 0, v = 0, \varphi = 0, \psi = 0$) is a singular trajectory and the singular control equals zero. Let us calculate the order of the singular trajectory:

$$\frac{\mathrm{d}H_1}{\mathrm{d}t} = -C\varphi, \quad \frac{\mathrm{d}^2 H_1}{\mathrm{d}t^2} = -Cs - C\lambda\psi, \quad \frac{\mathrm{d}^3 H_1}{\mathrm{d}t^3} = -Cv + C\lambda\varphi, \quad \frac{\mathrm{d}^4 H_1}{\mathrm{d}t^4} = 2C\lambda s + C\lambda^2\psi - C^2 u$$

Thus, a control term appears for the first time at the 4th differentiation. Hence, the origin is a singular trajectory of order 2. Kelley's condition for optimality holds in the strict form

$$(-1)^2 \frac{\partial}{\partial u} \frac{\mathrm{d}^4}{\mathrm{d}t^4} H_1 = -C^2 < 0$$

Using the results [6,7], we have for the problem (10), (11) the statements (2a) and (2b). Theorem 4.1 is proved. □

5. Construction of a suboptimal solution for control the harmonic oscillator with quadratic cost functional

Optimal trajectories for the control problem of an oscillator have an infinite number of control switchings in a finite time interval. In practice, we need to construct suboptimal trajectories with a finite number of control switchings that approximate the optimal chattering solutions. In order to construct the suboptimal solution we consider two auxiliary problems.

5.1. Time-optimal problem

$$T \to \min, \quad \ddot{s}(t) = u(t), \quad s(0) = \tau_1^0, \quad \dot{s}(0) = \tau_1, \quad s(T) = \dot{s}(T) = 0 \tag{12}$$

A scalar control u is bounded $-\alpha \leqslant u \leqslant \beta$, $\alpha > 0$, $\beta > 0$. It was shown [8] that the optimal control of this problem is the bang-bang control and that it has, at most, one switch.

5.2. Time-optimal problem for oscillator

$$T \to \min, \quad \ddot{s}(t) + \lambda s(t) = u(t), \quad s(0) = \tau_1^0, \quad \dot{s}(0) = \tau_1, \quad s(T) = \dot{s}(T) = 0 \tag{13}$$

A scalar control u is bounded $-\alpha \leqslant u \leqslant \beta$, $\alpha > 0$, $\beta > 0$. The optimal control has a finite number of switchings, and the number of switchings depends on an initial point. The further an initial point is removed from the origin, the larger will be the number of control switchings [8].

Now we can construct the suboptimal solution for (10), (11). Let $(s^*(t), \dot{s}^*(t))$ be an optimal trajectory for the problem (10), (11) such that $(s^*(0), \dot{s}^*(0)) = S_0$. Consider a part of this trajectory that contains exactly N points of control switches. Suppose S_N is the end point of this arc. Let us consider the optimal trajectory of the problem (12) or of the problem (13) with the initial condition S_N. By $\hat{J}_N(S_0)$ we denote the value of the functional (10) on this trajectory. Let $J^*(S_0)$ be the optimal value of the functional of the problem (10), (11). Then we have the following estimation of the loss in the functional for the suboptimal regimes:

Theorem 5.1. [9]. *There exists a neighborhood O_ε of the origin and two constants $M > 0$, $\nu > 0$ such that for any point $S_0 \in O_\varepsilon$ the following estimation holds*:

$$\widehat{J}_N(S_0) - J^*(S_0) < M\,\mathrm{e}^{-\nu N}$$

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