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Analytic calculation of the stresses in an ensiled granular medium

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Abstract

We present in this Note a new analytic approach, of continuous medium type, which improves the Janssen theory and enables us to calculate the stresses in an ensiled granular medium. This approach is based on the two dimensional equilibrium equations, coupled with the Mohr–Coulomb criterion and a slip condition at the walls of the silo. An analytic resolution is developed to compute the stresses for cohesive and non cohesive materials in the whole silo. ***To cite this article: O. Millet et al., C. R. Mecanique 334 (2006).***

Résumé

Calcul analytique des contraintes dans un matériau granulaire ensilé. On présente dans cette Note une nouvelle approche analytique, de type milieu continu, qui améliore la théorie de Janssen et permet de calculer les contraintes dans un milieu granulaire ensilé. Cette approche est basée sur les équations d'équilibre bidimensionnelles couplées à un critère de Mohr–Coulomb et à une condition de glissement aux parois. Une résolution analytique est développée pour calculer les contraintes pour les matériaux cohésifs et non cohésifs en tout point du silo. ***Pour citer cet article : O. Millet et al., C. R. Mecanique 334 (2006).***

Keywords: Granular media; Continuum mechanics; Janssen theory; Mohr–Coulomb criterion

Mots-clés : Milieux granulaires ; Milieux continus ; Théorie de Janssen ; Critère de Mohr–Coulomb

Version française abrégée

La théorie de Janssen [1] est une des plus anciennes approches permettant de calculer les contraintes dans un matériau granulaire ensilé. Basée sur le principe fondamental de la statique, elle repose cependant sur des hypothèses contradictoires : les contraintes σ_{XX} et σ_{ZZ} sont supposées principales alors que la contrainte de cisaillement σ_{XZ} est non nulle. D'autre part, elle nécessite a priori de savoir si le matériau est dans un état actif ou passif qui est supposé être le même dans tout le silo. Enfin, la théorie de Janssen est limitée aux matériaux non cohésifs.

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Nous proposons une approche de type milieu continu qui améliore les approches existantes et en particulier la théorie de Janssen. Cette approche permet de calculer les contraintes en tout point du milieu granulaire ensilé, et ceci quelques soient les paramètres physiques intervenant. Elle est basée sur une résolution des équations d'équilibre bidimensionnelles des milieux continus couplées à un critère de Mohr–Coulomb et à une condition de glissement aux parois. Elle permet de prendre en compte la variation des contraintes par rapport au diamètre du silo ainsi que la cohésion. Cette approche permet également de déterminer l'état du matériau granulaire (actif ou passif) en fonction des données du problème (angle de frottement interne, angle de frottement aux parois, cohésion, dimensions du silo). Nous limiterons ici notre analyse à un modèle linéaire en cisaillement et pour des angles de frottement interne ϕ et aux parois ϕ_p égaux.

Dans le cas des matériaux non cohésifs, des comparaisons avec les solutions analytiques existantes permettent de valider l'approche proposée. D'autre part, on constate qu'en l'absence de cohésion, le matériau granulaire semble toujours être dans un état actif. De plus, les contraintes aux parois sont bornées par les états actifs et passifs de la théorie de Janssen (Fig. 2). En particulier la contrainte aux parois σ_x^0 est supérieure à celle prédite par la théorie de Janssen dans le cas actif. Or, il est connu que dans ce cas, la théorie de Janssen sous-estime les contraintes horizontales aux parois.

Dans le cas des matériaux cohésifs, il apparaît un nouvel état mixte de contraintes à l'intérieur du silo : on passe d'un état passif à un état actif lorsque la profondeur augmente. De plus, lorsque la cohésion augmente, les autres paramètres étant fixés, le matériau devient passif dans tout le silo (voir Fig. 3). D'autre part, toujours pour les matériaux cohésifs, on montre que le rapport des contraintes aux parois $\lambda_p = \sigma_x^0/\sigma_z^0$ n'est plus constant (Fig. 4(b)). Il dépend, en plus de l'angle de frottement interne, de la cohésion et de l'angle α^0, solution de l'équation différentielle (8).

1. Introduction

The Janssen theory [1] is historically one of the oldest approaches which enables one to compute the stresses inside an ensiled granular material. This approach, based on the principle of the statics, is, however, grounded on contradictory assumptions: the stresses σ_{XX} and σ_{ZZ} are assumed to be the main stresses, whereas the shear stress σ_{XZ} is different from zero. On the other hand, Janssen theory, as most of the other existing theories in soil mechanics, needs to know a priori if the material is in an active or passive state, which is supposed to be the same in all the silo. Other limitations of the Janssen theory have been underlined in [2,3].

The aim of this Note is to improve the Janssen theory and other existing ones [4], by developing a new analytic approach of the continuum mechanics type, which enables one to compute accurately the stresses inside an ensiled granular material. This approach is based on the resolution of the two dimensional equilibrium equations modelling the granular media, coupled with a Mohr–Coulomb criterion and a slip condition at the walls. It takes into account the variation of the stresses with respect to the diameter of the silo and the cohesion of the material. We limit here our analysis to a linear horizontal distribution of shear stress and to the event of equal internal and wall fiction angles.

2. The analytic approach proposed

In this approach, the granular material is modelled as a continuous medium, which verifies the two-dimensional equilibrium equations[1] in the plane $(\vec{x},\vec{z})$, where the horizontal and vertical directions $\vec{x}$ and $\vec{z}$ correspond to the silo axes, $\vec{z}$ being oriented towards the bottom. If we denote σ_x and σ_z instead of σ_{xx} and σ_{zz} to simplify the notations, we have:

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xz}}{\partial z} = 0 \tag{1}$$

$$\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\sigma_z}{\partial z} = \rho g \tag{2}$$

Moreover, we assume that the Mohr–Coulomb criterion is satisfied on the lateral walls of the silo. Then, we have $s^0 = (p^0 + H)\sin\phi$, where s^0 and p^0 denote respectively the radius and the center of Mohr circle at the wall and

[1] This corresponds to the analogical Schneebeli's model [5].

$H/\tan\phi$ the cohesion of the material.[2] Let us denote $\alpha = (\vec{x}, \vec{e_I})$ the angle between the horizontal axis $\vec{x}$ and the main stress direction $\vec{e_I}$. Then the stresses at the walls can be expressed as follows:

$$\sigma_x^0 = p^0 + s^0 \cos 2\alpha^0, \qquad \sigma_z^0 = p^0 - s^0 \cos 2\alpha^0, \qquad \tau_{xz}^0 = s^0 \sin 2\alpha^0 \tag{3}$$

On the other hand, we assume that along the walls of the silo, the matter is in a slip state[3] $\tau_{xz}^0 = \mu_p \sigma_x^0$, where $\mu_p = \tan\phi_p$ is the wall friction coefficient. Finally, we limit our analysis to the framework of a linear distribution of shear stress in the horizontal direction:

$$\tau_{xz}(x,z) = \frac{x}{x_0}\tau_{xz}^0(z) \tag{4}$$

This supplementary assumption is classical [4] and enables us to reduce the system of partial derivative equations to a system of differential equations, easier to solve. Using (4), Eq. (1) leads to:

$$\sigma_x = \sigma_x^0(z) + \frac{1}{2}\left(1 - \frac{x^2}{x_0^2}\right) x_0 \frac{\partial \tau_{xz}^0}{\partial z} \tag{5}$$

On the other hand, introducing the boundary condition of a free surface without load at $z = 0$, the equilibrium equation (2) leads to $\sigma_z = \sigma_z^0(z)$ which verifies the equilibrium equation of the Janssen theory:

$$\frac{\partial \sigma_z^0}{\partial z}(z) = \rho g - \frac{1}{x_0}\tau_{xz}^0(z) \tag{6}$$

In order to compute the wall stresses, we use the wall slip condition and the relations of Mohr circle written near the walls. We easily get the value of p^0:

$$p^0 = \frac{H \sin\phi(\mu_p \cos 2\alpha^0 - \sin 2\alpha^0)}{\sin\phi \sin 2\alpha^0 - \mu_p(1 + \sin\phi \cos 2\alpha^0)}$$

and the wall stresses expressions:

$$\begin{cases} \sigma_x^0 = -\dfrac{H \sin\phi \sin 2\alpha^0}{\sin\phi \sin 2\alpha^0 - \mu_p(1 + \sin\phi \cos 2\alpha^0)} \\ \sigma_z^0 = \dfrac{H \sin\phi(2\mu_p \cos 2\alpha^0 - \sin 2\alpha^0)}{\sin\phi \sin 2\alpha^0 - \mu_p(1 + \sin\phi \cos 2\alpha^0)} \\ \tau_{xz}^0 = -\dfrac{\mu_p H \sin\phi \sin 2\alpha^0}{\sin\phi \sin 2\alpha^0 - \mu_p(1 + \sin\phi \cos 2\alpha^0)} \end{cases} \tag{7}$$

where α^0 is the angle $(\vec{x}, \vec{e_I})$ near the walls. Then, introducing the previous expressions in Eq. (6), we obtain the following differential equation with respect to $\alpha^0(z)$:

$$\left(2\mu_p \sin 2\alpha^0 + \cos 2\alpha^0 - \sin\phi\right)\frac{\mathrm{d}\alpha^0}{\mathrm{d}z} = \frac{D}{2R_h}\left(D\frac{\xi}{2\mu_p \cos\phi} + \sin 2\alpha^0\right) \tag{8}$$

where $D = \sin\phi \sin 2\alpha^0 - \mu_p(1 + \sin\phi \cos 2\alpha^0)$.

In (8), a dimensionless number $\xi = (2\rho g R_h)/(\mu H)$ naturally appears. Its depends on the density ρ of the grains, on the gravity g, on the cohesion of the material, on the internal friction angle and on the hydraulic radius R_h of the silo. Two cases can occur. If the solution $\alpha^0(z)$ of Eq. (8) is such that:

- $\alpha^0 < \pi/4$, then we adopt the classical notations $\sigma_I = \sigma_X$ and $\sigma_{II} = \sigma_Z$, where the main direction $\vec{X}$ is those closer to $\vec{x}$. In this case we have $\sigma_X > \sigma_Z$ and $\sigma_x^0 > \sigma_z^0$ according to (3), where σ_X and σ_Z are the stresses in the main directions, respectively $\vec{X}$ and $\vec{Z}$. This corresponds to the passive state of the Janssen theory.
- $\alpha^0 > \pi/4$, then we adopt the notations $\sigma_I = \sigma_Z$ and $\sigma_{II} = \sigma_X$. In this case we have $\sigma_X < \sigma_Z$ and $\sigma_x^0 < \sigma_z^0$ according to (3). This corresponds to the active state of the Janssen theory.

[2] The superscript ° denotes the values of the physical parameters on the wall interface.

[3] In Janssen theory, the matter is assumed to be in a slip state in all the silo.

3. Resolution of the differential equation

In order to develop a general resolution for $\phi = \phi_p$, we make the following change of variables $u = \tan\alpha^0$ in (8). If we set $u_0 = (1 + \sin\phi)/\cos\phi$, we obtain:

$$D = \sin\phi \sin 2\alpha^0 - \mu_p\big(1 + \sin\phi\cos 2\alpha^0\big) = \frac{-\mu(1-\sin\phi)}{1+u^2}(u-u_0)^2$$

and the differential equation (8) becomes:

$$\tilde{k}\frac{(u-u_1)(u-u_2)}{(u-u_0)^2(u-u_5)(u-u_6)}\,\mathrm{d}u = \mathrm{d}z \tag{9}$$

with

$$u_5 = \frac{2u_0}{\xi}\left[1 + \frac{\xi}{2} - \sqrt{1+\xi}\right], \qquad u_1 = \frac{2\mu + \sqrt{4\mu^2 + \cos^2\phi}}{1+\sin\phi}$$

$$u_6 = \frac{2u_0}{\xi}\left[1 + \frac{\xi}{2} + \sqrt{1+\xi}\right], \qquad u_2 = \frac{2\mu - \sqrt{4\mu^2 + \cos^2\phi}}{1+\sin\phi}$$

and $\tilde{k} = -(2u_0^3 H)/(\rho g)$. Noticing that $u_1 = u_0$ in the case $\phi = \phi_p$, the decomposition of (9) in simple elements leads to:

$$\tilde{k}\left(\frac{A}{u-u_0} + \frac{B}{u-u_5} + \frac{C}{u-u_6}\right)\mathrm{d}u = \mathrm{d}z \tag{10}$$

with

$$A = \frac{u_0 - u_2}{(u_0-u_5)(u_0-u_6)}, \qquad B = \frac{u_5 - u_2}{(u_5-u_0)(u_5-u_6)}, \qquad C = \frac{u_6 - u_2}{(u_6-u_0)(u_6-u_5)}$$

On the other hand, since $\sigma_z(z) = \sigma_z^0(z)$, the boundary condition $\sigma_z(z) = 0$ in $z = 0$ and Eqs. (7) give us the value $\alpha_l = \frac{1}{2}\arctan(2\tan\phi)$ of α^0 in $z = 0$. Then we get $u_l = \tan\alpha_l$, and the integration of the Eq. (10) leads to:

$$q(u) - q(u_l) = z \tag{11}$$

where $q(u)$ is defined as $q(u) = \tilde{k}(A\ln|u-u_0| + B\ln|u-u_5| + C\ln|u-u_6|)$. Because of the existence of singularities of the function $q(u)$, we need to determine the interval of resolution of Eq. (11) for z fixed. The first boundary of the interval is necessarily u_l (corresponding to $z = 0$), and the other one corresponds to a singularity where the limit of $q(u)$ is $+\infty$. Once $u(z)$ and $\alpha^0(z) = \arctan(u(z))$ numerically computed for a given depth z, we calculate the stresses in the ensiled granular material using (7), then (4) and (5).

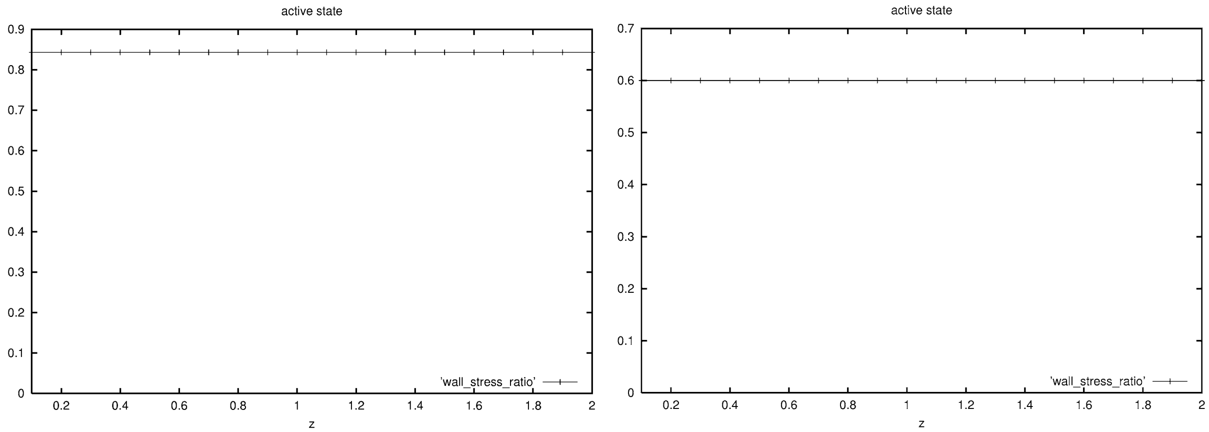
4. Validation for non cohesive materials and comparison with the Janssen theory

To validate the approach proposed, let us consider the particular case of non cohesive materials. Then the wall stress ratio $\lambda_p = \sigma_x^0/\sigma_z^0$ can be calculated analytically from (7). Using the relations of Mohr circles, we obtain the following expression[4] of λ_p:

$$\lambda_p = \frac{1-\sin^2\phi}{1+\sin^2\phi} = \frac{1}{1+2\tan^2\phi} \tag{12}$$

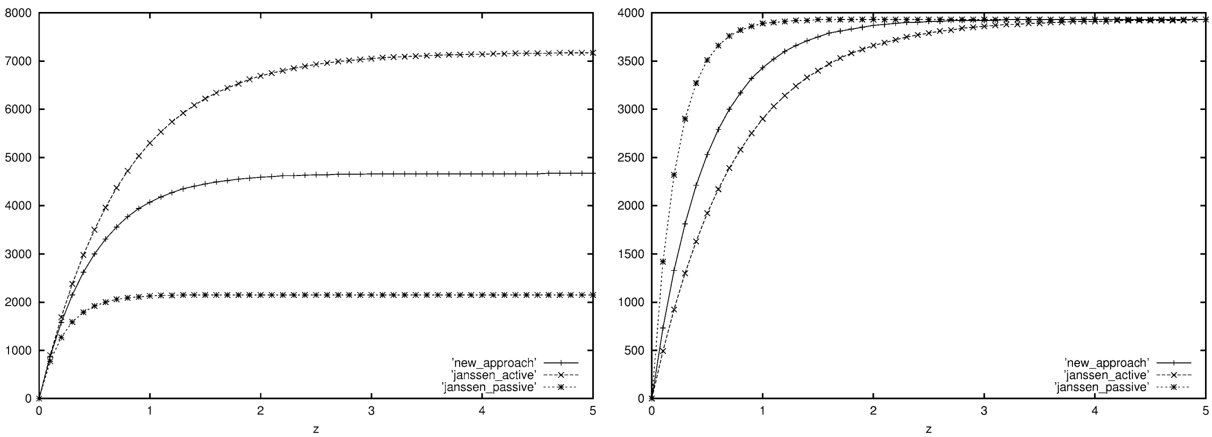
Now let us compare the results given by our approach with the theoretical solution (12). Figs. 1(a) and 1(b) represent the values of λ_p given by (7) after resolution of Eq. (11) for a very weak cohesion $H = 10^{-6}$ N/m^2. The obtained results coincide with (12) for different values of the internal friction angle (here $\phi = 17°$ and 30°). These results validate our approach for non cohesive materials. Moreover, we notice that whatever the friction angle ϕ, the granular material seems to be always in an active state. Finally, let us compare our approach with the Janssen theory. Figs. 2(a)

[4] This is a classical result for non cohesive materials (see [4] for example).



(a) (b)

Fig. 1. (a) $\lambda_p(z)$ for $H = 10^{-6}$ N/m^2 and $\phi = 17°$, (b) $\lambda_p(z)$ for $H = 10^{-6}$ N/m^2 and $\phi = 30°$.



(a) (b)

Fig. 2. (a) $\sigma_z^0(z)$ for $H = 10^{-6}$ N/m^2 and $\phi = 17°$, (b) $\sigma_x^0(z)$ for $H = 10^{-6}$ N/m^2 and $\phi = 17°$.

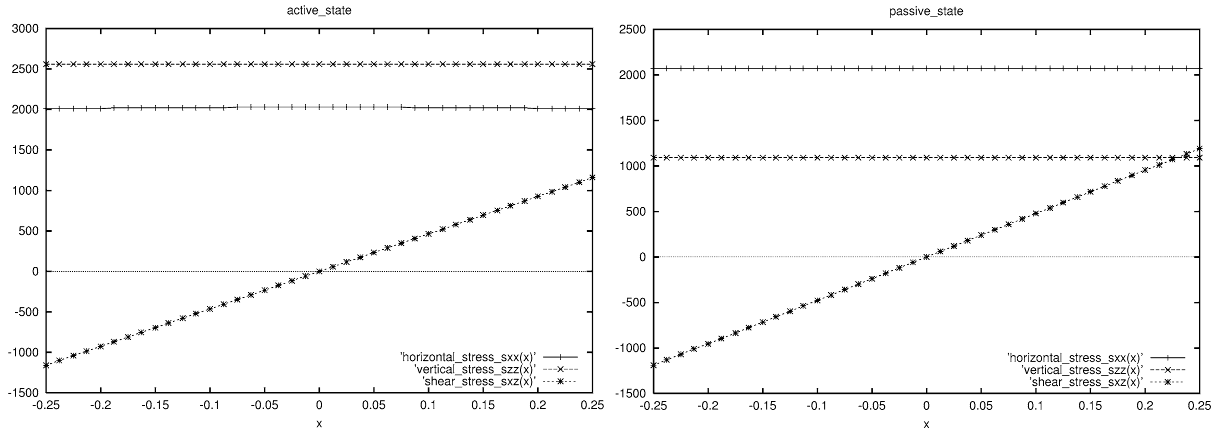
and 2(b) represent the variation of the stresses σ_x^0 and σ_z^0 given by our approach for $H = 10^{-6}$ N/m^2, and by the Janssen theory in the active and passive states.[5] We notice that with the approach developed here, the stresses σ_x^0 and σ_z^0 are framed by the active and passive states of Janssen theory. Moreover, the stress σ_x^0 given by our approach is greater than that obtained by the Janssen theory in the active case. Thus the approach developed here improves the Janssen theory which is known to underestimate the wall horizontal stresses in this case.

5. Study of cohesive materials

For cohesive materials, we observe the apparition of new mixed states of the stresses inside the silo: the material changes from a passive to an active state when the depth increases (see Fig. 4(a)). These mixed states are reached when the cohesion increases until a threshold value which depends on ϕ (see Figs. 3 and 4(a)). Beyond this threshold value, the material is in a passive state in all the silo (Fig. 3(b)). Finally, we observe in Fig. 4(b) that λ_p is no longer constant and depends on the cohesion, on the internal friction angle ϕ and on the angle[6] α^0.

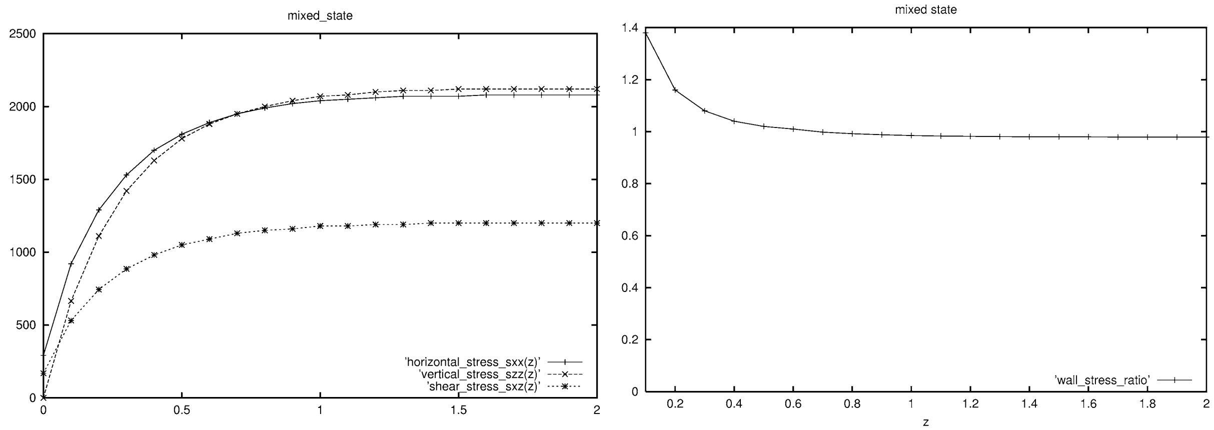
[5] We recall that this distinction in Janssen theory is fundamental to determine the coefficient λ_p. Here this coefficient is obtained after resolution of Eq. (8).

[6] This angle depends on z for cohesive materials.



(a) (b)

Fig. 3. $\sigma(x)$ for $z = 1$ m and $\phi = 30°$ (a) $H = 100$ N/m^2, (b) $H = 1000$ N/m^2.



(a) (b)

Fig. 4. (a) $\sigma^0(z)$ for $H = 300$ N/m^2 and $\phi = 30°$, (b) $\lambda_p(z)$ for $H = 300$ N/m^2 and $\phi = 30°$

6. Conclusion

The approach of continuous medium type proposed here improves the existing ones and in particular the Janssen theory. This approach enables to compute the stresses in all the ensiled granular material, for all the physical parameters involved (internal friction angle, wall friction angle, cohesion, ...). Some comparisons with analytic solutions existing for non cohesive materials have enabled to validate the approach proposed. For non cohesive materials, the stresses at the walls are bounded by the active and the passive states of the Janssen theory. For cohesive materials, the wall stress ratio λ_p is not constant anymore and depends on the depth of the silo. We observe also the apparition of mixed states for the stresses (passive then active) inside the silo.

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