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A generalized continuum approach to describe instability pattern formation by a multiple scale analysis

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Abstract

Macroscopic descriptions of instability pattern formation can be obtained by the generic amplitude equations of Ginzburg–Landau type. In the simple example of beam buckling, a variant of this approach is established, that permits one to account for the coupling between local and global instabilities. The mean field and the amplitude of the fluctuations are governed by similar equations. The resulting model is a generalized continuum, where the generalized stresses are Fourier coefficients of the microscopic stress. ***To cite this article: N. Damil, M. Potier-Ferry, C. R. Mecanique 334 (2006).***

Résumé

Une approche de milieu continu généralisé pour décrire des instabilités spatio-temporelles utilisant une analyse d'échelles multiples. L'évolution des instabilités spatio-temporelles peut se décrire macroscopiquement par des équations d'amplitude génériques de type Ginzburg–Landau. Dans l'exemple élémentaire du flambage d'une poutre, on établit une variante de cette approche, qui permet de prendre en compte des couplages entre instabilités locales et globales et qui traite de la même manière le champ moyen et la fluctuation. Le modèle final est un milieu continu généralisé, où les contraintes généralisées sont des coefficients de Fourier de la contrainte microscopique. ***Pour citer cet article : N. Damil, M. Potier-Ferry, C. R. Mecanique 334 (2006).***

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Mots-clés : Milieux continus ; Équation de Ginzburg–Landau ; Échelles multiples ; Instabilités locales ; Couplage local–global ; Flambage

1. Introduction

The appearance of cellular structures due to instability is very wide-spread within nonlinear systems [1,2]. Buckling of long plates [3] or Rayleigh–Bénard rolls [4,5] are typical examples. The response of such systems is spatially almost

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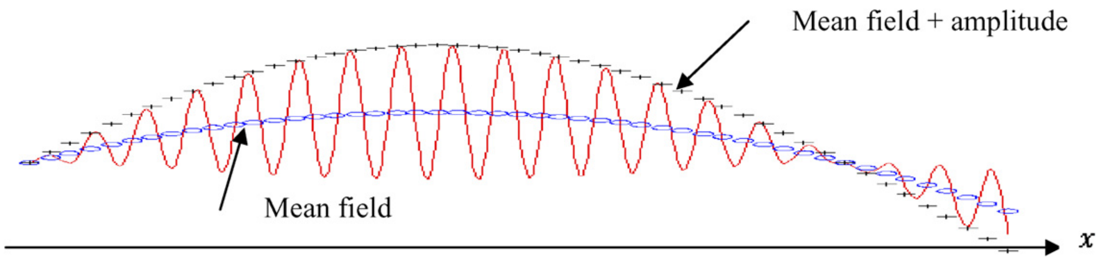

Fig. 1. Two main unknowns to describe nearly periodic patterns: the mean value and the amplitude of the oscillation.

periodic near the instability threshold. Classically, the disturbance from the bifurcation point U_0 is sought in the form of product of an envelop function and of the linear instability mode, that is harmonic in the x-direction:

$$U(x,y) - U_0(x,y) = a(x)U_{\text{mode}}(y)e^{iqx} + cc \tag{1}$$

In (1), cc denotes the complex conjugate of the preceding expression. The amplitude $a(x)$ is complex valued and is assumed to vary slowly with respect to the mode e^{iqx}. According to the asymptotic bifurcation analysis, $a(x)$ satisfies an equation of Ginzburg–Landau, whose simplest form is [6]:

$$\frac{d^2a}{dx^2} + \alpha_1(\lambda - \lambda_c)a - \alpha_2 a|a|^2 = 0 \tag{2}$$

Within this bifurcation approach, the amplitude is only a scalar multiplier that is the same for all components of the unknown fields. Furthermore, this bifurcation equation (2), which has the advantage to be generic, has the drawback to be independent of the underlying physical model.

In this Note, a physically based approach of cellular instabilities is presented, where the slowly variable unknowns are the mean field U_0 and a few Fourier coefficients, see Fig. 1. This leads to macroscopic models that are generalized continua, the macroscopic stress being defined by the Fourier coefficients of the microscopic stress. In this sense, the present technique is similar to homogenization theory, where a double scale analysis permits to deduce macroscopic generalized continua from microscopic classical ones [7,8]. From a practical point of view, the present method could permit to describe coupling between local and global instabilities, as in the buckling of stiffened plates [9] and of sandwich structures [10], or to account for the influence of wrinkles on the behaviour of membranes [11].

2. An example: nonlinear beam buckling

Let us consider a nonlinear beam model, within Von Karman approximation. The material domain is the interval $0 \leqslant x \leqslant L$. The unknowns are the axial displacement $u(x)$, the deflection $v(x)$, the normal force $n(x)$, the bending moment $m(x)$, the axial strain $\gamma(x) = u' + \frac{1}{2}v'^2$ and the curvature $k(x) = v''$. If $f(x)$ and $g(x)$ are the components of the external forces, the differential equations of the system are:

$$n' + f = 0 \tag{3}$$

$$\frac{n}{ES} = u' + \frac{1}{2}v'^2 \tag{4}$$

$$EI\,v'''' - (nv')' - g = 0 \tag{5}$$

where the material data are axial and bending stiffness ES and EI. If boundary forces are disregarded, the variational form of (3)–(5) can be written as (Principle of Virtual Work):

$$\int_0^L (n\delta\gamma + m\delta k)\,dx = \int_0^L (f\delta u + g\delta v)\,dx \tag{6}$$

In this Note, $f(x)$ is given and the lateral force follows a nonlinear elastic law $-g(v) = Cv + C_3v^3$.

It is well known that nearly harmonic instability patterns can occur in regions where the normal force is compressive. For instance, let us consider the stability of the straight state with a uniform compressive stress $\lambda = -n$ (i.e. $f \approx 0$). Hence the linearized form of the bending equation (5) becomes

$$EI\,v'''' + \lambda v'' + Cv = 0 \tag{7}$$

The latter equation has harmonic solutions $v(x) = \mathrm{e}^{\mathrm{i}qx}$, $\lambda(q) = EI\,q^2 + C/q^2$. The critical stress and the critical wave number correspond to the minimum of the curve $\lambda(q)$:

$$q^4 = \frac{C}{EI}, \qquad \lambda_c = 2\,EI\,q^2 = 2\frac{C}{q^2} \tag{8}$$

If the prebifurcation state is not homogeneous, a similar analysis can be done locally.

3. A generalized beam model including cellular instabilities

3.1. Generalized stress and principle of virtual work

The micro-macro modelization is deduced from the asymptotic double scale method [12]. The nearly periodic unknowns $U = (u, v, n, m, \gamma, k)$ are written as a Fourier series

$$U = \begin{pmatrix} u \\ v \\ n \\ m \\ \gamma \\ k \end{pmatrix} = \sum_{j=-\infty}^{+\infty} U_j(x)\mathrm{e}^{j(\mathrm{i}qx)} = \sum_{j=-\infty}^{+\infty} \begin{pmatrix} u_j(x) \\ v_j(x) \\ n_j(x) \\ m_j(x) \\ \gamma_j(x) \\ k_j(x) \end{pmatrix} \mathrm{e}^{j(\mathrm{i}qx)} \tag{9}$$

where the Fourier coefficients $U_j(x)$ varies more slowly than the harmonic part $\mathrm{e}^{j(\mathrm{i}qx)}$. So, they can be assumed constant on any basic cell $[x, x + 2\pi/q]$. If $\langle\ \rangle$ denotes the mean value on a basic cell, these rules permit to define generalized displacements, stresses and strains:

$$U_j(x) = \big\langle U(x)\mathrm{e}^{-j(\mathrm{i}qx)}\big\rangle, \quad -\infty \leqslant j \leqslant +\infty \tag{10}$$

Let $b(x)$ and $c(x)$ be two nearly periodic functions, as in (9). By using, first a splitting of the integral, second Parseval identity and by accounting that the Fourier coefficients $b_j(x)$ and $c_j(x)$ are constant on a cell, we get

$$\begin{aligned} \int_0^L b(x)c(x)\,\mathrm{d}x &= \sum_{\text{cell}} \int_{\text{cell}} b(x)c(x)\,\mathrm{d}x = \sum_{\text{cell}} \frac{2\pi}{q} \sum_{j=-\infty}^{+\infty} b_j c_{-j} = \sum_{\text{cell}} \sum_{j=-\infty}^{+\infty} \int_{\text{cell}} b_j c_{-j}\,\mathrm{d}x \\ \int_0^L b(x)c(x)\,\mathrm{d}x &= \int_0^L \sum_{j=-\infty}^{+\infty} b_j c_{-j}\,\mathrm{d}x \end{aligned} \tag{11}$$

Next the application of the rule (11) to the principle of virtual work (6) leads to a macroscopic version of the principle of virtual work

$$\int_0^L \left(\sum_{j=-\infty}^{+\infty} (n_j \delta\gamma_{-j} + m_j \delta k_{-j}) \right) \mathrm{d}x = \int_0^L \left(\sum_{j=-\infty}^{+\infty} (f_j \delta u_{-j} + g_j \delta v_{-j}) \right) \mathrm{d}x \tag{12}$$

In this way, a generalized continuum model has been defined. In the homogenization literature the equality of the left-hand sides of (6) and (12) is called Hill–Mandel macro homogeneity condition [8].

3.2. A few calculations rules

If $b(x)$ and $c(x)$ are two functions as in (9), the following identities hold:

$$\left(\frac{\mathrm{d}a}{\mathrm{d}x}\right)_j = (a')_j = \left(\frac{\mathrm{d}}{\mathrm{d}x} + j\mathrm{i}q\right)a_j = (a_j)' + j\mathrm{i}qa_j \tag{13}$$

$$\left(\frac{\mathrm{d}^2a}{\mathrm{d}x^2}\right)_j = (a'')_j = \left(\frac{\mathrm{d}}{\mathrm{d}x} + j\mathrm{i}q\right)^2 a_j = (a_j)'' + 2j\mathrm{i}q(a_j)' - j^2q^2a_j \tag{14}$$

$$(ab)_j = \sum_{j_1=-\infty}^{+\infty} a_{j_1} b_{j-j_1} \tag{15}$$

For example, let us apply these rules to the first term in the left-hand side of (12), where the strain is given by $\gamma(x) = u' + \frac{1}{2}v'^2$. By letting $a = nv'$, one gets the identities:

$$\sum_{j=-\infty}^{+\infty} n_j\delta\gamma_{-j} = \sum_{j=-\infty}^{+\infty} n_j\left(\frac{\mathrm{d}}{\mathrm{d}x} - j\mathrm{i}q\right)\delta u_{-j} + \sum_{j=-\infty}^{+\infty} a_j\left(\frac{\mathrm{d}}{\mathrm{d}x} - j\mathrm{i}q\right)\delta v_{-j} \tag{16}$$

$$a_j = \sum_{j_1=-\infty}^{+\infty} n_{j_1}\left(\frac{\mathrm{d}}{\mathrm{d}x} + (j-j_1)\mathrm{i}q\right)v_{j-j_1} \tag{17}$$

3.3. Equilibrium equations

Because of the identities (17) and (14), the virtual work equation (12) can be rewritten as:

$$\int_0^L \left(\sum_{j=-\infty}^{+\infty}\left(n_j\left(\frac{\mathrm{d}}{\mathrm{d}x} - j\mathrm{i}q\right)\delta u_{-j} + a_j\left(\frac{\mathrm{d}}{\mathrm{d}x} - j\mathrm{i}q\right)\delta v_{-j} + m_j\left(\frac{\mathrm{d}}{\mathrm{d}x} - j\mathrm{i}q\right)^2\delta v_{-j}\right)\right)\mathrm{d}x \tag{18}$$

From (18), the equilibrium equations of the extended beam model are

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} + j\mathrm{i}q\right)n_j + f_j = 0 \tag{19}$$

$$-\left(\frac{\mathrm{d}}{\mathrm{d}x} + j\mathrm{i}q\right)^2 m_j + \left(\frac{\mathrm{d}}{\mathrm{d}x} + j\mathrm{i}q\right)a_j + g_j = 0 \tag{20}$$

3.4. Constitutive laws

In the same way, the identities (13)–(15) are applied to the constitutive laws (4), $m = EI\,v''$, and to the cubic law for the foundation $-g(v) = Cv + C_3v^3$. The constitutive laws of the generalized beam model are:

$$n_j = ES\left(\frac{\mathrm{d}}{\mathrm{d}x} + j\mathrm{i}q\right)u_j + \frac{ES}{2}\sum_{j_1=-\infty}^{+\infty}\left(\frac{\mathrm{d}}{\mathrm{d}x} + j_1\mathrm{i}q\right)v_{j_1}\left(\frac{\mathrm{d}}{\mathrm{d}x} + (j-j_1)\mathrm{i}q\right)v_{j-j_1} \tag{21}$$

$$m_j = EI\left(\frac{\mathrm{d}}{\mathrm{d}x} + j\mathrm{i}q\right)^2 v_j \tag{22}$$

$$-g_j = Cv_j + C_3\left(v^3\right)_j = Cv_j + C_3\sum_{j_1=-\infty}^{+\infty}\sum_{j_2=-\infty}^{+\infty} v_{j_1}v_{j_2}v_{j-j_1-j_2} \tag{23}$$

3.5. A simplified analysis with three harmonics

The simplest model coupling global and local instabilities involves three harmonics $j = -1, 0, +1$. This model is limited to a real set of unknowns U_0 for harmonic $j = 0$ and a complex set of unknowns U_1 for the harmonic $j = 1$ ($U_{-1} = \overline{U_1}$ because U is real). Moreover, one assumes that the linearized force does not fluctuate on the basic cell ($f(x) = f_0(x)$) so that the normal stress $n(x)$ and the horizontal displacement $u(x)$ do not fluctuate ($n(x) = n_0(x)$, $u(x) = u_0(x)$, $n_1 = 0$, $u_1 = 0$). According to these principles, the simplified version of the proposed model is

$$\frac{\mathrm{d}n_0}{\mathrm{d}x} + f_0 = 0 \tag{24}$$

$$\frac{n_0}{ES} = u_0' + \frac{1}{2}v_0'^2 + |v_1' + \mathrm{i}qv_1|^2 \tag{25}$$

$$EI\, v_0'''' - (n_0 v_0')' + Cv_0 + C_3\big(v_0^3 + 6v_0|v_1|^2\big) = 0 \tag{26}$$

$$EI\big(v_1'''' + 4\mathrm{i}qv_1''' - 6q^2v_1'' - 4\mathrm{i}q^3v_1' + q^4v_1\big) - \left(\frac{\mathrm{d}}{\mathrm{d}x} + \mathrm{i}q\right)\big(n_0(v_1' + \mathrm{i}qv_1)\big) + Cv_1 + 3C_3\big(v_1|v_1|^2\big) \tag{27}$$

4. Discussion and comments

One checks that the model (24)–(27) permits to recover the starting equations (3)–(5) when the local mode is neglected. The amplitude of the fluctuation appears in the mean constitutive law (25): the last term $|v_1' + \mathrm{i}qv_1|^2$ accounts for the strain increase, due to the local instability. This term could permit to explain the reduction of compressive stresses in wrinkled membrane. Another coupling local–global appears in the nonlinear foundation term, see (26).

The nonlinear complex equation (27) models the appearance of the local instabilities and it is obviously coupled with the mean field unknowns v_0 and n_0. So it plays the same role as the Ginzburg–Landau equation (2). The latter equation (2) can be recovered if the prebuckling stress is uniform ($n_0 = -\lambda = \text{constant}$) and if there is no global deflection ($v_0 = 0$). In other words, the Ginzburg–Landau equation is valid only if the cellular pattern takes place from a uniform state. With these hypotheses, the equation for the fluctuation (27) is reduced to:

$$EI\, v_1'''' + 4\,EI\,\mathrm{i}qv_1''' - \big(6\,EI\,q^2 - \lambda\big)v_1'' - 2\mathrm{i}q\big(2\,EI\,\mathrm{i}q^2 - \lambda\big)v_1' + \big(EI\,q^4 - \lambda q^2 + C\big)v_1 + 3C_3v_1|v_1|^2 \tag{28}$$

Furthermore, Ginzburg–Landau equation is generally deduced from an asymptotic approach, and it describes only the onset of the instability. If the load increment is chosen as a perturbation parameter ($\lambda - \lambda_c = \varepsilon$), the asymptotic technique requires $\frac{\mathrm{d}}{\mathrm{d}x} = 0(\sqrt{\varepsilon})$, $v_1 = 0(\sqrt{\varepsilon})$. With these orders of magnitudes, (28) leads to

$$\varepsilon^{3/2}\big(-4\,EI\,q^2v_1'' - (\lambda - \lambda_c)v_1 + 3C_3v_1|v_1|^2\big) + \mathrm{O}\big(\varepsilon^2\big) = 0 \tag{29}$$

and (29) coincides exactly with the amplitude equation established in [13].

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