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Computational AeroAcoustics: from acoustic sources modeling to farfield radiated noise prediction

ADER discontinuous Galerkin schemes for aeroacoustics

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Abstract

In this paper we apply the ADER approach to the Discontinuous Galerkin (DG) framework for the two-dimensional linearized Euler equations. The result is an efficient high order accurate single-step scheme in time which uses less storage than Runge–Kutta DG schemes, especially for very high order of accuracy. The aim is to obtain an arbitrarily accurate scheme in space and time on unstructured grids for accurate noise propagation in the time domain in very complex geometries. We will present numerical convergence rates for ADER-DG methods up to 10th order of accuracy in *space and time* on structured and unstructured meshes. ***To cite this article: M. Dumbser, C.-D. Munz, C. R. Mecanique 333 (2005).***

Résumé

Schémas de Galerkine discontinu ADER pour l'aéroacoustique. Nous appliquons l'approche ADER au cadre des éléments finis discontinus pour les équations d'Euler linéarisées bidimensionnelles. Le résultat sont des schémas de haute précision tout en utilisant moins de mémoire que les schémas du type Runge–Kutta Galerkin discontinus, spécialement pour les ordres trés élevés. Le but est d'obtenir un schéma de précision arbitraire en temps et en espace sur des maillages non-structurés pour le calcul précis du bruit dans les géometries très complexes. Nous présentons des études de convergence numériques pour des méthodes ADER-DG sur des maillages structurés et non-structurés jusqu'à l'ordre 10 en *temps et en espace*. ***Pour citer cet article : M. Dumbser, C.-D. Munz, C. R. Mecanique 333 (2005).***

Keywords: Acoustics; Discontinuous Galerkin schemes; Very high order methods; ADER approach; Unstructured grids

Mots-clés : Acoustique ; Schémas Galerkin discontinus ; Méthodes d'ordre très élevé ; Approche ADER ; Maillages non-structurés

1. Introduction

For the accurate simulation of noise propagation in the near field, which may still contain convection effects and geometrically complex obstacles, very accurate numerical schemes for solving the linearized Euler equations in

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the time domain are necessary. In order to be able to treat complex geometries, the method should be designed to run also on unstructured grids. For this purpose, quadrature-free Runge–Kutta Discontinuous Galerkin methods [1] are very attractive since they can easily be implemented for unstructured grids. Concerning the accuracy of the scheme in time, however, one is more or less limited since the Runge–Kutta schemes become complicated for orders of accuracy higher than four and they need more and more intermediate stages. The recently developed ADER approach of Toro et al. [2–4] may overcome these difficulties. It is essentially based on the solution of generalized Riemann problems using Taylor expansions and the Lax–Wendroff procedure, which is also known as Cauchy–Kovalewski procedure. It can be programmed in such a way that it really runs at arbitrary order of accuracy in space and time for any linear hyperbolic system [5]. The method presented in [5] is based on the finite volume concept and thus needs a reconstruction procedure which may become very cumbersome on unstructured grids. Our proposed scheme applies the ADER approach to the Discontinuous Galerkin finite element method where the high order polynomials are directly stored and evolved in time in each element.

2. The ADER discontinuous Galerkin scheme

In the following, we consider the two-dimensional linearized Euler equations (1) where we subsequently use classical tensor notation

$$\frac{\partial u_p}{\partial t} + A_{pq}\frac{\partial u_p}{\partial x} + B_{pq}\frac{\partial u_p}{\partial y} = 0 \tag{1}$$

The fluctuation quantities are $u_p = (\rho', u', v', p')^{\mathrm{T}}$, the Jacobians of the fluxes are A_{pq} and B_{pq}, where linearization is performed about a mean flow denoted by the subscript 0

$$A_{pq} = \begin{pmatrix} u_0 & \rho_0 & 0 & 0 \\ 0 & u_0 & 0 & \dfrac{1}{\rho_0} \\ 0 & 0 & u_0 & 0 \\ 0 & \gamma p_0 & 0 & u_0 \end{pmatrix}, \qquad B_{pq} = \begin{pmatrix} v_0 & 0 & \rho_0 & 0 \\ 0 & v_0 & 0 & 0 \\ 0 & 0 & v_0 & \dfrac{1}{\rho_0} \\ 0 & 0 & \gamma p_0 & v_0 \end{pmatrix} \tag{2}$$

The numerical solution of (1) is written in terms of pure spatial basis functions Φ_l with associated time-dependent degrees of freedom. The basis functions are chosen such that they are orthogonal, see e.g. [6].

$$u_p(x,t) = \hat{u}_{pl}(t)\Phi_l(x) \tag{3}$$

Multiplication of (1) by a test function Φ_k and integration by parts over a triangle $T^{(m)}$ yields

$$\int_{T^{(m)}} \Phi_k \frac{\partial u_p}{\partial t}\,\mathrm{d}V + \int_{\partial T^{(m)}} \Phi_k F_p^h\,\mathrm{d}S - \int_{T^{(m)}} \left(\frac{\partial \Phi_k}{\partial x}A_{pq}u_q + \frac{\partial \Phi_k}{\partial y}B_{pq}u_q\right)\mathrm{d}V = 0 \tag{4}$$

where a suitable numerical flux F_p^h must be introduced for the surface integral because the numerical solution is discontinuous at an element interface. In the following, we use the Courant–Isaacson–Rees (CIR) flux [7] for the locally linearized Euler equations, supposing a piecewise constant background flow in each element, which produces Godunov's flux in the case of constant Jacobians. Application to unstructured grids needs a transformation of each triangle $T^{(m)}$ in the physical x–y space to a reference triangle T^E in ξ–η space, since all Φ_k are defined in T^E. We also transform (1) to the ξ–η system:

$$\frac{\partial u_p}{\partial t} + A^*_{pq}\frac{\partial u_q}{\partial \xi} + B^*_{pq}\frac{\partial u_q}{\partial \eta} = 0 \quad \text{with } A^*_{pq} = A_{pq}\xi_x + B_{pq}\xi_y \quad \text{and} \quad B^*_{pq} = A_{pq}\eta_x + B_{pq}\eta_y \tag{5}$$

Following the ADER approach, we first expand the vector of state in a Taylor series in time up to order N and then apply the Lax–Wendroff procedure to the modified system (5). This procedure expresses time-derivatives of

u_p by pure space-derivatives of u_p, making successive use of the modified governing PDE (5), finally yielding the following result:

$$u_p(x,t) = \sum_{k=0}^{N} \frac{t^k}{k!} \frac{\partial^k}{\partial t^k} u_p(x,0) = \sum_{k=0}^{N} \frac{t^k}{k!} (-1)^k (A^*_{pq} \partial_\xi + B^*_{pq} \partial_\eta)^k u_q(x,0) \tag{6}$$

It is the key point of the method to apply the Lax–Wendroff procedure to the transformed system (5). Otherwise the application to unstructured grids would be extremely expensive. Then the basis function expansion (3) of the numerical solution for the current time level n is introduced into (6) for each element. Now we project this expression onto the basis functions in order to get an approximation of the temporal evolution of the degrees of freedom in each element during one timestep

$$\int_0^{\Delta t} \hat{u}_{pl}(t)\,\mathrm{d}t = \frac{\langle \Phi_n, \sum_{k=0}^{N} \frac{\Delta t^{(k+1)}}{(k+1)!} (-1)^k (A^*_{pq} \partial_\xi + B^*_{pq} \partial_\eta)^k \Phi_m(\xi) \rangle}{\langle \Phi_n, \Phi_l \rangle} \hat{u}_{qm}(0) \tag{7}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product over the reference triangle T^E and the division by the mass-matrix stands for the multiplication by the inverse mass-matrix. This approximation can now be introduced into (4) and due to the linearity of the governing equations (1) the system can be analytically integrated in space and time. Many parts of (7) can be evaluated beforehand since the basis functions are known a priori, i.e. we only need to calculate once and then store the projections of the basis functions and all their spatial derivatives onto the DG basis. The Lax–Wendroff procedure (6) can be coded in a completely automatic manner which only needs as input the Jacobian matrices and the desired order of accuracy in time. For Cartesian grids, the ADER-DG scheme is similar and even more terms in (7) can be pre-computed.

We summarize that the ADER-DG schemes are quadrature-free finite elements which perform time integration in one single step, making use only of the degrees of freedom of the element and its direct neighbors, thus being ideal for parallelization. They also need less memory compared to RK-DG schemes since no intermediate RK stages have to be stored. Numerical experiments have also shown that the ADER-DG method is faster compared to a RK-DG scheme. Since Eq. (7) can be coded in a completely general manner, the ADER-DG schemes are really of arbitrary high order of accuracy in *space and time* on structured and unstructured grids.

3. Numerical convergence rates

In this section, we present numerical convergence rates which have been obtained for the linearized Euler equations in a very simple setting. We consider the advection of an initial Gaussian density fluctuation

$$\rho'(x) = \mathrm{e}^{-\frac{1}{2}(x^2+y^2)/\sigma^2}, \qquad u'(x) = v'(x) = p'(x) = 0 \tag{8}$$

in a rectangular computational domain with periodic boundary conditions and with extents $[100 \times 100]$. The background velocity is $u = v = 1$. The density fluctuation is transported along the diagonal of the domain for one period so that the exact reference solution is just the initial condition. We perform convergence studies on a Cartesian grid ($\sigma = 3$) and on a regular unstructured grid ($\sigma = 5$) which is derived from the Cartesian grid by subdividing each square in four equilateral triangles, each consisting of two corners and the barycentre of the square. In Tables 1–4 the errors in L_∞ and L_2 norm are given as well as the corresponding convergence rates, where $\mathcal{O}_{L_\infty}$ and $\mathcal{O}_{L_2}$ denote the measured order of convergence in L_∞-norm and L_2-norm, respectively, between two successive grids. N_G is the number of squares in each dimension and N_d is the total number of degrees of freedom. Tables 1–4 clearly show that the high-order schemes need less degrees of freedom N_d in order to reach the same precision as the low order schemes. This favours high-order schemes when high precision is needed and leads to considerable memory savings, which are important when big aeroacoustical problems are to be tackled in very complex geometries.

Table 1
Numerical convergence rates for ADER-DG $\mathcal{O}2$ and $\mathcal{O}4$ schemes (P1 and P3) on the Cartesian grid

Tableau 1
Résultats de convergence numériques pour les schémas ADER-DG $\mathcal{O}2$ et $\mathcal{O}4$ (P1 et P3) sur un maillage Cartésien

ADER-DG $\mathcal{O}2$						ADER-DG $\mathcal{O}4$					
N_G	N_d	L_∞	L_2	$\mathcal{O}_{L_\infty}$	$\mathcal{O}_{L_2}$	N_G	N_d	L_∞	L_2	$\mathcal{O}_{L_\infty}$	$\mathcal{O}_{L_2}$
100	30 000	1.5660E–01	6.4463E–01			25	6250	7.8971E–02	2.8840E–01		
150	67 500	7.2868E–02	2.8628E–01	1.9	2.0	50	25 000	4.5254E–03	1.4428E–02	4.1	4.3
200	120 000	4.0392E–02	1.5835E–01	2.1	2.1	75	56 250	6.9786E–04	2.2745E–03	4.6	4.6
300	270 000	1.7306E–02	6.8851E–02	2.1	2.1	100	100 000	2.0581E–04	6.4561E–04	4.2	4.4

Table 2
Numerical convergence rates for ADER-DG $\mathcal{O}6$ and $\mathcal{O}10$ schemes (P5 and P9) on the Cartesian grid

Tableau 2
Résultats de convergence numériques pour les schémas ADER-DG $\mathcal{O}6$ et $\mathcal{O}10$ (P5 et P9) sur un maillage Cartésien

ADER-DG $\mathcal{O}6$						ADER-DG $\mathcal{O}10$					
N_G	N_d	L_∞	L_2	$\mathcal{O}_{L_\infty}$	$\mathcal{O}_{L_2}$	N_G	N_d	L_∞	L_2	$\mathcal{O}_{L_\infty}$	$\mathcal{O}_{L_2}$
25	13 125	1.0116E–03	4.6516E–03			10	5500	2.3342E–03	9.1034E–03		
50	52 500	1.6321E–05	7.0422E–05	6.0	6.0	25	34 375	7.0918E–07	2.8242E–06	8.8	8.8
75	118 125	1.5928E–06	5.4450E–06	5.7	6.3	50	137 500	1.0632E–09	4.1396E–09	9.4	9.4
100	210 000	2.6423E–07	9.1726E–07	6.2	6.2	75	309 375	2.6470E–11	7.1228E–11	9.1	10.0

Table 3
Numerical convergence rates for ADER-DG $\mathcal{O}2$ and $\mathcal{O}4$ scheme (P1 and P3) on the regular unstructured mesh

Tableau 3
Résultats de convergence numériques pour les schémas ADER-DG $\mathcal{O}2$ et $\mathcal{O}4$ (P1 et P3) sur un maillage non-structuré régulier

ADER-DG $\mathcal{O}2$						ADER-DG $\mathcal{O}4$					
N_G	N_d	L_∞	L_2	$\mathcal{O}_{L_\infty}$	$\mathcal{O}_{L_2}$	N_G	N_d	L_∞	L_2	$\mathcal{O}_{L_\infty}$	$\mathcal{O}_{L_2}$
25	7500	1.0143E–01	8.4504E–01			10	4000	2.6614E–02	1.7035E–01		
40	19 200	4.3561E–02	3.5315E–01	1.8	1.9	20	16 000	2.1150E–03	1.1262E–02	3.7	3.9
50	30 000	2.9151E–02	2.2921E–01	1.8	1.9	40	64 000	1.7118E–04	7.7531E–04	3.6	3.9
75	67 500	1.3540E–02	1.0488E–01	1.9	1.9	60	144 000	3.7768E–05	1.6343E–04	3.7	3.8

Table 4
Numerical convergence rates for ADER-DG $\mathcal{O}6$ and $\mathcal{O}10$ schemes (P5 and P9) on the regular unstructured mesh

Tableau 4
Résultats de convergence numériques pour les schémas ADER-DG $\mathcal{O}6$ et $\mathcal{O}10$ (P5 et P9) sur un maillage non-structuré régulier

ADER-DG $\mathcal{O}6$						ADER-DG $\mathcal{O}10$					
N_G	N_d	L_∞	L_2	$\mathcal{O}_{L_\infty}$	$\mathcal{O}_{L_2}$	N_G	N_d	L_∞	L_2	$\mathcal{O}_{L_\infty}$	$\mathcal{O}_{L_2}$
5	2100	2.0367E–02	1.6683E–01			4	3520	1.1908E–03	7.5866E–03		
10	8400	8.1399E–04	4.5373E–03	4.6	5.2	8	14 080	7.1357E–06	2.1936E–05	7.4	8.4
20	33 600	2.3346E–05	8.3467E–05	5.1	5.8	12	31 680	1.4006E–07	4.4864E–07	9.7	9.6
30	75 600	2.3879E–06	7.8982E–06	5.6	5.8	16	56 320	1.2676E–08	2.8100E–08	8.4	9.6

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References

[1] H.L. Atkins, C.-W. Shu, Quadrature-free implementation of discontinuous Galerkin method for hyperbolic equations, AIAA J. 36 (1998) 775–782.
[2] E.F. Toro, R.C. Millington, L.A.M. Nejad, Towards very high order Godunov schemes, in: E.F. Toro (Ed.), Godunov Methods: Theory and Applications, Kluwer Academic, 2001, pp. 905–937.
[3] E.F. Toro, V.A. Titarev, Solution of the generalized Riemann problem for advection-reaction equations, Proc. Roy. Soc. London 458 (2002) 271–281.
[4] T. Schwartzkopff, C.-D. Munz, E.F. Toro, ADER: A high-order approach for linear hyperbolic systems in 2D, J. Sci. Comput. 17 (2002) 231–240.
[5] T. Schwartzkopff, M. Dumbser, C.-D. Munz, ADER: A high-order approach for linear hyperbolic systems in 2D, J. Comput. Phys. 197 (2004) 532–539.
[6] B. Cockburn, G.E. Karniadakis, C.-W. Shu, Discontinuous Galerkin Methods, Lecture Notes in Comput. Sci. Engrg., Springer, 2000.
[7] R. Courant, E. Isaacson, M. Rees, On the solution of nonlinear hyperbolic differential equations by finite differences, Comm. Pure Appl. Math. 5 (1952) 243–255.