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C. R. Mecanique 333 (2005) 528–533

COMPTES RENDUS MECANIQUE

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On some localized waves described by the extended KdV equation

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Received 6 October 2004; accepted after revision 14 June 2005

Available online 14 July 2005

Presented by Évariste Sanchez-Palencia

Abstract

The influence of higher-order nonlinear terms on the shape of solitary waves is studied for mechanical systems governed by a generalization of the 5th order Korteweg–de Vries equation. New localized travelling wave with intrinsic oscillations (not breathers) is shown to arise from arbitrary initial pulse thanks only to the higher-order quadratic nonlinearity, while cubic nonlinearity is responsible for the formation of so-called 'fat' solitary wave. ***To cite this article: A.V. Porubov et al., C. R. Mecanique 333 (2005).***

Résumé

Sur quelques ondes localisées gouvernées par une équation de KdV généralisée. On étudie l'influence des termes non linéaires d'ordre élevé sur la forme d'ondes solitaires dans des systèmes mécaniques gouvernés par une équation de KdV d'ordre cinq. On montre que de nouvelles solutions d'ondes localisées présentant des oscillations intrinsèques (pas des 'breathers') sont engendrées par une impulsion initiale arbitraire grâce aux non linéarités quadratiques, alors que la non linéarité cubique est responsable de la formation d'une onde solitaire dite « épaisse » (ou « grasse »). ***Pour citer cet article : A.V. Porubov et al., C. R. Mecanique 333 (2005).***

Keywords: Waves; Nonlinear wave; Solitary wave; Numerical solution

Mots-clés : Ondes ; Ondes non linéaires ; Ondes solitaires ; Solution numérique

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1631-0721/$ – see front matter
doi:10.1016/j.crme.2005.06.003

1. Introduction

One of the important problems of nonlinear wave propagation in fluids and solids is to know whether an initial finite amplitude localized pulse will evolve into a sequence of solitary waves in a dispersive medium. The solitary waves propagate keeping their shape and they transfer energy over long distances. They usually exist as a result of a balance between nonlinearity and dispersion. However, the features of the solitary wave depend upon the type of nonlinear and dispersive terms in the governing equation.

We consider the following nonlinear equation

$$u_t + 2buu_x + 3cu^2u_x + ruu_{xxx} + su_xu_{xx} + du_{xxx} + fu_{xxxxx} = 0 \tag{1}$$

This equation is often called extended Korteweg–de Vries (KdV) equation. It appears, in particular, in the shallow water theory, see [1] and references therein. Eq. (1) may be used for a modeling of weak nonlocality in solids [2]. It may account for a continuum limit of discrete models with far neighbour interactions [3]. We get from Eq. (1) the fifth-order (in space derivatives) KdV (5th KdV) equation when $c = r = s = 0$. This equation arises for water waves when surface tension is rather strong [4]. When $c = r = s = d = 0$, the resulting equation models the LC ladder electrical transmission lines [5,6].

Known exact travelling wave solutions obtained in [4] for the 5th KdV equation, and in the general case in [7], account for a familiar bell-shaped solitary wave like in the classic KdV case. At the same time, the phase portrait analysis performed for the 5th KdV equation in [8], revealed a solitary wave that oscillatorilly vanishes far from its core. Also a two-hump exact travelling solitary wave solution was found in [9] in the case $c = 0$. Hence, analytical solutions demonstrate the dependence of the wave shape upon the type of nonlinear and dispersive terms. However, they require special initial conditions, thus they cannot describe a *formation* of localized wave structures, a problem which is very important in applications.

We study how higher-order dispersion and nonlinear terms in Eq. (1) affect the features of the solitary waves solutions and their *appearance* from rather arbitrary localized finite amplitude initial pulse. Previously [10] we have found that generated solitary waves may decay at infinity (far from their core) either monotonically (like KdV solitons) or oscillatory (like found by Kawahara [8]) thanks to the influence of the fifth-order dispersive term, fu_{xxxxx}. Now attention is paid to an appearance of rather unusual localized waves due to the influence of the higher-order quadratic nonlinear term ruu_{xxx} at $c = s = 0$. This was first mentioned in [10], but important features of this solution were not dutifully recognized there. Also a formation of so-called 'fat' solitary wave is found thanks to the cubic nonlinear term, $3cu^2u_x$, when $r = s = 0$.

We used two methods for computations, finite-difference and pseudospectral, in order to prove the validity of the numerical results. The difference scheme that we applied to Eq. (1) is similar to those used in [5,8]; the hybrid fourth-order Runge–Kutta method has been chosen for numerical solution. Vanishing boundary conditions are imposed at the ends of the computation interval. The accuracy of the method at each time step is $O(\Delta t^4, \Delta x^2)$. The computation code for the pseudospectral simulation was designed in [11]. We adopted a slightly modified code as given in [11] for our purposes, in particular, to process the terms ruu_{xxx} and su_xu_{xx}. The program computes solutions of 1D scalar PDEs with periodic boundary conditions. It evaluates spatial derivatives in Fourier space by means of the Fast Fourier Transform, while the time discretization is performed using the fourth-order Runge–Kutta method. This scheme appears to have a good stability with respect to the time step. Below only those results are shown that were obtained using both numerical methods.

2. Formation of multi-hump localized structures

We obtained in the case $s = c = 0$ [10] that an increase in positive values of r yields a decrease in the velocity and an increase in the amplitude of the solitary waves arising from the Gaussian input. The number of solitary waves decreases, but the train of generated solitary waves looks similar to the KdV case. However, at negative values of r

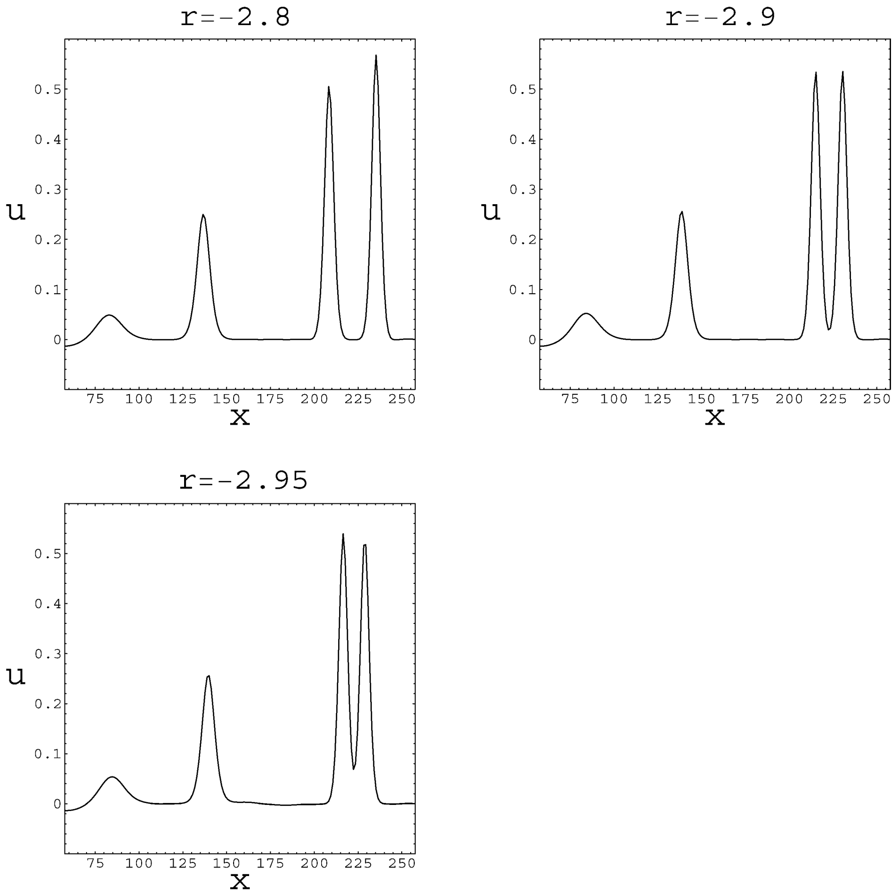

Fig. 1. Equalization of the first and the second solitary waves and subsequent exceeding of the second wave due to the alteration of the negative values of the coefficient r. Other coefficients are fixed: $b = 1$, $d = 1$, $f = -1$.

we found that at the initial stage of the splitting of the Gaussian profile the amplitude of the second solitary wave becomes equal to that of the first one at $r = -2.9$, see Fig. 1. Comparing with previous results from [10] we see that the critical value of r increases as the value of d decreases, in particular, it is equal to -1.57 for $d = 0.5$ [10]. At smaller r the second solitary wave becomes higher, and two solitary waves form an unusual multi-hump localized structure. This multi-hump travelling structure was already presented in [10] but an important question remains whether it is a regular or a chaotic solution. The structure itself moves with permanent velocity, but what about positions of the humps inside it? Is there any periodicity in their movement? That is why we consider the evolution of the multi-hump structure more carefully. Shown in Fig. 2 are 21 stages of generation and evolution of this localized structure. The first three stages in Fig. 2 demonstrate usual splitting of the Gaussian initial pulse into the solitary waves. Then the first two waves form a two-hump travelling wave. It is no longer quasistationary since the positions of the humps vary in time inside the wave structure. However, there is a periodicity in the humps relative movement.

Indeed, we see five different shapes: the first appears at the stages 5, 10, 15, 20 in Fig. 2; the second – at the stages 6, 11, 16, 21; the third – at the stages 7, 12, 17; the fourth – at the stages 8, 13, 18; and the fifth – at the stages 9, 14 and 19. Hence a sequence of shapes at the stages 5–9 repeats at the stages 10–14 and in the following. This is an evidence of periodic *regular* behavior of the humps movement. The number of humps depends both upon the parameters of the initial pulse and the value of the coefficient r. Decreasing r we achieve an increase in the

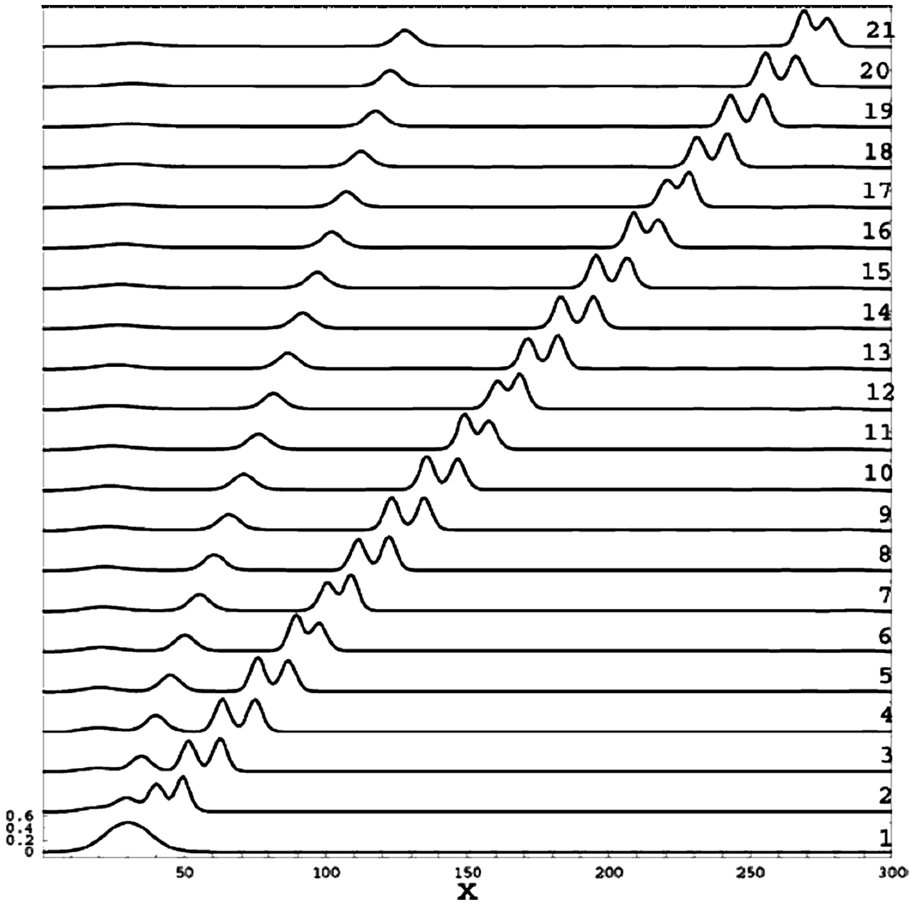

Fig. 2. Two-hump solitary wave formation for $r = -3$, $b = 1$, $d = 1$, $f = -1$. Initial pulse parameters are: $A = 0.5$, $\sigma = 0.008$.

number of humps in the multi-hump localized wave [10]. Hence this localized multi-hump wave is not the usual *quasistationary* multi-hump wave shown, in particular, in [9,12]. Now the shape of this travelling wave varies in time, and its evolution is not governed by the ODE reduction of Eq. (1). That is why we can use neither the phase-portrait analysis nor the known exact solutions to explain the numerical results. On the other hand, one cannot call these waves breathers since a breather usually contains intrinsic oscillations around the zero level.

3. Generation of the 'fat' solitary wave

We studied the influence of the amplitude and the width of the initial condition on the shape of the emerging solitary waves. The most interesting results have been found in the case $r = s = 0$ when only cubic nonlinearity is added to the 5th-order KdV equation. Earlier, an action of the cubic nonlinear term on the type of decay of solitary waves was exhibited, depending on the sign of c [10]. Also the solution is sensitive to the ratio between nonlinear contributions, b/c, and the value of the amplitude of an initial pulse. Indeed we have found that for $b \sim c$, $b > 0$, $f < 0$, the train of solitary waves arises only from a positive initial pulse while a negative one is dispersed. Dependence of the amplitude of the exact solutions of the equation with quadratic nonlinearity on the sign of b is quite typical. However, at smaller b, $b < c$, the formation of solitary waves no longer depends upon the sign of

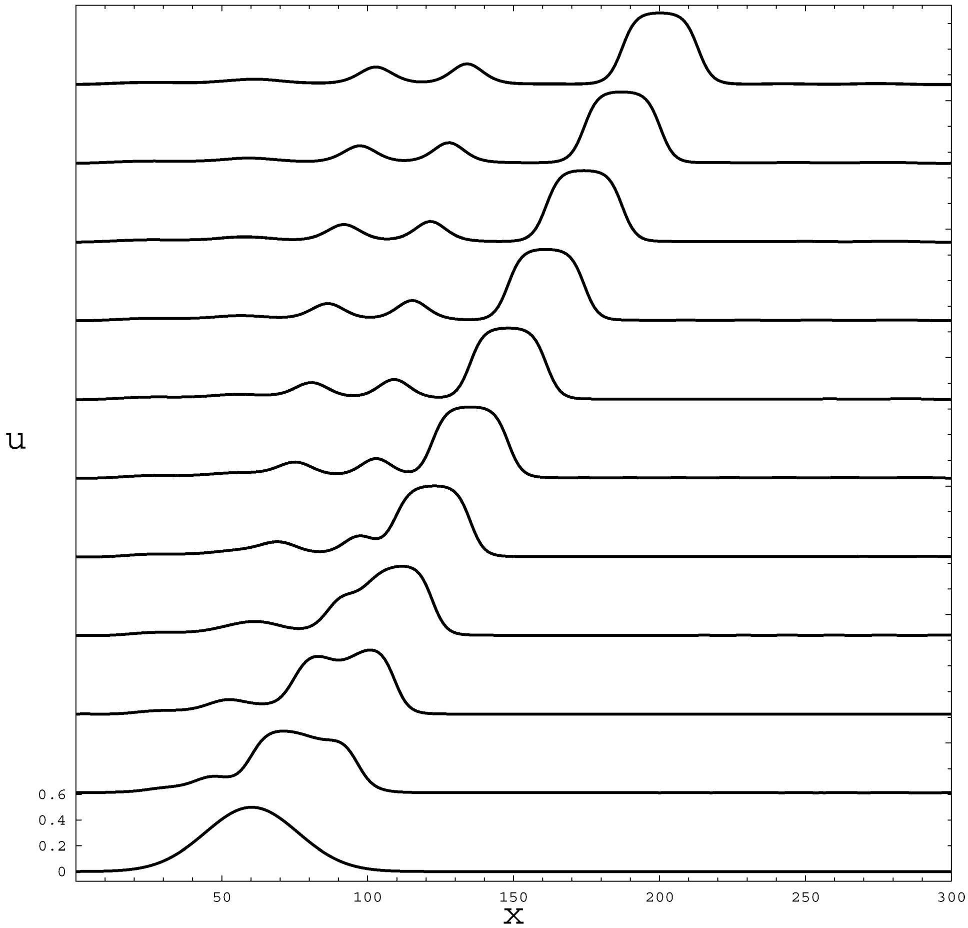

Fig. 3. Formation of the 'fat' solitary wave from the wide Gaussian input for $c = -1$, $b = 1$, $d = 1$, $f = -1$. Initial pulse parameters are: $A = 0.5$, $\sigma = 0.002$.

the initial pulse amplitude and on the sign of b. A similar tendency is observed for a higher initial amplitude when predominant cubic nonlinearity excludes an influence of the quadratic one on the sign of the wave amplitude like in the exact solution.

The shape of the solitary waves may also depend upon the parameters of the input. Indeed, numerical simulations show us that usually increasing the width or the amplitude of the initial pulse provides for $c < 0$ the formation of the so-called 'fat' solitary wave, see Fig. 3, whose shape differs from other usual monotonic solitary waves in the train. The difference is that widening of the wave due to the widening of the input (or growth of its amplitude) is accompanied by much smaller growth of the amplitude in comparison with what happens for $c > 0$. Only one 'fat' solitary wave may be generated, and no alternate transition from monotonic to oscillatory 'fat' solitary wave is observed varying values of f in contrast to the case of the 5th order KdV equation [10].

It is interesting to note that the shape of the solitary wave in Fig. 3 is similar to the shape of the exact travelling wave solution for $c < 0$ found in [13,14] for the Gardner equation, a particular case of Eq. (1) for $f = r = s = 0$. Also one can find a similarity with the generation of the Gardner equation solitary wave from Gaussian input for $c < 0$ in [15]. Hence we have found that the Gardner 'fat' solitary wave may exist in a more general case and that the 5th order derivative term, $f u_{xxxxx}$, does not prevent a formation of it.

Acknowledgements

Part of this work was completed when A.V.P. held a CNRS Visiting Research position at the LMM, UPMC. G.A.M. benefits from a Max Planck Award for International Cooperation.

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