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Eshelby tensor for a crack in an orthotropic elastic medium

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Abstract

In the present Note, we provide new analytical expressions of the components of Hill tensor $\mathbb{P}$ (or equivalently the Eshelby tensor $\mathbb{S}$) associated to an arbitrarily oriented crack in orthotropic elastic medium. The crack is modelled as an infinite cylinder along a symmetry axis of the matrix, with low aspect ratio. The three dimensional results obtained show explicitly the interaction between the primary (structural) anisotropy and the crack-induced anisotropy. They are validated by comparison with existing results in the case where the crack is in a symmetry plane. ***To cite this article: C. Gruescu et al., C. R. Mecanique 333 (2005).***

Résumé

Tenseur d'Eshelby pour une fissure en milieu élastique orthotrope. On présente une méthode permettant de déterminer analytiquement le tenseur d'Eshelby $\mathbb{S}$ (ou, de manière équivalente, du tenseur de Hill $\mathbb{P}$) associé à une fissure arbitrairement orientée dans un milieu orthotrope. La fissure est modélisée comme un cylindre infini suivant un des axes d'orthotropie, de faible rapport d'aspect. Les résultats analytiques obtenus mettent en évidence l'intéraction entre l'anisotropie structurale du matériau et l'anisotropie induite par la fissuration. Ils sont validés par comparaison avec des résultats existants dans le cas où la fissure se trouve dans un plan de symétrie. ***Pour citer cet article : C. Gruescu et al., C. R. Mecanique 333 (2005).***

Keywords: Damage; Homogenization; Micromechanics; Cracks; Anisotropy; Eshelby tensor

Mots-clés : Endommagement ; Homogénéisation ; Micromécanique ; Fissuration ; Anisotropie ; Tenseur d'Eshelby

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1. Introduction

Microcracking is one of the basic mechanisms of inelastic deformation for a large class of materials ranging from geologic materials (Sedimentary Rocks) to man-made materials (Concrete, Ceramics Matrix Composites). The modelling of the mechanical behavior for such deteriorating materials is classically performed in the framework of Continuum Damage Mechanics, using either a purely macroscopic approach or a micromechanics technique. In the latter, the determination of the overall properties of the cracked material requires primarily the calculation of the Eshelby tensor $\mathbb{S}$ (or equivalently the Hill tensor $\mathbb{P}$) associated to a crack. Existing works concern linear isotropic solid matrix containing penny-shaped cracks (see for instance [1–3]). For materials exhibiting a primary (structural) anisotropy, only few results exist and correspond to the case when the crack lies in one plane of symmetry of the solid matrix (see [4–6]). The main difficulty comes from the lack of an analytical expression for $\mathbb{P}$ for arbitrarily oriented cracks. The objective of this note is to provide new results for the case of arbitrarily oriented cracks embedded in an orthotropic solid matrix. *Summation convention on repeated indices is adopted*.

2. Determination of $\mathbb{P}$ for an arbitrarily oriented crack in an orthotropic medium

2.1. Introduction

Let us consider an orthotropic solid matrix (with a stiffness tensor $\mathbb{C}^s$), weakened by a crack. The geometrical modelling of the crack in it's associated local frame is given by:

$$\frac{z_1^2}{a^2} + \frac{z_2^2}{b^2} = 1, \quad -\infty < z_3 < \infty \tag{1}$$

That is, the crack is modelled as an infinite cylinder in the direction 3, having an elliptical section and low aspect ratio $X = b/a$ (see Fig. 1). Let us first note that in the local frame of the crack the components of the stiffness tensor are obtained from usual transformation rules: $C_{ijkl} = U_{ip}U_{jq}U_{kr}U_{ls}C^s_{pqrs}$, with:

$$U = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

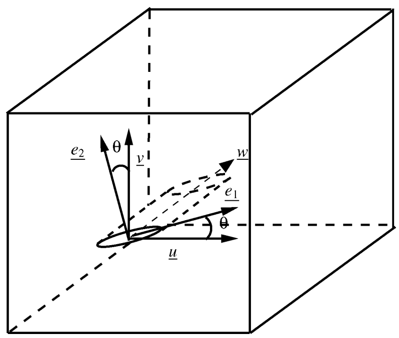

Fig. 1. Crack modelled as an infinite cylinder with elliptical section; the aspect ratio is $X = b/a$.

We note that the stiffness tensor of the crack depends on the angle θ which defines the crack plane and presents an apparent monoclinic symmetry. The determination of $\mathbb{P}$ is based on the work of Kinoshita and Mura [7] (see also Faivre [8], Laws [4] or Willis [9]) who proved that:

$$P_{ijkl} = \frac{ab}{\pi} \int_0^{2\pi} \frac{T_{ijkl}(\xi_1, \xi_2)}{a^2\xi_1^2 + b^2\xi_2^2} \, \mathrm{d}\psi \tag{3}$$

which implies an integration over the unit circle centered at the origin, in the plane (ξ_1, ξ_2): $|\underline{\xi}| = 1$, i.e. for $\underline{\xi} = \cos\psi\, \underline{e}_1 + \sin\psi\, \underline{e}_2$. The components of the fourth order tensor $\mathbb{T}$ can be expressed:

$$T_{ijkl}(\xi_1, \xi_2) = D_{ijkl}(\xi_1, \xi_2, 0), \quad \text{with:} \tag{4}$$

$$D_{ijkl} = \frac{1}{4}\left(\xi_i K_{jk}^{-1}\xi_l + \xi_j K_{ik}^{-1}\xi_l + \xi_i K_{jl}^{-1}\xi_k + \xi_j K_{il}^{-1}\xi_k\right) \tag{5}$$

$\mathbf{K} = \underline{\xi} \cdot \mathbb{C} \cdot \underline{\xi}$ represents the acoustic tensor associated with $\mathbb{C}$ and the $\underline{\xi}$ direction: $K_{ik}(\underline{\xi}) = C_{ijkl}\xi_j\xi_l$.

Note that the anisotropy of the elastic solid matrix affects the determination of $\mathbb{P}$ through the acoustic tensor $\mathbf{K}$. Besides, the dependence of $\mathbb{P}$ on the crack's orientation is understood from the fact that the above expressions imply calculation in the local frame of the crack. We note that the components of $\mathbb{P}$ can be put in the following form:

$$P_{ijkl} = \frac{1}{4}[\mathcal{M}_{ijkl} + \mathcal{M}_{jikl} + \mathcal{M}_{ijlk} + \mathcal{M}_{jilk}], \quad \text{where:} \tag{6}$$

$$\mathcal{M}_{ijkl} = \frac{ab}{\pi} \int_0^{2\pi} \frac{\xi_i K_{jk}^{-1}\xi_l}{a^2\xi_1^2 + b^2\xi_2^2} \, \mathrm{d}\psi \tag{7}$$

2.2. Developed methodology for the determination of $\mathbb{P}$

Based on the recent study of Suvorov and Dvorak [10] and following the procedure described by Ting and Lee [11],[1] we consider the two fixed unit orthogonal vectors $\underline{e}_1$ and $\underline{e}_2$ in the plane $\xi_3 = 0$. Recalling that any unit vector in this plane is defined as $\underline{\xi} = \cos\psi\, \underline{e}_1 + \sin\psi\, \underline{e}_2$, $\mathbf{K} = \underline{\xi} \cdot \mathbb{C} \cdot \underline{\xi}$ takes the form:

$$\mathbf{K} = (\cos\psi)^2\mathbf{Q} + \cos\psi \sin\psi(\mathbf{R} + \mathbf{R}^T) + (\sin\psi)^2\mathbf{T} \tag{8}$$

By the substitution $z = \cot\psi$, (8) can be rewritten in the following form:

$$\mathbf{K}(\psi) = (\sin\psi)^2\left[\mathbf{Q}z^2 + z(\mathbf{R} + \mathbf{R}^T) + \mathbf{T}\right] = (\sin\psi)^2\mathbf{K}(z), \quad \text{with:} \tag{9}$$

$$\mathbf{K}(z) = z^2\mathbf{Q} + z(\mathbf{R} + \mathbf{R}^T) + \mathbf{T} \tag{10}$$

The second order tensors $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{T}$ are defined as: $\mathbf{Q} = \underline{e}_1 \cdot \mathbb{C} \cdot \underline{e}_1$; $\mathbf{R} = \underline{e}_1 \cdot \mathbb{C} \cdot \underline{e}_2$; $\mathbf{T} = \underline{e}_2 \cdot \mathbb{C} \cdot \underline{e}_2$. In order to evaluate $\mathbb{P}$ (Eq. (4)) one needs to invert $\mathbf{K}(z)$. Denoting $|\mathbf{K}(z)|$ the determinant of $\mathbf{K}(z)$ and $\widetilde{\mathbf{K}}(z)$ it's adjoint, one has: $\mathbf{K}(z) \cdot \widetilde{\mathbf{K}}(z) = |\mathbf{K}(z)|\mathbf{1}$; it follows that:

$$\mathbf{K}^{-1}(z) = \frac{\widetilde{\mathbf{K}}(z)}{|\mathbf{K}(z)|} \tag{11}$$

[1] Note that these authors have not studied the case of an arbitrarily oriented inclusion in an orthotropic solid matrix.

Similarly, for the tensor $\Delta = \underline{\xi} \otimes \underline{\xi}$ which, with $\mathbf{K}^{-1}$, enters in definition (4), it can be verified that: $\Delta(\psi) = (\sin\psi)^2 \Delta(z)$ with $\Delta(z) = z^2 \underline{e}_1 \otimes \underline{e}_1 + z(\underline{e}_1 \otimes \underline{e}_2 + \underline{e}_2 \otimes \underline{e}_1) + \underline{e}_2 \otimes \underline{e}_2$. Moreover, we note that $a^2\xi_1^2 + b^2\xi_2^2 = (\sin\psi)^2(a^2z^2 + b^2)$. By taking into account the variable change, $z = \cot\psi$, (7) can be written:

$$\mathcal{M}_{ijkl} = \frac{ab}{\pi} \int_{-\infty}^{\infty} \frac{\widetilde{K}_{jk}(z)\Delta_{il}(z)}{(a^2z^2 + b^2)|\mathbf{K}(z)|} \, \mathrm{d}z = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{X\widetilde{K}_{jk}(z)\Delta_{il}(z)}{(z^2 + X^2)|\mathbf{Q}| f(z)} \, \mathrm{d}z \tag{12}$$

where Eq. (9), identity $|\mathbf{K}(z)| = |\mathbf{Q}| f(z)$ and the aspect ratio $X = \frac{b}{a}$ were used.

The interest of (12) lies in the fact that it allows the evaluation of $\mathcal{M}_{ijkl}$ with the help of the residues theorem, the function to be integrated being holomorphic out of the poles that have to be determined.

An interesting observation is that $z = \pm \mathrm{i}X$ are two of these poles. It is also useful to note that in the general case $f(z)$ is a polynomial function of degree 6, with three 'pairs' of complex conjugate roots. Obviously, these are the roots of $\mathbf{K}(z) = 0$. Noting z_p ($p = 1, 2, 3$), the roots whose imaginary part is positive, one has:

$$f(z) = (z - z_1)(z - \bar{z}_1)(z - z_2)(z - \bar{z}_2)(z - z_3)(z - \bar{z}_3) \tag{13}$$

The application of the residues theorem for the evaluation of $\mathcal{M}$ tensor is detailed in the next section.

3. Analytical expression of the $\mathbb{P}$ tensor and validation

3.1. Analytical expression of the components of Hill's tensor $\mathbb{P}$

In the case of a crack with an arbitrary orientation (represented by an angle θ) with respect to the symmetry axis 1 of the material, the calculation performed in the local frame of the crack leads to a polynomial function (of degree 6) which allows to write $f(z)$ as the product of a polynomial function of degree 4 ($f_1(z)$) and a polynomial function of degree 2 ($f_2(z)$). Therefore:

$$\left|\mathbf{K}(z)\right| = |\mathbf{Q}| f(z) = f_1(z) \cdot f_2(z) \tag{14}$$

The determinant $|\mathbf{Q}|$ which appears in this expression takes the form:

$$|\mathbf{Q}| = C^s_{1111} C^s_{1212} (\cos^2\theta + \alpha \sin^2\theta)(\cos^2\theta + \beta \sin^2\theta)(C^s_{3232} \sin^2\theta + C^s_{3131} \cos^2\theta) \tag{15}$$

The analytical expressions of $f_1(z)$, $f_2(z)$ are presented in Appendix A.1. The solutions of $f_1(z) = 0$ are:

$$\begin{aligned} z_1 &= \frac{\mathrm{i}\sqrt{\alpha}\cos\theta - \sin\theta}{\cos\theta + \mathrm{i}\sqrt{\alpha}\sin\theta}; \qquad \overline{z_1} = \frac{\mathrm{i}\sqrt{\alpha}\cos\theta + \sin\theta}{-\cos\theta + \mathrm{i}\sqrt{\alpha}\sin\theta} \\ z_2 &= \frac{\mathrm{i}\sqrt{\beta}\cos\theta - \sin\theta}{\cos\theta + \mathrm{i}\sqrt{\beta}\sin\theta}; \qquad \overline{z_2} = \frac{\mathrm{i}\sqrt{\beta}\cos\theta + \sin\theta}{-\cos\theta + \mathrm{i}\sqrt{\beta}\sin\theta} \end{aligned} \tag{16}$$

with α and β the complex conjugate roots of the characteristic equation of the orthotropic 2D solid:

$$C^s_{1111} C^s_{1212} x^2 - \left(C^s_{1111} C^s_{2222} - {C^s_{1122}}^2 - 2C^s_{1122} C^s_{1212}\right)x + C^s_{2222} C^s_{1212} = 0 \tag{17}$$

The roots of $f_2 = 0$ are:

$$\begin{aligned} z_3 &= -\frac{\sin(2\theta)(C^s_{3232} + C^s_{3131}) + 2\mathrm{i}\sqrt{C^s_{3232} C^s_{3131}} \cos(2\theta)}{2(C^s_{3131} \cos^2\theta + C^s_{3232} \sin^2\theta)} \\ \overline{z_3} &= -\frac{\sin(2\theta)(C^s_{3232} + C^s_{3131}) - 2\mathrm{i}\sqrt{C^s_{3232} C^s_{3131}} \cos(2\theta)}{2(C^s_{3131} \cos^2\theta + C^s_{3232} \sin^2\theta)} \end{aligned} \tag{18}$$

Then, using (16) and (18), $|\mathbf{K}(z)|$ can be rewritten in the form:

$$\left|\mathbf{K}(z)\right| = |\mathbf{Q}|(z-z_1)(z-\overline{z_1})(z-z_2)(z-\overline{z_2})(z-z_3)(z-\overline{z_3}) \tag{19}$$

The poles being assumed to be different, the expression of $\mathcal{M}$ provided by (12) can be expressed:

$$\mathcal{M}_{ijkl} = 2\mathrm{i}\left\{\frac{\widetilde{K}_{jk}(\mathrm{i}X)\Delta_{il}(\mathrm{i}X)}{2\mathrm{i}|\mathbf{Q}|f(\mathrm{i}X)} + \sum_{i=1}^{3}\frac{\widetilde{K}_{jk}(z_i)\Delta_{il}(z_i)X}{|\mathbf{Q}|f'(z_i)(X^2+z_i^2)}\right\} \tag{20}$$

It is recalled here that $\widetilde{K}_{jk}$ are the components of the adjoint of $\mathbf{K}$.

Considering only the approximation to first order in X at the value $X=0$, (20) gives:

$$\mathcal{M}_{ijkl} = \frac{\widetilde{K}_{jk}(0)\Delta_{il}(0)}{f(0)|\mathbf{Q}|} + \frac{2\mathrm{i}X}{|\mathbf{Q}|}\left\{\sum_{i=1}^{3}\frac{\tilde{K}_{jk}(z_i)\Delta_{il}(z_i)}{f'(z_i)z_i^2} + \frac{1}{2f(0)}\left[\widetilde{K}'_{jk}(0)\Delta_{il}(0) + \widetilde{K}_{jk}(0)\Delta'_{il}(0) - \widetilde{K}_{jk}(0)\Delta_{il}(0)\frac{f'(0)}{f(0)}\right]\right\} \tag{21}$$

It is shown that the imaginary part of $\mathcal{M}$ is null:

$$\Im(\mathcal{M}_{ijkl}) = \frac{2X}{|\mathbf{Q}|}\left\{\Re\left[\frac{\widetilde{K}_{jk}(z_1)\Delta_{il}(z_1)}{f'(z_1)z_1^2}\right] + \Re\left[\frac{\widetilde{K}_{jk}(z_2)\Delta_{il}(z_2)}{f'(z_2)z_2^2}\right] + \Re\left[\frac{\widetilde{K}_{jk}(z_3)\Delta_{il}(z_3)}{f'(z_3)z_3^2}\right]\right\} + \frac{X}{f(0)|\mathbf{Q}|}\left\{\left[\widetilde{K}'_{jk}(0)\Delta_{il}(0) + \widetilde{K}_{jk}(0)\Delta'_{il}(0) - \widetilde{K}_{jk}(0)\Delta_{il}(0)\frac{f'(0)}{f(0)}\right]\right\} = 0 \tag{22}$$

and its real part gives:

$$\mathcal{M}_{ijkl} = \Re(\mathcal{M}_{ijkl}) = \frac{\tilde{K}_{jk}(0)\Delta_{il}(0)}{f(0)|\mathbf{Q}|} - \frac{2X}{|\mathbf{Q}|}\Im\sum_{i=1}^{3}\frac{\widetilde{K}_{jk}(z_i)\Delta_{il}(z_i)}{f'(z_i)z_i^2} \tag{23}$$

These results are associated to the local frame of the crack. A last change of basis allows to express the tensor $\mathbb{P}$ in the global frame defined by the symmetry axes of the solid matrix. The detailed expressions of the nine components of $\mathbb{P}$ in this global frame are given in Appendix A (Eqs. (A.4)).

3.2. Validation of obtained results

As a first validation of the results presented in Appendix A we consider the case of a parallel crack whose orientation coincides with the symmetry axis 1 of the solid matrix. This configuration has been studied by Laws [4], who gives the following non zero components of $\mathbb{P}$:

$$\begin{aligned}
&P_{1111} = \frac{C^s_{2222} + C^s_{1212}\sqrt{\alpha\beta}}{C^s_{1111}C^s_{1212}\sqrt{\alpha\beta}(\sqrt{\alpha}+\sqrt{\beta})}X; && P_{2222} = \frac{1}{C^s_{2222}} + \frac{C^s_{2222} - C^s_{1212}(\alpha+\beta+\sqrt{\alpha\beta})}{C^s_{2222}C^s_{1212}\sqrt{\alpha\beta}(\sqrt{\alpha}+\sqrt{\beta})}X \\
&P_{1122} = P_{2211} = -\frac{C^s_{1122} + C^s_{1212}}{C^s_{1111}C^s_{1212}\sqrt{\alpha\beta}(\sqrt{\alpha}+\sqrt{\beta})}X; && P_{3232} = \frac{1}{4C^s_{3232}} - \frac{\sqrt{C^s_{3131}}}{4{C^s_{3232}}^{3/2}}X \\
&P_{3131} = \frac{1}{4\sqrt{C^s_{3232}C^s_{3131}}}X; && P_{1212} = \frac{1}{4C^s_{1212}} - \frac{C^s_{1111}C^s_{2222} - {C^s_{1122}}^2}{4C^s_{1111}{C^s_{1212}}^2\sqrt{\alpha\beta}(\sqrt{\alpha}+\sqrt{\beta})}X
\end{aligned} \tag{24}$$

which corresponds to the particular case ($\theta = 0$) of the general expressions (A.4) established in the present Note. The general analytical new expressions provided are easily incorporated into various homogenization schemes (dilute scheme, Mori–Tanaka model, self-consistent scheme) in order to determine the macroscopic properties of an orthotropic solid matrix weakened by cracks. For application purposes in [12] a more general validation of pure 2D solutions (see [13]) is also provided.

Appendix A. Analytical expressions of $|\mathbf{K}(z)|$ and the components of the $\mathbb{P}$ tensor

A.1. Expressions of $f_1(z)$, $f_2(z)$ in $|\mathbf{K}(z)|$

The two polynomial functions $f_1(z)$, $f_2(z)$ entering in (14) are respectively:

$$\begin{aligned} f_1(z) = {} & \big[C^s_{1111}C^s_{1212}(\cos\theta)^4 + C^s_{2222}C^s_{1212}(\sin\theta)^4 + \chi(\cos\theta)^2(\sin\theta)^2\big]z^4 \\ & + \big\{\big[2C^s_{1111}C^s_{1212}(\cos\theta)^2 - 2C^s_{2222}C^s_{1212}(\sin\theta)^2\big] - \chi\cos 2\theta\big\}\sin 2\theta z^3 \\ & + \big\{\chi\big[1 - 6(\cos\theta)^2(\sin\theta)^2\big] + 6C^s_{1212}(C^s_{1111} + C^s_{2222})(\cos\theta)^2(\sin\theta)^2\big\}z^2 \\ & + \big[2C^s_{1111}C^s_{1212}(\sin\theta)^2 + 2C^s_{2222}C^s_{1212}(\cos\theta)^2 + \chi\cos 2\theta\big]\sin 2\theta\, z \\ & + C^s_{1111}C^s_{1212}(\sin\theta)^4 + C^s_{2222}C^s_{1212}(\cos\theta)^4 + \chi(\sin\theta)^2(\cos\theta)^2 \end{aligned} \tag{A.1}$$

where: $\chi = (C^s_{1111}C^s_{2222} - {C^s_{1122}}^2 - 2C^s_{1122}C^s_{1212})$, and:

$$f_2(z) = (C^s_{3232}\sin^2\theta + C^s_{3131}\cos^2\theta)z^2 + (C^s_{3232} + C^s_{3131})\sin 2\theta\, z + C^s_{3232}\cos^2\theta + C^s_{3131}\sin^2\theta \tag{A.2}$$

Note that, in order to make the search of the roots of $f_1(z) = 0$ easier, we introduced the variable change $z = (u\cos\theta - \sin\theta)/(\cos\theta + u\sin\theta)$, which allowed (A.1) to be written as:

$$f_1(z) = \frac{C^s_{1111}C^s_{1212}u^4 + (C^s_{1111}C^s_{2222} - {C^s_{1122}}^2 - 2C^s_{1122}C^s_{1212})u^2 + C^s_{2222}C^s_{1212}}{(\cos\theta + u\sin\theta)^4} \tag{A.3}$$

A.2. Components of $\mathbb{P}$

The determination of the components P_{ijkl} is performed by combination of (6) and (23). In the frame defined by the material symmetry axes, the following expressions are obtained:

$$\begin{aligned} P_{1111} &= \frac{(\sin\theta)^2[C^s_{22}(\cos\theta)^2 + C^s_{66}(\sin\theta)^2]}{n} + \frac{X[p\sqrt{\alpha}(\alpha C^s_{66} - C^s_{22}) - q\sqrt{\beta}(\beta C^s_{66} - C^s_{22})]}{C^s_{11}C^s_{66}(\alpha - \beta)} \\ P_{1122} &= -\frac{(C^s_{12} + C^s_{66})(\cos\theta)^2(\sin\theta)^2}{n} + \frac{X(C^s_{12} + C^s_{66})(p\sqrt{\alpha} - q\sqrt{\beta})}{C^s_{11}C^s_{66}(\alpha - \beta)} \\ P_{2222} &= \frac{(\cos\theta)^2[C^s_{11}(\sin\theta)^2 + C^s_{66}(\cos\theta)^2]}{n} + \frac{X[(p/\sqrt{\alpha})(C^s_{66} - \alpha C^s_{11}) - (q/\sqrt{\beta})(C^s_{66} - \beta C^s_{11})]}{C^s_{11}C^s_{66}(\alpha - \beta)} \\ P_{1112} &= \sin 2\theta\left\{\frac{C^s_{22}(\cos\theta)^2 - C^s_{12}(\sin\theta)^2}{4n} + \frac{X[r\sqrt{\alpha}(\alpha C^s_{12} + C^s_{22}) - s\sqrt{\beta}(\beta C^s_{12} + C^s_{22})]}{C^s_{11}C^s_{66}(\alpha - \beta)}\right\} \\ P_{2212} &= \sin 2\theta\left\{\frac{[C^s_{11}(\sin\theta)^2 - C^s_{12}(\cos\theta)^2]}{4n} - \frac{X[r(\alpha C^s_{11} + C^s_{12}) - s\sqrt{\beta}(\beta C^s_{11} + C^s_{12})]}{C^s_{11}C^s_{66}(\alpha - \beta)}\right\} \\ P_{1212} &= \frac{[C^s_{11}(\sin\theta)^4 + C^s_{22}(\cos\theta)^4 - 2C^s_{12}(\sin\theta)^2(\cos\theta)^2]}{4n} + \frac{X(C^s_{11}C^s_{22} - {C^s_{12}}^2)(p\sqrt{\alpha} - q\sqrt{\beta})}{4C^s_{11}{C^s_{66}}^2(\alpha - \beta)} \\ P_{3131} &= \frac{(\sin\theta)^2}{m} + \frac{X}{m^2}\sqrt{\frac{C^s_{44}}{C^s_{55}}}\big[C^s_{44}(\cos\theta)^2 - C^s_{55}(\sin\theta)^2\big] \\ P_{3232} &= \frac{(\cos\theta)^2}{m} - \frac{X}{m^2}\sqrt{\frac{C^s_{55}}{C^s_{44}}}\big[C^s_{44}(\cos\theta)^2 - C^s_{55}(\sin\theta)^2\big] \\ P_{3132} &= \frac{\sin\theta\cos\theta}{m} - \frac{X\sqrt{C^s_{55}C^s_{44}}\sin 2\theta}{m^2} \end{aligned} \tag{A.4}$$

In previous expressions the following notations were used for the elastic coefficients:

$$m = 4\big[C^s_{44}(\cos\theta)^2 + C^s_{55}\sin^2\theta\big]^2; \qquad n = C_{11}C_{66}\big[(\sin\theta)^4 + \alpha\beta(\cos\theta)^4 + (\alpha+\beta)(\cos\theta)^2(\sin\theta)^2\big]$$
$$p = \big[\alpha(\cos\theta)^2 - (\sin\theta)^2\big]\big/\big[\alpha(\cos\theta)^2 + (\sin\theta)^2\big]^2; \qquad q = \big[\beta(\cos\theta)^2 - (\sin\theta)^2\big]\big/\big[\beta(\cos\theta)^2 + (\sin\theta)^2\big]^2$$
$$r = \sqrt{\alpha}\big/\big[\alpha(\cos\theta)^2 + (\sin\theta)^2\big]^2; \qquad s = \sqrt{\beta}\big/\big[\beta(\cos\theta)^2 + (\sin\theta)^2\big]^2$$

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