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An example of entropy balance in natural convection, Part 2: the *thermodynamic* Boussinesq equations

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Presented by Sébastien Candel

This work is dedicated to the memory of Bernard Spinner

Abstract

Numerical simulations of natural convection performed with the *usual* Boussinesq equations result in unbalanced irreversibility budget. The *thermodynamic* Boussinesq equations solve this problem, especially because they simulate production of kinetic energy within the fluid through its expansion and contraction. These fluid volume changes, without which natural convection would not occur, also induce heat transfer by *piston effect*. The piston effect, which appears then as an intrinsic component of buoyancy-induced natural convection, introduces the non-dimensional adiabatic temperature gradient as a control parameter of natural convection. ***To cite this article: M. Pons, P. Le Quéré, C. R. Mecanique 333 (2005).***

Résumé

Un exemple de bilan d'entropie pour la convection naturelle, Partie 2 : les équations de Boussinesq *thermodynamiques*. Les simulations numériques réalisées avec les équations de Boussinesq *usuelles* ne peuvent pas donner un bilan d'irréversibilité fermé. Les équations de Boussinesq *thermodynamiques* apportent une solution au problème, en particulier parce qu'elles incluent la production d'énergie cinétique à l'intérieur du fluide, par sa dilatation et contraction. Ces variations de volume du fluide, sans lesquelles la convection naturelle n'existerait pas, provoquent aussi un transfert de chaleur par *effet piston*. L'effet piston, qui apparaît alors consubstantiel à la convection naturelle, fait du gradient de température adiabatique adimensionné un des paramètres de contrôle de la convection naturelle. ***Pour citer cet article : M. Pons, P. Le Quéré, C. R. Mecanique 333 (2005).***

Keywords: Heat transfer; Fluid mechanics; Natural convection; Boussinesq; Thermodynamics; Irreversibility; Piston effect

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1. Introduction

For reasons of publication, the work presented here is divided in two parts, Part 1 [1] and this article (Part 2).

It has been shown in Part 1 that simulating natural convection with the *usual* Boussinesq (UB) equations leads to fundamental contradictions with the second law. Indeed, the UB equations correspond to the energy diagram Fig. 1(b), i.e. artificially introducing exchanges of work between the simulated system and its surroundings while discarding the internal exchanges between heat and work. One example of the obtained contradictions is given for a very simple case: air at $T_0 = 300$ K ($Pr = 0.71$) in a square two-dimensional differentially heated cavity ($A_r = 1$), with $Ra = 10^6$, i.e. in steady-state, with a cavity height H of 1.2866 m, and an extremely small ΔT (4.890 mK). The results of that numerical experiment are recalled in the first line of Table 1. Actually, a careful reading of literature, e.g. [2,3], shows that this case should not be calculated with UB equations. Indeed, these equations are valid when the parameter ϕ [$\phi = \beta g H T_0/(C_p \Delta T)$] is small compared to 1. In the present case, ϕ is larger than 2, and the model must take this feature into account.

2. The *thermodynamic* Boussinesq equations

2.1. A modified heat equation

Unlike the so-called *non-Boussinesq* flows with very large temperature differences (requiring *Low Mach number* models), the ΔT herein is small enough for validating the assumptions of solenoidal flow and constant fluid properties, so that the model remains in the general Boussinesq framework. The usual continuity and momentum equations (2) and (3) of [1] still hold. However, the heat equation (4) of [1] must be rewritten. We start from its enthalpic form, which permits avoiding Boussinesq's paradox (stating $\nabla \cdot \mathbf{v} = 0$ for a fluid the density of which obviously changes). This equation is:

$$\frac{\mathrm{D}T}{\mathrm{D}t} = \alpha_0 \nabla^2 T + q_v + \frac{T}{C_p}\left(\frac{\partial(\rho^{-1})}{\partial T}\right)_P \frac{\mathrm{D}P}{\mathrm{D}t} \tag{1}$$

where t is the time, q_v the density of heat released by viscous friction (not neglected for sake of energy conservation), and P the pressure (the other quantities are defined in [1]). The viscous heat-source term q_v is given by the Newton law. As only steady-states are considered herein, the total derivative DP/Dt reduces to $\mathbf{v} \cdot \nabla P$. Moreover, the first order approximation of the pressure gradient ∇P is the hydrostatic one, $-\rho_0 g \mathbf{z}$ (where $\mathbf{z}$ is the vertical upward unit vector). Indeed, natural convection is mainly due to density variations in the hydrostatic pressure field (Archimed forces), and effects such as thermoacoustics cancel in steady-state. The resulting non-dimensional heat equation is:

$$\frac{\partial\theta}{\partial\tau} + u\frac{\partial\theta}{\partial x} + w\frac{\partial\theta}{\partial z} = \frac{1}{Ra^{1/2}}\nabla^2\theta + \frac{1}{Ra^{1/2}}\frac{\beta g H}{C_p}\Phi - \frac{\beta g H}{C_p} w\left(\theta + \frac{T_0}{\Delta T}\right) \tag{2}$$

with $\Phi = 2(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})^2 + 2(\frac{\partial w}{\partial z})^2$.

This is practically the heat equation called *extended* in [3], *deep convection*, and *thermodynamic* in [4,5]. The latter expression is retained in the followings. Several thermodynamic comments can be done on Eq. (2). First, the kinetic energy lost in viscous friction is now correctly transformed into heat [$Ra^{-1/2}(\beta g H/C_p)\Phi$]. Second, Eq. (2) accounts for the work of pressure (hydrostatic field) on the flow [$-(\beta g H/C_p) w(\theta + T_0/\Delta T)$]. When the heat

Table 1
Thermodynamic balances calculated with the *usual* and *thermodynamic* Boussinesq equations, for a cavity height H of 1.2866 m at $Ra = 10^6$. In both cases, one has $\phi = 2.568$, and $\Delta T/T_0 = 16.30 \times 10^{-6}$

Boussinesq model	Nu	$N_{\Sigma q} + N_{\Sigma v}$	$N_{\Sigma q}$	$N_{\Sigma v}$	N_{Wm}	Nu_{mid}
Usual	8.8407	17.6431	8.8407	8.8024	143.48×10^{-6}	8.8407
Thermodynamic	13.1514	13.1514	9.9046	3.2468	52.92×10^{-6}	1.4156

equation is formulated with internal energy, the specific heat arising is that at constant volume C_{v}, and the equation contains one term in $\nabla \cdot \mathbf{v}$. Indeed, expansion produces work ($p\,\mathrm{d}v$ is positive) out of internal energy, contraction transforms work into internal energy. When $\nabla \cdot \mathbf{v}$ is strictly assigned to zero, like in (4) of [1], these transfers are discarded. The enthalpic formulation accounts for these internal transfers between heat and work without explicitly involving $\nabla \cdot \mathbf{v}$ while fully respecting the thermodynamic relation between C_{v}, C_p, and $(\partial P/\partial T)_\rho$ as defined by the fluid state-equation. It can also be noticed that the thermal diffusivity α appearing in the Rayleigh and Prandtl numbers involves C_p and not C_{v}, making it preferable to work with the former rather than the latter. Now, is the correct irreversibility budget recovered? Is the conversion efficiency $\eta = N_{Wm}/Nu$ fairly smaller than the Carnot factor?

2.2. Numerical results

The two source terms of (2) are implemented in the numerical model among the non-linear terms, i.e. explicitly with a linear extrapolation. The overcost in terms of CPU-time is negligible. It has been checked that the model yields $Nu_h = Nu_c$, and $N_{Wm} = N_{Wv}$, as it must. The other results of the calculations are presented in the second line of Table 1 where they can be compared to the UB calculations. First, the irreversibility budget ($N_{\Sigma q} + N_{\Sigma v} = Nu$) is now balanced, and this is true for any calculation done with the *thermodynamic* Boussinesq (TB) equations. Second, in the present case, the conversion efficiency is $\eta = 4.024 \times 10^{-6}$, a value slightly less than 25% of the Carnot efficiency, a reasonable ratio. It can be seen that the system simulated with the TB equations fulfils the four thermodynamic milestones stated in Part 1, demonstrating its thermodynamic consistency with the physical phenomenon. This consistency also justifies a posteriori considering only the hydrostatic field as the pressure gradient appearing in (1). The results can now be studied in some more detail.

2.3. Involvement of piston effect

Table 1 shows that choosing the UB equations instead of the TB equations has the following consequences (confirmed by all the other calculations we have done). The heat transfer is underestimated, the distribution between the conductive and viscous irreversibilities is not correct, the latter being overestimated altogether with the mechanical energy N_{Wm}. This feature indicates that the flow intensity is overestimated (in the present case, the maximal speed is twofold that obtained with the TB equations).

It must be noticed that the departure from the UB equations most frequently investigated in the literature is the non-Boussinesq problem, i.e. when the temperature difference is so large that the assumptions of solenoidal flow and constant fluid properties no longer hold. Among recent results, Fröhlich et al. investigate the Rayleigh–Bénard configuration at $Ra = 2000$, with $\Delta T = 300$ K [6], or Vierendeels et al. investigate the square differentially-heated cavity with at $Ra = 10^6$–10^7, with $\Delta T = 600$ K [7]; both find Nusselt numbers differing from the UB solution by some 2–4%. The present problem is completely different: the temperature difference is very small (some milliKelvin) and the cavity is rather high (about one meter), but still under the domain of geophysics (where the UB approximation is accepted not to hold). It can be seen that the effects evidenced herein are much more significant than the ones studied in the non-Boussinesq problem.

From the description above, one question also arises: how can the heat transfer be underestimated by the UB system, when convection is overestimated?

In the UB case, the heat flux transferred by conduction plus advection through any vertical line crossing the cavity equates the heat flux at the active walls. Considering the heat transfer through the vertical line at mid-cavity, and non-dimensionalising it like the other energy fluxes, one obtains the Nusselt number at mid-cavity:

$$Nu_{\mathrm{mid}} = \frac{1}{A_r}\int_0^1 \left(Ra^{1/2} u\theta - \frac{\partial \theta}{\partial x} \right) \mathrm{d}z$$

where the quantities u, θ, and $\partial\theta/\partial x$ are taken at $x = x_c/2$. In the UB framework, one always obtains $Nu_{\mathrm{mid}} = Nu_h = Nu_c$. On the opposite, with the TB equations Nu_{mid} is always smaller than Nu (in our example $Nu_{\mathrm{mid}} = 1.4156$), so that a certain proportion of the total heat flux (about 89% herein) is transferred from one wall to the other by a process which is neither conduction nor advection through the mid-line. None of the two models includes radiation, and viscous friction cannot be invoked for explaining that additional heat transfer. Among the terms of the heat equation, remains the work of the pressure field, given by the last term $-(\beta g H/C_p)(\theta + T_0/\Delta T)w$ in Eq. (2). This term can be split as $-(\beta g H/C_p)\theta w - \phi w$, the first part of which can readily be combined with the viscous heat-source term. Eq. (2) can then be rewritten as:

$$\frac{\partial \theta}{\partial \tau} + u\frac{\partial \theta}{\partial x} + w\frac{\partial \theta}{\partial z} = \frac{1}{Ra^{1/2}}\nabla^2\theta + \frac{\beta g H}{C_p}\left(\frac{\Phi}{Ra^{1/2}} - \theta w\right) - \phi w \tag{3}$$

Integrating $(\beta g H/C_p)\Phi Ra^{-1/2}$ yields the total heat flux released by viscous friction, forcedly equal to the lost kinetic energy N_{Wv}, and integrating $-(\beta g H/C_p)\theta w$ over the whole fluid domain yields $-N_{Wm}$, see Eq. (8) in [1]. $[(\beta g H/C_p)(\Phi Ra^{-1/2} - \theta w)]$ has thus a zero integral; in addition its order of magnitude is $(\beta g H/C_p)$ (4×10^{-5} herein), it is locally very small. The last term $(-\phi w)$ also has a zero integral, but as $\mathrm{T}_0/\Delta T \gg 1$, it is by far the dominant part of the work related to the pressure forces. Talking of orders of magnitude, that last term is, like ϕ, not *necessarily negligible* [8–10]. It is even significant herein ($\phi = 2.568$). Talking of energy, $-\phi w$ is a heat-sink term there where the fluid climbs up, i.e. close to the hot wall, transforming thus part of the thermal energy transmitted to the fluid into mechanical energy (flow against gravity). Close to the cold wall, where the fluid falls down, $-\phi w$ is a net heat-source term due to the transformation of mechanical energy (flow along gravity) into thermal energy. In the present case, the integral of $Ra^{1/2}\phi w$ over the hot half of the cavity (i.e. between the hot wall and the mid-line) yields 11.7357, i.e. the difference $Nu - Nu_{\mathrm{mid}}$, within a quantity that compares to N_{Wm}. The term $-\phi w$ is responsible for the additional heat transfer. It can be here noticed that $-(\beta g H/C_p)\theta w$ and $-\phi w$ both originate from the same term of (1), showing that the net mechanical energy responsible for the fluid motion (N_{Wm}) results from a very weak unbalance in the exchanges between heat and work above-described. These exchanges between heat and work also result in a global heat transfer, from the vicinity of the hot wall directly to that of the cold wall, operating in parallel to conduction and convection, and much more significant than the mechanical energy generated in the fluid or dissipated in viscous friction. According to the literature, this mode of heat transfer is the *piston effect*, well known in the field of near-critical fluids, see [11] and the bibliography therein. Studies of piston effect in fluids away from their near-critical region are quite recent and consider only transient situations [12]. The piston effect is indeed mainly known as a transient phenomenon. However the present study evidences its involvement in steady-state (without thermoacoustic waves). The paradox is only apparent. Indeed, from a Lagrangian point of view, a single particle is alternatively heated up and cooled down along the streamline it follows, it alternatively expands and contracts. In the cavity, there are always fluid particles that expand somewhere while other particles contract somewhere else, a double process that results in an apparent heat transfer from the former to the latter by the so-called piston effect. In other words, the volume changes undergone by the fluid because of temperature changes induce *altogether* natural convection *and* piston effect. Because of this common origin, the piston effect is an intrinsic component of natural convection (when induced by buoyancy). Fig. 1(c) shows the diagram energy of natural convection, correctly simulated by the TB model.

The heat equation (3) shows that the piston effect is ruled by the non-dimensional parameter $\phi = (\beta g H/C_p) \times T_0/\Delta T$. Tritton mentions ϕ as the adiabatic temperature gradient $(\beta g T_0/C_p)$ non-dimensionalised in the problem

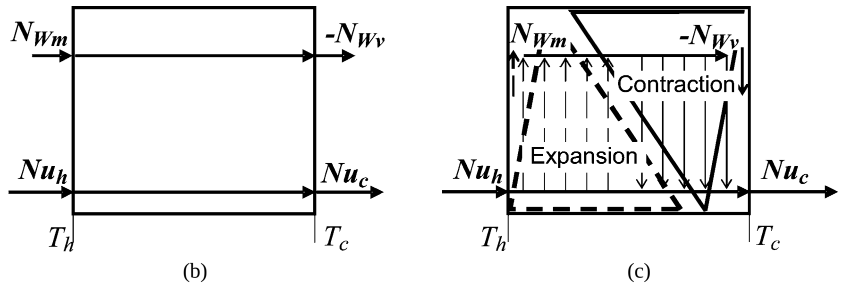

Fig. 1. Energy diagram describing natural convection in steady-state. The lower level represents thermal energy, the upper level mechanical energy. The fluxes of thermal and mechanical energies are indicated in non-dimensional form (Nu_h, Nu_c, N_{Wm}, N_{Wv}). For sake of clarity, the heat generated by viscous friction is not mentioned. (b) *Usual* Boussinesq system: the two heat fluxes are equal, the fluid also receives mechanical energy from outside, and viscous dissipation corresponds to a sink of kinetic energy not transformed into heat. (c) Real case and *thermodynamic* Boussinesq system: expansion and contraction induce significant transfers between thermal and mechanical energy, the result of which is net production of kinetic energy AND heat transfer by piston effect, viscous friction generates some heat in the fluid.

framework (i.e. divided by $\Delta T/H$) [2]. The parameter ϕ also appears in the Schwarzchild criterion for the onset of natural convection in the Rayleigh–Bénard problem near critical point [13]. The present study leads to the conclusion that ϕ is a control parameter of buoyancy-induced natural convection.

3. Conclusion

The thermodynamic inconsistencies appearing in the *usual* Boussinesq calculations are solved when the heat equation includes the work of pressure forces, leading to the *thermodynamic* Boussinesq equations. As a consequence, the piston effect, which also contributes to heat transfer, is an intrinsic component of natural convection. As the piston effect is ruled by the non-dimensional adiabatic temperature gradient $\phi = (\beta g H/C_p)T_0/\Delta T$, ϕ is a control parameter of buoyancy-induced natural convection.

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