High-Order Methods for the Numerical Simulation of Vortical and Turbulent Flows

A penalization method applied to the wave equation

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Abstract

In this Note we investigate the mathematical properties of the volume penalization method applied to the one-dimensional wave equation. Generally speaking, the penalization method allows one to handle complex geometries by simply adding a term to the equation to impose the boundary conditions. We study the convergence of the method with regards to the penalization parameter and we present error and stability analyses for the wave equation. Numerical simulations using a finite difference scheme illustrate the results. ***To cite this article: A. Paccou et al., C. R. Mecanique 333 (2005).***
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Résumé

Une méthode de pénalisation appliquée à l'équation des ondes. Nous étudions une méthode de pénalisation pour l'équation des ondes unidimensionelle. Nous présentons une analyse de convergence théorique et une vérification numérique dans le cadre d'une discrétisation uniforme par différences finies. ***Pour citer cet article : A. Paccou et al., C. R. Mecanique 333 (2005).***
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Une des difficultés principale des simulations numériques de phénomènes physiques réels réside dans la prise en compte de géométries complexes. De nombreuses méthodes ont été développées (changements de coordonnées, maillages adaptés,...) mais l'une des plus simple à mettre en oeuvre en pratique, est la méthode de pénalisation

introduite par Arquis et Caltagirone [3]. Elle consiste à modéliser un obstacle solide par un milieu de porosité η, puis à faire tendre vers zéro cette porosité que l'on appelle aussi le paramètre de pénalisation.

De nombreuses simulations numériques utilisant cette méthode ont été réalisées depuis à partir de techniques de discrétisation aussi diverses que les différences finies, les volumes finis [4,5,16], les méthodes pseudo-spectrales [6,2,14,7,8] ou encore les méthodes ondelettes [9], pour ne citer que quelques références.

L'analyse mathématique de la convergence de cette méthode a été effectuée par Angot et al. [15] pour les équations de Navier–Stokes. Une étude très détaillée à également été réalisée par Kevlahan et Ghidaglia [6] sur l'équation de la chaleur unidimensionelle.

Nous nous proposons dans cet article d'étudier les propriétés de la méthode de pénalisation sur une équation hyperbolique simple, l'équation de ondes. Nous présentons une analyse théorique de la convergence que nous comparons aux résultats numériques obtenus par un schéma aux différences finies appliqué à cette équation.

Nous considérons le système (1) décrivant le mouvement d'une corde vibrante fixée en ses extrémités $x = 0$ et $x = \pi$. La solution exacte de ce système est connue, (2), ce qui permettra de comparer avec la solution du système pénalisé.

La condition en $x = 0$, $U(0, t) = 0$, est alors remplacée par l'ajout dans le système (1) du terme $\frac{1}{\eta}\chi_{\mathbb{R}^-}(x)$. Le système pénalisé est alors (3), et la solution est maintenant recherchée sur $]-\infty, \pi[$.

En utilisant une transformée de Laplace en temps, il est possible de calculer analytiquement la solution exacte du système (3) et par la suite l'erreur avec la solution du système (1).

Une majoration de cette erreur en norme L^∞, (5), et en norme L^1, (6), met en évidence une décroissance en $\sqrt{\eta}$ et une propagation dans $[0, \pi]$ à la vitesse c.

Le système (3) est ensuite discrétisé par un schéma aux différences finies et la solution est comparée à la solution exacte (2). La propagation de l'erreur à la vitesse c est visible sur la Fig. 2, alors que la convergence en fonction de η est représentée en Fig. 3.

Le comportement théorique de l'erreur L^1 en $\sqrt{\eta}$ est observé pour $\eta \geqslant \mathrm{d}x^2$ suivi d'une décroissance plus rapide, en $\mathrm{O}(\eta)$, pour $\eta \leqslant \mathrm{d}x^2$.

Ce phénomène a également été observé par Kevlahan et Ghidaglia [6] pour l'équation de la chaleur et pourrait être lié à la résolution de la couche limite au voisinage de $x = t$.

En conclusion, nous avons analysé de façon théorique le comportement de la méthode de pénalisation sur une équation hyperbolique simple. Les résultats ont été vérifiés sur un modèle numérique pour lequel une convergence meilleure que prévue à été observée pour une certaine gamme de paramètres.

Ces travaux sont préliminaires à des simulations numériques des équations d'Euler pénalisées pour des géométries complexes.

1. Introduction

Complex geometries and the treatment of boundary conditions are among the main challenges in modern computational fluid dynamics (CFD). Grid generation plays hereby an important role but different strategies have been developed so far, such as body fitted grids, coordinate transforms, fictious domain approaches (see for example [1]) and surface [2] or volume penalization methods [3].

The latter can be interpreted as modeling solid obstacles in viscous flows by porous media which permeability tends to zero with the so-called penalization parameter η. Using such a method, the geometry of the obstacles can therefore simply be taken into account using a spatially varying permeability coefficient. The motivation to use the penalization method to compute flows in complex geometries comes from the fact that existing numerical codes can be used as the geometry of the flow is simply described by adding a penalization term to the equations. Hence a classical grid, e.g. a Cartesian one, can still be used to compute flows past obstacles of arbitrary shape even moving in time or interacting with the fluid.

From a mathematical point of view the penalization method is a suitable tool for imposing different types of boundary conditions in a partial differential equation. Homogeneous or inhomogeneous Dirichlet, Neumann or even Robin type of conditions can be treated thus. The mathematical properties of the volume penalization method, and particularly the convergence of U_η, the solution of the penalized equations, towards U, the solution of the original equations, when η goes to 0, has been analyzed in [4] for the Navier–Stokes equations with no-slip boundary conditions. Among many others, different numerical simulations of viscous flows using finite difference/volume methods [4,5,16], pseudospectral methods [6,2,14,7,8] and adaptive wavelet methods [9] have shown the efficiency of the method.

In [6] the mathematical properties of the penalization method applied to the heat equation have been investigated in some detail, and an explicit error analysis has been conducted. It has been shown that the solution of the penalized equation converges towards the solution of the exact equation with an L^∞ error of the order $\sqrt{\eta}$ when $\eta \to 0$. Numerical simulations have shown an improved convergence rate of order η [6].

In the present article we follow the approach of Kevlahan and Ghidaglia [6] and analyze the penalization method applied to an hyperbolic equation, i.e. the one-dimensional wave equation. We perform a theoretical error analysis solving the penalized equation explicitly and check numerically the results using a finite difference discretization of the equation. Finally, we conclude and present some perspectives for future work.

2. Application of the volume penalization to an hyperbolic equation

We consider the one-dimensional wave-equation describing the motion of a vibrating string of size π fixed at $x = 0$ and $x = \pi$. The governing equation for the position U of the string is given by d'Alembert's equation,

$$\begin{cases} \partial_{tt} U - c^2 \partial_{xx} U = 0, & (x, t) \in]0, \pi[\times \mathbb{R}^+ \\ U(x, 0) = \sin(x) \\ \partial_t U(x, 0) = 0 \\ U(0, t) = U(\pi, t) = 0 \end{cases} \tag{1}$$

where c stands for the propagation velocity of the wave.

The solution of the above system is given by,

$$U(x, t) = \cos ct \sin x \tag{2}$$

Applying the penalization method for the boundary condition $U(0, t) = 0$ to Eq. (1) consists in adding a penalization term $\frac{1}{\eta} \chi_{\mathbb{R}^-}(x) U(x, t)$ to the equation, where χ_Ω is the characteristic function of the set Ω, replacing $]0, \pi[$ by $]-\infty, \pi[$ and removing the boundary condition at $x = 0$.

The resulting penalized equation can therefore be expressed, as:

$$\begin{cases} \partial_{tt} U_\eta - c^2 \partial_{xx} U_\eta + \frac{1}{\eta} \chi_{\mathbb{R}^-} U_\eta = 0, & (x, t) \in]-\infty, \pi[\times \mathbb{R}^+, \\ U_\eta(x, 0) = \sin(x) \chi_{[0, \pi]}(x) \\ \partial_t U_\eta(x, 0) = 0 \\ U_\eta(\pi, t) = 0 \end{cases} \tag{3}$$

3. Error calculation and estimate

Following [6], the solutions of Eqs. (3) can be reached using a Laplace transformation in time (4) that is written:

$$\mathcal{L}(f)(p) = \int_0^\infty f(t) e^{-pt} \, dt \quad \text{for } f \in L^1_{\text{loc}}(\mathbb{R}) \tag{4}$$

The solutions of the Laplace transformed equation are calculated separately on each subset, $]-\infty, 0[$ and $]0, \pi[$. Then $\mathcal{L}(U_\eta)$ and $\partial_x \mathcal{L}(U_\eta)$ are matched continuously at $x = 0$. On $]0, \pi[$ we obtain $\mathcal{L}(u)(p) = \sin(x)\frac{p}{p^2+c^2} + \frac{pc\sqrt{\eta}}{(p^2+c^2)(p\sqrt{\eta}+\sqrt{p^2\eta+1})}\exp(-px/c)$. Using the basic properties of a Laplace transform [10], one gets:

$$U_\eta(x, t) = U(x, t) + \epsilon(x, t, \eta) \quad \text{for } x \in [0, \pi],$$

$$\text{and with } \epsilon(x, t, \eta) = \delta_{t-x/c} * \frac{1}{t}J_1\left(\frac{t}{\sqrt{\eta}}\right)\chi_{\mathbb{R}^+}(t) * \left(c\sqrt{\eta}(\cos ct)\right)\chi_{\mathbb{R}^+}(t) \tag{5}$$

where J_1 is the first order Bessel function and $*$ stands for the convolution with respect to t.

Introducing the pointwise error (L^∞) and using [10], $\forall z \in \mathbb{C}$, $|J_1(z)| \leqslant \left|\frac{z\exp\sqrt{1-z^2}}{1+\sqrt{1-z^2}}\right|$, one gets:

$$\left|\epsilon(x, t, \eta)\right| \leqslant c\chi_{\{t-x/c>0\}}\left(\frac{\pi}{2}\exp(1)\sqrt{\eta} + \sqrt{\eta}\ln\frac{t-x/c}{\sqrt{\eta}}\chi_{\{t-x/c>\sqrt{\eta}\}}\right)$$

which leads to

$$\epsilon(x, t, \eta) \overset{\eta\to 0}{=} \mathrm{O}\left(\sqrt{\eta}\ln(\eta)\right) \quad \text{and} \quad \ln\left(\left|\epsilon(x, t, \eta)\right|\right) \overset{\eta\to 0}{=} \mathrm{O}\left(\ln(\eta)/2\right) \tag{6}$$

Similarly, if $\tilde{\epsilon}$ stands for the $L^1(]0, \pi[)$ error, one gets:

$$\tilde{\epsilon}(t, \eta) \overset{\eta\to 0}{=} \mathrm{O}\left(\sqrt{\eta}\ln(\eta)\right) \quad \text{and} \quad \ln\left(\tilde{\epsilon}(t, \eta)\right) \overset{\eta\to 0}{=} \mathrm{O}\left(\ln(\eta)/2\right) \tag{7}$$

4. Numerical investigation

Numerical simulations of Eq. (3) have been performed using a second order finite difference approximation on $]-l, \pi[\times]0, T[$ with a regular grid with space step δx and time step δt.

The time discretization of the second order term $\partial_{xx}U_\eta$ is explicit. Two different series of computations have been performed using either an explicit or an implicit discretization for the penalization term.

For instance, for an explicit discretization of the penalization term, using $r^2 = c^2\delta t^2/(\delta x^2)$, we get for U_i^n standing for the approximation of $U_\eta(i\delta x, n\delta t)$,

$$U_i^{n+1} = \left(r^2\left(U_{i+1}^n + U_{i-1}^n - 2U_i^n\right) + 2U_i^n - \frac{dt^2}{\eta}\chi_i U_i^n\right) - U_i^{n-1}$$

where $\chi_i = 0$ if $i\delta x > 0$ and $\chi_i = 1$ elsewhere.

The explicit discretization of the second order term $\partial_{xx}U_\eta$ imposes a stability condition that can be written

$$r^2 \leqslant \frac{1}{2}$$

When the discretization of the penalization term is explicit, a supplementary stability condition (see [11] for details) has to be imposed and is given by:

$$\frac{\delta t^2}{2} \leqslant \eta$$

while no extra stability condition is required otherwise.

Remark 1. In this simple one-dimensional case the implicit discretization for the penalization term is obviously the good choice. However, it has been shown in [12] that, in multi-dimensions, the system associated to an implicit discretization of the penalization term is ill-conditioned.
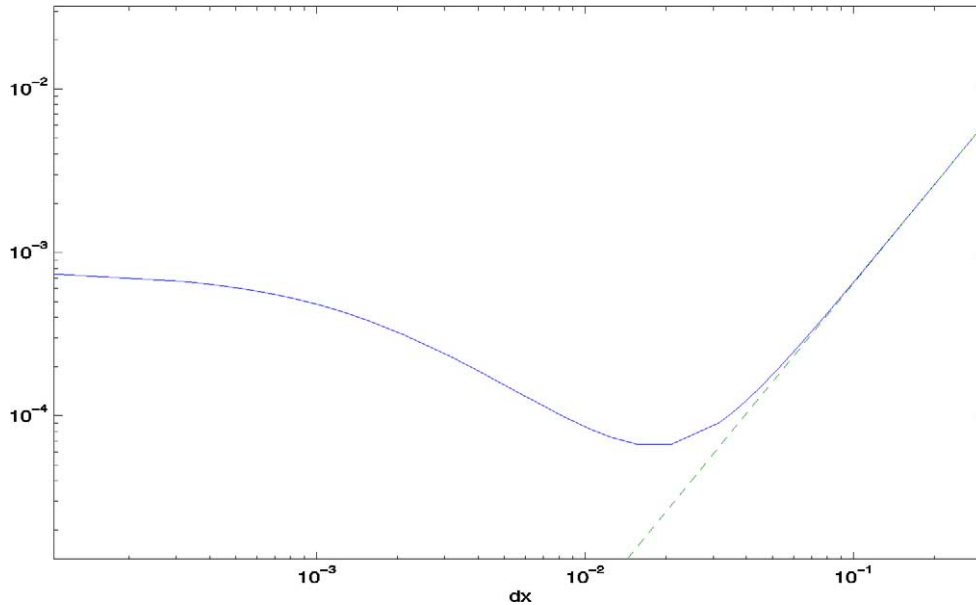
Fig. 1. L^1 error (solid curve) and δx^2 (dashed curve) with regards to δx for $\eta = 10^{-6}$, $\mathrm{d}t = 2 \times 10^{-5}$.

Remark 2. Using an explicit discretization of the penalization term, a scheme of order higher or equal to two is useless, as $\sqrt{\eta}$ could at most be of order δt.

In our numerical simulations we used $T = 1$, $l = 100 \times \delta x$, $10^{-4} \leqslant \delta x \leqslant 10^{-1}$ and $10^{-8} \leqslant \eta \leqslant 10^{-1}$. The propagation velocity c is fixed to 1. The convergence of the numerical solution of the penalized equation towards the solution of the initial equation has been investigated. The convergence with respect to the space step δx can be observed in Fig. 1. For δx above 10^{-2} we observe a second order convergence, while for δx below 10^{-2} the error is increasing due to the convergence toward $\tilde{\epsilon}$.

The error at time $t = 1$ between the numerical solution of the penalized equation and the solution of the original equation (1) is plotted on Fig. 2, where we observe the propagation of the penalization error with velocity $c = 1$.

The convergence with respect to the penalization parameter η is shown in Fig. 3.

Remark 3. We observe two different regimes: a $\sqrt{\eta}$ decay when η is larger than δx^2, and a linear decay, for η smaller than δx^2. As η determines the thickness of the 'boundary layer' around $x = t$, our conjecture is that, if the boundary layer is not resolved numerically, i.e. for δx^2 larger than η, an improved convergence is found. Similar results have also been reported in the case of a one-dimensional heat-equation in [6].

5. Conclusion and perspectives

We presented an analysis of the volume penalization method applied to the 1-D wave equation. Boundary conditions are taken into account for arbitrary domains by a simple mask function without modifying the numerical scheme. We studied theoretically the error between the exact solution and the solution of the penalized equations for a one-dimensional wave equation and validated the results numerically. We have shown that the theoretical error decays like $\sqrt{\eta}$, where η is the penalization parameter. Numerical results may exhibit a stronger decay for special choices of the different parameters that can be linked to the resolution of the boundary layer around the penalised
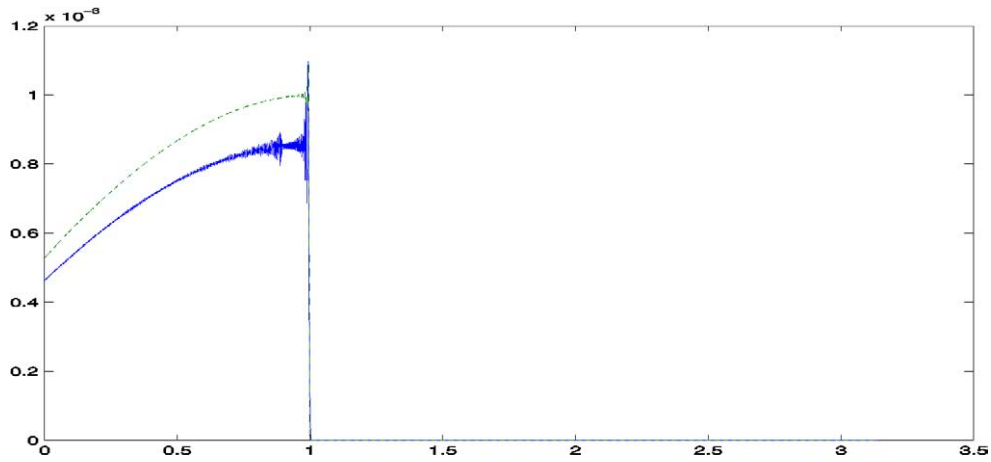
Fig. 2. Error at $t = 1$: $U_\eta(x, t = 1) - U(x, t = 1)$ (dashed curve); $U_i^n - U(i\delta_x, t = 1)$ (solid curve) for $r^2 = 0.1$, $\eta = 10^{-6}$, $dx = \pi \times 10^{-4}$. Relative L^1 error $= \dfrac{\tilde{\epsilon}(t=1,\eta=10^{-6})}{||U(\cdot, t=1, \eta=10^{-6})||_{L^1}} = 6 \times 10^{-4}$.
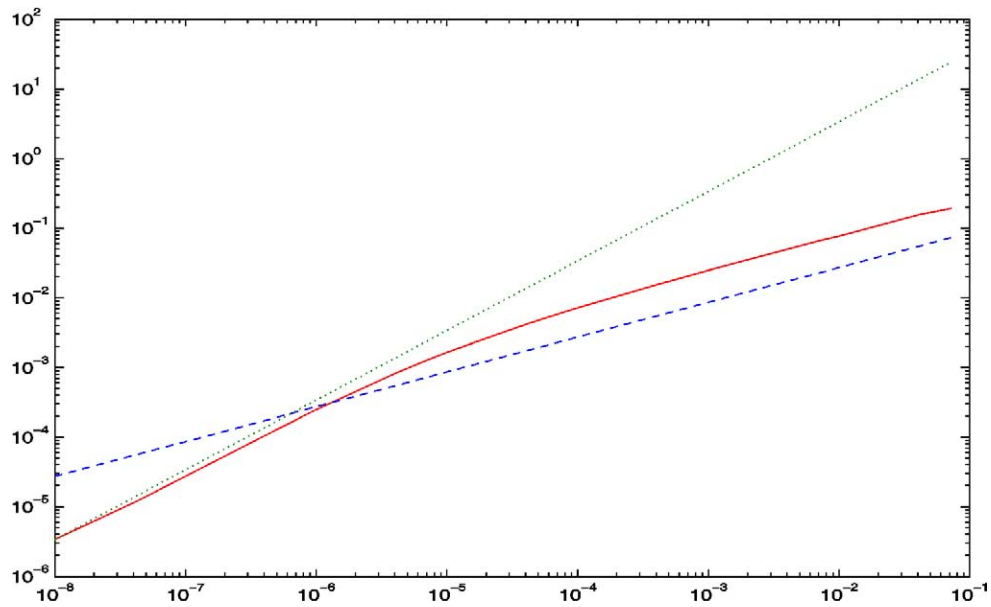

Fig. 3. Relative L^1 error of $\tilde{\epsilon}$ (solid curve), η (dotted curve) and $\sqrt{\eta}$ (dashed curve) versus η at $t = 1$, $dx = \pi \times 10^{-4}$, $r^2 = 10^{-4}$.

boundary condition. Links between the different parameters of the computations, implied by stability conditions and convergence rate may restrict the domain of application of such methods to first order time scheme.

In future work we will couple the penalization approach with the adaptive multiscale method for the Euler equations developed in [13] to compute compressible flows in complex geometries. This will allow us to adapt the grid of the numerical computation automatically, not only to the solution of the equation, but also to the geometry of the flow, as shown in [9] for the case of incompressible viscous flows.

From theoretical point of view we expect that nonlinear approximation theory results will probably improve the error estimations in the case of adaptive discretizations since the mask function has a much higher Besov regularity than its Sobolev regularity of 1/2.

Acknowledgements

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