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On regular and singular perturbations of acoustic and quantum waveguides

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Abstract

We consider regular and singular perturbations of the Dirichlet and Neumann boundary value problems for the Helmholtz equation in n-dimensional cylinders. The existence of eigenvalues and their asymptotics are studied. ***To cite this article: R.R. Gadyl'shin, C. R. Mecanique 332 (2004).***

Résumé

Sur des perturbations régulières et singulières dans des guides d'ondes acoustiques et quantiques. On considère des perturbations régulières et singulières des problèmes aux limites de Dirichlet et de Neumann pour l'équation de Helmholtz dans les cylindres n-dimensionnels. Sont étudies l'existence des valeurs propres et de leur comportement asymptotique. ***Pour citer cet article : R.R. Gadyl'shin, C. R. Mecanique 332 (2004).***

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Mots-clés : Acoustique ; Guide d'ondes ; Perturbation ; Valeur propre

1. Introduction

We consider regular and singular perturbations of the Neumann and Dirichlet boundary value problems for $\mathcal{H}_0^{(m)} := -(\Delta + \mu_m)$ in the n-dimensional cylinder $\Pi = (-\infty, \infty) \times \Omega$, where $\Omega \subset \mathbb{R}^{n-1}$ is a simply connected bounded domain with C^∞-boundary for $n \geqslant 3$ and is an interval (a, b) for $n = 2$. Hereinafter, μ_j and ϕ_j are the eigenvalues and eigenfunctions of $-\Delta' := -(\frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2})$ in Ω subject to the same type of the boundary condition on $\partial\Omega$ as in the original unperturbed boundary value problem for $\mathcal{H}_0^{(m)}$ on $\partial\Pi$, $\mu_j < \mu_{j+1}$, $j = 1, 2, \ldots$. The functions ϕ_j are assumed to be normalized in $L^2(\Omega)$. The Neumann problem is a mathematical model

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describing acoustic waveguide while the Dirichlet one corresponds to a quantum waveguide. It is known that unperturbed boundary value problems have no eigenfunctions in $H^1(\Pi)$. At the same time such eigenfunctions and eigenvalues (bounded states) can emerge under perturbations. We study the questions on existence and absence of such emerging eigenvalues and construct their asymptotic expansions. Both cases of regular and singular perturbations of these boundary value problems are considered. The regular perturbation treated in the next section is performed by a small localized linear operator of second order. The example of such operator is a small complex potential as well as other perturbations considered in [1] for the Schrödinger operator on the axis. Other examples are small deformations of strips and cylinders, which can be reduced to the case we consider by a change of variables [2–5]. As a singular perturbation of the Dirichlet and Neumann boundary value problems in Π we consider the switching of type of boundary condition at a small segment of the boundary. Such a choice is motivated by a number of articles having appeared recently and containing both rigorous results for quantum waveguides [3,6] as well as non-rigorous asymptotic results (see [7,8] and other articles of these authors on singularly perturbed two- and three-dimensional quantum waveguides given in the bibliography of [7,8]. These formal asymptotics were derived by the method of matching of asymptotic expansions [9] on the basis of scheme employed in [10–12] for constructing the asymptotics for scattering frequencies of Helmholtz resonator. However, rigorous justification of the asymptotics for these scattering frequencies adduced in [10–12] is based on the compactness of obstacle (boundary) and due to this fact it cannot be applied to the case of a waveguide. The question of an estimation of the inverse operator for singularly perturbed waveguides (providing a possibility to justify formal asymptotics) is treated in the third section. In two concluding sections we construct the leading terms for asymptotics of the eigenvalues and poles for the singularly perturbed quantum and acoustic waveguides.

2. Regular perturbation

Hereafter $H^j_{\mathrm{loc}}(\Pi)$ denote a set of functions defined on Π whose restriction to any bounded domain $D \subset \Pi$ belongs to $H^j(D)$, $\|\cdot\|_G$ and $\|\cdot\|_{j,G}$ are norms in $L^2(G)$ and $H^j(G)$, respectively. Next, let $Q = (-R, R) \times \Omega$, where $R > 0$ is an arbitrary fixed number, $L^2(\Pi; Q)$ be the subset of functions in $L^2(\Pi)$ with supports in $\overline{Q}$, $\mathcal{L}_\varepsilon$ be linear operators mapping $H^2_{\mathrm{loc}}(\Pi)$ into $L^2(\Pi; Q)$ such that $\|\mathcal{L}_\varepsilon[u]\|_Q \leqslant C(\mathcal{L})\,\|u\|_{2,Q}$, where the constant $C(\mathcal{L})$ is independent of ε, $0 < \varepsilon \ll 1$. In this section we study the existence and the asymptotics of the eigenvalues of the Dirichlet and Neumann boundary value problems for $\mathcal{H}^{(m)}_\varepsilon := \mathcal{H}^{(m)}_0 - \varepsilon\mathcal{L}_\varepsilon$ in Π. For a small complex k, we define a linear operator $A^{(m)}(k) : L^2(\Pi; Q) \to H^2_{\mathrm{loc}}(\Pi)$ as

$$A^{(m)}(k)g := \left(\sum_{j=1}^{m-1} + \sum_{j=m}^{\infty}\right) \frac{\phi_j(x')}{2K^{(m)}_j(k)} \int_\Pi \mathrm{e}^{-K^{(m)}_j(k)|x_1 - t_1|} \phi_j(t') g(t)\,\mathrm{d}t \tag{1}$$

where $x' = (x_2, \ldots, x_n)$, $K^{(m)}_j(k) = \mathrm{i}\sqrt{\mu_m - \mu_j - k^2}$ for $j < m$, $K^{(m)}_m(k) = k$ and $K^{(m)}_j(k) = \sqrt{\mu_j - \mu_m + k^2}$ for $j > m$. By analogy with [1] for $f \in L^2(\Pi; Q)$ we search for a solution of

$$\mathcal{H}^{(m)}_\varepsilon u_\varepsilon = -k^2 u_\varepsilon + f, \quad \text{as } x \in \Pi, \qquad u_\varepsilon = 0 \quad \left(\text{or } \frac{\partial u_\varepsilon}{\partial \nu} = 0\right) \quad \text{as } x \in \partial\Pi \tag{2}$$

(where ν is normal) as

$$u_\varepsilon = A^{(m)}(k) g_\varepsilon \tag{3}$$

where $g_\varepsilon \in L^2(\Pi; Q)$. By definition, (3) is the solution of the equation $\mathcal{H}^{(m)}_0(k)u_\varepsilon = -k^2 u_\varepsilon + g_\varepsilon$ in Π and satisfies the boundary condition in (2). Substituting (3) into (2), we get that (3) gives a solution for (2) if

$$\bigl(I - \varepsilon\mathcal{L}_\varepsilon A^{(m)}(k)\bigr) g_\varepsilon = f \tag{4}$$

where I is identity mapping. If $\mathcal{L}_\varepsilon[\phi_m]=0$, due to (1), (3) and (4) it follows that the pole $k_\varepsilon^{(m)}$ of (3) is equal to, zero and $g_\varepsilon \to 0$ as $\varepsilon \to 0$. Thus, there is no small eigenvalue in this case. Assume $\mathcal{L}_\varepsilon[\phi_m]\neq 0$,

$$\langle F\rangle := \int_\Pi F\,\mathrm{d}x,\quad \widetilde{T}_\varepsilon^{(m)}(k)g := \mathcal{L}_\varepsilon\bigl[A^{(m)}(k)g\bigr]-\frac{\langle g\phi_m\rangle}{2k}\mathcal{L}_\varepsilon[\phi_m],\quad S_\varepsilon^{(m)}(k) := \bigl(I-\varepsilon\widetilde{T}_\varepsilon^{(m)}(k)\bigr)^{-1}$$

Applying the operator $S_\varepsilon^{(m)}(k)$ to both sides of Eq. (4), we obtain that

$$\left(g_\varepsilon-\varepsilon\frac{\langle g_\varepsilon\phi_m\rangle}{2k}S_\varepsilon^{(m)}(k)\mathcal{L}_\varepsilon[\phi_m]\right)=S_\varepsilon^{(m)}(k)f \tag{5}$$

$$\langle g_\varepsilon\phi_m\rangle\left(1-\frac{\varepsilon}{2k}\bigl\langle\phi_m S_\varepsilon^{(m)}(k)\mathcal{L}_\varepsilon[\phi_m]\bigr\rangle\right)=\bigl\langle\phi_m S_\varepsilon^{(m)}(k)f\bigr\rangle \tag{6}$$

The equality (6) allows us to determine $\langle g_\varepsilon\phi_m\rangle$. Substituting its value into (5), we easily get the formula

$$g_\varepsilon=\varepsilon\frac{2k\langle S_\varepsilon^{(m)}(k)f\rangle S_\varepsilon^{(m)}(k)\mathcal{L}_\varepsilon[\phi_m]}{2k-\varepsilon\langle\phi_m S_\varepsilon^{(m)}(k)\mathcal{L}_\varepsilon[\phi_m]\rangle}+S_\varepsilon^{(m)}(k)f \tag{7}$$

Formulas (7) and (3) imply, that, if $k_\varepsilon^{(m)}$ is a solution of the equation

$$2k-\varepsilon\bigl\langle\phi_m S_\varepsilon^{(m)}(k)\mathcal{L}_\varepsilon[\phi_m]\bigr\rangle=0 \tag{8}$$

then the residue of (3) at $k_\varepsilon^{(m)}$:

$$\psi_\varepsilon^{(m)}=A^{(m)}\bigl(k_\varepsilon^{(m)}\bigr)S_\varepsilon^{(m)}\bigl(k_\varepsilon^{(m)}\bigr)\mathcal{L}_\varepsilon[\phi_m] \tag{9}$$

is the solution of the equation $\mathcal{H}_\varepsilon^{(m)}\psi_\varepsilon^{(m)}=\lambda_\varepsilon^{(m)}\psi_\varepsilon^{(m)}$ in Π (with the corresponding homogeneous Dirichlet or Neumann boundary conditions), where $\lambda_\varepsilon^{(m)}=-(k_\varepsilon^{(m)})^2$. Formulas (1), (9) show that if $\operatorname{Re}k_\varepsilon^{(1)}>0$, then $\psi_\varepsilon^{(1)}\in L^2(\Pi)$ and, hence, $\lambda_\varepsilon^{(1)}$ is the eigenvalue which due to (8) has the asymptotics

$$\lambda_\varepsilon^{(m)}=-\varepsilon^2\frac{1}{4}\bigl\langle\phi_m\mathcal{L}_\varepsilon[\phi_m]\bigr\rangle^2+\mathrm{O}\bigl(\varepsilon^3\bigr) \tag{10}$$

with $m=1$ (and the function (9) is the associated eigenfunction). For $m\geqslant 2$, the formulas (1), (8), (9) imply, that if $\operatorname{Re}k_\varepsilon^{(m)}>0$ and $\operatorname{Im}k_\varepsilon^{(m)}>0$, then $\psi_\varepsilon^{(m)}\in L^2(\Pi)$, too, and, hence, $\lambda_\varepsilon^{(m)}$ is the eigenvalue of the perturbed problem with asymptotics (10). In particular, Eq. (8) allows us to maintain that in the case $\langle\phi_1\mathcal{L}_\varepsilon[\phi_1]\rangle\geqslant\delta>0$ there exists a small eigenvalue.

3. Singular perturbations: convergence of poles and representation of solutions near poles

Assume for simplicity in describing the of perturbations that the domain Ω coincides with the half-space $x_n>0$ in some neighborhood of the origin (in variables x'), ω is a $(n-1)$-dimensional bounded domain in the hyperplane $x_n=0$ having smooth boundary, $\omega_\varepsilon=\{x\colon x\varepsilon^{-1}\in\omega\}$, $\Gamma_\varepsilon=\partial\Pi\setminus\overline{\omega_\varepsilon}$. For a given $f\in L^2(\Pi;Q)$, we consider two singularly perturbed boundary value problems

$$\mathcal{H}_0^{(m)}u_\varepsilon=-k^2u_\varepsilon+f,\quad x\in\Pi,\quad u_\varepsilon=0,\quad x\in\Gamma_\varepsilon\ (\text{or }x\in\omega_\varepsilon),\quad \frac{\partial u_\varepsilon}{\partial\nu}=0,\quad x\in\omega_\varepsilon\ (\text{or }x\in\Gamma_\varepsilon) \tag{11}$$

Let $\Gamma_0^R=\partial\Pi\cap\partial Q$, $\Omega^R=\partial Q\setminus\overline{\Gamma_0^R}$, $\Gamma_\varepsilon^R=\Gamma^R\setminus\overline{\omega_\varepsilon}$. For each $V\in H^2(Q)$, we denote by $\sigma_\varepsilon: H^2(Q)\to H^1(Q)$ the inverse operator for the following boundary value problems

$$\Delta W_\varepsilon=\Delta V,\quad x\in Q,\quad W_\varepsilon=V,\quad x\in\Omega^R$$
$$W_\varepsilon=0,\quad x\in\Gamma_\varepsilon^R\ (\text{or }x\in\omega_\varepsilon),\qquad \frac{\partial W_\varepsilon}{\partial\nu}=0,\quad x\in\omega_\varepsilon\ \bigl(\text{or }x\in\Gamma_\varepsilon^R\bigr)$$

Let $\chi^{\pm}(x_1)$ be an infinitely differentiable mollifier function equalling to one for $\pm x_1 \leqslant R/2$ and vanishing for $\pm x_1 \geqslant R$, $\Pi_{\pm} = \{x\colon\ x \in \Pi,\ \pm x_1 > 0\}$, $p_{\pm}$ be the restriction operator from Π to $\Pi_{\pm}$, $p_{\pm}^{Q}$ be the restriction operator from $\Pi_{\pm}$ to $\Pi_{\pm} \cap Q$,

$$A_{\pm}^{(m)}(k)g^{\pm} := \sum_{j=1}^{\infty} \frac{\phi_j(x')}{2K_j^{(m)}(k)} \int\limits_{\Pi_{\pm}} \left(\mathrm{e}^{-K_j^{(m)}(k)|x_1-t_1|} - \mathrm{e}^{-K_j^{(m)}(k)|x_1+t_1|}\right)\phi_j(t')g^{\pm}(t)\,\mathrm{d}t, \quad x \in \Pi_{\pm}$$

$$\begin{aligned}\mathcal{A}_{\varepsilon}^{(m)}(k)g :={}& (1-\chi^{+})A_{+}^{(m)}(k)p_{+}g + (1-\chi^{-})A_{-}^{(m)}(k)p_{-}g \\ &+ \chi^{+}\chi^{-}\sigma_{\varepsilon}\left(p_{+}^{Q}A_{+}^{(m)}(k)p_{+}g + p_{-}^{Q}A_{-}^{(m)}(k)p_{-}g\right), \quad g \in L^2(\Pi; Q)\end{aligned}$$

We construct the solution of (11) in the form

$$u_{\varepsilon} = \mathcal{A}_{\varepsilon}^{(m)}(k)g_{\varepsilon} \tag{12}$$

where g_{ε} is a some function belonging to $L^2(\Pi; Q)$. Substituting (12) into (11), by analogy with [13] we deduce that this function is a solution of (11) in the case

$$g_{\varepsilon} = \left(I + T_{\varepsilon}^{(m)}(k)\right)^{-1} f \tag{13}$$

where, for any fixed ε, $T_{\varepsilon}^{(m)}(k)$ is a holomorphic operator-valued function and, for any fixed k, $T_{\varepsilon}^{(m)}(k)$ is a compact operator in $L^2(\Pi; Q)$. The analysis of this family with respect to ε (which is similar to [14] and based on [13]) and the representations (12), (13) show that there exists one pole $k_{\varepsilon}^{(m)} \to 0$ of the solution of (11) and for small k, this solution meet the representation

$$u_{\varepsilon}(x,k) = \frac{\psi_{\varepsilon}^{(m)}(x)}{2(k - k_{\varepsilon}^{(m)})} \int\limits_{\Pi} \psi_{\varepsilon}^{(m)}(y)\, f(y)\,\mathrm{d}y + \widetilde{u}_{\varepsilon}(x,k), \qquad \|\tilde{u}_{\varepsilon}\|_{1,D} \leqslant C(D,Q)\|f\|_{\Pi} \tag{14}$$

for any bounded domain $D \subset \Pi$. The residue $\psi_{\varepsilon}^{(m)}$ at this pole is a solution to the equation $\mathcal{H}_0^{(m)}\psi_{\varepsilon}^{(m)} = \lambda_{\varepsilon}^{(m)}\psi_{\varepsilon}^{(m)}$ in Π, where $\lambda_{\varepsilon}^{(m)} = -\left(k_{\varepsilon}^{(m)}\right)^2$, satisfies the boundary conditions from (11) and for any fixed x_1 converges to ϕ_m (up to a multiplicative constant) as $\varepsilon \to 0$. This convergence, the representation (12) and the definition of $\mathcal{A}_{\varepsilon}^{(m)}(k)$ imply that

$$\psi_{\varepsilon}^{(m)}(x) = \sum_{j=1}^{m-1} a_j^{\varepsilon}\phi_j(x')\,\mathrm{e}^{-|x_1|K_j^{(m)}(k_{\varepsilon}^{m})} + a_m^{\varepsilon}\phi_m(x')\,\mathrm{e}^{-|x_1|k_{\varepsilon}^{(m)}} + \mathrm{o}\left(\mathrm{e}^{-|x_1|\delta}\right) \quad \text{as } |x_1| \to \infty$$

where $a_m^{\varepsilon} = 1 + \mathrm{o}(1)$ as $\varepsilon \to 0$ and $\delta > 0$ some fixed number. In partially, this asymptotics implies that

$$\text{there exists eigenvalue } \lambda_{\varepsilon}^{(1)} \text{ provided } \operatorname{Re} k_{\varepsilon}^{(1)} > 0 \tag{15}$$

$$\text{if } m \geqslant 2,\ \operatorname{Re} k_{\varepsilon}^{(m)} > 0 \text{ but } \operatorname{Im} k_{\varepsilon}^{(m)} < 0 \text{ and } a_1^{\varepsilon} \neq 0, \text{ then there is no an eigenvalue} \tag{16}$$

$$\text{there is no an eigenvalue if } \operatorname{Re} k_{\varepsilon}^{(m)} \leqslant 0 \tag{17}$$

Thus, in fact we need to construct and to justify asymptotics of the pole $k_{\varepsilon}^{(m)}$ (and, an additional, asymptotics of the residue $\psi_{\varepsilon}^{(m)}$ in the case (16)) which generates the eigenvalue or does not. As above mentioned in the case of regular perturbation the asymptotics for pole can obtained by simple calculations in (8), while dealing with singular perturbation, we have no such an equation. On the other hand, the representation (14) allows us to justify the method of matching asymptotic expansions in constructing the asymptotics for the poles $k_{\varepsilon}^{(m)}$ and for the residue $\psi_{\varepsilon}^{(m)}$.

As it has been mentioned above, the formal construction of complete asymptotics of poles for the boundary valued problems (11) and for Helmholtz resonator [10–12] is similar. This is why in the next two section we will construct first perturbed terms of poles only.

4. Singular perturbation of quantum waveguide: asymptotics of poles and eigenvalues

Let S_n be the unit sphere in $\mathbb{R}^n$, $G_m^{(\mathcal{D})}(x,y,k)$ be the Green function of the unperturbed Dirichlet boundary value problem in Π, $\Phi_m = \frac{\partial}{\partial x_n}\phi_m(x')|_{x'=0} \neq 0$, $\Psi_m^{(\mathcal{D})}(x,k) = -2k\Phi_m^{-1}\frac{\partial}{\partial y_n}G_m^{(\mathcal{D})}(x,y,k)|_{y=0}$. By definition

$$\Psi_m^{(\mathcal{D})}(x,k) \to \phi_m(x'), \quad k \to 0 \quad \text{for any fixed } x \neq 0 \tag{18}$$

$$\Psi_m^{(\mathcal{D})}(x,k) = \Phi_m x_n + \frac{4k}{\Phi_m|S_n|}\frac{x_n}{r^n} + \mathrm{O}\big(kr^{-n+2}\big), \quad r = |x| \to 0,\ k \to 0 \tag{19}$$

Taking into account (18), outside small neighborhood of ω_ε we construct the residue $\psi_\varepsilon^{(m)}$ in the form $\psi_\varepsilon^{(m)}(x) \sim \Psi_m^{(\mathcal{D})}(x,k_\varepsilon^{(m)})$. Near ω_ε we construct asymptotics by using the method of matching asymptotic expansions [9–12] in the variables $\xi = \varepsilon^{-1}x$. The structure of the expansions of $\psi_\varepsilon^{(m)}$ in this zone and of the pole $k_\varepsilon^{(m)}$ are inspired by the following consideration. When $x = \varepsilon\xi$ and $k = k_\varepsilon^{(m)}$, both terms in right-hand side of (19) must have the same order with respect to ε. This degree determines the first term in the interior layer for $\psi_\varepsilon^{(m)}$, while the right-hand side of (19) (rewritten in variables ξ and for $k = k_\varepsilon^{(m)}$) determines the asymptotics of this term as $\rho = |\xi| \to \infty$. Due to these reasons, we construct asymptotics as

$$k_\varepsilon^{(m)} = \varepsilon^n\tau_n^{(m)} + \cdots, \qquad \psi_\varepsilon^{(m)}(x) = \varepsilon v_1^{(m)}(\xi) + \cdots \tag{20}$$

$$v_1^{(m)}(\xi) = \Phi_m\xi_n + 4\tau_n^{(m)}\big(\Phi_m|S_n|\big)^{-1}\xi_n\rho^{-n} + \mathrm{o}\big(\rho^{-n+1}\big), \quad \rho \to \infty \tag{21}$$

Substituting (20) in (11) (with $f = 0$ and $k = k_\varepsilon^{(m)}$), we obtain the boundary value problem for $v_1^{(m)}$:

$$\Delta_\xi v_1^{(m)} = 0, \quad \xi_n > 0, \qquad v_1^{(m)} = 0, \quad \xi \in \Gamma, \qquad \frac{\partial v_1^{(m)}}{\partial \xi_n} = 0, \quad \xi \in \omega \tag{22}$$

where $\Gamma = \{\xi\colon\ \xi_n = 0,\ \xi \notin \omega\}$. It is known, there exists the solution X_n of (22) with asymptotics $X_n(\xi) = \xi_n + c_n(\omega)\xi_n\rho^{-n} + \mathrm{o}(\rho^{-n+1})$ as $\rho \to \infty$, where $c_n(\omega) > 0$. Thus it follows from (21) that

$$v_1^{(m)}(\xi) = \Phi_m X_n(\xi), \qquad \tau_n^{(m)} = 4^{-1}c_n(\omega)|S_n|\Phi_m^2 > 0 \tag{23}$$

By (20), (23) we have $\operatorname{Re} k_\varepsilon^{(m)} > 0$ and, hence (see (15)), there exists eigenvalue

$$\lambda_\varepsilon^{(1)} = -\varepsilon^{2n}\left(\frac{c_n(\omega)|S_n|\Phi_1^2}{4}\right)^2 + \mathrm{o}\big(\varepsilon^{2n}\big)$$

For $m \geqslant 2$, constructing next terms for expansions $k_\varepsilon^{(m)}$ and $\psi_\varepsilon^{(m)}$ (similar [10–12]) one can obtain that

$$\operatorname{Im} k_\varepsilon^{(m)} = -\varepsilon^{2n}\left(\frac{c_n(\omega)|S_n|\Phi_m}{4}\right)^2\sum_{j=1}^{m-1}\frac{\Phi_j^2}{\sqrt{\mu_m - \mu_j}} + \mathrm{o}\big(\varepsilon^{2n}\big) < 0, \qquad a_1^\varepsilon \sim \frac{k_\varepsilon^{(m)}\Phi_1}{K_1^{(m)}(k_\varepsilon^{(m)})\Phi_m} \neq 0$$

where $\Phi_j = \frac{\partial}{\partial x_n}\phi_j(x')|_{x'=0}$. Therefore, the pole $k_\varepsilon^{(m)}$ admits the asymptotics (20), (23), but (see (16)) does not generate an eigenvalue of the considered singular perturbation of the Dirichlet boundary value problem.

5. Singular perturbation of acoustic waveguide: asymptotics of poles

Let $G_m^{(\mathcal{N})}(x,y,k)$ be the Green function of the unperturbed Neumann boundary value problem, $\phi_m(0) \neq 0$, $\Psi_m^{(\mathcal{N})}(x,k) = -2k\phi_m^{-1}(0)G_m^{(\mathcal{N})}(x,0,k)$, $\alpha_n(r) = r^{-n+2}$ for $n \geqslant 3$ and $\alpha_2(r) = -\ln r$. By definition

$$\begin{aligned}&\Psi_m^{(\mathcal{N})}(x,k) \to \phi_m(x'), \quad k \to 0 \quad \text{for any fixed } x \neq 0\\ &\Psi_m^{(\mathcal{N})}(x,k) = \phi_m(0) + 4k\big(\phi_m(0)|S_n|\big)^{-1}\alpha_n(r) + \mathrm{O}\big(kr^{-n+3-\delta_n^2}\big), \quad r \to 0,\ k \to 0\end{aligned} \tag{24}$$

where δ^s_J is the Kronecker delta. Taking into account (24) and following the method of matching asymptotic expansions similar the previous section we obtain that

$$\begin{aligned} &k_\varepsilon^{(m)} = \varepsilon^{n-2}\tau_{n-2}^{(m)} + \cdots, \quad n \geqslant 3, \qquad k_\varepsilon^{(m)} = -\ln^{-1}\varepsilon\,\tau_0^{(m)} + \cdots, \quad n = 2 \\ &\tau_{n-2}^{(m)} = -\frac{C_n(\omega)|S_n|\phi_m^2(0)}{4} < 0, \quad n \geqslant 3, \quad \tau_0^{(m)} = -\frac{\pi\phi_m^2(0)}{2} < 0, \quad n = 2 \end{aligned} \tag{25}$$

where $C_n(\omega) > 0$ is the capacity of the disk ω. Thus, $\operatorname{Re} k_\varepsilon^{(m)} < 0$. Therefore, the pole $k_\varepsilon^{(m)}$ meets the asymptotics (25), but (see (17)) it does not generate an eigenvalue of the considered singular perturbation of the Neumann boundary value problem.

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References

[1] R.R. Gadyl'shin, On local perturbations of Shrödinger operator in axis, Teor. Mat. Fiz. 132 (1) (2002) 97–104; English translation: Theor. Math. Phys. 132 (1) (2002) 976–982.

[2] P. Duclos, P. Exner, Curvature-induced bound states in quantum waveguides in two and three dimensions, Rev. Math. Phys. 7 (1995) 73–102.

[3] W. Bulla, F. Gesztesy, W. Renger, B. Simon, Weakly coupled bound states in quantum waveguides, Proc. Amer. Math. Soc. 127 (1997) 1487–1495.

[4] P. Exner, S.A. Vugalter, Bound states in a locally deformed waveguide: the critical case, Lett. Math. Phys. 39 (1997) 59–68.

[5] D. Borisov, P. Exner, R. Gadyl'shin, D. Krejcirik, Bound states in weakly deformed strips and layers, Ann. Inst. H. Poincaré 2 (3) (2001) 553–572.

[6] P. Exner, S.A. Vugalter, Asymptotic estimates for bound states in quantum waveguides coupled laterally through a narrow window, Ann. Inst. H. Poincaré: Phys. Théor. 65 (1996) 109–123.

[7] I.Yu. Popov, Asymptotics of bounded states for laterally coupled three-dimensional waveguides, Rep. Math. Phys. 48 (2001) 277–288.

[8] I.Yu. Popov, S.V. Frolov, Three laterally coupled quantum waveguiedes: breaking of symmetry and resonance asymptotics, J. Phys. A 36 (2003) 1655–1670.

[9] A.M. Il'in, Matching of Asymptotic Expansions of Solutions of Boundary Value Problems, American Mathematical Society, Providence, RI, 1992.

[10] R.R. Gadyl'shin, Surface potentials and the method of matching asymptotic expansions in the problem of the Helmholtz resonator, Algebra Anal. 4 (2) (1992) 88–115; English translation: St. Petersburg Math. J. 4 (1) (1993) 273–296.

[11] R.R. Gadyl'shin, On acoustic Helmholtz resonator and on its electromagnetic analog, J. Math. Phys. 35 (7) (1994) 3464–3481.

[12] R.R. Gadyl'shin, Existence and asymptotics of poles with small imaginary part for the Helmholtz resonator, Uspekhi Mat. Nauk 52 (1) (1997) 3–76; English translation: Russian Math. Surveys 52 (1) (1997) 1–72.

[13] É. Sanchez-Palencia, Non-Homogeneous Media and Vibration Theory, Springer-Verlag, New York, 1980.

[14] R.R. Gadyl'shin, On analogs of Helmholtz resonator in averaging theory, Mat. Sb. 193 (11) (2002) 43–70; English translation: Sb. Math. 193 (11) (2002) 1611–1638.