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Selection of two-dimensional nonlinear strain waves in micro-structured media

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Abstract

It is shown that the micro-displacement gradient allows the propagation of two-dimensional localized long nonlinear strain waves in a medium with microstructure. These waves may exist even in the presence of dissipation and energy input in the microstructured medium but with selected values of the wave amplitude and velocity. An increase or a decrease in the wave amplitude and velocity happens faster at the initial stage than that of the plane localized wave. However, their steady values selected by the energy input/output, are higher for the plane waves. ***To cite this article: A.V. Porubov et al., C. R. Mecanique 332 (2004).***

Résumé

Selection d'ondes de déformation non linéaires à deux dimensions dans des milieux à microstructure. On montre que dans un milieu élastique à microstructure le gradient de micro-déplacement permet la propagation de longues ondes localisées non linéaires de déformation en deux dimensions. Ces ondes penvent exister même en présence de dissipation et d'apport d'énergie mais pour des valeurs précises de l'amplitude de l'onde et sa vitesse de propagation. Une augmentation on une diminution de ces deux quantités se produit plus rapidement dans la phase initiale de propagation que pour une onde plane localisée. Cependont, les valeurs stationnaires sélectionnées par l'apport et perte d'énergie sont plus élevées que pour les ondes planes. ***Pour citer cet article : A.V. Porubov et al., C. R. Mecanique 332 (2004).***

Keywords: Computational solid mechanics; Nonlinear wave; Microstructure

Mots-clés : Mécanique des solides numérique ; Ondes non linéaires ; Microstructure

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1. Introduction

One of the most important problem in microstructured medium is to define the values of the parameters of a microstructure or, more generally, to verify the model used to describe it [1,2]. One possibility is to use the measurements of the parameters of strain waves propagating in such a medium. Indeed the amplitude and the velocity of the wave depend upon the parameters of the microstructure. Certainly waves that keep their shapes and velocities while propagating are of special interest. Usually these waves may exist under balance conditions between nonlinearity, dispersion, dissipation/energy input. When the wave propagation is described by two-dimensional equations, an additional factor affecting the wave behavior is diffraction. The balances define the shape of the wave. Thus the balance between nonlinearity and dispersion allows the existence of a bell-shaped wave, while diffraction supports either a plane localized wave or a two-dimensional fully localized wave. Nonlinearity, dispersion and the other terms in the governing equations describe various features of the model of a microstructure. Hence even the shape of the wave qualitatively reflects the presence or absence of these features. More detailed information may be obtained having analytical solutions of the governing equations since this allows us to establish relationships between the amplitude and velocity of the strain wave and the parameters of the microstructure.

The above mentioned possibility has been illustrated recently in [3] for a one-dimensional model of a microstructured medium. Here we introduce weak transverse variations, and consider the evolution of two-dimensional (2D) localized strain waves. First the governing nonlinear equation is obtained for longitudinal strain waves. In the absence of dissipation and energy input it is nothing but the well-known Kadomtsev–Petviashvili equation [4,5] that admits 2D localized wave solution having the shape of a Mexican hat. Then an asymptotic solution is obtained to describe the influence of weak dissipation/energy input on the 2D localized wave. The possibility of a selection of the wave is exhibited when its amplitude and velocity tend to the finite values prescribed by the coefficients of the governing equation. A comparison is presented with the evolution of the plane localized strain wave.

2. Derivation of the governing equations

The governing equations are obtained using the model developed in [3]. The micromorphic materials [6,7] are considered when in a reference configuration, the fundamental strains are given by the Cauchy–Green macro-strain tensor, the distortion tensor, and the micro-displacement gradient tensor. The macro-motion is supposed to be small but finite, and the Murnaghan model [8] is used to describe the so-called physical nonlinearity in the expansion of free or potential energy. The microstructure is assumed sufficiently weak to be considered in the linear approximation. A dissipation and an energy input are introduced through the additive linear terms in all three tensors similar to the Voigt model [9], the simplest extension of the Hooke law to viscoelastic media.

Now we are interested in the weak transverse variations. Since it is assumed that the influence of a microstructure is weak, we can modify the 1D equation from [3] adding only transverse macro-terms following from the classic theory of elasticity. Considering only long waves with characteristic length $L \gg 1$ we choose L as a scale for x, the direction of the wave propagation, while Y denotes a scale for the transverse variable y. Then the parameter $\kappa = L^2/Y^2 \ll 1$ characterizes weak transverse variations. Let us denote displacements along x and y axis by $U(x, y, t)$, $V(x, y, t)$ respectively. Then a scale W is introduced as for longitudinal strains $v = U_x$, and $W \ll 1$ that is natural for the Murnaghan materials. The scale for another strain $w = V_y$ is chosen equal to κW. Also L/c_0 is used as a scale for time t, $c_0^2 = (\lambda + 2\mu)/\rho$ is a characteristic velocity, λ, μ are the Lamé coefficients, ρ is the macro-density. We also introduce a typical size p of a microstructure element and the dissipation parameter d having the dimension of a length. Three positive dimensionless parameters will be used in the following: $\varepsilon = W \ll 1$ accounting for elastic strains; $\delta = p^2/L^2 \ll 1$, characterizing the ratio between the microstructure size and the wavelength; $\gamma = d/L$, characterizing the influence of the dissipation.

Then the governing nonlinear equations for long waves in non-dimensional form from [3] may be generalized to the weakly transverse case as

$$v_{tt} - v_{xx} - \kappa b_1 w_{xx} - \varepsilon\alpha_1\left(v^2\right)_{xx} - \gamma\alpha_2 v_{xxt} + \delta\alpha_3 v_{xxxx} - \delta\alpha_4 v_{xxtt} + \gamma\delta(\alpha_5 v_{xxxxt} + \alpha_6 v_{xxttt}) = \mathrm{O}\left(\varepsilon^2, \delta^2, \gamma^2, \kappa^2\right) \tag{1}$$

$$(b_1+\mu)w_{tt} - \mu w_{xx} = b_1 v_{yy} + \mathrm{O}(\varepsilon, \delta, \gamma, \kappa) \tag{2}$$

where the nonlinear term coefficient α_1 depends upon the Murnaghan moduli, and dispersion terms v_{xxxx} and v_{xxtt} describe a tribute of the micro-displacement gradient and micro-inertia respectively. Transverse terms coefficient $b_1 = (\lambda+\mu)/(\lambda+2\mu)$. Dissipative term v_{xxt} arises thanks to dissipative additions in the macro-strain tensor, while remaining terms account for the influence of dissipative parts in the distortion tensor and the micro-displacement gradient tensor. Relationships for the coefficients α_i may be found in [3].

3. Two-dimensional localized wave selection

Let us consider weak transverse variations when a balance between nonlinearity and dispersion is achieved, $\delta = \mathrm{O}(\varepsilon)$, $\varepsilon \ll \gamma \ll 1$. Assume that $\alpha_2 = \varepsilon\alpha_2^*$, and all dissipative terms are of the same order. Introducing fast and slow variables $\theta = x - t$, y, $\tau = \varepsilon t$, $T = \gamma\tau$, we obtain from Eqs. (1), (2):

$$2v_{\theta\tau} + \alpha_1\left(v^2\right)_{\theta\theta} - (\alpha_3-\alpha_4)v_{\theta\theta\theta\theta} + b_1 v_{yy} = \gamma\left[\alpha_2^* v_{\theta\theta\theta} - (\alpha_5+\alpha_6)v_{5\theta} - 2v_{\theta T}\right] + \mathrm{O}\left(\gamma^2\right) \tag{3}$$

At $\gamma = 0$ we obtain the Kadomtsev–Petviashvili (KP) equation [4,5] that admits two-dimensional travelling localized wave solutions at $\alpha_3 - \alpha_4 > 0$. The asymptotic solution at nonzero γ is sought as a function of ζ, y and T, where $\zeta_\theta = 1$, $\zeta_\tau = c(T)$,

$$v = v_0(\zeta, y, T) + \gamma v_1(\zeta, y, T) + \cdots \tag{4}$$

Then the following reduction of the KP equation for v_0 holds in the leading order,

$$2cv_{0,\zeta\zeta} + \alpha_1\left(v_0^2\right)_{\zeta\zeta} - (\alpha_3-\alpha_4)v_{0,4\zeta} + b_1 v_{0,yy} = 0 \tag{5}$$

whose known exact two-dimensional localized travelling wave solution is [5]:

$$v_0 = \frac{24b_1c(\alpha_4-\alpha_3)[3b_1(\alpha_3-\alpha_4) - 2b_1c\zeta^2 + 4c^2y^2]}{\alpha_1(3b_1(\alpha_3-\alpha_4) + 2b_1c\zeta^2 + 4c^2y^2)^2} \tag{6}$$

The shape of the solution shown in Fig. 1 is similar to a Mexican hat along the direction of propagation, x-axis, but the solution decays monotonically in the transverse direction. The following linear equation for v_1 holds in the next order,

$$2cv_{1,\zeta\zeta} + 2\alpha_1(v_0v_1)_{\zeta\zeta} - (\alpha_3-\alpha_4)v_{1,4\zeta} + b_1v_{1,yy} = \alpha_2^* v_{0,\zeta\zeta\zeta} - (\alpha_5+\alpha_6)v_{0,5\zeta} - 2v_{0,\zeta T} \tag{7}$$

The solvability condition may be obtain as follows. Let us integrate Eqs. (5), (7) over ζ from $-\infty$ to ζ. Multiplying the first equation by v_1 and subtracting the second equation multiplied by v_0, one obtains

$$\int_{-\infty}^{\infty} \mathrm{d}\zeta \int_{-\infty}^{\infty} v_0\left(\alpha_2^* v_{0,\zeta\zeta} - (\alpha_5+\alpha_6)v_{0,4\zeta} - 2v_{0,T}\right)\mathrm{d}y = 0 \tag{8}$$

Substituting from Eq. (6) into Eq. (8), one obtains the following equation for $c(T)$,

$$3(\alpha_3-\alpha_4)^2 c_T = -8c^2\left(4[\alpha_5+\alpha_6]c + \alpha_2^*[\alpha_3-\alpha_4]\right) \tag{9}$$

The behavior of the solution of this equation depends upon the signs of α_2^* and $\alpha_5+\alpha_6$. Thus a vanishing of c occurs at positive signs of the coefficients while an unbounded growth takes place when both of them are negative.

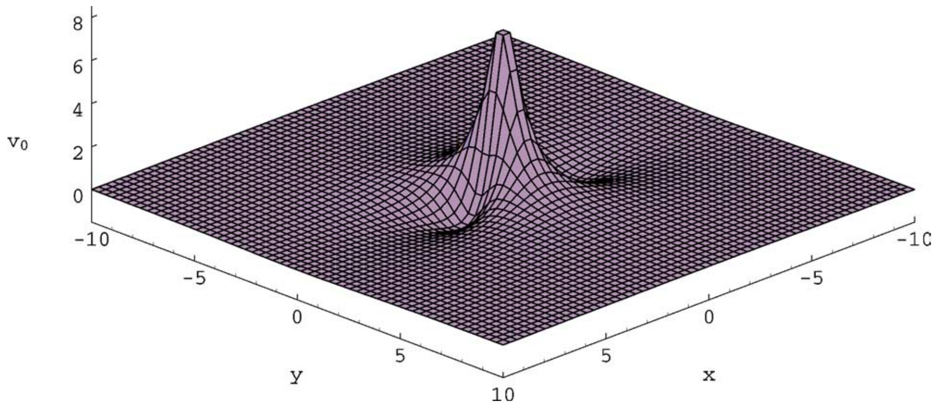

Fig. 1. Two-dimensional localized wave solution.

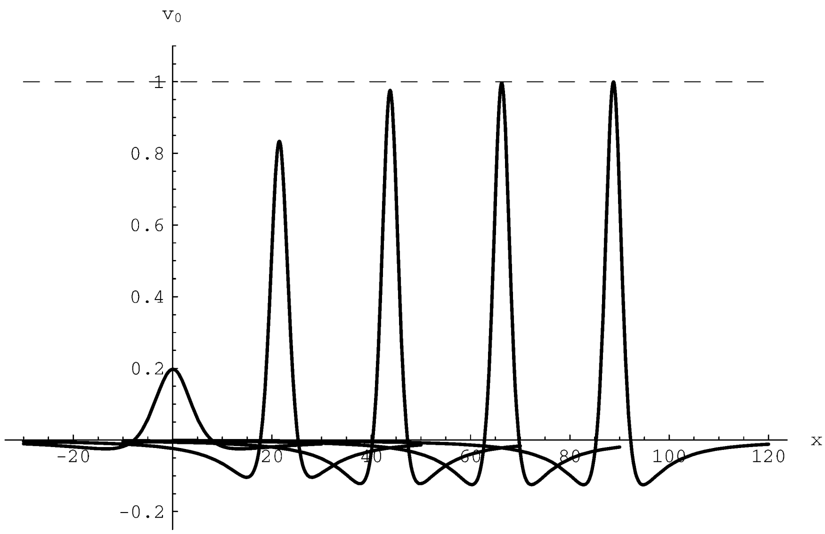

Fig. 2. Amplification and selection of the 2D localized wave along the direction of propagation in the plane $y = 0$.

When $\alpha_5 + \alpha_6 < 0$, $\alpha_2^* > 0$ then c vanishes if its initial value is less then $c^* = -\alpha_2^*[\alpha_3 - \alpha_4]/(4(\alpha_5 + \alpha_6))$. If $c(0) > c^*$ unbounded growth occurs. An interesting scenario takes place when $\alpha_5 + \alpha_6 > 0$, $\alpha_2^* < 0$. In this case the amplification of an initial localized wave with velocity $c_0 < c^*$ happens by a *finite* value of the amplitude equal to $8c^*/\alpha_1$, while the attenuation of the initial wave to the same amplitude takes place when $c_0 > c^*$. We call this phenomenon of the selection of the localized wave, since the value of c^* is defined by the equation coefficients or by the features of the microstructure. The amplitude of the 2D localized wave tends to the value $v^* = 2\alpha_2^*[\alpha_4 - \alpha_3]/(\alpha_1(\alpha_5 + \alpha_6))$. The amplification is shown in Figs. 2 and 3 in the planes $y = 0$, $\zeta = 0$ respectively. One can see that the increase in amplitude is accompanied by the decrease in the width of the wave.

When $\alpha_3 - \alpha_4 < 0$, Eq. (3) does not admit 2D localized wave solutions at $\gamma = 0$ but a plane localized wave solution may exist that depends upon a variable ζ, $\zeta_x = 1$, $\zeta_y = m$ and $\zeta_\tau = c(T)$. When the plane wave moving along the x-direction is studied, one can assume $m = 0$. Then we get in the leading order from Eq. (3) the Korteweg–de Vries equation whose localized travelling wave solution is

$$v_0 = -\frac{3c}{\alpha_1}\cosh^{-2} b\zeta \tag{10}$$

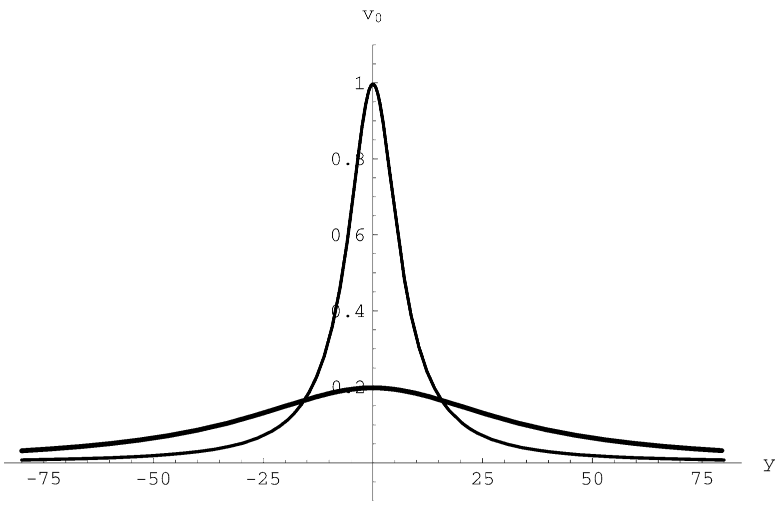

Fig. 3. Transverse amplification of the 2D localized wave in the plane $\zeta = 0$.

where $b = \sqrt{c/(2(\alpha_3 - \alpha_4))}$, hence $c < 0$. In the next order the solvability condition is [5]

$$\int_{-\infty}^{\infty} v_0\big(\alpha_2^* v_{0,\zeta\zeta} - (\alpha_5 + \alpha_6) v_{0,4\zeta} - 2v_{0,T}\big)\,\mathrm{d}\zeta = 0 \tag{11}$$

The equation for $c(T)$ follows from Eq. (11) of the form

$$105(\alpha_3 - \alpha_4)^2 c_T = -4c^2\big(10[\alpha_5 + \alpha_6]c + 7\alpha_2^*[\alpha_3 - \alpha_4]\big) \tag{12}$$

One can see that the selected value of c, $\tilde{c} = 7\alpha_2^*[\alpha_4 - \alpha_3]/(10[\alpha_5 + \alpha_6])$, and the selected amplitude $\tilde{v} = 21\alpha_2^*[\alpha_3 - \alpha_4]/(10\alpha_1[\alpha_5 + \alpha_6])$ differ from those of the 2D localized wave. Also c is negative now, hence the wave velocity is greater than c_0. The features of the wave amplification or attenuation are similar to those shown in Fig. 2 with the exception of the evolution of depressions near the core of the 2D localized wave solution.

4. Discussion

First we note the significant role played by micro-inertia and micro-displacement gradient gradient in the sign of $\alpha_3 - \alpha_4$ [3]. A negative sign allowing plane wave propagation is always achieved in presence of micro-inertia only, while micro-displacement gradient is responsible for a positive sign through α_3, hence the existence of 2D localized wave. Nonlinearity affects the sign of the wave amplitude thus allowing either a compressive or a tensile strain wave to move. The sign of α_1 is defined by the values of the Murnaghan moduli. It is difficult to measure the long wave parameters but it is easy to qualitatively distinguish the features of the evolution 2D localized and plane localized waves. Therefore even qualitative differences in the 2D waves behavior allow us to establish the above mentioned features of the microstruture.

A more important problem is to verify dissipative part of the model of a microstructure. Again the signs of the dissipative terms coefficients may be estimated studying the evolution of localized waves. One can see that the amplitude of the selected plane wave is higher than that of the 2D localized wave. However, an increase or decrease of the amplitude is higher for the 2D wave. It is easy to see it considering a particular case corresponding to the Voigt model, $\alpha_2 > 0$ or $\alpha_5 = \alpha_6 = 0$. Then we get the solutions of Eqs. (9), (12) in the form

$$c = \frac{1}{Q + PT}$$

where $Q = 1/c_0$, while $P = 8\alpha_2^*/(3(\alpha_3 - \alpha_4))$ for 2D localized wave and $P = 4\alpha_2^*/(15(\alpha_3 - \alpha_4))$ for plane wave. We see a decay of c in both cases but a 2D wave decreases faster at an initial stage by the time prescribed by an initial velocity and the value of $\alpha_3 - \alpha_4$. Presumably, this happens because of the change of the width of the wave in the transverse direction.

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