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Effective flow of a viscous liquid through a helical pipe

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Abstract

We study the flow of a viscous fluid through a pipe with helical shape parameterized with $\mathbf{r}_\varepsilon(x_1) = (x_1, a\cos\frac{x_1}{\varepsilon}, a\sin\frac{x_1}{\varepsilon})$, where the small parameter ε stands for the distance between two coils of the helix. The pipe has small cross-section of size ε. Using the asymptotic analysis of the microscopic flow described by the Navier–Stokes system, with respect to the small parameter ε that tends to zero, we find the effective fluid flow described by an explicit formula of the Poisseuile type including a small distorsion due to the particular geometry of the pipe. ***To cite this article: E. Marušić-Paloka, I. Pažanin, C. R. Mecanique 332 (2004).***

Résumé

Ecoulement effectif d'un liquide visqueux dans un tube hélicoïdal. On considère un écoulement dans un tube de section circulaire et de forme hélicoïdale paramétré par $\mathbf{r}_\varepsilon(x_1) = (x_1, a\cos\frac{x_1}{\varepsilon}, a\sin\frac{x_1}{\varepsilon})$, où ε est la distance entre deux tours de la spirale. Le rayon de la section du tube est lui aussi supposé égal à ε. A partir de l'écoulement microscopique décrit par le système de Navier–Stokes et en utilisant l'analyse asymptotique par rapport à ce petit paramètre ε on obtient l'écoulemment effectif décrit par une formule explicite de type Poiseuille associée à une petite déviation due à la géometrie du tube. ***Pour citer cet article : E. Marušić-Paloka, I. Pažanin, C. R. Mecanique 332 (2004).***

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Mots-clés : Mécanique des fluides ; Tube hélicoïdale ; Équations de Navier–Stokes

Version française abrégée

Nous allons étudier un écoulement dans un tube de section circulaire et de forme hélicoïdale, mince (ou long) pour déterminer la loi effective qui décrit cette situation physique. Dans ce but on étudie par l'analyse asymptotique

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le système de Navier–Stokes incompressible dans un tube hélicoïdal dont la courbe centrale est une hélice décrite par la paramétrisation $\mathbf{r}_\varepsilon(x_1) = (x_1, a\cos\frac{x_1}{\varepsilon}, a\sin\frac{x_1}{\varepsilon})$. Le tube est de petite section égale aussi à ε. Ce problème apparaît dans de nombreuses applications, comme par exemple le réfrigérant de Liebig.

Comme précisé plus haut, l'écoulement microscopique est régi par les équations de Navier–Stokes (2). On définit d'abord la base de Frénet $(\mathbf{t}_\varepsilon, \mathbf{n}_\varepsilon, \mathbf{b}_\varepsilon)$ de l'hélice (1). En omettant les termes $O(\varepsilon^2)$, on ecrit les équations de Navier–Stokes dans cette base dans (3). Ainsi, on fait apparaître le petit paramètre ε explicitement dans les équations, ce qui est pratique pour notre analyse. On a deux échelles spatiales et on cherchera donc $(\mathbf{u}_\varepsilon, p_\varepsilon)$ sous la forme d'un développement asymptotique (4). On obtient une suite d' Éqs. (5) et (7) définissant les termes dans le développement (4). En utilisant la séparation des variables, on trouve des solutions pour les Éqs. (5) et (7) sous la forme (6), (9), (10) et (11). On peut alors écrire notre approximation asymptotique sous une forme de type Poiseuille $V_\varepsilon = \varepsilon^2 \ell_\varepsilon \frac{q_l - q_0}{4\ell^2 a^2 \mu}(x_2^2 + x_3^2 - \varepsilon^2)(1 - \frac{x_2}{4a})\mathbf{t}_\varepsilon(x_1)$, $Q_\varepsilon = q_0 + \frac{q_\ell - q_0}{\ell}(x_1 + \varepsilon^2 \frac{x_3}{a^2})$. Finalement on obtient l'estimation d'erreur :

Théorème 0.1. *Soit* $(\mathbf{u}_\varepsilon, p_\varepsilon)$ *une solution du système des équations de Navier–Stokes* (2) *et soit* $(V_\varepsilon, Q_\varepsilon)$, $V_\varepsilon = \varepsilon^3 \mathcal{V}_\varepsilon$ *une approximation donnée par* (11) *et* (12). *Alors il existe une constante* $C > 0$ *indépendante de* ε *telle que*

$$\left\| \frac{\mathbf{u}_\varepsilon}{\varepsilon^3} - \mathcal{V}_\varepsilon \right\|_\varepsilon \leqslant C\varepsilon^2, \qquad \|p_\varepsilon - P_0\|_\varepsilon \leqslant C\varepsilon$$

ou $\|\cdot\|_\varepsilon = \frac{1}{\sqrt{|P_\varepsilon|}} |\cdot|_{L^2(P_\varepsilon)}$.

Corollaire 0.2. *Supposons que la vitesse soit prolongée par zéro en dehors du tube. Alors*

$$\varepsilon^{-4}\mathbf{u}_\varepsilon \rightharpoonup \lambda_1 \quad \text{faible* dans } \mathcal{M}\big(]0,\ell[\times \mathbf{R}^2\big)$$
$$\varepsilon^{-5}(\mathbf{u}_\varepsilon \cdot \mathbf{i}) \rightharpoonup \lambda_2 \quad \text{faible* dans } \mathcal{M}\big(]0,\ell[\times \mathbf{R}^2\big)$$

où λ_1 *et* λ_2 *sont des méasures concentrées sur le cylindre de rayon* a, *définie par*

$$\langle \lambda_1 | \psi \rangle = \frac{q_\ell - q_0}{8\ell\mu}\pi \int_0^\ell \int_0^{2\pi} \big(\psi_2(x_1, a\cos t, a\sin t)\sin t - \psi_3(x_1, a\cos t, a\sin t)\cos t\big)\,\mathrm{d}t\,\mathrm{d}x_1$$
$$\langle \lambda_2 | \psi \rangle = -\frac{q_\ell - q_0}{8\ell a\mu}\pi \int_0^\ell \int_0^{2\pi} \psi(x_1, a\cos t, a\sin t)\,\mathrm{d}t\,\mathrm{d}x_1, \quad \psi \in C_0\big(]0,\ell[\times \mathbf{R}^2\big)^3$$

1. Introduction

The goal of this Note is to study the fluid flow through a thin or long pipe with a helical shape (Fig. 1). When the problem is written in an adimensionalised form, suitable for an asymptotic analysis, three important geometric parameters appear: the pipe's thickness, the distance between two coils of the helix (helix step) and the helix diameter. We suppose that the pipe's thickness and the helix step have the same small order $\varepsilon \ll 1$ while the diameter of the helix is larger, of order 1. To get the result we generalize the method proposed in [1–3]. Such pipes appear in many devices as for instance the Liebig cooler. They are frequently coupled with the heat conduction equation, but such coupling takes place on a higher order asymptotics, due to the smallness of the viscous friction. Thus we concentrate here on the hydrodynamic part only and we suppose that the flow through the pipe is governed by the pressure drop between pipe's ends. Our goal is to find the effective behavior of the flow using the asymptotic analysis with respect to the small parameter ε. The idea is to write the equations in a Frenet basis related to a helix and to perform a classical asymptotic analysis on such equations. However, such basis depends on our small parameter ε too, and proving the convergence of the appropriate components would be meaningless. Only the

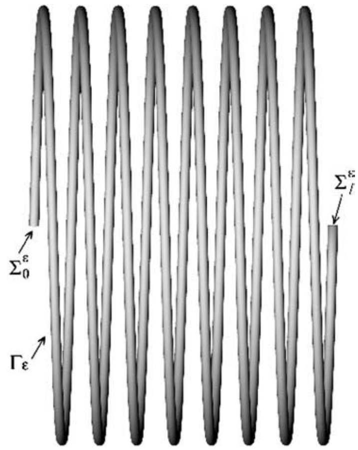

Fig. 1. Helical pipe.

convergence of the complete vector of the velocity has a meaning. Indeed the tangent and the normal vectors on such helix are weakly converging to zero. Only the binormal is constant (up to a term uniformly converging to zero) and it is pointing in the direction of the helix central axis. Thus, in the weak limit the helix transforms into the curved part of the cylinder's surface over which the helix was wrapped and the weak limit of the velocity gives the averaged mass flux along that surface.

1.1. The geometry

Let $(\mathbf{i},\mathbf{j},\mathbf{k})$ be the standard Cartesian basis. We consider a helix given by a parameterization $\mathbf{r}_\varepsilon(x_1) = x_1\mathbf{i} + a\cos\frac{x_1}{\varepsilon}\mathbf{j} + a\sin\frac{x_1}{\varepsilon}\mathbf{k}$, where $x_1 \in [0,\ell]$. We can now compute the Frenet basis

$$\begin{aligned}
&\mathbf{t}_\varepsilon(x_1) = \frac{\mathbf{r}'_\varepsilon(x_1)}{|\mathbf{r}'_\varepsilon(x_1)|} = \frac{1}{\sqrt{a^2+\varepsilon^2}}\left(\varepsilon\mathbf{i} - a\sin\frac{x_1}{\varepsilon}\mathbf{j} + a\cos\frac{x_1}{\varepsilon}\mathbf{k}\right) \\
&\mathbf{n}_\varepsilon(x_1) = \frac{\mathbf{t}'_\varepsilon(x_1)}{|\mathbf{t}'_\varepsilon(x_1)|} = -\cos\frac{x_1}{\varepsilon}\mathbf{j} - \sin\frac{x_1}{\varepsilon}\mathbf{k}, \qquad \mathbf{b}_\varepsilon(x_1) = \frac{1}{\sqrt{a^2+\varepsilon^2}}\left(a\mathbf{i} + \varepsilon\sin\frac{x_1}{\varepsilon}\mathbf{j} - \varepsilon\cos\frac{x_1}{\varepsilon}\mathbf{k}\right)
\end{aligned} \tag{1}$$

With a unit circle $B(0,1) \subset \mathbf{R}^2$ and a small parameter $0 < \varepsilon \ll 1$ we define an undeformed pipe

$$T_\varepsilon = \left\{x = (x_1,x_2,x_3) \in \mathbf{R}^3\colon\ x_1 \in\,]0,\ell[,\ (x_2,x_3) \in \varepsilon B(0,1)\right\} = \,]0,\ell[\, \times B(0,\varepsilon)$$

We now define the mapping $\mathbf{\Phi}_\varepsilon : T_\varepsilon \to \mathbf{R}^3$ by $\mathbf{\Phi}_\varepsilon(x) = \mathbf{r}_\varepsilon(x_1) + x_2\mathbf{n}_\varepsilon(x_1) + x_3\mathbf{b}_\varepsilon(x_1)$ and we put $P_\varepsilon = \mathbf{\Phi}_\varepsilon(T_\varepsilon)$. Such domain is our helical thin pipe, filled with Newtonian viscous incompressible fluid, governed by a pressure drop between the pipe's ends. It is important to notice that (x_i) are not Cartesian but curvilinear coordinates.

1.2. The equations

The equations of motion can now be expressed:

$$\begin{aligned}
&-\mu\Delta\mathbf{u}_\varepsilon + (\mathbf{u}_\varepsilon\nabla)\mathbf{u}_\varepsilon + \nabla p_\varepsilon = 0, \quad \operatorname{div}\mathbf{u}_\varepsilon = 0 \quad \text{in } P_\varepsilon \\
&\mathbf{u}_\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon = \mathbf{\Phi}_\varepsilon\big(]0,\ell[\, \times \partial B(0,\varepsilon)\big) \\
&\mathbf{u}_\varepsilon \times \mathbf{t}_\varepsilon = 0,\ p_\varepsilon = q_i \quad \text{on } \Sigma_i^\varepsilon = \mathbf{\Phi}_\varepsilon\big(\{i\} \times B(0,\varepsilon)\big),\ i = 0,\ell
\end{aligned} \tag{2}$$

Such problem has a solution $\mathbf{u}_\varepsilon \in H^1(P_\varepsilon)^3$, $p_\varepsilon \in L^2(P_\varepsilon)$. The pressure is determined only up to a constant, regardless of the boundary condition. Furthermore such solution is unique in some ball $B_{R_0} = \{\phi \in H^1(P_\varepsilon)^3;\ \operatorname{div}\phi = 0,\ \phi = 0 \text{ on } \Gamma_\varepsilon,\ |\nabla\phi|_{L^2(P_\varepsilon)} \leqslant R_0\}$ for $\varepsilon \ll 1$ (see, e.g., [4,5] or [6]). Furthermore R_0 remains bounded as $\varepsilon \to 0$.

2. Asymptotic analysis

2.1. A careful choice of basis

We now write system (2) in a basis that will allow an asymptotic analysis with respect to the small parameter ε. In this context, the usual idea is to use a pair of covariant and contravariant basis for the mapping $\mathbf{\Phi}_\varepsilon$ and to write our system using those two dual basis. However, both basis turn out to depend on the small parameter ε in an inconvenient way (contain negative powers of ε). Thus we go back to the Frenet's basis $(\mathbf{t}_\varepsilon, \mathbf{n}_\varepsilon, \mathbf{b}_\varepsilon)$ and modify it in a natural way. Indeed $\mathbf{b}_\varepsilon \to \mathbf{i}$ uniformly in x_1 and $|\mathbf{t}_\varepsilon + \mathbf{e}_\varepsilon| \to 0$ uniformly in x_1, where

$$\mathbf{e}_\varepsilon = \mathbf{e}_\varepsilon(x_1) = \sin\frac{x_1}{\varepsilon}\mathbf{j} - \cos\frac{x_1}{\varepsilon}\mathbf{k}$$

Therefore, the natural choice for our analysis is a basis $(\mathbf{e}_\varepsilon, \mathbf{n}_\varepsilon, \mathbf{i})$ which is the one appropriate for further asymptotic analysis. The Navier–Stokes system in such basis (we write only the terms of order 1 and ε, i.e. neglect the terms with higher powers of ε) reads:

$$\begin{aligned}
&-\mu\left(\frac{\partial^2 V_\varepsilon^1}{\partial x_2^2} + \frac{\partial^2 V_\varepsilon^1}{\partial x_3^2} + \frac{a}{a^2+\varepsilon^2}\frac{\partial V_\varepsilon^1}{\partial x_2} - \frac{1}{a^2+\varepsilon^2}V_\varepsilon^1 + \frac{2\varepsilon}{a^2+\varepsilon^2}\left(\frac{\partial V_\varepsilon^2}{\partial x_1} - \frac{a}{\sqrt{a^2+\varepsilon^2}}\frac{\partial V_\varepsilon^2}{\partial x_3}\right)\right)\\
&\quad - \frac{a^3}{(a^2+\varepsilon^2)^2}V_\varepsilon^2 V_\varepsilon^1 + \frac{\partial V_\varepsilon^1}{\partial x_2}V_\varepsilon^2 + \frac{a}{a^2+\varepsilon^2}V_\varepsilon^1 V_\varepsilon^2 + \frac{a}{\sqrt{a^2+\varepsilon^2}}\frac{\partial V_\varepsilon^1}{\partial x_3}V_\varepsilon^3 - \frac{a\varepsilon}{a^2+\varepsilon^2}\frac{\partial V_\varepsilon^1}{\partial x_1}V_\varepsilon^1\\
&\quad + \frac{\varepsilon}{\sqrt{a^2+\varepsilon^2}}\frac{\partial V_\varepsilon^1}{\partial x_3}V_\varepsilon^1 - \frac{a\varepsilon}{a^2+\varepsilon^2}\frac{\partial P_\varepsilon}{\partial x_1} + \frac{\varepsilon}{\sqrt{a^2+\varepsilon^2}}\frac{\partial P_\varepsilon}{\partial x_3} = 0\\
&-\mu\left(\frac{\partial^2 V_\varepsilon^2}{\partial x_2^2} + \frac{\partial^2 V_\varepsilon^2}{\partial x_3^2} - \frac{a}{a^2+\varepsilon^2}\frac{\partial V_\varepsilon^2}{\partial x_2} - \frac{1}{a^2+\varepsilon^2}V_\varepsilon^2 - \frac{2\varepsilon}{a^2+\varepsilon^2}\frac{\partial V_\varepsilon^1}{\partial x_1}\right)\\
&\quad + \frac{a}{(a^2+\varepsilon^2)^2}V_\varepsilon^1 V_\varepsilon^1 + \frac{\partial V_\varepsilon^2}{\partial x_2}V_\varepsilon^2 + \frac{a}{\sqrt{a^2+\varepsilon^2}}\frac{\partial V_\varepsilon^2}{\partial x_3}V_\varepsilon^3 + \frac{\varepsilon}{\sqrt{a^2+\varepsilon^2}}\frac{\partial V_\varepsilon^2}{\partial x_3}V_\varepsilon^1\\
&\quad - \frac{a\varepsilon}{\sqrt{a^2+\varepsilon^2}}\frac{\partial V_\varepsilon^2}{\partial x_1}V_\varepsilon^1 - \frac{\varepsilon}{a^2+\varepsilon^2}V_\varepsilon^1 V_\varepsilon^3 + \frac{\partial P_\varepsilon}{\partial x_2} = 0\\
&-\mu\left(\frac{\partial^2 V_\varepsilon^3}{\partial x_2^2} + \frac{\partial^2 V_\varepsilon^3}{\partial x_3^2} - \frac{a}{a^2+\varepsilon^2}\frac{\partial V_\varepsilon^3}{\partial x_2} - \frac{2\varepsilon}{a^2+\varepsilon^2}\frac{\partial V_\varepsilon^1}{\partial x_2}\right)\\
&\quad + \frac{\partial V_\varepsilon^3}{\partial x_2}V_\varepsilon^2 + \frac{a}{\sqrt{a^2+\varepsilon^2}}\frac{\partial V_\varepsilon^3}{\partial x_3}V_\varepsilon^3 - \frac{a\varepsilon}{a^2+\varepsilon^2}\frac{\partial V_\varepsilon^3}{\partial x_1}V_\varepsilon^1 + \frac{\varepsilon}{\sqrt{a^2+\varepsilon^2}}\frac{\partial V_\varepsilon^3}{\partial x_3}V_\varepsilon^1\\
&\quad - \frac{\varepsilon}{a^2+\varepsilon^2}V_\varepsilon^1 V_\varepsilon^2 + \frac{a}{\sqrt{a^2+\varepsilon^2}}\frac{\partial P_\varepsilon}{\partial x_3} = 0\\
&\frac{\partial V_\varepsilon^2}{\partial x_2} + \frac{a}{\sqrt{a^2+\varepsilon^2}}\frac{\partial V_\varepsilon^3}{\partial x_3} - \frac{a}{a^2+\varepsilon^2}V_\varepsilon^2 - \frac{a\varepsilon}{a^2+\varepsilon^2}\frac{\partial V_\varepsilon^1}{\partial x_1} + \frac{\varepsilon}{\sqrt{a^2+\varepsilon^2}}\frac{\partial V_\varepsilon^1}{\partial x_3} = 0
\end{aligned} \tag{3}$$

2.2. Asymptotic expansion

We look for an asymptotic expansion in the form

$$\begin{cases}
V_\varepsilon^1(x) = \varepsilon^3 V_0^1\left(x_1, \frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\right) + \varepsilon^4 V_1^1\left(x_1, \frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\right) + \cdots\\
V_\varepsilon^i(x) = \varepsilon^4 V_0^i\left(x_1, \frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\right) + \varepsilon^5 V_1^i\left(x_1, \frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\right) + \cdots, \; i = 2, 3\\
P_\varepsilon(x) = P_0(x_1) + \varepsilon^3 P_1\left(x_1, \frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\right) + \cdots
\end{cases} \tag{4}$$

Denoting $y_\alpha = \frac{x_\alpha}{\varepsilon}$, $\alpha = 2, 3$, substituting the expansion in system (3) and collecting the terms with equal powers of ε we get

$$\begin{aligned}
&\varepsilon\colon\ -\mu\Delta_{y'} V_0^1 - \frac{1}{a}\frac{\partial P_0}{\partial x_1} = 0, \quad \varepsilon^3\colon\ \mathrm{div}_{y'} V_0 + \frac{1}{a}\frac{\partial V_0^1}{\partial y_3} = 0\\
&\varepsilon^2\colon\ \Delta_{y'} V_1^1 + \frac{1}{a}\frac{\partial V_0^1}{\partial y_2} = 0, \qquad -\mu\Delta_{y'} V_0^i + \frac{\partial P_1}{\partial y_i} = 0, \quad i = 2, 3
\end{aligned} \tag{5}$$

where $y' = (y_2, y_3)$, $\mathrm{div}_{y'} W = \frac{\partial W^2}{\partial y_2} + \frac{\partial W^3}{\partial y_3}$, $\Delta_{y'} W = \frac{\partial^2 W}{\partial y_2^2} + \frac{\partial^2 W}{\partial y_3^2}$, for $W = (W^1, W^2, W^3)$. Taking into account the boundary conditions, we obtain

$$\begin{aligned}
&V_0^1 = \frac{q_\ell - q_0}{\ell} w(y'), \qquad w(y') = \frac{1}{4a\mu}\bigl(1 - y_2^2 - y_3^2\bigr), \qquad P_0 = q_0 + \frac{x_1}{\ell}(q_\ell - q_0), \qquad V_0^2 = 0\\
&V_0^3 = -\frac{q_\ell - q_0}{a\ell} w(y') = -\frac{1}{a} V_0^1, \qquad P_1 = \frac{q_\ell - q_0}{a^2\ell} y_3 + P_1^0(x_1), \qquad V_1^1 = -\frac{1}{4a} y_2 V_0^1
\end{aligned} \tag{6}$$

We need to compute one more term in the incompressibility equation

$$\varepsilon^4\colon\ \mathrm{div}_{y'} V_1 - \frac{1}{a} V_0^2 - \frac{1}{a}\frac{\partial V_0^1}{\partial x_1} + \frac{1}{a}\frac{\partial V_1^1}{\partial y_3} = 0 \tag{7}$$

According to (6) functions $V_0^2 = 0$ and V_0^1 do not depend on x_1 so that (7) implies

$$\frac{\partial V_1^2}{\partial y_2} + \frac{\partial V_1^3}{\partial y_3} = \frac{1}{4a^2} y_2 \frac{\partial V_0^1}{\partial y_3} \tag{8}$$

Eq. (8) will be satisfied if we take

$$V_1^2 = 0, \qquad V_1^3 = \frac{1}{4a^2} y_2 V_0^1 = -\frac{1}{a} V_1^1 \tag{9}$$

Following (4), (6) and (9) we get the expansion of the form

$$V_\varepsilon^1 = \varepsilon^3 \frac{q_\ell - q_0}{\ell} w(y')\left(1 - \frac{\varepsilon}{4a} y_2\right), \qquad V_\varepsilon^2 = 0 \tag{10}$$

$$V_\varepsilon^3 = -\varepsilon^4 \frac{q_\ell - q_0}{a\ell} w(y')\left(1 - \frac{\varepsilon}{4a} y_2\right), \qquad Q_\varepsilon = q_0 + \frac{q_\ell - q_0}{\ell}\left(x_1 + \varepsilon^2 \frac{x_3}{a^2}\right) \tag{11}$$

To get a better idea of our approximation we write it in the form

$$V_\varepsilon = \varepsilon^2 \ell_\varepsilon \frac{q_\ell - q_0}{4\ell^2 a^2 \mu}\bigl(x_2^2 + x_3^2 - \varepsilon^2\bigr)\left(1 - \frac{x_2}{4a}\right)\mathbf{t}_\varepsilon(x_1) \tag{12}$$

where $\ell_\varepsilon = \ell\sqrt{1 + \frac{a^2}{\varepsilon^2}}$ stands for the length of the pipe.

Remark 1. We see that (12) has the form of the Poiseuille flow in direction tangential to the central line of the pipe, except for the small deviation caused by the term $(1 - \frac{x_2}{4a}) = 1 + O(\varepsilon)$. Since $\ell_\varepsilon = O(\frac{1}{\varepsilon})$, that small deviation term is of order ε^4, as it can be seen from (4). Due to the $O(\varepsilon)$ term the flow profile is not exactly parabolic as in the case of Poiseuille flow. The pressure Q_ε is the Poiseuille pressure, except for the term $\varepsilon^2 \frac{x_3}{a^2} = O(\varepsilon^3)$. It is worth noticing that the pressure $Q_\varepsilon = O(1)$ while that velocity $V_\varepsilon = O(\varepsilon^3)$, so that the difference in orders is $O(\varepsilon^3)$ and not $O(\varepsilon^2)$ as in the case of classical Poiseuille flow through a straight thin pipe.

3. Convergence

It remains to prove the error estimate for the formally derived asymptotic approximation. We use the similar tools as in [2] but adapted to our complex geometry. It leads to the following result:

Theorem 3.1. *Let* $(\mathbf{u}_\varepsilon, p_\varepsilon)$ *be the solution of the Navier–Stokes system* (2) *and let* $(V_\varepsilon, Q_\varepsilon)$, $V_\varepsilon = \varepsilon^3 \mathcal{V}_\varepsilon$ *be the approximation given by* (11) *and* (12). *Then there exists a constant* $C > 0$ *independent from* ε *such that*

$$\left\| \frac{\mathbf{u}_\varepsilon}{\varepsilon^3} - \mathcal{V}_\varepsilon \right\|_\varepsilon \leqslant C\varepsilon^2, \qquad \| p_\varepsilon - P_0 \|_\varepsilon \leqslant C\varepsilon \tag{13}$$

where $\| \cdot \|_\varepsilon = \frac{1}{\sqrt{|P_\varepsilon|}} | \cdot |_{L^2(P_\varepsilon)}$.

As an easy consequence we get:

Corollary 3.2. *Suppose that the velocity is extended by zero outside the pipe. Then*

$$\varepsilon^{-4} \mathbf{u}_\varepsilon \rightharpoonup \lambda_1 \quad \textit{weak* in } \mathcal{M}\big(]0, \ell[\times \mathbf{R}^2\big)$$
$$\varepsilon^{-5} (\mathbf{u}_\varepsilon \cdot \mathbf{i}) \rightharpoonup \lambda_2 \quad \textit{weak* in } \mathcal{M}\big(]0, \ell[\times \mathbf{R}^2\big)$$

where λ_1 *and* λ_2 *are the measures concentrated on cylinder of radius* a, *defined by*

$$\langle \lambda_1 | \psi \rangle = \frac{q_\ell - q_0}{8\ell\mu} \pi \int_0^\ell \int_0^{2\pi} \big(\psi_2(x_1, a\cos t, a\sin t) \sin t - \psi_3(x_1, a\cos t, a\sin t) \cos t \big) \, \mathrm{d}t \, \mathrm{d}x_1$$

$$\langle \lambda_2 | \psi \rangle = -\frac{q_\ell - q_0}{8\ell a \mu} \pi \int_0^\ell \int_0^{2\pi} \psi(x_1, a\cos t, a\sin t) \, \mathrm{d}t \, \mathrm{d}x_1, \quad \psi \in C_0\big(]0, \ell[\times \mathbf{R}^2\big)^3$$

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