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Analysis of the thermodynamic conditions for brittle fracture

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Abstract

The analysis of the thermodynamic conditions for brittle fracture is given for the two known models of an isolated defect. In the first model the stresses on the external surface of the solid remain the same as before and after formation of a defect. In the second model the displacements on the external surface of the solid at a defect formation remain the same as in the solid without a defect. It is shown that the first model in the isothermal case of deformation leads to Griffith condition and the second model leads to the other proposed energy condition of fracture which, unlike the Griffith condition, contains an increment of the entropy component of the internal energy which is not zero in a general case. ***To cite this article: I.M. Dunaev, V.I. Dunaev, C. R. Mecanique 332 (2004).***

Résumé

Analyse des conditions thermodynamique pour la rupture fragile. On présente une analyse des conditions thermodynamiques pour la rupture fragile, dans le cadre des deux modèles du défaut isolé bien connus. Dans le premier modèle, on maintient les mêmes contraintes à la surface extérieure du solide avant et après l'apparition du défaut. Dans le deuxième modèle, ce sont les déplacements que l'on garde constants. Nous démontrons que le premier modèle conduit à la condition de Griffith dans le cas d'une déformation isothermique, tandis que le deuxième modèle produit une condition bien différente. Selon cette condition, les variations de l'entropie ne sont plus nulles, contrairement à la condition de Griffith. ***Pour citer cet article : I.M. Dunaev, V.I. Dunaev, C. R. Mecanique 332 (2004).***

Keywords: Rupture; Energy condition; Internal energy; Entropy

Mots-clés : Rupture ; Condition sur l'énergie ; Energie interne ; Entropie

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1. Introduction

Griffith [1,2] was the first to approach the theory of brittle fracture from the point of view of energy considerations and he proposed an energy condition of brittle fracture

$$\mathrm{d}U_p - \gamma_p \,\mathrm{d}\Sigma = \mathrm{d}A \tag{1}$$

Here, $U_p = U_p^{(0)} - U_p^{(1)}$ is the change of a potential energy caused by the defect formation; $U_p^{(0)}$ and $U_p^{(1)}$ are potential energies of a solid without and with defect, respectively; γ_p is the energy necessary for the formation of a unit area of defect surface Σ, which is equal to the specific surface energy; A is the change of the work of external forces during a defect formation. It is well-known that condition (1) is inapplicable for strength assessment at compression. In particular, for problems of material fracture under uniaxial and biaxial tension/compression of a plate with a defect, the condition (1) leads to the temperature independent and identical in absolute value critical stresses, contradicting experimental data for practically all known materials. This short-coming was not eliminated using assumption [2,3] that defects are closing at compression, consequently the sliding friction appears on the defect surface. However, to explain the compressive strength exceeding (in absolute value) the tensile strength, in the calculations using this assumption [2,3] the value of sliding friction coefficient on defect surface has to be accepted improbably large in comparison with experimental data. In numerous investigations [2,4,5] based on thermodynamics, a generalized energy approach to the fracture was proposed, and in particular condition (1) obtained. However, in these investigations the authors did not state explicitly the fundamental physical and mathematical assumptions of their models of fracture leading to condition (1). In particular, they did not precise the physical hypotheses and mathematical substantiations in their models of fracture on which basis the increment of the entropy component of internal energy was assumed equal to zero.

In this Note the necessary energy conditions of fracture are formulated on the basis of thermodynamics for thermoelastic deformation of solids (Section 2). Analyzing the conditions for the two known models of a solid with a defect (Section 3), it is shown that one of the models leads to a Griffith type energy condition of fracture, which, unlike condition (1), contains an increment of an entropy component of the internal energy which is nonzero in the general case. This condition allows the elimination of the above mentioned short-comings of the original Griffith theory. Another model of a solid with a defect leads to a thermodynamically incomplete (increment of the entropy component of the internal energy is zero) condition (1) and, therefore, to physically groundless and experimentally unsupported results.

2. Necessary energy conditions of fracture

Let us consider a linear thermoelastic solid where a defect (a crack) is forming and propagating. Then, we introduce notations for values with indexes $^{(0)}$ and $^{(1)}$ for the solid without and with a defect respectively: $u^{(0)}$, $\eta^{(0)}$, $\varepsilon_{ij}^{(0)}$, $\sigma_{ij}^{(0)}$, $u_i^{(0)}$, $T^{(0)}$, V_0 is a volume, S_0 is surface area, $u^{(1)}$, $\eta^{(1)}$, $\varepsilon_{ij}^{(1)}$, $\sigma_{ij}^{(1)}$, $u_i^{(1)}$, $T^{(1)}$, V_1, $S_1 = S_0 + \Sigma$, where Σ is the surface area of the defect. Here, indexes $^{(0)}$ and $^{(1)}$ are used for: u the specific internal energy, η the specific entropy, ε_{ij} the components of the strain tensor, σ_{ij} the components of the stress tensor, u_i the components of the displacement vector, T the absolute temperature. For the thermoelastic solid we have [6]:

$$u = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} + \frac{3}{2}\alpha_0 K_0 \Theta (T - T_0) + 3\alpha_0 K_0 T_0 \Theta + c_\varepsilon (T - T_0) \tag{2}$$

$$\eta = 3\alpha_0 K_0 \Theta + c_\varepsilon \ln\frac{T}{T_0} \approx 3\alpha_0 K_0 \Theta + c_\varepsilon \frac{T - T_0}{T_0}, \quad \text{at } \frac{T - T_0}{T_0} \ll 1 \tag{3}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad \sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\Theta\delta_{ij} - 3\alpha_0 K_0 (T - T_0)\delta_{ij}, \quad i, j = 1, 2, 3 \tag{4}$$

where x_i are Cartesian coordinates of the solid points. In the expression (2) and beyond, the summation is performed over repeating indices i, j, $\Theta = \varepsilon_{ij}\delta_{ij}$ is the first invariant of the strain tensor, δ_{ij} is the Kronecker delta, α_0 is the linear coefficient of thermal expansion, c_ε is the specific heat at constant strain, μ and λ are Lame's coefficients, $2\mu = E/(1+\nu)$, $\lambda = \nu E/[(1+\nu)(1-2\nu)]$, $3K_0 = E/(1-2\nu)$, E is the Young's modulus, ν is the Poisson's ratio. Using the integral form of the first and second laws of thermodynamics and the conservation mechanical energy theorem [3,6–8], let us write the energy conditions for formation and propagation of the defect (crack)

$$\mathrm{d}K + \mathrm{d}U - \mathrm{d}U^* = \mathrm{d}A + \mathrm{d}Q \tag{5}$$

$$\mathrm{d}S - \mathrm{d}S^* = \int\limits_{V_0} \frac{\mathrm{d}Q^{(0)}}{T^{(0)}}\,\mathrm{d}V - \int\limits_{V_1} \frac{\mathrm{d}Q^{(1)}}{T^{(1)}}\,\mathrm{d}V \tag{6}$$

In expressions (5) and (6)

$$K = K^{(0)} - K^{(1)}, \quad U = U^{(0)} - U^{(1)}, \quad A = A^{(0)} - A^{(1)}, \quad Q = Q^{(0)} - Q^{(1)}, \quad S = S^{(0)} - S^{(1)} \tag{7}$$

$$U^{(q)} = \int\limits_{V_q} u^{(q)}\,\mathrm{d}V, \quad S^{(q)} = \int\limits_{V_q} \eta^{(q)}\,\mathrm{d}V, \quad q = 0, 1$$

are the respective changes of the kinetic energy, internal energy, work of the external forces, influx of heat and entropy, caused by defect propagation during time t,

$$U^* = \oint\limits_{\Sigma} u^*\,\mathrm{d}s = \gamma\Sigma, \qquad S^* = \int\limits_{\Sigma} \eta^*\,\mathrm{d}s \tag{8}$$

are the internal energy of defect formation of a surface Σ and the entropy of defect formation, ds is an element of surface area of defect, $u^* > 0$ is a specific internal energy, γ is an average specific internal energy, η^* is the specific entropy. The condition (5) is the thermodynamic requirement to define crack propagation. The entropy of defect formation (8) could be obtained from Eq. (6). To solve applied problems, the conditions (5) and (6) must be complemented by geometrical, physical and mathematical model of the defect, loading conditions and also by the condition which determines the relative position of the defect and it trajectory. In static loading and $T^{(0)} = T^{(1)} = T_0 = \mathrm{const}$, $\mathrm{d}K = 0$, $\mathrm{d}Q = 0$ from conditions (5) and (6) considering (8) we find:

$$\mathrm{d}U - \gamma\,\mathrm{d}\Sigma = \mathrm{d}A \tag{9}$$

$$\mathrm{d}S - \mathrm{d}S^* = 0 \tag{10}$$

With these conditions, unlike condition (1), the internal energy (2) includes the entropy component (3) and $\gamma = \mathrm{const}$ has another, above mentioned, physical meaning. Further it will be shown with what additional assumptions the conditions (9) and (10) lead to the condition (1). In the case when the defect surface may be determined by only one parameter a (e.g. a half-length of a crack or large half-axis of an ellipse form defect in plain deformation) let us rewrite condition (9) as:

$$\frac{\mathrm{d}W}{\mathrm{d}a}\dot{a} = 0, \quad W = U - \gamma\Sigma - A \tag{11}$$

If $\dot{a} = 0$ then there is no crack propagation. Thus the criteria of probable propagation of the crack is:

$$\frac{\mathrm{d}W}{\mathrm{d}a} = 0, \quad \dot{a} \geqslant 0 \tag{12}$$

but we do not know whether it will actually happen or not (so that $\dot{a} = 0$ cannot be excluded) [7].

3. Energy condition for fracture using two models for an isolated defect

Let us consider in detail the energy condition (12) at plane stressed (strained) state for the two known models of an isolated defect. In model (A) the same stresses are prescribed on the external surface of the solid S_0 before and after formation of the defect, and stresses on the surface of the defect Σ are equal to zero. External forces in this model produce work on the external surface S_0 on displacement, caused by the formation of the defect. In model (B), on the external surface of the solid S_0 before and after formation of the defect, displacements which correspond to the applied load are fixed, but before the defect was formed. Stresses are also zero on the surface of the defect Σ in model (B). The work of external forces on the external surface of the solid S_0 during formation of the defect is $\mathrm{d}A = 0$ as displacements are fixed. The integrals of the internal energy (7) taking into consideration (2)–(4) at $T^{(0)} = T^{(1)} = T_0$ may be written

$$\begin{aligned} U &= \left(U_p^{(0)} - U_p^{(1)}\right) + T_0\left(S^{(0)} - S^{(1)}\right) \\ &= \frac{1}{2}\left(\int\limits_{V_0} \sigma_{ij}^{(0)}\varepsilon_{ij}^{(0)}\,\mathrm{d}V - \int\limits_{V_1} \sigma_{ij}^{(1)}\varepsilon_{ij}^{(1)}\,\mathrm{d}V\right) + \alpha_0 T_0 k_1\left(\int\limits_{V_0} \varepsilon_{ij}^{(0)}\delta_{ij}\,\mathrm{d}V - \int\limits_{V_1} \varepsilon_{ij}^{(1)}\delta_{ij}\,\mathrm{d}V\right) \end{aligned} \tag{13}$$

where $k_1 = E/(1-\nu)$ is for the plane stressed state, $k_1 = E/(1-2\nu)$ is for the plane deformation, $i, j = 1, 2$. Let us compute the increment of the total energy W (11). For model (A), using relations (13) and (4) at $T = T_0$, Betti's reciprocal theorem [6], the expression for the work of external forces, and Airy stress function F,

$$\sigma_{ij}^{(0)}\varepsilon_{ij}^{(1)} = \sigma_{ij}^{(1)}\varepsilon_{ij}^{(0)}, \quad A = \oint\limits_{S_0} \sigma_{ij}^{(0)}\left(u_i^{(0)} - u_i^{(1)}\right)n_j\,\mathrm{d}s, \quad \sigma_{11}^{(q)} = \frac{\partial^2 F^{(q)}}{\partial x_2^2}, \quad \sigma_{22}^{(q)} = \frac{\partial^2 F^{(q)}}{\partial x_1^2} \tag{14}$$

respectively, we obtain

$$\begin{aligned} W = {} & \frac{1}{2}\int\limits_{V_1} \left(\sigma_{ij}^{(0)} + \sigma_{ij}^{(1)}\right)\left(\varepsilon_{ij}^{(0)} - \varepsilon_{ij}^{(1)}\right)\mathrm{d}V + \frac{1}{2}\int\limits_{V} \sigma_{ij}^{(0)}\varepsilon_{ij}^{(0)}\,\mathrm{d}V - \oint\limits_{S_0} \sigma_{ij}^{(0)}\left(u_i^{(0)} - u_i^{(1)}\right)n_j\,\mathrm{d}s \\ & + \alpha_0 T_0 \chi\left[\int\limits_{V_0}\left(\frac{\partial^2 F^{(0)}}{\partial x_1^2} + \frac{\partial^2 F^{(0)}}{\partial x_2^2}\right)\mathrm{d}V - \int\limits_{V_1}\left(\frac{\partial^2 F^{(1)}}{\partial x_1^2} + \frac{\partial^2 F^{(1)}}{\partial x_2^2}\right)\mathrm{d}V\right] - \gamma\Sigma \end{aligned} \tag{15}$$

Here $V = V_0 - V_1$, $\chi = 1$ and $\chi = 1 + \nu$ for the plane stressed state and the plane deformation, respectively, n_j are direction cosines of the external normal $\mathbf{n}$ to the solid boundary. Using [6] on the boundaries S_0 and $S_0 + \Sigma$ expressions

$$\begin{aligned} & F_{x_1}^{(q)}(s) = \frac{\partial F^{(q)}}{\partial x_1} = -\int\limits_0^s \sigma_{nx_2}^{(q)}(s)\,\mathrm{d}s + A_q, \qquad F_{x_2}^{(q)}(s) = \frac{\partial F^{(q)}}{\partial x_2} = \int\limits_0^s \sigma_{nx_1}^{(q)}(s)\,\mathrm{d}s + B_q \\ & \sigma_{nx_1}^{(q)} = \sigma_{11}^{(q)}n_1 + \sigma_{12}^{(q)}n_2, \quad \sigma_{nx_2}^{(q)} = \sigma_{12}^{(q)}n_1 + \sigma_{22}^{(q)}n_2, \quad q = 0, 1 \end{aligned} \tag{16}$$

Green's formula and equations of equilibrium

$$\frac{\partial\sigma_{ij}^{(0)}}{\partial x_i} = 0, \quad \frac{\partial\sigma_{ij}^{(1)}}{\partial x_i} = 0, \quad i, j = 1, 2 \tag{17}$$

the expression (15), considering the formula (4), may be written

$$W = \frac{1}{2}\oint\limits_{S_0+\Sigma} \left(\sigma_{ij}^{(0)} + \sigma_{ij}^{(1)}\right)\left(u_i^{(0)} - u_i^{(1)}\right)n_j\,\mathrm{d}s + \frac{1}{2}\oint\limits_{\Sigma} \sigma_{ij}^{(0)}u_i^{(0)}n_j\,\mathrm{d}s - \oint\limits_{S_0} \sigma_{ij}^{(0)}\left(u_i^{(0)} - u_i^{(1)}\right)n_j\,\mathrm{d}s$$

$$+\alpha_0 T_0 \chi \left\{ \oint_{S_0} \left[\int_0^s \left(\sigma_{nx_1}^{(1)} - \sigma_{nx_1}^{(0)}\right) \mathrm{d}s \right] \mathrm{d}x_1 + \left[\int_0^s \left(\sigma_{nx_2}^{(1)} - \sigma_{nx_2}^{(0)}\right) \mathrm{d}s \right] \mathrm{d}x_2 + \oint_{\Sigma} \left(\int_0^s \sigma_{nx_1}^{(1)} \, \mathrm{d}s \right) \mathrm{d}x_1 + \left(\int_0^s \sigma_{nx_2}^{(1)} \, \mathrm{d}s \right) \mathrm{d}x_2 \right\} - \gamma \Sigma, \quad \oint_{\Sigma} A_q \, \mathrm{d}x_1 + B_q \, \mathrm{d}x_2 = 0 \tag{18}$$

So, since far from the boundary conditions (16) in a solid with and without the defect in model (A) we have

$$\sigma_{nx_1}^{(0)} = \sigma_{nx_1}^{(1)}, \quad \sigma_{nx_2}^{(0)} = \sigma_{nx_2}^{(1)} \quad \text{at } (x_1, x_2) \in S_0; \qquad \sigma_{nx_1}^{(1)} = \sigma_{nx_2}^{(1)} = 0 \quad \text{at } (x_1, x_2) \in \Sigma \tag{19}$$

then the last two integrals in the expression (18) are equal to zero. Therefore, the entropic component of the internal energy (13) also is equal to zero. Then, on account of the condition (19) and the energy condition (12) we finally obtain the increment of the total energy W (11) and Griffith criterion (1) in the form

$$W = \frac{1}{2} \oint_{\Sigma} \sigma_{ij}^{(0)} u_i^{(1)} n_j \, \mathrm{d}s - \gamma \Sigma, \qquad \frac{\mathrm{d}W}{\mathrm{d}a} = \frac{\mathrm{d}}{\mathrm{d}a} \left(\frac{1}{2} \oint_{\Sigma} \sigma_{ij}^{(0)} u_i^{(1)} n_j \, \mathrm{d}s - \gamma \Sigma \right) = 0 \tag{20}$$

Let us compute the increment of the total energy (11) for model (B), using the conditions on the boundary S_0 of a solid with and without a defect and on the boundary of the defect Σ

$$u_i^{(0)} = u_i^{(1)} \quad \text{at } (x_1, x_2) \in S_0, \qquad \sigma_{nx_1}^{(1)} = \sigma_{nx_2}^{(1)} = 0 \quad \text{at } (x_1, x_2) \in \Sigma \tag{21}$$

In this case the work of external forces is equal to zero. Using the relations (13) and (14), Green's formula and Eqs. (17), (4) for expression (11) we obtain

$$W = U - \gamma \Sigma = \frac{1}{2} \oint_{S_0 + \Sigma} \left(\sigma_{ij}^{(0)} + \sigma_{ij}^{(1)}\right)\left(u_i^{(0)} - u_i^{(1)}\right) n_j \, \mathrm{d}s + \frac{1}{2} \oint_{\Sigma} \sigma_{ij}^{(0)} u_i^{(0)} n_j \, \mathrm{d}s + \alpha_0 T_0 k_1 \left[\oint_{S_0 + \Sigma} \left(u_i^{(0)} - u_i^{(1)}\right) \delta_{ij} n_j \, \mathrm{d}s + \oint_{\Sigma} u_i^{(0)} \delta_{ij} n_j \, \mathrm{d}s \right] - \gamma \Sigma \tag{22}$$

Then, on account of conditions (21) we reduce the expressions (22) and (12) to

$$\frac{\mathrm{d}W}{\mathrm{d}a} = \frac{\mathrm{d}}{\mathrm{d}a} \left(\frac{1}{2} \oint_{\Sigma} \sigma_{ij}^{(0)} u_i^{(1)} n_j \, \mathrm{d}s + \alpha_0 T_0 k_1 \oint_{\Sigma} u_i^{(1)} \delta_{ij} n_j \, \mathrm{d}s - \gamma \Sigma \right) = 0 \tag{23}$$

Therefore, the model of the defect (B) leads to condition (23) in which increment of the entropy component of the internal energy in a general case is not zero. This is the main essential difference of the proposed condition (23) from the Griffith criterion (20).

4. Test problem

Let us consider the problem of the determination of critical stresses in a circular plate, radius b with a round shaped defect radius a under a symmetrical load P in the case of a plane deformation. The solution of the problem of elasticity theory is:

$$\begin{aligned}
&u_1^{(1)} = c_1 r + \frac{c_2}{r}, \quad u_2^{(1)} = 0, \quad \sigma_{12}^{(1)} = \sigma_{r\vartheta}^{(1)} = 0 \\
&\sigma_{11}^{(1)} = \sigma_r^{(1)} = \frac{E}{1+\nu}\left(\frac{c_1}{1-2\nu} - \frac{c_2}{r^2}\right), \quad \sigma_{22}^{(1)} = \sigma_\vartheta^{(1)} = \frac{E}{1+\nu}\left(\frac{c_1}{1-2\nu} + \frac{c_2}{r^2}\right) \\
&u_1^{(0)} = \frac{P(1-2\nu)r}{1+\nu}, \quad u_2^{(0)} = 0, \quad \sigma_{11}^{(0)} = \sigma_r^{(0)} = \sigma_\vartheta^{(0)} = P, \quad \sigma_{12}^{(0)} = \sigma_{r\vartheta}^{(0)} = 0
\end{aligned} \tag{24}$$

where r, ϑ are the polar coordinates of the solid points, c_1, c_2 are the integration constants which we determine from the boundary conditions in models (A) and (B). For model (A), using the boundary conditions at $r = a$, $\sigma_{11}^{(1)} = 0$, $r = b$, $\sigma_{11}^{(1)} = P$ and the solution (24), from condition (20) at $n_1 = 1$, $n_2 = 0$, $\Sigma = 2\pi a$, after integration we obtain equal (in absolute value) critical stresses at tension P^+ and compression P^-

$$P^{\pm} = \pm\sqrt{\frac{\gamma E(1 - a^2/b^2)}{2a(1 - \nu^2)}}$$

For model (B) using the boundary conditions at $r = a$, $\sigma_{11}^{(1)} = 0$, at $r = b$, $u_1^{(1)} = u_1^{(0)}$, the solution (24) and the condition (23) we obtain similarly

$$P^{\pm} = -\alpha_0 T_0 k_1 \pm \sqrt{(\alpha_0 T_0 k_1)^2 + \frac{\gamma E}{2a(1 - \nu^2)}\left[1 + \frac{a^2}{b^2}\frac{1}{(1 - 2\nu)}\right]^2} \tag{25}$$

The critical stresses (25) at tension and compression are different and obviously depend on T_0 and α_0. In [9], the solution similar to (25) was obtained for a defect in the form of a crack. In [10], using condition (23), a curve of fracture in the form of an ellipse in the space of principle stresses P_1, P_2 has been obtained with additional limitations imposed upon the physical and mathematical models of the defect, conditions of isotropy and convexity

$$P_1^2 + P_2^2 - 2\nu_* P_1 P_2 + 2\alpha_0 T_0 k_1 (1 - \nu_*)(P_1 + P_2) = 32\mu(1 - \nu_*)\gamma/\big[(æ + 1)\pi a_*\big] \tag{26}$$

where a_* is a half-length of a crack critical dimension, $\nu_* \in (-1, 1)$, $æ = 3 - 4\nu$, $æ = (3 - \nu)/(1 + \nu)$ for plain deformation and plane stressed state, respectively.

5. Conclusion

The work proposes an energy condition (23) for the brittle fracture of solids on the basis of thermodynamics of a thermoelastic deformation, considering an increment of the entropy component of the internal energy.

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