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On resonances in a homogenization problem

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Abstract

We consider a two-dimensional boundary value problem for the Helmholtz equation with Neumann boundary condition on a set of arcs. This set is obtained from a closed curve by cutting out small holes situated closely each to other and having locally periodic structure. We construct asymptotics of scattering frequencies (poles of analytic continuation of solutions) with small imaginary parts and show that these scattering frequencies imply resonances. ***To cite this article: R.R. Gadyl'shin, C. R. Mecanique 331 (2003).***

Résumé

Sur les résonances dans un problème d'homogénéisation. On considère un problème aux limites bidimensionnel pour l'équation de Helmholtz avec des conditions de Neumann sur un ensemble d'arcs. Cet ensemble s'obtient à partir d'une courbe fermée en supprimant des petites parties très proches les unes des autres et disposées de façon périodique. Nous déduisons le comportement asymptotique des fréquences de diffusion (pôles des prolongements analytiques des solutions) avec une petite partie imaginaire et nous montrons qu'elles impliquent l'existence des résonances. ***Pour citer cet article : R.R. Gadyl'shin, C. R. Mecanique 331 (2003).***

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Mots-clés : Vibrations ; Asymptotique ; Un petit paramètre ; Homogénéisation

1. Introduction

It is known that the scattering of H-polarized electromagnetic field with an ideally conducting cylindrical surface the cross-section of which is the curve $\varGamma_\delta$, and vibrations of a membrane which is not fixed on the cut $\varGamma_\delta$, are described by the solution of the following boundary value problem in $\varOmega_\delta = \mathbb{R}^2 \setminus \overline{\varGamma_\delta}$:

$$\big(\Delta + k^2\big)u_\delta = f, \quad x \in \varOmega_\delta, \quad \frac{\partial u_\delta}{\partial \nu} = 0, \quad x \in \varGamma_\delta, \quad \frac{\partial u_\delta}{\partial r} - \mathrm{i}ku_\delta = \mathrm{o}\big(r^{-1/2}\big), \quad r \to \infty \tag{1}$$

where $x = (x_1, x_2)$, $r = |x|$, ν is the normal to $\varGamma_\delta$, k is a positive number. Hereinafter we assume that the function f belongs to $L_2(\mathbb{R}^2)$ and has bounded support, $\varGamma_0 \in C^\infty$ is a boundary of a bounded simply-connected

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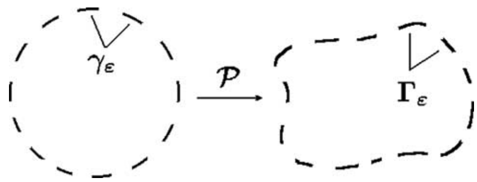

Fig. 1. The perturbed problem.

domain Ω, and for $\delta = \varepsilon > 0$ the curves Γ_ε is obtained from Γ_0 by cutting out a great number of openings of small diameter located closely to each other. Namely (see Fig. 1), let ω be a unit circle with center at the origin. Suppose that $\gamma_0 = \partial\omega$, $N \gg 1$ is an integer number, $\varepsilon = 2N^{-1}$, $\gamma_\varepsilon = \{(r,\theta)\colon\ r = 1,\ \varepsilon(a(\varepsilon) + m\pi) < \theta < \varepsilon(\pi(m+1) - a(\varepsilon)),\ m = 1, \ldots, N\}$, where θ is a polar angle and $\mathcal{P}$ is a diffeomorphism $\mathbb{R}^2$ into $\mathbb{R}^2$. We denote $\Omega = \mathcal{P}(\omega)$, $\Gamma_\delta = \mathcal{P}(\gamma_\delta)$. For $\delta = \varepsilon$ we will call the problem (1) the perturbed problem. Since $\Omega_0 = \Omega \cup (\mathbb{R}^2 \setminus \overline{\Omega})$, it follows that for $\delta = 0$ the problem (1) decomposes into two Neumann problems, in Ω and in $\mathbb{R}^2 \setminus \overline{\Omega}$. We will call them the limit internal and the limit external problems, respectively. In what follows we assume that:

$$a(\varepsilon) = \exp\left(-\frac{1}{\varepsilon A(\varepsilon)}\right), \quad A(\varepsilon) > 0,\ \lim_{\varepsilon \to 0} A(\varepsilon) = 0,\ \varepsilon \to 0 \tag{2}$$

For this case it is known [1] (see also [2]) that if $\lambda = k^2$ is not the eigenvalue of the limit internal problem then a solution of the perturbed problem converges to solutions of the limit problems in Ω and $\mathbb{R}^2 \setminus \overline{\Omega}$. Assume that $k_0 > 0$ and $\lambda_0 = k_0^2$ is an eigenvalue of the Neumann problem for the operator $-\Delta$ in Ω. In [3] one can find that the analytic continuation (with respect to k) of the solution to a boundary value problem (1) has a pole τ^ε with small imaginary part (a scattering frequency), converging to k_0 as $\varepsilon \to 0$. This pole lies in the lower complex half-plane $\operatorname{Im} k < 0$, and the residue of the analytic continuation of the solution is a solution of the boundary value problem

$$\bigl(\Delta + (\tau^\varepsilon)^2\bigr)\psi^\varepsilon = 0, \quad x \in \Omega_\varepsilon, \qquad \frac{\partial \psi^\varepsilon}{\partial \nu} = 0, \quad x \in \Gamma_\varepsilon \tag{3}$$

Let us emphasize that for fixed ε the function ψ^ε increases exponentially as $r \to \infty$. At the same time, it being a solution to the homogeneous boundary value problem (3), we will call it a quasi-eigenfunction. Let $\lambda_0 = k_0^2 > 0$ be a simple eigenvalue of the limit problem, and ψ_0 be the associated eigenfunction normalized in $L_2(\Omega)$. For this case in [3] one can find that for k, close to k_0, solutions of the problem (1) and their analytic continuation meet the representation

$$u_\varepsilon(x,k) = \frac{\psi^\varepsilon(x)}{k^2 - (\tau^\varepsilon)^2} \int\limits_{\mathbb{R}^2} \psi^\varepsilon(y)\, f(y)\, \mathrm{d}y + \tilde{u}_\varepsilon(x,k) \tag{4}$$

$$\psi^\varepsilon \underset{\varepsilon \to 0}{\to} \psi_0 \quad \text{in } H^1(\Omega), \quad \psi^\varepsilon \underset{\varepsilon \to 0}{\to} 0, \quad \tilde{u}_\varepsilon \underset{\varepsilon \to 0}{\to} u \quad \text{in } H^1_{\text{loc}}\bigl(\mathbb{R}^2 \setminus \overline{\Omega}\bigr), \quad \|\tilde{u}_\varepsilon\|_{H^1(\Omega)} \leqslant C\|f\|_{L_2(\mathbb{R}^2)} \tag{5}$$

where $u(x;k)$ is a solution (and its analytic continuation) of the boundary value problem

$$\bigl(\Delta + k^2\bigr)u = f, \quad x \in \mathbb{R}^2 \setminus \overline{\Omega}, \quad \frac{\partial u}{\partial \nu} = 0, \quad x \in \Gamma_0, \quad \frac{\partial u}{\partial r} - \mathrm{i}ku = \mathrm{o}\bigl(r^{-1/2}\bigr), \quad r \to \infty \tag{6}$$

From (4) it follows that for the real frequencies k the first term in the right-hand side causes most affect in the case

$$k = k(\varepsilon) = \operatorname{Re}\tau^\varepsilon + \mathrm{O}\bigl(\operatorname{Im}\tau^\varepsilon\bigr) \tag{7}$$

The main goal of this paper is to construct the leading terms of the asymptotics for $\operatorname{Im}\tau^\varepsilon$ and ψ^ε in $\mathbb{R}^2 \setminus \overline{\Omega}$ allowing to employ formula (4) to establish the resonance nature of the first term in this formula for the peak frequencies (7).

2. Asymptotics of quasi-eigenelements

The leading terms of the asymptotics for the pole τ^ε and associated quasi-eigenfunction ψ^ε outside a neighborhood of Γ_0 are constructed in the form

$$\tau^\varepsilon = \tau_0\big(A(\varepsilon)\big) + \cdots, \quad \tau_0 \underset{A\to 0}{\to} k_0 \tag{8}$$

$$\psi^\varepsilon(x) = \Psi_0^+\big(x; A(\varepsilon)\big) + \cdots, \quad x \in \Omega,\ \Psi_0^+ \underset{A\to 0}{\to} \psi_0 \tag{9}$$

$$\psi^\varepsilon(x) = A(\varepsilon)\Psi_0^-\big(x; \tau^\varepsilon; A(\varepsilon)\big) + \cdots, \quad x \in \mathbb{R}^2 \setminus \overline{\Omega} \tag{10}$$

where $\Psi_0^-(x; k; A)$ is a solution (and its analytic continuation) of the equation and radiation condition

$$\big(\Delta + k^2\big)\Psi_0^- = 0, \quad x \in \mathbb{R}^2 \setminus \overline{\Omega}, \quad \frac{\partial \Psi_0^-}{\partial r} - \mathrm{i}k\Psi_0^- = \mathrm{o}\big(r^{-1/2}\big), \quad r \to \infty \tag{11}$$

Denote $\Lambda_0(A) = \tau_0^2(A)$. Substituting (8), (9) in (3), we obtain the following equation for Ψ_0^+:

$$(\Delta + \Lambda_0)\Psi_0^+ = 0, \quad x \in \Omega \tag{12}$$

In a small neighborhood of Γ_0 (moreover, outside smaller neighborhood of the openings) the asymptotics of ψ^ε are constructed in another form, employing the method of matching asymptotic expansions [4–6]. In order to do so, we introduce local coordinates (s, t) in a neighborhood of Γ_0. Here t is the distance from the point to Γ_0 measured in the direction of the internal normal, containing this point, and s is an arc length of the curve Γ_0. Since $\Gamma_0 = \mathcal{P}(\gamma_0)$, it is obvious that the angle θ parameterized the curve γ_0, can be expressed by s: $\theta = \theta(s),\ \theta'(s) > 0,\ \theta(0) = 0$. We denote

$$\phi_0^+ = \Psi_0^+\big|_{\Gamma_0}, \quad \phi_1^+ = \frac{\partial \Psi_0^+}{\partial \nu}\bigg|_{\Gamma_0}, \quad \phi_0^- = \Psi_0^-\big|_{\Gamma_0, k=\tau_0}, \quad \phi_1^- = \frac{\partial \Psi_0^-}{\partial \nu}\bigg|_{\Gamma_0, k=\tau_0} \tag{13}$$

$\xi = (\xi_1, \xi_2) = (\theta(s)\varepsilon^{-1}, t\theta'(s)\varepsilon^{-1})$. Hereafter we consider ν as an outer normal to Ω. By the matching condition of the series (9), (10) with new series in the ξ variables in a neighborhood of Γ_0 we deduce that these series have the form

$$\psi^\varepsilon(x) = V_0^\pm(\xi; s; A) + \varepsilon V_1^\pm(\xi; s; A) + \cdots, \quad \pm\xi_2 > 0 \tag{14}$$

$$\begin{aligned} &V_0^+(\xi; s; A) \sim \phi_0^+(s; A), \quad V_1^+(\xi; s; A) \sim -\frac{\phi_1^+(s; A)}{\theta'(s)}\xi_2, \quad \xi_2 \to +\infty \\ &V_0^-(\xi; s; A) \sim A\phi_0^-(s; A), \quad V_1^-(\xi; s; A) \sim -A\frac{\phi_1^-(s; A)}{\theta'(s)}\xi_2, \quad \xi_2 \to -\infty \end{aligned} \tag{15}$$

Here s is a 'slow' variable. All the openings expressed in ξ being exponentially small, we will introduce another expansion at each opening. Therefore, the coefficients of the series (14) should obey the homogeneous boundary condition

$$\frac{\partial V_j^\pm}{\partial \xi_2} = 0, \quad \xi_2 = 0,\ \xi_1 \neq \pi m,\ m \in \mathbb{Z} \tag{16}$$

Substituting (14) and (8) in (3) and passing to ξ, we obtain the equation for $V_j^\pm$ and, for instance:

$$\Delta_\xi V_j^\pm = 0, \quad \pm\xi_2 > 0 \tag{17}$$

for $j = 0$. It is clear that the functions

$$V_0^+(\xi; s; A) \equiv \phi_0^+(s; A), \qquad V_0^-(\xi; s; A) \equiv A\phi_0^-(s; A) \tag{18}$$

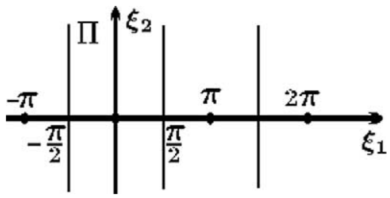

Fig. 2. The half-strip, Π.

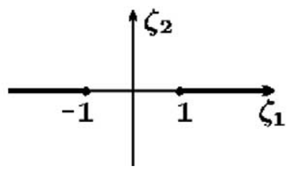

Fig. 3. The boundary value problem.

satisfy (17) and have the asymptotics (15). Keeping in mind (18) we conclude that the equations for $V_1^{\pm}$ have the form (17) where $j=1$. Denote $X(\xi)=\operatorname{Re}\ln\sin z+\ln 2$, where $z=\xi_1+\mathrm{i}\xi_2$ is a complex variable. By definition the function X is harmonic in a half-strip $\Pi=\{\xi\colon -\pi/2<\xi_1<\pi/2,\ \xi_2>0\}$ (see Fig. 2), can be continued over the period π on the half-plane $\xi_2>0$ and satisfies the condition

$$\frac{\partial X}{\partial \xi_2}=0, \quad \xi_2=0,\ \xi_1\neq \pi m \tag{19}$$

Let us extend this function on the semi-plane $\xi_2<0$ in a even way and keep the same notation X for this extension. As a result, we conclude that the function X has the asymptotics

$$X(\xi)=\pm\xi_2+\mathrm{O}\big(\exp\{\mp 2\xi_2\}\big), \quad \xi_2\to\pm\infty, \qquad X(\xi)=\ln\rho+\ln 2+\mathrm{O}\big(\rho^2\big), \quad \rho=|\xi|\to 0 \tag{20}$$

Using (19) and (20), we obtain that the functions

$$V_1^{+}(\xi;s;A)=-\frac{\phi_1^{+}(s;A)}{\theta'(s)}X(\xi), \qquad V_1^{-}(\xi;s;A)=A\frac{\phi_1^{-}(s;A)}{\theta'(s)}X(\xi) \tag{21}$$

are the solutions of Eqs. (17) satisfying the boundary condition (16) and having the asymptotics (15).

The coefficients of $V_j^{\pm}$ have the jump on the openings, that is why in a neighborhood of the m-th opening the asymptotics of the quasi-eigenfunction ψ^{ε} is sought as a new series whose coefficients depend on the variables $\zeta^{(m)}=(\zeta_1^{(m)},\zeta_2)=((\xi_1-\pi m)a^{-1},\xi_2 a^{-1})$, where $a(\varepsilon)$ is defined by formula (2). Rewriting the asymptotics of the series (14) as $(\xi_1-\pi m,\xi_2)\to 0$ in the variables $\zeta^{(m)}$ and according to the method of matching asymptotic expansions, by (18), (20) and (21) we obtain that in the neighborhood of the m-th opening the asymptotics of the function ψ^{ε} has the form

$$\psi^{\varepsilon}(x)=W_0^{(m)}\big(\zeta^{(m)};s;A\big)+\varepsilon W_1^{(m)}\big(\zeta^{(m)};s;A\big)+\cdots \tag{22}$$

where $W_j^{(m)}$ satisfies the asymptotics as $|\zeta^{(m)}|\to\infty$

$$\begin{aligned}
&W_0(\zeta;s;A)\sim\phi_0^{+}(s;A)+A^{-1}\frac{\phi_1^{+}(s;A)}{\theta'}, \quad W_1(\zeta;s;A)\sim-\frac{\phi_1^{+}(s;A)}{\theta'}\big(\ln|\zeta|+\ln 2\big), \quad \zeta_2>0\\
&W_0(\zeta;s;A)\sim A\phi_0^{-}(s;A)-\frac{\phi_1^{-}(s;A)}{\theta'}, \quad W_1(\zeta;s;A)\sim A\frac{\phi_1^{-}(s;A)}{\theta'}\big(\ln|\zeta|+\ln 2\big), \quad \zeta_2<0
\end{aligned} \tag{23}$$

Hereinafter for the sake of brevity we omit the indexes m of $W_j^{(m)}$ and $\zeta^{(m)}$. Substituting (22) and (8) in (3) and passing to ζ, we obtain the boundary value problems for W_j, and, in particular:

$$\Delta_\zeta W_0=0, \quad \zeta\in\mathbb{R}^2\setminus\overline{\Gamma^0}, \qquad \frac{\partial W_0}{\partial\zeta_2}=0, \quad \zeta\in\Gamma^0 \tag{24}$$

where Γ^0 is an axis $O\zeta_1$ without the segment $[-1,1]$ (see Fig. 3). Clearly, under the condition

$$\phi_0^{+}(s;A)+\big(A\theta'(s)\big)^{-1}\phi_1^{+}(s;A)=A\phi_0^{-}(s;A)-\big(\theta'(s)\big)^{-1}\phi_1^{-}(s;A) \tag{25}$$

the solution to the boundary value problem (24), having the asymptotics (23), has the form

$$W_0(\zeta; s; A) \equiv \phi_0^+(s; A) + \frac{\phi_1^+(s; A)}{A\theta'(s)} \tag{26}$$

and it is independent of ζ. By (26) the boundary value problem for W_1 is

$$\Delta_\zeta W_1 = 0, \quad \zeta \in \mathbb{R}^2 \setminus \overline{\Gamma^0}, \qquad \frac{\partial W_1}{\partial \zeta_2} = 0, \quad \zeta \in \Gamma^0 \tag{27}$$

Denote $Y(\zeta) = \operatorname{Re}\ln(w + \sqrt{w^2 - 1})$, where $w = \zeta_1 + \mathrm{i}\zeta_2$ is a complex variable. It is easily checked that the harmonic in $\mathbb{R}^2 \setminus \overline{\Gamma^0}$ function $Y \in H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \overline{\Gamma^0}) \cap C^\infty(\mathbb{R}^2 \setminus \overline{\Gamma^0})$ satisfies the boundary condition

$$\frac{\partial Y}{\partial \zeta_2} = 0, \quad \zeta \in \Gamma^0 \tag{28}$$

and has the following asymptotic at infinity:

$$Y = \pm\big(\ln|\zeta| + \ln 2\big) + \mathrm{O}\big(|\zeta|^{-2}\big), \quad \pm\zeta_2 > 0 \tag{29}$$

By (28), (29) the function

$$W_1(\zeta; s; A) = -\frac{\phi_1^+(s; A)}{\theta'(s)} Y(\zeta)$$

is a solution to the boundary value problem (27) and meets the asymptotics (23) if

$$\frac{\phi_1^+(s; A)}{\theta'} = A\frac{\phi_1^-(s; A)}{\theta'} \tag{30}$$

From (30), (25) and (13) for Eqs. (11), (12) we obtain the boundary conditions of the conjugation type:

$$\begin{aligned}
&\frac{\partial}{\partial \nu}\big(\Psi_0^+(x; A) + A\Psi_0^-\big(x; \tau_0(A); A\big)\big) + A\theta'(s)\big(\Psi_0^+(x; A) - A\Psi_0^-\big(x; \tau_0(A); A\big)\big) = 0 \\
&\frac{\partial}{\partial \nu}\big(\Psi_0^+(x; A) - A\Psi_0^-\big(x; \tau_0(A); A\big)\big) = 0, \quad x \in \Gamma_0
\end{aligned} \tag{31}$$

From the solvability conditions of (11) and (12), the boundary conditions (31) and the conditions $\tau_0 \to k_0$ and $\Psi_0^+ \to \psi_0$ as $A \to 0$ by (8) and (9) we determine $\tau_0(A)$, $\Psi_0^\pm$ and, in particular, we get that

$$\tau_0(A) = k_0 + A\tau_{0,1} + A^2\tau_{0,2} + \mathrm{O}\big(A^3\big), \qquad \Psi_0^-(x; k; A) \underset{A\to 0}{\to} \psi_0^-(x; k) \tag{32}$$

where $\psi_0^-(x; k)$ is a solution of the boundary value problem

$$\big(\Delta + k^2\big)\psi_0^- = 0, \quad x \in \mathbb{R}^2 \setminus \overline{\Omega}, \qquad \frac{\partial \psi_0^-}{\partial \nu} = -\frac{\theta'\psi_0}{2}, \quad x \in \Gamma_0, \qquad \frac{\partial \psi_0^-}{\partial r} - \mathrm{i}k\psi_0^- = \mathrm{o}\big(r^{-1/2}\big), \quad r \to \infty \tag{33}$$

if $k > 0$ and its analytic continuation in the complex plane with the cut along the imaginary axis,

$$\tau_{0,1} = \frac{1}{4k_0}\int_{\Gamma_0} \psi_0^2\big(x(s)\big)\,\mathrm{d}\theta(s) > 0, \quad \operatorname{Im}\tau_{0,2} = -\frac{\sigma}{2} < 0, \quad \sigma = \lim_{R\to\infty}\int_{|x|=R} \big|\psi_0^-(x; k_0)\big|^2\,\mathrm{d}s > 0 \tag{34}$$

Thus from (8)–(10) and (32) it follows that the pole of the analytic continuation and the associated quasi-eigenfunction have the asymptotics

$$\tau^\varepsilon \sim k_0 + A\tau_{0,1} + A^2\tau_{0,2}, \quad \psi_\varepsilon(x) \sim \psi_0(x), \quad x \in \Omega, \qquad \psi_\varepsilon(x) \sim A\psi_0^-(x; k_0), \quad x \in \mathbb{R}^2 \setminus \overline{\Omega} \tag{35}$$

where $\tau_{0,j}$ is defined by (34), ψ_0 is a normalized in $L_2(\Omega)$ eigenfunction of the Neumann problem for $-\Delta$ in Ω, associated with the eigenvalue $\lambda_0 = k_0^2 > 0$, and ψ_0^- is the solution of (33).

3. Resonances of solutions to problem (1)

Let t be an arbitrary real number. From (7), (35), (34) it follows that the peak frequencies have the form

$$k(\varepsilon) = k_0 + A(\varepsilon)\tau_{0,1} + A^2(\varepsilon)t + \mathrm{o}\big(A^2(\varepsilon)\big) \tag{36}$$

Denote $T(t) = 2k_0(t - \tau_{0,2})$. Suppose that $\operatorname{supp} f \subset \Omega$ (the radiation problem). Then substituting (36) in (4), using (34), (35) and (5) we obtain that for real k satisfying (36) the leading terms of the asymptotics of the solution to the boundary value problem (1) have the form

$$u_\varepsilon(x;k) \sim \frac{\psi_0(x)}{A^2(\varepsilon)T(t)} \int\limits_\Omega \psi_0 f \,\mathrm{d}y, \quad x \in \Omega, \qquad u_\varepsilon(x;k) \sim \frac{\psi_0^-(x;k_0)}{A(\varepsilon)T(t)} \int\limits_\Omega \psi_0 f \,\mathrm{d}y, \quad x \in \mathbb{R}^2 \setminus \overline{\Omega} \tag{37}$$

Let $\operatorname{supp} f \subset \mathbb{R}^2 \setminus \overline{\Omega}$ (the scattering problem) and $u(x;k)$ be a solution of the boundary value problem (6). Then from the same formulae and due to (6) and (33) it follows that

$$u_\varepsilon(x;k) \sim -\frac{\psi_0(x)}{2A(\varepsilon)T(t)} \int\limits_{\Gamma_0} \psi_0\big(x(s)\big)u\big(x(s);k_0\big)\,\mathrm{d}\theta(s), \quad x \in \Omega$$

$$u_\varepsilon(x,k) \sim -\frac{\psi_0^-(x;k_0)}{2T(t)} \int\limits_{\Gamma_0} \psi_0\big(x(s)\big)u\big(x(s);k_0\big)\,\mathrm{d}\theta(s) + u(x;k_0), \quad x \in \mathbb{R}^2 \setminus \overline{\Omega}$$

From (37) it follows that in the case of the radiation problem the solution increases unboundedly in $\mathbb{R}^2$ on the peak frequencies. In the case of the scattering problem on the peak frequencies the solution increases in Ω only, while in $\mathbb{R}^2 \setminus \overline{\Omega}$ it differs from the solution of the limit external problem (6) up to $\mathrm{O}(1)$. Exactly this phenomenon was discovered by Rayleigh for the classic acoustic resonator with one opening in [7].

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