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Joseph Boussinesq and his approximation: a contemporary view

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Abstract

A hundred years ago, in his 1903 volume II of the monograph devoted to 'Théorie Analytique de la Chaleur', Joseph Valentin Boussinesq observes that: "The v*ariations of density can be ignored except were they are multiplied by the acceleration of gravity in equation of motion for the vertical component of the velocity vector*." A spectacular consequence of this Boussinesq observation (called, in 1916, by Rayleigh, the 'Boussinesq approximation') is the possibility to work with a quasi-incompressible system of coupled dynamic, (Navier) and thermal (Fourier) equations where buoyancy is the main driving force. After a few words on the life of Boussinesq and on his observation, the applicability of this approximation is briefly discussed for various thermal, geophysical, astrophysical and magnetohydrodynamic problems in the framework of 'Boussinesquian fluid dynamics'. An important part of our contemporary view is devoted to a logical (100 years later) justification of this Boussinesq approximation for a perfect gas and an ideal liquid in the framework of an asymptotic modelling of the full fluid dynamics (Euler and Navier–Stokes–Fourier) equations with especially careful attention given to the validity of this approximation. ***To cite this article: R.Kh. Zeytounian, C. R. Mecanique 331 (2003).***

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Résumé

Joseph Boussinesq et son approximation : un aperçu actuel. En 1903, Gauthier-Villars éditait à Paris le tome II, du traité de Joseph Boussinesq intitulé : « Théorie Analytique de la Chaleur ». A la page VII de l'Avertissement à ce tome II Boussinesq écrit :

« *.. il fallait encore observer que, dans la plupart des mouvements provoqués par la chaleur sur nos fluides pesants, les volumes ou les densités se conservent à très peu près, quoique la variation correspondante du poids de l'unité de volume soit justement la cause des phénomènes qu'il s'agit d'analyser.*

De là résulte la possibilité de négliger les variations de la densité, là où elles ne sont pas multipliées par la gravité g, tout en conservant, dans les calculs, leur produit par celle-ci ».

Cette observation est, ce que l'on appelle, aujourd'hui : « l'approximation de Boussinesq » (en accord avec l'appellation, en 1916, de Rayleigh), et une conséquence spectaculaire en est la possibilité de considérer un système d'équations quasi-incompressible couplé pour la dynamique (équation de Navier) et la température (équation de Fourier) pour lequel la poussée d'Archimède est la force active principale régissant le mouvement. Après un bref aperçu sur la vie de Boussinesq et sur son observation, l'application de l'approximation de Boussinesq (dans le cadre d'une « dynamique des fluides de Boussinesq ») pour les problèmes thermiques, géophysiques, astrophysiques et magnétohydrodynamiques fait l'objet de divers commentaires. Une part importante de notre aperçu actuel est consacrée à une justification logique de cette approximation de Boussinesq (100 ans après) pour un gaz parfait et un liquide ideal, dans le cadre d'une modélisation asymptotique des équations (d'Euler

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et de Navier–Stokes–Fourier) de la dynamique des fluides, avec une attention toute particulière pour ce qui concerne la validité de cette approximation. ***Pour citer cet article : R.Kh. Zeytounian, C. R. Mecanique 331 (2003).***
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Keywords: Fluid mechanics; Asymptotic modelling in fluid dynamics; Thermal convection; Geo-astro physical fluid dynamics; Magneto-hydrodynamics

Mots-clés : Mécanique des fluides ; Modélisation asymptotique en dynamique des fluides ; Convection thermique ; Dynamique des fluides ; Géo-astro physiques : Magnéto-hydrodynamique

1. A few lines concerning Joseph Boussinesq and his approximate equations with an additional gravity term

The French mathematician, Joseph Valentin Boussinesq was born on 15 March 1842 in Saint-André de Sangonis (in the Hérault department, about 30 km from the town of Montpellier). In 1867, when he was 25, Boussinesq received the scientific degree of Docteur de la Faculté des Sciences de Paris (Sorbonne) for a thesis entitled: 'Sur la propagation de la chaleur dans les milieux hétérogènes'. Thanks to the protection of Saint-Venant, in 1873 Boussinesq (now 31) became Professeur de Calcul Différentiel et Intégral in the Science Faculty in Lille. In 1886 (at 44), Boussinesq was elected member of the Académie des Sciences de Paris (Mechanics Section) and became Professeur with the Chair of 'Mécanique Physique et Expérimentale' at Science Faculty of Paris, a position which he held until 1896. He then moved to the Chair of 'Physique Mathématique et Calcul des Probabilités'. All things considered, Boussinesq kept the position of Professeur at the Sorbonne during more than 30 years. Amongst the various and numerous important scientific contributions of Boussinesq in hydraulics and hydrodynamics in this period, we shall mention only the fundamental contribution of Joseph Boussinesq to the theory of long surface waves on water. In particular, Boussinesq (see [3]), with the help of his famous approximation, so-called, 'Boussinesq equations' resolved the conflict between Russell's observation of the solitary wave and the Airy's shallow water theory, according to which a wave of finite amplitude cannot propagate without changing its profile. Moreover, we observe that, from these Boussinesq equations in [3, p. 354], Boussinesq himself derived the single $K\mathrm{d}V$ equation as a particular case (15 years before Korteweg and de Vries [4]). For a detailed review concerning these long surface waves on water, solitons and the Boussinesq contributions, see our 1995 review [5] paper. It is only from 1900 that Boussinesq became seriously interested by the influence of temperature on various fluid motions and he wrote, in particular, volume II, [1], of his monograph 'Théorie Analytique de la Chaleur'. During 40 years, Boussinesq was every day a frequent reader at the library of the Institut de France, across the river Seine from the Louvre, from 3:00 pm until closing time. By the importance of his works, as well as by the nobleness and modesty of his person, Boussinesq has honoured the French Académie des Sciences. Boussinesq died on 19 February 1929 at almost 87, and for a detailed biography, concerning the life and the work of Joseph Boussinesq, the reader can consult the 'Lecture', [6], given by Emile Picard. In Boussinesq's monograph [1], the reader can find (see p. 174) a set of approximate equations, written according to the discussion of section 261 (pp. 172–174). If we use u, v, w, for the velocity components, π for the perturbation of the pressure (relative to hydrostatic pressure $p_0(z)$), and θ for the heating (linked with the Archimedean force), then the following approximate '*à la Boussinesq*' equations (with a constant density ρ_0) can be written:

$$
\begin{aligned}
&\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0\\
&(1/\rho_0)\partial\pi/\partial x = -\mathrm{d}u/\mathrm{d}t, \quad (1/\rho_0)\partial\pi/\partial y = -\mathrm{d}v/\mathrm{d}t, \quad (1/\rho_0)\partial\pi/\partial z = \varGamma_0\theta - \mathrm{d}w/\mathrm{d}t\\
&\mathrm{d}\theta/\mathrm{d}t = (K_0/C_0)\big[\partial^2\theta/\partial x^2 + \partial^2\theta/\partial y^2 + \partial^2\theta/\partial z^2\big]
\end{aligned}
\tag{1}
$$

with $\mathrm{d}/\mathrm{d}t \equiv \partial/\partial t + u\partial/\partial x + v\partial/\partial y + w\partial/\partial z$, where $\mathrm{d}u/\mathrm{d}t$, $\mathrm{d}v/\mathrm{d}t$, $\mathrm{d}w/\mathrm{d}t$, are the three components of the acceleration. Both the coefficients K_0 (thermal conductibility) and C_0 (specific caloric of the unit of volume at constant pressure), are assumed constant. In the equation for w the term $\Gamma_0\theta$ (according to Boussinesq) is an additional gravity term, proportional to the heating θ, but directed from below in the upwards direction, where $\Gamma_0 = \alpha_0 g$, and α_0 is the constant coefficient of the thermal cubic dilatation (expansion) of the fluid. In particular, for a perfect gas, $p = R\rho T$, where T is the temperature and R the gas constant, we have $\alpha_0 = 1/T_0$ with T_0 a constant temperature. Let us notice that Boussinesq deduced the set of approximate equations (1) mainly on the basis of pertinent physical considerations from the exact equations for a compressible and heat conductor fluid subject to force of gravity (but do not "as a consequence of a list of assertions" as this is claimed in Joseph's book [7, p. 4]). Namely, first, Boussinesq observes that the small variations of density ρ are approximately related to a constant pressure, ρ being, in a such case, a function only of the heating θ, and the equation for θ, in Boussinesq system (1), is a direct consequence of this observation. Then, Boussinesq observed that an important consequence of the heating is the reduction of the weight of particles during their ascending movement, and in this case the weight $g\rho$ is divided by $(1+\alpha\theta)$! From this last observation Boussinesq deduced the emergence of an additional gravity term, $\rho_0 g\alpha_0\theta$, proportional to heating θ, but directed from below in the upwards direction, in the momentum equation for w. Finally, Boussinesq observed that, in thermal convection problems, the velocity changes appreciably the form of particles but without important modifications of the volume (and as a consequence of the density which is replaced by $\rho_0 = \text{const}$), and the term: $-(1/\rho)\,\mathrm{d}\rho/\mathrm{d}t$ is very small in the exact compressible continuity equation. Obviously, if the fluid is viscous, then it is only necessary (in (1)) to add in right-hand side of the three momentum equations for u, v and w a term with their Laplacian multiplied by the kinematic viscosity ν_0 (see, for example, below Eqs. (2a), (2b)). A spectacular consequence of the Boussinesq observation, which leads to the approximate Boussinesq equations, is the possibility to work with a quasi-incompressible system of coupled dynamics (Euler or Navier, with a buoyancy term) and thermal (Fourier) simplified equations, where buoyancy is the main driving force. It is true that the above mentioned additional gravity term proportional to heating also emerges easily in the approximate *linear* equations derived in 1879 by Oberbeck [8] and the reader can find in the paper by Eckart and Ferris [9, p. 50] a remark concerning these Oberbeck equations and their relation with the Boussinesq equations. Rayleigh [2], who has used Boussinesq's 1903 observation (in 1916), called this observation the 'Boussinesq approximation'. Indeed, from an ad-hoc approach (as, for instance, in Drazin and Reid [10, §7] or in Landau and Lifshitz [11, §56] books), it is very easy to derive the Oberbeck–Boussinesq approximate equations following the standard scheme of perturbation theory. In Landau and Lifshitz [11, §56] the Boussinesq approximate equations derived (in ad hoc manner) are called 'free convection equations' and are written in the following form, for the velocity vector $\mathbf{v}$ and thermodynamic perturbations T', p':

$$\nabla.\mathbf{v} = 0, \quad \rho = \rho_0 = \text{const}, \quad \partial T'/\partial t + \mathbf{v}.\nabla T' = \chi\,\Delta T' \tag{2a}$$

$$\partial\mathbf{v}/\partial t + (\mathbf{v}.\nabla)\mathbf{v} + \nabla(p'/\rho_0) + \alpha_0\mathbf{g}T' = \nu_0\Delta\mathbf{v} \tag{2b}$$

where T' and p' are the perturbations of temperature and pressure relative to a constant T_0 averaged temperature and hydrostatic pressure $p_0(z)$ such that:

$$\mathrm{d}p_0(z)/\mathrm{d}z + \rho_0 g = 0 \tag{2c}$$

with z the vertical coordinate with unit vector $\mathbf{k}$ ($\mathbf{g} = -g\mathbf{k}$).

2. Boussinesquian fluid dynamics

The Boussinesq approximation, which gives the possibility to consider a Boussinesquian (à la Boussinesq) fluid, is actually, perhaps, the most widely used simplification in various fluid dynamics problems. A very good illustration of this plurality of 'Boussinesquian fluid dynamics' is the numerous survey papers in volumes of the Annual Review of Fluid Mechanics where this Boussinesq approximation is the basis for mathematical formulation in various problems (for example, as in: convection in mushy layers [12]; solar convection [13];

magnetoconvection [14] and magnetic buoyancy [13]; mantle convection [15]; atmospheric lee waves [16]; Rayleigh–Bénard (RB) instabilities [17]; oceanic general circulation [18]; buoyancy-driven flows in crystal-growth melts [19]; environmental fluid mechanics [20]; fluid-dynamical problems in Galaxies [21]; gravity currents in rotating systems [22]; internal waves in the atmosphere and ocean [23]; convection involving thermal and salt fields [24]; dynamic of Jovian atmospheres [25]; etc. ...) whether for gases and liquids or for more complicated fluids with various complementary effects. It is interesting to observe that already in 1891 Oberbeck uses a Boussinesq type approximation in meteorological studies of the Hadley thermal regime for the trade-winds arising from the deflecting effect of the Earth's rotation. Among the early investigations of the applicability of the Boussinesq approximation we note, first, the applicability to flow in a thin layer of a compressible fluid by Jeffreys [26], for infinitesimal steady motions and, then, the book by Joseph [7, Chapter VIII; pp. 4, 5], where it is observed that the crux of the Boussinesq approximation for a non-homogeneous (stratified), heat conducting viscous and compressible (dilatable) fluid (liquid) motion in the gravity field, is that: (i) the variation of the density perturbation is neglected in the mass continuity equation and in the equation for the horizontal motion; (ii) however, this density perturbation is taken into account in the equation for the vertical motion through its influence as a buoyancy term; (iii) the influence of a pressure perturbation on the buoyancy and in the equation of energy (written for the temperature perturbation) can be neglected; (iv) the influence of a perturbation of pressure in the equation of state can be also neglected and the rate of viscous dissipation is neglected in the equation for the temperature perturbation. When all of these simplifying factors are present, the Navier–Stokes–Fourier (NSF) exact equations for compressible heat conducting and diffusive flow of a viscous, nonhomogeneous fluid can be approximated by the following set of (the so-called *Oberbeck–Boussinesq* (OB)) equations:

$$\rho = \rho_0\big[1 - \alpha_0(T - T_0) + \Gamma_0(C - C_0)\big], \quad \nabla.\mathbf{U} = 0 \tag{3a}$$

$$\rho_0\big[\partial\mathbf{U}/\partial t + (\mathbf{U}\cdot\nabla)\mathbf{U}\big] + \nabla P - \rho_0\big[1 - \alpha_0(T - T_0) + \Gamma_0(C - C_0)\big]\mathbf{g} = \nabla\cdot\mathbf{S} \tag{3b}$$

$$\partial T/\partial t + \mathbf{U}.\nabla T = \kappa_T\nabla^2 T + Q_T(t,\mathbf{x}) \tag{3c}$$

$$\partial C/\partial t + \mathbf{U}.\nabla C = \kappa_C\nabla^2 C + Q_C(t,\mathbf{x}) \tag{3d}$$

In (3b) $\mathbf{T} = -P\mathbf{I} + \mathbf{S}$ is the stress, $\mathbf{S} = 2\mu\mathbf{D}[\mathbf{U}]$ is the extra stress, $\mathbf{U}$ is the (solenoidal) velocity and $\mathbf{g}$ is a body-force field (typically gravity). In Eq. (3c) for the temperature $T(t,\mathbf{x})$, κ_T is the thermal diffusivity and $Q_T(t,\mathbf{x})$ is a prescribed heat source field. Finally, in Eq. (3d) for the solute concentration $C(t,\mathbf{x})$, κ_C is the solute diffusivity and $Q_C(t,\mathbf{x})$ is a prescribed field specifying the distribution of solute sources. Indeed, for a *dilatable* (expansible) liquid layer heated from below, the application of the Boussinesq approximation is more subtle, especially when it is necessary to consider the influence of an *upper* (*deformable*) free surface separating this liquid layer from the air above, as in the case of the Bénard thermal problem [27]. Concerning this aspect of the Boussinesq approximation see, the discussion in next Section 3, and here we observe only that for a *very thin* layer of the liquid (of order of a millimetre) the *buoyancy force* at the leading-order is *negligible* but the *deformations* of the free surface are *operative*. On the other hand, for the RB instability problem, the main operative force is the buoyancy, and deformations of the free surface are negligible. For an ad-hoc justification of the Boussinesq approximation, when the equation of state is $\rho = \rho(T, p)$, and the derivation of the OB equations, see, for instance, Spiegel and Veronis [28], Mihaljan [29], and Dutton and Fichtl [30] among other papers. Concerning the *atmospheric flow* in the Kotchin, Kibel and Roze [31, §§36, 38] book, the reader can again find an ad hoc presentation of the Boussinesq equations. The main difference is the necessity to take into account the existence of a hydrostatic (so-called 'standard') reference state, $p^*(z^*)$, $\rho^*(z^*)$, $T^*(z^*)$, which is a function only of the standard altitude z^*. Namely:

$$\mathrm{d}p^*/\mathrm{d}z^* + g\rho^*(z^*) = 0, \quad p^* = R\rho^*T^* \quad \text{and} \quad \Theta^*(z^*) = -\mathrm{d}T^*/\mathrm{d}z^* \tag{4a}$$

where in the adiabatic case (considered below) $\Theta^*(z^*)$ is a given function of z^*.

Then, for the thermodynamic functions dependent of $(t,\mathbf{x})$ we write:

$$p = p^*(z^*) + p', \quad \rho = \rho^*(z^*) + \rho', \quad T = T^*(z^*) + T' \tag{4b}$$

with

$$|p'| \ll p^*, \quad |\rho'| \ll \rho^*, \quad |T'| \ll T^* \tag{4c}$$

In this case we can derive for the velocity $\mathbf{u}$, $\Phi = RT^*(0)(p'/p^*)$ and T' the following approximate *inviscid* Boussinesq equations:

$$\nabla.\mathbf{u} = 0, \quad \partial\mathbf{u}/\partial t + (\mathbf{u}.\nabla)\mathbf{u} + \nabla\Phi - \big[g/T^*(0)\big]T'\mathbf{k} = 0 \tag{5a}$$

$$\partial T'/\partial t + u.\nabla T' + \big\{(g/R)\big[(\gamma-1)/\gamma\big] + \Theta^*(0)\big\}w = 0 \tag{5b}$$

with $w = \mathbf{u}.\mathbf{k}$ and γ is the ratio of specific heats. The above system of Eqs. (5a), (5b) is very frequently used in the investigations of the *lee waves phenomenon downstream of a mountain* in the stratified (when the term proportional to w in (5b) is different from zero) baroclinic atmosphere. An another application of the Boussinesq approximation in the atmosphere is linked with the *local circulations phenomena above a thermally non-homogeneous ground*. In this case for the velocity (horizontal and vertical) components $\mathbf{v}$ and w, thermodynamic perturbations θ and π we obtain the following *quasi-static system*, à la Boussinesq written with *dimensionless* quantities [32, Section 29]:

$$\mathbf{D}\cdot\mathbf{v} + \partial w/\partial z = 0, \quad \partial\pi/\partial z = \tau\theta \tag{6a}$$

$$S\partial\mathbf{v}/\partial t + (\mathbf{v}.\mathbf{D})\mathbf{v} + w\partial v/\partial z + (1/Ro)(\mathbf{k}\wedge\mathbf{v}) + \mathbf{D}\pi = Gr^{-1/2}\,\partial^2\mathbf{v}/\partial z^2 \tag{6b}$$

$$S\partial\theta/\partial t + \mathbf{v}\cdot\mathbf{D}\theta + w\partial\theta/\partial z + \tau\Lambda(0)w = Gr^{-1/2}\,\partial^2\theta/\partial z^2 \tag{6c}$$

where S, Ro and Gr are the Strouhal, Rossby and Grashof numbers, respectively, $\tau = \mathrm{O}(1)$ is a similarity parameter such that $\tau = \beta/M$ with $\beta = \Delta T^0/T^*(0) \ll 1$, where ΔT^0 is a measure of the heating of the non-homogeneous ground and $M \ll 1$ the Mach number. Finally, $\Lambda^*(0) = \{[(\gamma-1)/\gamma] + \Theta^*(0)\}$, $\gamma = C_p/C_v$, and we observe that this system is obtained in an asymptotically consistent way ([33], as in Section 3).

Concerning the application of the Boussinesq approximation for the dynamics of the (upper) ocean see, for instance, the very pertinent book by Phillips [34, §2.4]. In Section 4 we consider the so-called isochoric system (which is very judicious in the inviscid case for oceanic motions) and its relation with the Boussinesq system. However, in fact, many authors use the system of Boussinesq equations (3a)–(3d) for ocean motions, with the quasi-hydrostatic approximation (a boundary layer type simplification).

In general, for the application of the Boussinesq approximation to the mathematical formulation of various problems, where the buoyancy plays an important role, it is necessary to state some *similarity rules* between the main small Mach number (low compressibility) and other small parameters which characterize the considered physical problem. For example, in magnetoconvection the ratio of the (small) Alfven number to Mach number must be of order 1.

A very important area of the application of the Boussinesq approximation concerns the thermal convection in the Earth's mantle [35,36], numerical study and experimental investigation in magneo-convection [37,38], atmospheric circulation and climate of various planets of the Solar system as well as the atmosphere of the Sun [39,40] and Hadley circulation [41]. In recent book by Getling [42] the reader can find an analysis of the structures and dynamics of RB convection.

3. The asymptotic justification of the Boussinesq approximation

We observe that it is only during the last 25 years that the development in asymptotic modelling [43] gives the possibility to reveal consistently the asymptotic character of the Boussinesq approximation, first, for a polytropic gas [44], and, then, for a dilatable (expansible) liquid [45,46]. Indeed, the approximate equations derived via the Boussinesq approximation can be obtained in the same unique form for any fluid [47] at order zero with respect to the small parameter: $\varepsilon = U_c/(C_pT_c)^{1/2}$, where U_c is a characteristic speed of the medium, C_p and T_c being a characteristic (constant) heat capacity with constant pressure and a characteristic temperature of this medium, respectively. Below, we consider, first, the case of a perfect gas (with $\gamma = C_p/C_v$ and R both constant) when $M \ll 1$ ($U_c \ll (\gamma R T_c)^{1/2}$) and, then, the case of a weakly expansible liquid when $\varepsilon = \Delta T^0 \alpha_0 \ll 1$, where ΔT^0 is a difference of temperatures in the classical Bénard thermal problem and α_0 the constant coefficient of thermal cubic dilatation of the liquid.

3.1. The case of a perfect gas low-Mach number flow

When the gas is perfect:

$$p = R\rho T \tag{7}$$

then we have an intrinsic vertical height $H^* = p^*(0)/g\rho^*(0)$ implied by the hydrostatic reference state (4a). As a consequence we can consider the ratio $H_0/H^* = Bo$ (the so-called *Boussinesq* number), where H_0 is the characteristic vertical length scale of the considered low-Mach ($M \ll 1$) number perfect gas flow, and write the following similarity rule:

$$Bo/M = B^* = \mathrm{O}(1) \tag{8}$$

The height H^* is a judicious characteristic vertical length scale for the standard altitude z^* such that, with dimensionless quantities: $H^* z^{*\prime} = H_0 z' \Rightarrow z^{*\prime} = Bo\, z'$, and in place of the first two relations of (4a) we obtain (in dimensionless form): $p^{*\prime} = \rho^{*\prime} T^{*\prime}$, $\mathrm{d}p^{*\prime}/\mathrm{d}z^{*\prime} + \rho^{*\prime}(z^{*\prime}) = 0$ with $\mathrm{d}/\mathrm{d}z^{*\prime} = (1/Bo)\partial/\partial z'$. For the derivation of *approximate Boussinesq inviscid adiabatic* equations it is necessary to consider the following *Boussinesq limit*

$$Bo \downarrow 0 \text{ and } M \downarrow 0 \quad \text{such that} \quad Bo/M = B^* = \mathrm{O}(1) \tag{9}$$

which is the more significant limit between two other particular (and more degenerate-less, significant) limits of flows at low Mach and Boussinesq numbers in the presence of gravity (see, for instance, our recent book, Zeytounian [48, pp. 150, 151]. To apply the above Boussinesq limit (9) with the low-Mach-number expansions:

$$p' = p^{*\prime}(z^{*\prime})\big(1 + M^2\pi_B + \cdots\big), \quad \rho' = \rho^{*\prime}(z^{*\prime})(1 + M\omega_B + \cdots) \tag{10a}$$

$$T' = T^{*\prime}(z^{*\prime})(1 + M\theta_B + \cdots) \tag{10b}$$

we derive at the leading-order, from the exact Euler (inviscid adiabatic) dimensionless equations, the following Boussinesq approximate equations:

$$\begin{aligned} &\nabla.\mathbf{u}_B = 0; \quad \omega_B = -\theta_B \\ &D_B\mathbf{u}_B/Dt + \nabla(\pi_B/\gamma) - (B^*/\gamma)\theta_B\mathbf{k} = 0 \\ &D_B\theta_B/Dt + B^*\Lambda^*(0)w_B = 0 \end{aligned} \tag{11}$$

with $D_B/Dt = \partial/\partial t + \mathbf{u}_B.\nabla$, where $\mathbf{u}_B$ is the Boussinesq velocity vector as a limit of the Eulerian velocity $\mathbf{u}$ by (9). We observe that, in dimensionless form, we obviously have: $T^{*\prime}(0) \equiv 1$, but, in general, $\Lambda^*(0)$ is *different from zero*. The choice of (8) and (10a), (10b)), which give, with (9), the limit Boussinesq equations (11) follows from a carefully asymptotic analysis of the various degeneracies of the exact dimensionless Euler equations [49, Chapter 8]. Namely, it is necessary to observe that the above way, for the derivation of inviscid adiabatic Boussinesq model equations (11), is the only rational way for a consistent derivation of second-order, 'à la Boussinesq' linear model equations (with the non-Boussinesq effects). In Zeytounian [49, Chapter 8], the reader can find a tentative theory of the Boussinesq approximation, for atmospheric motion. In fact, in Volume II of the book 'Mécanique' by Paul Germain [50, pp. 225, 226] the reader can find our above Boussinesq equations (11), but with, in the right-hand side of the second equation (for $\mathbf{u}_B$) a viscous (incompressible, 'à la Navier' – since we have $\nabla\cdot\mathbf{u}_B = 0$) term of the form: $(1/Re)\nabla^2\mathbf{u}_B$, and in right-hand side of the third equation (for θ_B) a dissipative term of the form $(1/Pe)\nabla^2\theta_B$, where *Pe* is the Péclet number (the product of the Reynolds number *Re*, with the Prandtl *Pr*, number). In particular, when $Re \to \infty$, but $Pr \to 0$ such that $Pe = \mathrm{O}(1)$ and $\Lambda^*(0) \equiv 0$, we rediscover the Boussinesq equations (1) written in the dimensionless form. When we consider a *steady two-dimensional flow* in the plane of gravity (x, z), such that: $u_B = \partial\psi_B/\partial z$, and $w_B = -\partial\psi_B/\partial x$ then the following single Boussinesq equation for $\psi_B(x, z)$ is derived:

$$\partial^2\psi_B/\partial x^2 + \partial^2\psi_B/\partial z^2 - (B^*/\gamma)z\,\mathrm{d}H(\psi_B)/\mathrm{d}\psi_B = F(\psi_B) \tag{12}$$

where $H(\psi_B)$ and $F(\psi_B)$, in (12), are two arbitrary functions of ψ_B only. In a particular case, in the framework of the airflow over mountains, if at upstream infinity of a mountain, when $x \to -\infty$, we have a 2D steady uniform constant flow in the direction of $x > 0$, with $\pi_B = \theta_B \equiv 0$, then we derive the following *linear Helmholtz* equation for the function $\Delta_B(x, z) = z - \psi_B$:

$$\partial^2\Delta_B/\partial x^2 + \partial^2\Delta_B/\partial z^2 + (B^{*2}/\gamma)\Lambda^*(0)\Delta_B = 0 \tag{13}$$

The dominant feature, from a mathematical point of view, is that the linearity of Eq. (13) is do not related to any one hypothesis of small perturbations. But, from exact slip condition on the considered mountain, simulated by the dimensionless equations: $z = \delta\eta(x)$, where δ is an amplitude parameter, we must write the following boundary (slip, non-linear!) condition for Δ_B on the surface of the mountain: $\Delta_B(x, \delta\eta(x)) = \delta\eta(x)$.

3.2. *The case of a weakly expansible liquid*

When the fluid is an *expansible liquid*, with an equation of state of the general form (tri-variate baroclinic fluid):

$$\rho = \rho(T, p), \tag{14}$$

the justification of the Boussinesq approximation for the derivation of the OB model equations for the RB shallow convection is more subtle. Below we consider the classical Bénard thermal problem of an infinite horizontal layer of viscous, thermally conducting, and (weakly) expansible liquid, of density ρ, heated from below (at $x_3 = 0$, $T = T_w$) and when at the level $x_3 = d$, is a free surface which separates the liquid from a passive atmosphere (at rest of constant temperature T_a and pressure p_a, having negligible viscosity and density). In motionless *conduction* steady state the temperature is $T_S(x_3) = T_w - b_\perp x_3$ and we introduce a perturbation of the temperature $\theta = (T - T_0)/\Delta T$, with $\Delta T = (T_w - T_0) = b_\perp d$, and $T_0 = T_S(d)$. On the other hand, we observe that from the asymptotic analysis of the considered Bénard problem, if we want derive consistently the OB approximate equations, then it is necessary to introduce a pressure perturbation $\pi = (1/Fr_d^2)\{(p - p_a)/g\,d\rho_0 + (x_3/d) - 1\}$, the density ρ_0 being a reference constant density and $Fr_d = (\nu_0/d)/(gd)^{1/2}$ is a Froude number with ν_0 the kinematic constant viscosity. For the liquid, according to Dutton and Fichtl [30, Section 2], we consider a *weakly expansible ideal liquid*, when in place of (14) we write as approximate equation of state:

$$\rho \approx \rho_0\big\{1 - \alpha_0\Delta T\theta + \beta_0 g\,d\rho_0\big[Fr_d^2\pi - (x_3/d) + 1\big]\big\} \tag{15}$$

where β_0 is the constant pressure expansion coefficient (or, isothermal compressibility coefficient). However, we have the following thermodynamic relation:

$$\alpha_0^2/\beta_0 = C_v(\gamma - 1)(\rho_0/T_0), \quad \text{with } \gamma = C_p/C_v \tag{16}$$

and we deduce the following similarity relation:

$$\beta_0 g\,d\rho_0 = K_0\varepsilon^2, \quad \text{where } K_0 = \big[g\,dT_0/(\Delta T)^2 C_v(\gamma - 1)\big] \tag{17}$$

Finally, if we assume that $K_0 = \mathrm{O}(1) \Rightarrow (\Delta T)^2 \approx [g\,dT_0/C_v(\gamma - 1)]$, then in the leading order for the shallow convection OB equations we can use as equation of state:

$$\rho \approx \rho_0[1 - \varepsilon\theta] \tag{18}$$

with and error of $\mathrm{O}(\varepsilon^2)$. According to Zeytounian [45,46], for the derivation of the OB approximate equations, governing the RB shallow convection model problem, it is necessary to consider the following *Boussinesq limit process*:

$$\varepsilon \downarrow 0 \text{ and } Fr_d \downarrow 0, \quad \text{such that} \quad \varepsilon/(Fr_d)^2 = Gr = \mathrm{O}(1) \tag{19}$$

where Gr is the Grashof number. In this case, since $Fr_d \downarrow 0$: "it is not consistent, in the leading order (from an asymptotic point of view), to take into account simultaneously in the RB thermal shallow convection model problem the buoyancy effect and the deformation effect of the free surface." More precisely if we associate to (19) the following asymptotic expansions for the velocity components u_i and thermodynamic perturbations π, θ:

$$u_i = u_{i0} + Fr^2 u_{i1} + \cdots, \quad \pi = \pi_0 + Fr^2 \pi_1 + \cdots, \quad \theta = \theta_0 + Fr^2 \theta_1 + \cdots \tag{20}$$

then, in place of the full exact NSF dimensionless equations, we derive asymptotically the following dimensionless OB equations for u_{i0}, and π_0, θ_0:

$$\begin{aligned} &\partial u_{i0}/\partial x_i = 0 \\ &D_0 u_{i0}/Dt + \partial \pi_0/\partial x_i - Gr\,\theta_0 \delta_{i3} = \Delta u_{i0} \\ &D_0 \theta_0/Dt = (1/Pr)\Delta\theta_0 \end{aligned} \tag{21}$$

In the limit equation for θ_0 the viscous dissipation term is negligible because

$$1\ \text{mm} \approx \left(\nu_0^2/g\right)^{1/3} \ll d \approx C_0/g\Delta T \tag{22}$$

with $C_0 = (dE/d\theta)_{\theta=0}$ where $E = E(\theta)$ is the internal specific energy of our ideal liquid with an approximate equation of state (18). The relation (22) is an estimate for the thickness of the liquid layer, d. As a sequel of a '*rigid-free*' 'exact' starting Bénard thermal problem, for the above OB model equations (21), we obtain in the Boussinesq limit (19) with (20) the following boundary conditions at the non-deformable surface $x_3 = 1$, with Marangoni and Biot effects:

$$v_{30} = 0, \quad \text{and} \quad \partial^2 v_{30}/\partial x_3^2 = Ma\left[\partial^2\theta/\partial x_1^2 + \partial^2\theta/\partial x_2^2\right] \tag{23a}$$

$$\partial\theta_0/\partial x_3 + Bi\,\theta_0 + 1 = 0 \tag{23b}$$

On the lower rigid flat plate we have the conditions:

$$u_{i0} = 0 \quad \text{and} \quad \theta_0 = 1, \quad \text{at } x_3 = 0 \tag{23c}$$

The deformation of the free surface $\eta(t, x_1, x_2)$ is then determined, when the perturbation of the pressure $\pi_1(t, x_1, x_2, x_3)$ is known at $x_3 = 1$, after the solution of the RB problem (21), (23(a)–(c)), by the equation:

$$\partial^2\eta/\partial x_1^2 + \partial^2\eta/\partial x_2^2 - (\delta^*/We)\eta = -(1/We)\pi_0(t, x_1, x_2, 1) \tag{24}$$

where $\delta^* = \delta/Fr_d^2$, with $\delta \ll 1$ the dimensionless amplitude parameter of the deformable free surface. The Marangoni (*Ma*) and Weber (*We*) numbers are linked with the surface tension, assumed temperature dependent, and the Biot (*Bi*) number is linked with the use of the Newton's law for the heat transfer between the air and liquid via the free surface. The above Eq. (24), for $\eta(t, x_1, x_2)$, seems do not have been derived in framework of classical, ad hoc, theory [10], and emerges very naturally in our asymptotic approach. The RB problem: (21), (23(a)–(c)), has been recently considered by Dauby and Lebon [51], but without Eq. (24). A final remark concerning the conclusions of the paper by Rajagopal, Ruzicka and Srinivasa [52], which curiously assert that their derivation of OB Eqs. (21): "is free from the additional assumptions usually added in various earlier works in order to obtain the correct equations"?

4. Some comments concerning the validity of the Boussinesq approximation

4.1. The problem of initial conditions

If we consider the above two main model system of equations, (11), and (21), derived in the above Sections 3, via the Boussinesq approximation, then we observe that in these approximate two model systems (derived when the time, t, is fixed and O(1)) we have a *velocity divergence-less equation*, in place of the full compressible continuity equation, and only two derivatives with respect to time: for $\mathbf{u}_B$ and θ_B in (11), and for $\mathbf{u}_{i0}$, and θ_0 in (21). On the other hand, in the full unsteady Euler and NSF equations, we have *three* derivatives with respect to time: for the velocity vector $\mathbf{u}$, density ρ and temperature T. As a consequence, if we want resolve a pure initial-value or *Cauchy* (*prediction*) *problem* (in the L^2-norm, for example), it is necessary (for the well-posedness) to have a complete set of initial conditions (data):

$$t = 0\colon\ \mathbf{u} = \mathbf{u}^0(\mathbf{x}), \quad \rho = \rho^0(\mathbf{x}), \quad T = T^0(\mathbf{x}) \tag{25}$$

where $\rho^0(\mathbf{x}) > 0$ and $T^0(\mathbf{x}) > 0$. Now, if we consider, for instance, the system of *Boussinesq equations*, (11), then we have the possibility to assume only *two* initial data, namely:

$$t = 0\colon\ \mathbf{u}_B = \mathbf{u}_B^0(\mathbf{x}) \quad \text{and} \quad \theta = \theta_B^0(\mathbf{x}) \tag{26}$$

This is due to the fact that, for instance, the Boussinesq limit process (t and $\mathbf{x}$ both fixed in (9)), which leads to the approximate equations (11), *filters out* some time derivatives – these corresponding to *acoustic fast waves* – because such waves are of no importance for low speed (hyposonic) motions considered. Due to this; one encounters the problem of deciding what initial conditions $(\mathbf{u}_B^0, \theta_B^0)$ one may prescribe for the approximate ('à la Boussinesq') equations (11), and in what way these two initial conditions (26) are related to the given initial conditions (25) associated with the starting Euler exact equations. It is important to note that the exact initial conditions for the Euler (and also NSF) equations, are not, in general, consistent with the estimates of basic orders of magnitude implied by the approximate (without acoustic waves!) model equations (11) and (21). A physical process of time evolution is necessary to bring the initial set to a consistent level as far as the orders of magnitude are concerned. Such a process is called one of '*unsteady adjustment*', and is short on the time scale (compared with the time characterizing the approximate simplified equations) and at the end of it, in an *asymptotic sense*, we obtain values for the set of initial conditions suitable to the simplified (via the Boussinesq approximation) equations, at $t = 0$: *The aim of the unsteady adjustment problem is to clarify just how a set of initial data associated with a determined (exact) starting system of equations can be related to another set of initial data associated with a simpler, approximate model equations, which is a significant degeneracy of the system of (exact) equations considered at the start.* More precisely, the obtaining of consistent initial conditions (at $t = 0$) for the approximate Boussinesq equations (11) is a consequence of a matching between the two asymptotic representations: the main one (*Boussinesq*, with t fixed; $t = 0$) and the local one (*acoustic*, near $t = 0$, with $\tau = t/M$ fixed; $\tau \to +\infty$), and Lim(Boussinesq, at $t = 0$) = Lim(Acoustics, at $\tau \to +\infty$). In order to solve a such problem, it is necessary to introduce an initial layer in the vicinity of $t = 0$ by distorting the time scale and the unknowns which were initially undefined. In [32, Chapter V], the reader can find the solution of this problem for the Boussinesq equations but the result is valid only when we assume, for the exact Euler equations, as initial conditions:

$$t = 0\colon\ u = u^0, \quad v = v^0, \quad w = w^0, \quad \pi = M\pi^0, \quad \omega = M\omega^0, \quad \text{and} \quad \theta = M\theta^0 \tag{27}$$

where the initial data are given functions of x, y and z, and the initial velocity vector is assumed be of the following form: $\mathbf{u}^0 = (u^0, v^0, w^0) = \nabla\phi^0 + \nabla \wedge \psi^0$.

4.2. The upper boundary condition at the top of the troposphere and the radiation (Sommerfeld) condition

If we assume that the upper flat plane, $x_3 = H^*$, bound (as a tropopause) the considered lee-waves phenomenon (in the troposphere), then we must write the following dimensionless upper boundary (slip) condition for the exact dimensionless Euler equations

$$w = 0 \quad \text{on } z = 1/Bo \tag{28}$$

which is *very singular* when $Bo \to 0$! As a consequence, *in reality*, for the 2D steady Boussinesq equation (13) we obtain, as upper condition the 'paradoxical' behaviour condition: $\Delta_B(x, z \uparrow +\infty) = +\infty$! Obviously, the infinity in altitude relative to z, for the Boussinesq equations (11), with (28), must be understood as a behaviour condition relative to the 'inner' vertical (Boussinesq) coordinate z (with the dimensionless quantities), which is matched with an outer, $\zeta = Mz$, coordinate. This outer vertical dimensionless coordinate ζ, taking into account the upper condition at the top of the troposphere, which is rejected at infinity in the framework of the Boussinesq (inner) problem (we observe that, as a consequence of $B^* = \mathrm{O}(1)$, the Boussinesq inner problem is significant only in a layer of the thickness $H_B = (U_c/g)[RT^*(0)/\gamma]^{1/2}$ of order of 10^3 m). The outer region is bounded by the plane $\zeta = 1/B^*$, with $B^* = \mathrm{O}(1)$, and in an unbounded atmosphere (which is in fact a boundary layer type inner region), it is necessary to impose for Δ_B, solution of (13), a radiation condition (à la Sommerfeld):

$$\Delta_B \approx [2K_0/\pi r]^{1/2} \sin\theta \,\mathrm{Real}\big\{G(\cos\theta)\exp\big[\mathrm{i}(K_0 r - \pi/4)\big]\big\} \tag{29}$$

when $r = [x^2 + z^2]^{1/2} \to \infty$, with $K_0 \equiv (B^{*2}/\gamma)\Lambda^*(0)$, where the function $G(\cos\theta)$ is arbitrary and depends on the form of the relief simulated by $z = \delta\eta(x)$. So as to satisfy the upstream infinity behaviour (for $x \to -\infty$), the condition $G(\cos\theta) = 0$, for $\cos\theta < 0$ must also be imposed. It is pointed out that the polar coordinates, r, θ, in the upper half-plane $z > 0$ are defined such that: $x = r\cos\theta$ and $z = r\sin\theta$. The inner Boussinesq model problem is, in fact, the problem considered by Miles [53] and also by Kozhevnikov [54], with the condition (29), which express that "no waves are radiated inwards". In Guiraud and Zeytounian [55] paper the associated outer problem is asymptotically analysed and it is shown that the upper and lower boundaries of the troposphere alternately reflect internal short gravity waves excited by the lee waves of the inner (Boussinesq) approximation, with a wavelength of the order of the Mach number, M, to the scale of the outer region. As a consequence, there is a double scale built into the solution and we must take care of it – the important point of GZ [55] analysis is that: "these short gravity excited waves propagate downstream and that not feedback occurs on the inner Boussinesq flow close to the mountain (to lower order at least!)". As a consequence, we should understand the imposed upper boundary at the top of the troposphere as an *artificial one*, having asymptotically *no effects* on the inner Boussinesq flow which is the only really interesting one. In Bois [56], it is shown that, *within a generalized Boussinesq approximation, the atmospheric linearized flow over a relief can, in realistic cases, be approximated by a confined flow*.

4.3. Isochoric model equation

If, first, we consider an Eulerian (nonviscous) motion with a *constant internal specific energy per unit mass* – a so-called '*isochoric*, $E = E_0 \equiv$ const' flow, then we observe that in the framework of Euler compressible nonviscous dimensionless equations for a perfect gas the corresponding isochoric model equations are derived under the following limiting process:

$$M \to 0 \text{ and } \gamma \to \infty, \quad \text{such that} \quad \gamma M^2 = M^* = \mathrm{O}(1) \tag{30}$$

and $\gamma \to \infty$, because for a perfect 'isochoric' gas $c_p = \mathrm{O}(1)$, but $c_v \to 0$. For instance, in the steady two-dimensional inviscid case, on the one hand, in place of dimensionless *linear Helmholtz–Boussinesq* equations (13), we derive, for the (isochoric) function $\Delta_{\mathrm{Is}}(x, z)$, the following dimensionless (quasi) *nonlinear* equation [48, pp. 104–108], where in the right-hand side we have the non-Boussinesq effects:

$$\begin{aligned}&\partial^2\Delta_{\mathrm{Is}}/\partial x^2 + \partial^2\Delta_{\mathrm{Is}}/\partial z^2 + (Bo^2/M^*)N_{\mathrm{Is}}(Bo\, z)\Delta_{\mathrm{Is}}\\ &\quad = (Bo/2)\big[2\partial\Delta_{\mathrm{Is}}/\partial z - (\partial\Delta_{\mathrm{Is}}/\partial x)^2 - (\partial\Delta_{\mathrm{Is}}/\partial z)^2\big]\end{aligned} \tag{31}$$

with $N_{\mathrm{Is}}(z^*) = (1/T^*(z^*))[1 + \mathrm{d}T^*(z^*)/\mathrm{d}z^*]$, $z^* = Bo\, z$, and, on the other hand, this isochoric equation (31) is valid in the whole thickness of the troposphere (*Bo being unity*). Moreover, when $Bo \to 0$, with $Bo = B^*M \to 0$, $\gamma = \mathrm{O}(1)$, we recover again a Helmholtz equation, similar to (13), for $\lim \Delta_{\mathrm{Is}}$ but with $K_{\mathrm{Is}}^2 = (B^{*2}/\gamma)N_{\mathrm{Is}}(0)$, in place of $(B^{*2}/\gamma)\Lambda^*(0) = K_0^2$.

4.4. Deep convection model equation

When $Bo = \mathrm{O}(1)$ we also have the possibility to assume a constraint on $\Theta^*(z^*)$ given by (4a). Indeed, first, in the whole thickness of the *troposphere*, $\Theta^*(z^{*\prime}) = -\mathrm{d}T^{*\prime}(z^{*\prime})/\mathrm{d}z^{*\prime}$ *is very close to* $(\gamma - 1)/\gamma$, and as a consequence we can assume that (without $'$, but in dimensionless form):

$$-\mathrm{d}T^*(z^*)/\mathrm{d}z^* = \big[(\gamma - 1)/\gamma\big] + M^2\chi(z^*) \tag{32}$$

where $|\chi(z^*)| = \mathrm{O}(1)$. In particular, again for the *steady, two-dimensional*, lee-waves problem in the whole troposphere, we derive the following single equation for the *deep stream function* Ψ_D:

$$\begin{aligned}&\partial^2\Psi_D/\partial x^2 + \partial^2\Psi_D/\partial z^2 + (Bo/\gamma)\big\{1 - \big[(\gamma - 1)\,Bo/\gamma\big]z\big\}^{-1}\partial\Psi_D/\partial z\\ &\quad = \big\{1 - \big[(\gamma - 1)\,Bo/\gamma\big]z\big\}^{-2/(\gamma-1)}\big\{F_D(\Psi_D) + (Bo/\gamma)z\big[\mathrm{d}H_D(\Psi_D)/\mathrm{d}\Psi_D\big]\big\}\end{aligned} \tag{33}$$

which is an extended form (with non-Boussinesq effects) of Boussinesq equation (12) written for Ψ_B. In Pekelis, [57], the reader can find some numerical results for the *deep lee-waves* in the troposphere, which are computed via an equation very similar to (33), but derived from the so-called 'anelastic' equations of Ogura and Phillips [58]. We observe that, in the two-dimensional steady case, the Boussinesq, (13), isochoric, (31), and deep convection, (33), equations, have been derived, in Zeytounian [59], directly from a single vorticity equation deduced from the exact Euler steady compressible two-dimensional non-viscous adiabatic dimensionless system. Finally, in Zeytounian [60] the reader can find a *deep OB system* of equations with the *non-Boussinesq* terms proportional to depth parameter: $\delta_d = \alpha_0 g\, d/C_0$.

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