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Homogenization of a degenerate parabolic problem in a highly heteregeneous medium

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Abstract

We consider the homogenization of a time-dependent heat transfer problem in a highly heteregeneous periodic medium made of two connected components having finite heat capacities $c_\alpha(x)$ and heat conductivities $a_\alpha(x)$, $\alpha = 1, 2$, of order one, separated by a third material with thickness of order ε the size of the basic periodicity cell, but with conductivity $\lambda a_3(x)$ where $a_3 = \mathrm{O}(1)$ and λ tends to zero with ε. Assuming only that $c_i(x) \geqslant 0$ a.e., such that the problem can degenerate (parabolic-elliptic), we identify the homogenized problem following the values of $\delta = \lim_{\varepsilon \to 0} \varepsilon^2/\lambda$. ***To cite this article: M. Mabrouk, A. Boughammoura, C. R. Mecanique 331 (2003).***

Résumé

Homogénéisation d'un problème parabolique dégénéré dans un milieu fortement hétérogène. Nous considérons un problème de propagation de chaleur dans un milieu périodique fortement hétérogène constitué de deux composantes connexes ayant des capacités calorifiques c_α et des conductivités a_α comparables, séparés par une couche d'épaisseur d'ordre ε, taille de la cellule de reférence, mais de conductivité λa_3, a_3 d'ordre un et λ tendant vers zéro avec ε. Sous l'hypothèse $c_i(y) \geqslant 0$ sur les capacités, autorisant une dégénérescence parabolique-elliptique, nous identifions les problèmes homogénéisés suivant les valeurs de $\delta = \lim_\varepsilon \varepsilon^2/\lambda$. ***Pour citer cet article : M. Mabrouk, A. Boughammoura, C. R. Mecanique 331 (2003).***

Keywords: Heat transfer; Heteregeneous periodic medium

Mots-clés : Transferts thermiques ; Milieu pédiodique fortement hétérogène

Version française abrégée

Dans ce travail, on s'intéresse à l'homogénéisation d'un problème parabolique dégénéré décrivant la propagation de la chaleur dans un milieu périodique fortement hétérogène. Il s'agit d'une suite au travail [1] de Mabrouk–

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Samadi consacré au même modèle mais dans le cas stationnaire, dont nous gardons les notations. Nous utilisons les idées et les résultats de [1] en les complétant dans le cas $\delta = +\infty$.

Soit $Y =]0, 1[^3 \subset \mathbb{R}^3$. On suppose que $Y = Y_1 \cup S_{12} \cup Y_2 \cup S_{23} \cup Y_3$ où Y_1, Y_2, Y_3 sont trois ouverts connexes tels que $\overline{Y_1} \cap \overline{Y_2} = \emptyset$ et où S_{ij} est la surface séparant Y_i et Y_j. On suppose, pour simplifier, que Y_3 est d'épaisseur constante, d'ordre un. On note χ_1, χ_2, χ_3 les fonctions caractéristiques de Y_1, Y_2, Y_3 et par θ_1, θ_2, θ_3 leurs mesures de Lebesgue respectives supposées être de même ordre. Les parties Y_1 et Y_2 sont occuppées par deux matériaux de conductivités a_1 et a_2 et de capacités calorifiques c_1, c_2 d'ordre un. Le matériau occupant Y_3 a une conductivité qui tend vers zéro suivant un ordre $\lambda = \lambda(\varepsilon)$, où ε est un nombre strictement positif tendant vers zéro. Si nous reproduisons la cellule Y par périodicité, géométriquement et matériellement à $\mathbb{R}^3$, nous obtenons les trois ouverts E_1, E_2, E_3 *que nous supposons connexes* et séparés par les surfaces régulières S_{13}, S_{23}. Plus précisément, chaque ouvert E_i est la réunion des $\mathbb{Z}^3$-translatés de $Y_i \cup (\overline{Y_i} \cap \partial Y)$. L'homothétie de rapport ε nous donne alors la cellule élémentaire $Y^\varepsilon = \varepsilon Y$, les trois ouverts élémentaires Y_i^ε de même ordre de grandeur $O(\varepsilon)$ et les *trois ouverts périodiques connexes* $E_i^\varepsilon = \varepsilon E_i$ séparés par les surfaces S_{13}^ε, S_{23}^ε. Soit $\Omega \subset \mathbb{R}^3$ ouvert borné connexe, de bord $\partial\Omega$ lipschitzien. On note $\Omega_i^\varepsilon = \Omega \cap E_i^\varepsilon$, $\Gamma_{ij}^\varepsilon = S_{ij}^\varepsilon \cap \Omega$. Etant donné $T > 0$, on définit aussi les domaines spatio-temporels $Q = (0, T) \times \Omega$, $Q_i^\varepsilon = (0, T) \times \Omega_i^\varepsilon$.

Soit c_i (resp. a_i) le capacité calorifique (resp. la conductivité) de la i-ème composante. Ce sont des fonctions Y-périodiques données sur la cellule de base, vérifiant (H1) : $0 \leqslant c_i(y) \leqslant C_i < +\infty$, $0 < a_0 \leqslant a_i(y) \leqslant A_i < +\infty$ p.p., avec C_i, A_i constantes données. Soient $c_i^\varepsilon(x) = c_i(\frac{x}{\varepsilon})$, $a_i^\varepsilon = a_i(\frac{x}{\varepsilon})$, $x \in \Omega_i^\varepsilon$, $a^\varepsilon(x) = \chi_1^\varepsilon(x)a_1^\varepsilon(x) + \chi_2^\varepsilon(x)a_2^\varepsilon(x) + \lambda\chi_3^\varepsilon(x)a_3^\varepsilon(x)$, $c^\varepsilon(x) = \sum_{i=1}^3 \chi_i^\varepsilon(x)c_i^\varepsilon(x)$. On considère le problème d'évolution pour la température u^ε

$$(\mathcal{P}_\varepsilon) =: \begin{cases} \frac{\partial}{\partial t}\bigl(c^\varepsilon(x)u^\varepsilon(t,x)\bigr) = \operatorname{div}\bigl(a^\varepsilon(x)\nabla u^\varepsilon(t,x)\bigr) + f^\varepsilon(t,x) \quad \text{dans } \Omega \times \mathbb{R}+^* \\ u^\varepsilon = 0 \quad \text{sur } \partial\Omega \times \mathbb{R}_+^*, \quad + \text{« Condition Initiale »} \end{cases} \tag{1}$$

où f^ε sont des sources de chaleur données. Ce problème met en jeu une condition initiale, dans un sens assez flou pour le moment, une condition au bord et des conditions de transmission exprimant la continuité de la température et du flux thermique à travers les surfaces de séparation. Dans le cas où c^ε n'est pas identiquement nulle, *nous donnons un sens précis à la condition initiale*, en utilisant l'approche fonctionnelle développée par Showalter ([2], pp. 136–139) pour les problèmes paraboliques dégénérés. La formulation obtenue (P_ε ou V_ε) continue à avoir un sens si $c^\varepsilon \equiv 0$ p.p., et permet ainsi d'englober le problème purement quasi-statique. Ceci est précisé dans la partie anglaise. Les résultats (Proposition 1) recouvrent ceux du problème stationnaire, mais mettent en lumière d'autres aspects, notamment un phénomène de mémoire dans le cas $0 < \delta < +\infty$. De plus, ils les complètent substantiellement dans le cas $\delta = +\infty$. Si l'échange à travers la couche interstitielle est quasi-statique ($c_3(y) = 0$ p.p.), nous retrouvons, dans le cas $0 < \delta < +\infty$, le modèle bien connu de Barenblatt–Rubinstein [4,3]. Pour prouver ces résultats, nous nous ramenons à un problème stationnaire via la transformation de Laplace. Le choix de cette approche a été dicté par la possibilité d'utiliser des résultats déjà établis dans [1]. Enfin, nous comparons avec le cas non dégnéré qui présente des simplifications importantes tant au niveau des méthodes que des résultats (Section 5).

1. Introduction

This paper is devoted to the homogenization of an evolution problem in a periodic composite medium made of three components, two of them having comparable (finite) conductivities while the third one, working as an isolating layer, have a weaker conductivity, of very low scale. In a preceding paper [1], Mabrouk and Samadi have addressed the stationary case which amounts to homogenize an elliptic problem. Here, we continue this investigations by studying the evolutionary time-dependent case. Let us remember the geometric and material assumptions used in [1].

We are given the basic cube $Y =]0, 1[^3$ of $\mathbb{R}^3$, partitioned as $Y = Y_1 \cup Y_{13} \cup Y_2 \cup Y_{23} \cup Y_3$ where Y_1, Y_2, Y_3 are three connected open subsets such that $\overline{Y_1} \cap \overline{Y_2} = \emptyset$, and where Y_{ij} is the interface between Y_i and Y_j; thus Y_3 separates Y_1 and Y_2. Without loss of generality, we assume that Y_3 have constant thickness of size $O(1)$. Let χ_1, χ_2,

χ_3 be the characteristic functions of Y_1, Y_2, Y_3 and θ_1, θ_2, θ_3 their respective Lebesgue measures supposed to be of the same order of magnitude. The parts Y_1 and Y_2 are made of two conducting media having finite heat capacities and conductivities. Oppositely, the material in Y_3 possesses a conductivity which becomes arbitrary small with a rate $\lambda = \lambda(\varepsilon)$, where ε is a small strictly positive number approaching zero. If we reproduce by periodicity the basic cell, geometrically and materially, to the whole of $\mathbb{R}^3$, we get the three open sets E_1, E_2, E_3 which we shall assume to be connected and separated by the smooth surfaces S_{13}, S_{23}. More precisely, every E_i is the union of the $\mathbb{Z}^3$-translates of $Y_i \cup (\overline{Y_i} \cap \partial Y)$. The homothety of ratio ε gives us the elementary cell $Y^\varepsilon = \varepsilon Y$, the three elementary subsets Y_i^ε having the same order of magnitude $\mathrm{O}(\varepsilon)$ and the three periodic connected open sets $E_i^\varepsilon = \varepsilon E_i$ separated by the surfaces S_{13}^ε, S_{23}^ε. Let then Ω be an open bounded connected subset of $\mathbb{R}^3$ with lipschitzian boundary $\partial\Omega$. We set $\Omega_i^\varepsilon = \Omega \cap E_i^\varepsilon$ and $\Gamma_{ij}^\varepsilon = S_{ij}^\varepsilon \cap \Omega$, $\forall i, j \in \{1, 2, 3\}$. Given $T > 0$, we define the space-time domains $Q = (0, T) \times \Omega$, $Q_i^\varepsilon = (0, T) \times \Omega_i^\varepsilon$. By $\mathcal{L}^{-1}$ we denote the inverse Laplace transform, the index α takes the values 1,2 and C_ε (resp. C) is a generic constant depending (resp. independent) on ε.

2. The mathematical model

We are now in position to state our problem.

For $i = 1, 2, 3$, let c_i (resp. a_i) be the heat capacity (resp. the heat conductivity) of the i-th component. They are Y-periodic functions given on the basic cell and verifying the following assumptions: (H1): $0 \leqslant c_i(y) \leqslant C_i < +\infty$, $0 < a_0 \leqslant a_i(y) \leqslant A_i < +\infty$ a.e., where C_i, A_i are given positive constants. The corresponding ε-periodic coefficients are defined by $c_i^\varepsilon(x) = c_i(\frac{x}{\varepsilon})$, $a_i^\varepsilon = a_i(\frac{x}{\varepsilon})$, $x \in \Omega_i^\varepsilon$. Let $a^\varepsilon(x) = \chi_1^\varepsilon(x)a_1^\varepsilon(x) + \chi_2^\varepsilon(x)a_2^\varepsilon(x) + \lambda\chi_3^\varepsilon(x)a_3^\varepsilon(x)$ and $c^\varepsilon(x) = \sum_{i=1}^3 \chi_i^\varepsilon(x)c_i^\varepsilon(x)$. We assume that the boundary of Ω is maintained at fixed temperature and that "an initial distribution of temperature" $u_0^\varepsilon(x)$ on Ω is given, in a sense to be precised, for every ε. Then the evolution of the temperature $u^\varepsilon(t, x)$ is governed by problem $(\mathcal{P}_\varepsilon)$. The coefficients c^ε, a^ε being discontinuous, the equation has to be interpreted in the sense of distributions on Ω. The transmission conditions express the continuity of the temperature and of the heat flow across the separating surfaces. The exact meaning of the initial condition will be made below. *This problem has two interesting features: the medium is highly heterogeneous, and the functions c_i may vanish.* It can degenerate to an elliptic-parabolic or even to a pure elliptic (quasi-static) one. We shall give a weak formulation of the problem which allows us to handle general data and especially to give a precise meaning to the initial and boundary conditions. For this, we recall shortly the functional framework developed by Showalter for degenerate parabolic equations (see [2], pp. 136–139).

We suppose first that c^ε does not vanish identically a.e., i.e., $\mathrm{meas}\{y\colon c(y) \neq 0\} > 0$. Let then $V = H_0^1(\Omega)$, $\mathcal{V} = L^2(0, T; V)$ and $V' = H^{-1}(\Omega)$, $\mathcal{V}' = L^2(0, T; V')$ their dual spaces. For every $\varepsilon > 0$, let $B^\varepsilon, A^\varepsilon : V \to V'$ the continuous operators defined by the bilinear forms on $V \times V$: $b^\varepsilon(u, v) = \int_\Omega c^\varepsilon(x)u(x)v(x)\,\mathrm{d}x$, $a^\varepsilon(u, v) = \int_\Omega a^\varepsilon(x)\nabla u(x)\nabla v(x)\,\mathrm{d}x$. B^ε is thus the operator of multiplication by the function c^ε. Let V_b^ε be the completion of V with the semi-norm $(b^\varepsilon(\cdot,\cdot))^{1/2}$ and let $V_b'^\varepsilon$ its dual. Then we have $V_b^\varepsilon = \{u$ measurable on $\mathrm{supp}(c^\varepsilon)$: $(c^\varepsilon)^{1/2}u \in L^2(\Omega)\}$ and $V_b'^\varepsilon = \{(c^\varepsilon)^{1/2}u,\ u \in L^2(\Omega)\}$. The operator B^ε admits a continuous extension from V_b^ε into $V_b'^\varepsilon$ denoted too by B^ε. Let $\mathcal{A}^\varepsilon$, $\mathcal{B}^\varepsilon$ be the realization of A^ε, B^ε as operators from $\mathcal{V}$ to $\mathcal{V}'$, that is precisely $(\mathcal{A}^\varepsilon u(t), \mathcal{B}^\varepsilon u(t)) = (A^\varepsilon(u(t)), B^\varepsilon(u(t)))$ for a.e. $t \in (0, T)$. For every $\varepsilon > 0$, we introduce the space $W_2^\varepsilon(0, T) =: \{u \in \mathcal{V}\colon \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{B}^\varepsilon \in \mathcal{V}'\}$. Then (see [2], Proposition 6.3), if u is in $W_2^\varepsilon(0, T)$, we have: $u \in C([0, T]; V_b^\varepsilon)$, $\mathcal{B}^\varepsilon u \in C([0, T]; V_b'^\varepsilon)$, $\|u\|_{C([0,T];V_b^\varepsilon)} \leqslant C_\varepsilon\|u\|_{W_2^\varepsilon(0,T)}$ and $\|\mathcal{B}^\varepsilon u\|_{C([0,T];V_b'^\varepsilon)} \leqslant C_\varepsilon\|u\|_{W_2^\varepsilon(0,T)}$. Conversely, if $(u_0, w_0) \in V_b^\varepsilon \times V_b'^\varepsilon$ are given, there exists a function $u \in W_2^\varepsilon(0, T)$ such that $\lim_{t\to 0+}(u(t), \mathcal{B}^\varepsilon u(t)) = (u_0, w_0)$. Thus, given u_0^ε in V_b^ε and w_0^ε in $V_b'^\varepsilon$ related by $w_0^\varepsilon = c^\varepsilon u_0^\varepsilon$, we can express the initial condition by one of the equivalent equalities

$$\big(\mathcal{B}^\varepsilon u^\varepsilon\big)(0) = B^\varepsilon\big(u^\varepsilon(0)\big) = w_0^\varepsilon \in V_b'^\varepsilon \quad \Longleftrightarrow \quad (c^\varepsilon)^{1/2}u^\varepsilon(0) = (c^\varepsilon)^{1/2}u_0^\varepsilon \text{ in } L^2(\Omega) \tag{2}$$

Given $f^\varepsilon \in L^2_{\mathrm{loc}}(\mathbb{R}; L^2(\Omega))$ or more generally in $\mathcal{V}'$ and u_0^ε in V_b^ε, problem $(\mathcal{P}_\varepsilon)$ can be written as an abstract Cauchy problem

$$P_\varepsilon: \quad \begin{cases} \text{Find } u \text{ in } W_2^\varepsilon(0,T)\text{: } \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{B}^\varepsilon u(t) + \mathcal{A}^\varepsilon u(t) = f^\varepsilon(t) \in V' \quad \text{in } (0,T) \\ (c^\varepsilon)^{1/2}u(0) = (c^\varepsilon)^{1/2}u_0^\varepsilon \quad \text{in } L^2(\Omega) \end{cases}$$

Thus, u_0^ε may be given only on the support of c^ε. The equivalent variational problem is:

$$V_\varepsilon := \begin{cases} \text{Find } u \in W_2^\varepsilon(0,T) \text{ s.t. } (c^\varepsilon)^{1/2}u(0) = (c^\varepsilon)^{1/2}u_0^\varepsilon \in L^2(\Omega) \text{ and} \\ \text{for a.e. } t\text{: } b^\varepsilon(u'(t), v) + a^\varepsilon(u(t), v) = \langle f^\varepsilon(t), v\rangle_{V',V} \ \forall v \in V \end{cases}$$

If $c^\varepsilon = 0$ a.e., then $\mathcal{B}^\varepsilon u = 0$, P_ε continue to be meaningful and reduces to an elliptic boundary-value problem (depending on the parameter t), V_ε being the corresponding variational formulation. The initial condition is then immaterial and plays no role. Thus, every time the initial condition is invoked, this will under hears that c^ε does not vanish identically. Since the form a^ε is symmetric, continuous and coercive on V, and the operator $\mathcal{B}^\varepsilon$ is continuous, linear, symmetric, monotone from V to V', our problem P_ε admits in any case a unique solution u^ε. Our objective is to study its behavior as $\varepsilon \to 0$.

For the sake of simplicity, we shall take here $u_0^\varepsilon = u_0 \in L^2(\Omega)$. This is allowable, since then $u_0^\varepsilon \in X_0 = \bigcap_{\varepsilon>0} V_b^\varepsilon$ by assumption (H1). The more general case $u_0^\varepsilon \in X_0$ needs slight modifications.

3. The results

Let w_i^1, w_i^2, $i = 1, \ldots, N$, be the $2N$ functions verifying the cellular problems:

$$(\mathbf{cell}\alpha) \quad \begin{cases} -\operatorname{div}_y(a_\alpha(y)[e_i + \nabla_y w_i^\alpha(y)]) = 0 \quad \text{in } Y_\alpha \\ a_\alpha(y)[e_i + \nabla_y w_i^\alpha] \cdot n(y) = 0 \quad \text{on } \partial Y_\alpha \cap \partial Y_3 \\ y \mapsto w_i^\alpha(y), \quad a_\alpha(y)\frac{\partial}{\partial n_y} w_i^\alpha(y)_{|\partial Y_\alpha \cap \partial Y} \quad Y\text{-periodic} \end{cases} \tag{3}$$

and let $\hat\eta_i$, $i = 1, 2, 3$, the solutions of the three following cellular problems:

$$(\widehat{\mathbf{cell}.\mathbf{31}}) \quad \begin{cases} pc_3(y)\hat\eta_1(p,y) - \frac{1}{\delta}\operatorname{div}_y(a_3(y)\nabla_y\hat\eta_1(p,y)) = 1 \quad \text{in } Y_3 \\ \hat\eta_1(p,y) = 0 \quad \text{on } \partial Y_1 \cap \partial Y_3, \quad \hat\eta_1(p,y) = 0 \quad \text{on } \partial Y_2 \cap \partial Y_3 \\ y \mapsto \hat\eta_1(p,y), \quad c_3(y)\frac{\partial}{\partial n}\hat\eta_1(p,y)_{|\partial Y_3 \cap \partial Y} \quad Y\text{-periodic} \end{cases} \tag{4}$$

$$(\widehat{\mathbf{cell}.\mathbf{32}}) \quad \begin{cases} pc_3(y)\hat\eta_2(p,y) - \frac{1}{\delta}\operatorname{div}_y(a_3(y)\nabla_y\hat\eta_2(p,y) = c_3(y) \quad \text{in } Y_3 \\ \hat\eta_2(p,y) = 0 \quad \text{on } \partial Y_1 \cap \partial Y_3, \quad \hat\eta_2(p,y) = 0 \quad \text{on } \partial Y_2 \cap \partial Y_3 \\ y \mapsto \hat\eta_2(p,y), \quad c_3(y)\frac{\partial}{\partial n}\hat\eta_2(p,y)_{|\partial Y_3 \cap \partial Y} \quad Y\text{-periodic} \end{cases} \tag{5}$$

$$(\widehat{\mathbf{cell}.\mathbf{33}}) \quad \begin{cases} pc_3(y)\hat\eta_3(p,y) - \frac{1}{\delta}\operatorname{div}_y(a_3(y)\nabla_y\hat\eta_3(p,y)) = 0 \quad \text{in } Y_3 \\ \hat\eta_3(p,y) = 0 \quad \text{on } \partial Y_1 \cap \partial Y_3, \quad \hat\eta_3(p,y) = 1 \quad \text{on } \partial Y_2 \cap \partial Y_3 \\ y \mapsto \hat\eta_3(p,y), \quad c_3(y)\frac{\partial}{\partial n}\hat\eta_3(p,y)_{|\partial Y_3 \cap \partial Y} \quad Y\text{-periodic} \end{cases} \tag{6}$$

Let $\widehat{K}_{\alpha 3}^i(p) = \int_{\partial Y_\alpha \cap \partial Y_3} a_3(y)\frac{\partial}{\partial n_3}\hat\eta_i(p,y)\,\mathrm{d}S_y$ and A_α^{hom} the matrices with entries $A_{\alpha ij}^{\text{hom}} = \int_{Y_\alpha} a_\alpha(y)[\mathrm{e}_y w_\alpha^i(y) + e_i] \times [\mathrm{e}_y w_\alpha^i(y) + e_j]\,\mathrm{d}y$, and let $f \in L^2(Q)$ be given. Then

Proposition 3.1. *Let $\delta = \lim_\varepsilon \varepsilon^2/\lambda$, $\tilde c = \int_Y c(y)\,\mathrm{d}y$, $\tilde c_\alpha = \int_{Y_\alpha} c_\alpha(y)\,\mathrm{d}y$. Then*

(1) $\delta = 0$, $f^\varepsilon = f$. *There exists $u \in \mathcal{D}'(\mathbb{R}; H_0^1(\Omega)) \cap L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega))$ with support in $[0, \infty[$ such that, for any fixed $T > 0$, the sequence u^ε converges weakly in $L^2(0,T; L^2(\Omega))$ to u, unique solution of the homogenized problem:*

$$\begin{cases} \tilde c u'(t,x) - \operatorname{div}_x[(A_1^{\text{hom}} + A_2^{\text{hom}})\nabla_x u(t,x)] = f(t,x) \quad \text{in } \mathbb{R}_+^* \times \Omega \\ \tilde c u(0,x) = \tilde c u_0(x) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \times \mathbb{R}_+^* \end{cases} \tag{7}$$

(2) $0<\delta<+\infty$, $f^\varepsilon=f$ *and* $\operatorname{meas}\{c_3(y)\neq 0\}>0$. *There exist* $u_\alpha\in\mathcal{D}'(\mathbb{R};H_0^1(\Omega))\cap L^2_{\mathrm{loc}}(\mathbb{R};L^2(\Omega))$, $\eta_i\in\mathcal{D}'(\mathbb{R};H^1_\sharp[Y_3]/\mathbb{R})\cap L^2_{\mathrm{loc}}(\mathbb{R};L^2(\Omega;H^1_\sharp[Y_3]/\mathbb{R}))$ *for* $i=1,2,3$ *with support in* $[0,\infty[$ *such that, for any fixed* $T>0$, *the sequences* $\chi^\varepsilon_\alpha u^\varepsilon(t,x)$ *converges weakly in* $L^2(0,T;L^2(\Omega))$ *to the functions* $u_\alpha(t,x)$, *unique solutions of the coupled homogenized problems*

$$\begin{cases}\tilde c_\alpha u'_\alpha(t,x)-\operatorname{div}_x[A^{\mathrm{hom}}_\alpha\nabla_x u_\alpha(t,x)]-K^{2'}_{\alpha 3}(t)*u_\alpha(t,x)-[u_\beta(t,x)-u_\alpha(t,x)]*K^3_{\alpha 3}(t)\\ \quad=\theta_\alpha f(t,x)-f(t,x)*K^1_{\alpha 3}(t)-K^2_{\alpha 3}(t)u_0(x)\quad \text{in } \mathbb{R}^*_+\times\Omega,\quad \beta\neq\alpha\\ \tilde c_\alpha u_\alpha(0,x)=\tilde c_\alpha u_0(x)\quad\text{in }\Omega,\quad u_\alpha=0\quad\text{on }\partial\Omega\times\mathbb{R}^*_+\end{cases}\tag{8}$$

Here $K^i_{\alpha 3}(t)=\mathcal{L}^{-1}\widehat{K}^i_{\alpha 3}(p)$, $\eta_i(t,y)=\mathcal{L}^{-1}\hat\eta_i(p,y)$ *and* $*$ *is the convolution in the time variable. The sequence* u^ε *converges weakly in* $L^2(0,T;L^2(\Omega))$ *to the function*

$$\begin{aligned}U(t,x)\,=\,&(1-\theta_2)u_1(t,x)+\theta_2u_2(t,x)+f(t,x)*\tilde\eta_1(t)+u_0(x)\tilde\eta_2(t)-u_1(t,x)*\tilde\eta'_2(t)\\&+\big(u_2(t,x)-u_1(t,x)\big)*\tilde\eta_3(t),\quad \text{with } \tilde\eta_i(t)=\int_{Y_3}\eta_i(t,y)\,\mathrm{d}y\end{aligned}$$

(3) $\delta=+\infty$. *Let* $v^\varepsilon=\frac{\sqrt\lambda}{\varepsilon}u^\varepsilon$, $z^\varepsilon=\frac{\sqrt\lambda}{\varepsilon}v^\varepsilon$ *and* $w^\varepsilon=\chi^\varepsilon_3c^\varepsilon_3u^\varepsilon$. *Then*

case 1: $f^\varepsilon=(1-\chi^\varepsilon_3)f$. *There exist* $u_\alpha(t,x)$ *in* $\mathcal{D}'(\mathbb{R};H^1_0(\Omega))\cap L^2_{\mathrm{loc}}(\mathbb{R};L^2(\Omega))$ *such that* $\chi^\varepsilon_\alpha u^\varepsilon_\alpha\rightharpoonup\theta_\alpha u_\alpha(t,x)$ *weakly in* $L^2(0,T;L^2(\Omega))$. *These functions are solutions of the two uncoupled problems*:

$$\begin{cases}\tilde c_\alpha u'_\alpha(t,x)-\operatorname{div}_x[A^{\mathrm{hom}}_\alpha\nabla_x u_\alpha(t,x)]=\theta_\alpha f(t,x)\quad\text{in }\Omega\times\mathbb{R}^*_+\\ \tilde c_\alpha u_\alpha(0,x)=\tilde c_\alpha u_0(x)\quad\text{in }\Omega,\quad u_\alpha=0\quad\text{on }\partial\Omega\times\mathbb{R}^*_+\end{cases}\tag{9}$$

Moreover:

- *If* $c_3(y)\geqslant 0$ *a.e., then* $v^\varepsilon(t,x)\to 0$ *strongly in* $L^2(Q)$,$w^\varepsilon(t,x)\rightharpoonup\tilde c_3u_0(x)$ *weakly in* $L^2(Q)$.
- *If* $c_3(y)=0$ *a.e., then* $v^\varepsilon(t,x)\rightharpoonup 0$ *weakly in* $L^2(Q)$,$z^\varepsilon(t,x)\to 0$ *strongly in* $L^2(Q)$.
- *If* $c_3(y)\geqslant c_0>0$ *a.e., then* $\chi^\varepsilon_3(x)u^\varepsilon(t,x)\rightharpoonup\theta_3u_0(x)$ *weakly in* $L^2(Q)$, $v^\varepsilon(t,x)\to 0$ *strongly in* $L^2(Q)$.

case 2: $f^\varepsilon=f$. *Then* $\chi^\varepsilon_\alpha v_\alpha\rightharpoonup 0$ *weakly in* $L^2(0,T;L^2(\Omega))$. *Moreover*:

- *If* $c_3(y)\geqslant 0$ *a.e., then* $w^\varepsilon(t,x)\rightharpoonup 0$ *weakly in* $L^2(Q)$, $z^\varepsilon(t,x)\to 0$ *strongly in* $L^2(Q)$.
- *If* $c_3(y)=0$ *a.e., then* $\chi^\varepsilon_3(x)z^\varepsilon(t,x)\rightharpoonup 0$ *weakly in* $L^2(Q)$, $\chi^\varepsilon_\alpha z^\varepsilon\to 0$ *strongly in* $L^2(Q)$, $(\sqrt\lambda/\varepsilon)^qv^\varepsilon(t,x)\to 0$ *strongly in* $L^2(Q)$ *for* $q>2$.
- *If* $c_3(y)\geqslant c_0>0$ *a.e., then* $\chi^\varepsilon_3(x)v^\varepsilon(t,x)\rightharpoonup 0$ *weakly in* $L^2(Q)$, $z^\varepsilon(t,x)\to 0$ *strongly in* $L^2(Q)$.

Remark. (i) If $c_3(y)=0$ a.e., the $\hat\eta_i$ are independent of p and we obtain, for $0<\delta<+\infty$, the simplified problems:

$$\tilde c_\alpha u'_\alpha(t,x)-\operatorname{div}_x\big[A^{\mathrm{hom}}_\alpha\nabla_xu_\alpha(t,x)\big]-K^3_{\alpha 3}\big(u_\beta(t,x)-u_\alpha(t,x)\big)=\big(\theta_\alpha-K^1_{\alpha 3}\big)f(t,x)-K^2_{\alpha 3}u_0(x)$$

$$\tilde c_\alpha u_\alpha(0,x)=\tilde c_\alpha u_0(x)\quad\text{in }\Omega,\quad u_\alpha(t,x)=0\quad\text{on }\partial\Omega,\ \textit{where now } K^1_{\alpha 3},\ K^2_{\alpha 3}\ \textit{are pure constants}.$$

Thus we recover, in the case of a quasi-static heat exchange in the interstitial layer, the model of Barenblatt–Rubinstein [4,3]. If moreover $c_\alpha=0$ a.e., the initial conditions disappear and we recover exactly the static model of [1].

(ii) If we rescale our data as $f^\varepsilon=\frac{\varepsilon}{\sqrt\lambda}f$, $u^\varepsilon_0=\frac{\varepsilon}{\sqrt\lambda}u_0$ then we obtain for v^ε the same results as for u^ε in the case $f^\varepsilon=(1-\chi^\varepsilon_3)f$.

4. Sketch of proof

The proof is achieved classically in three steps:

4.1. A priori estimates

We have
(1) $\delta < +\infty$, $f^\varepsilon = f$. Then $(\|\nabla_x u^\varepsilon\|_{L^2(Q_1^\varepsilon \cup Q_2^\varepsilon)}, \lambda\|\nabla_x u^\varepsilon\|_{L^2(Q_3^\varepsilon)}, \|u^\varepsilon\|_{L^2(Q)}) \leqslant C$.
(2) $\delta = +\infty$. Let $v^\varepsilon = \frac{\sqrt{\lambda}}{\varepsilon} u^\varepsilon$ and $z^\varepsilon = \frac{\sqrt{\lambda}}{\varepsilon} v^\varepsilon$. Four cases are distinguished:

- $f^\varepsilon = (1 - \chi_3^\varepsilon) f$, $0 \leqslant c_3(y)$. Then

$$\left(\|u^\varepsilon\|^2_{L^2(Q_1^\varepsilon \cup Q_2^\varepsilon)}, \|\nabla_x u^\varepsilon\|^2_{L^2(Q_1^\varepsilon \cup Q_2^\varepsilon)}, \lambda\|\nabla_x u^\varepsilon\|^2_{L^2(Q_3^\varepsilon)}\right) \leqslant C, \quad \|u^\varepsilon\|^2_{L^2(Q_3^\varepsilon)} \leqslant C + C\frac{\varepsilon^2}{\lambda}$$

$$\left(\|v^\varepsilon\|^2_{L^2(Q_1^\varepsilon \cup Q_2^\varepsilon)}, \|\nabla_x v^\varepsilon\|^2_{L^2(Q_1^\varepsilon \cup Q_2^\varepsilon)}, \lambda\|\nabla_x v^\varepsilon\|^2_{L^2(Q_3^\varepsilon)}\right) \leqslant C\frac{\lambda}{\varepsilon^2}$$

$$\|z^\varepsilon\|_{L^2(Q)} \leqslant C\frac{\sqrt{\lambda}}{\varepsilon}, \quad \left(\|v^\varepsilon\|_{L^2(Q_3^\varepsilon)}, \varepsilon\|\nabla v^\varepsilon\|_{L^2(Q_3^\varepsilon)}\right) \leqslant C$$

- $f^\varepsilon = (1 - \chi_3^\varepsilon) f$, $0 < c_0 \leqslant c_3(y)$. Then

$$\left(\|u^\varepsilon\|^2_{L^2(Q)}, \|\nabla_x u^\varepsilon\|^2_{L^2(Q_1^\varepsilon \cup Q_2^\varepsilon)}, \lambda\|\nabla_x u^\varepsilon\|^2_{L^2(Q_3^\varepsilon)}\right) \leqslant C, \quad \|v^\varepsilon\|_{L^2(Q)} \leqslant C\frac{\sqrt{\lambda}}{\varepsilon}$$

- $f^\varepsilon = f$, $0 \leqslant c_3(y)$. Then

$$\left(\|v^\varepsilon\|^2_{L^2(Q_1^\varepsilon \cup Q_2^\varepsilon)}, \|\nabla_x v^\varepsilon\|^2_{L^2(Q_1^\varepsilon \cup Q_2^\varepsilon)}, \lambda\|\nabla_x v^\varepsilon\|^2_{L^2(Q_3^\varepsilon)}\right) \leqslant C$$

$$\left(\|z^\varepsilon\|_{L^2(Q_3^\varepsilon)}, \varepsilon\|\nabla z^\varepsilon\|_{L^2(Q_3^\varepsilon)}\right) \leqslant C, \quad \|v^\varepsilon\|^2_{L^2(Q_3^\varepsilon)} \leqslant C + C\frac{\varepsilon^2}{\lambda}$$

$$\left(\|z^\varepsilon\|^2_{L^2(Q_1^\varepsilon \cup Q_2^\varepsilon)}, \|\nabla_x z^\varepsilon\|^2_{L^2(Q_1^\varepsilon \cup Q_2^\varepsilon)}, \lambda\|\nabla_x z^\varepsilon\|^2_{L^2(Q_3^\varepsilon)}\right) \leqslant C\frac{\lambda}{\varepsilon^2}$$

- $f^\varepsilon = f$, $0 < c_0 \leqslant c_3(y)$. Then

$$\left(\|v^\varepsilon\|^2_{L^2(Q)}, \|\nabla_x v^\varepsilon\|^2_{L^2(Q_1^\varepsilon \cup Q_2^\varepsilon)}, \lambda\|\nabla_x v^\varepsilon\|^2_{L^2(Q_3^\varepsilon)}\right) \leqslant C, \quad \|z^\varepsilon\|_{L^2(Q)} \leqslant C\frac{\sqrt{\lambda}}{\varepsilon}$$

4.2. Convergence in the Laplace space

Our analysis is based on the Laplace transform of problem P_ε. This is quite suitable for our purpose since we benefit from the results of the elliptic problem addressed in [1]. As in that work, our technique is the two-scale convergence.

4.3. Results in the physical space

The return to the physical space is done using the inverse Laplace transform.

5. Complements

(1) In the generic nondegenerate case where we suppose that $c_0 < c_i(y) < C_i \leqslant +\infty$ a.e., there are substantial simplifications. The problem P_ε becomes a classical parabolic problem and we do not have to use the complicated framework of Showalter. The functions u^ε are now in the space $\{u \in L^2(0,T; H_0^1(\Omega)),\ u' \in L^2(0,T; H^{-1}(\Omega))\}$ thus $u^\varepsilon \in C([0,T]; L^2(\Omega))$ and the initial condition reduce to $u^\varepsilon(0) = u_0^\varepsilon$ in $L^2(\Omega)$. The a priori estimates are easier to obtain. In particular, we do not need to use the results of [1], and we have the sharper estimate

$\|u^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C$ under the sole assumption $f^\varepsilon = f \in L^2(Q)$, u_0^ε bounded in $L^2(\Omega)$. We have less different situations. The proofs of convergence remains however globally the same. Proposition 3.1 remains true, but the convergences are now in $L^\infty(0,T;L^2(\Omega))$ weak-star and the case $\delta = +\infty$ becomes:

Proposition 5.1. *If $\delta = +\infty$, there exist $u_\alpha(t,x)$ in $\mathcal{D}'(\mathbb{R}; H_0^1(\Omega)) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}; L^2(\Omega))$ s.t.:*

(i) *$\chi_\alpha^\varepsilon u_\alpha^\varepsilon$ converges weak-star in $L^\infty(0,T;L^2(\Omega))$ to $\theta_\alpha u_\alpha(t,x)$, solutions of the two decoupled problems*:

$$\tilde{c}_\alpha u_\alpha'(t,x) - \mathrm{div}_x\big[A_\alpha^{\mathrm{hom}} \nabla_x u_\alpha(t,x)\big] = \theta_\alpha f(t,x), \quad u_\alpha(0,x) = u_0(x) \quad \text{in } \Omega, \quad u_\alpha = 0 \quad \text{on } \partial\Omega$$

(ii) *$\chi_3^\varepsilon(x) u^\varepsilon$ converges weak-star in $L^\infty(0,T;L^2(\Omega))$ to the function $\theta_3 u_0(x)$.*

(2) In the "asymptotically degenerate case", where $c_3(y) = \lambda c_3$, c_3 a strictly positive constant and λ goes to zero following some rate $\lambda(\varepsilon)$, our study applies too. *The results obtained are exactly the same but putting formally* $c_3(y) = 0$ a.e. In particular, we recover once more the Barenblatt–Rubinstein's model.

Remark. The mathematical problem adressed here can be used to model a lot of problems arising in transport phenomena, for example the problem of the fluid flow in a totally fissured porous medium. Then c_i and a_i are respectively the porosity and the permeability of the i-th component. The medium consists of two flow regions separated by a third one which represents the porous matrix and u^ε is the density of fluid.

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