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# Analysis of viscoelastic contact with normal compliance, friction and wear diffusion

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**Abstract**

We consider a quasistatic problem of frictional contact between a viscoelastic body and a moving foundation. The contact is with wear and is modeled by normal compliance and a law of dry friction. The novelty in the model is that it allows for the diffusion of the wear debris over the potential contact surface. Such kind of phenomena arise in orthopaedic biomechanics and influence the properties of joint prosthesis. We derive a weak formulation of the problem and state that, under a smallness assumption on the problem data, there exists a unique weak solution for the model. ***To cite this article: M. Shillor et al., C. R. Mecanique 331 (2003).***


**Résumé**

**Analyse d'un problème viscoélastique avec compliance normale, frottement et diffusion d'usure.** Nous considérons un problème quasistatique de contact entre un corps viscoélastique et une fondation mobile. Le contact est avec usure et se modélise avec compliance normale et une loi de frottement sec. La principale nouveauté du modèle réside dans le fait qu'il prend en considération la diffusion des débris d'usure sur la surface potentielle de contact. Ce phénomème est rencontré en biomécanique orthopédique et il influence les propriétés des prothèses. Nous présentons une formulation variationnelle du problème et, sous une hypothèse de petitesse, nous présentons l'existence et l'unicité de la solution faible du modèle. ***Pour citer cet article : M. Shillor et al., C. R. Mecanique 331 (2003).***
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## Version française abrégée

Soit un corps viscoélastique occupant un ouvert $\Omega \subset \mathbb{R}^3$, de frontière $\Gamma$, suffisamment régulière, divisée en trois parties disjointes et mesurables $\Gamma_D$, $\Gamma_N$ et $\Gamma_C$, telle que $\mathrm{mes}(\Gamma_D) > 0$. Soit $\boldsymbol{\nu}$ le vecteur unitaire de la normale sortante à $\Gamma$ et soit $[0, T]$ un intervalle de temps, $T > 0$. Le corps est supposé fixé sur la partie $\Gamma_D$ de la frontière alors que des forces volumiques et surfaciques de densités $\boldsymbol{f}_0$ et $\boldsymbol{f}_2$ agissent respectivement dans $\Omega$ et sur $\Gamma_N$. Sur la partie $\Gamma_C$ le corps est susceptible d'entrer en contact avec une fondation mobile. Afin de simplifier le modèle nous supposons que $\Gamma_C$ est un domaine régulier dans le plan $x_3 = 0$, de frontière $\gamma$, alors que la fondation est plane et se déplace dans le plan $x_3 = -g \leqslant 0$ avec une vitesse $\boldsymbol{v}^*$. Le contact est avec compliance normale et frottement et a pour conséquence l'usure d'une partie $D_w$ de $\Gamma_C$, accompagné de la diffusion des débris matériels dans tout le domaine $\Gamma_C$. Nous supposons que le processus est quasistatique et le comportement du matériau est viscoélastique du type Kelvin–Voigt.

Sous ces hypothèses, le problème mécanique considéré peut être formulé de la façon suivante :

**Problème *P*.** Trouver le champ des déplacements $\boldsymbol{u} : \Omega \times [0, T] \to \mathbb{R}^3$, le champ contraintes $\boldsymbol{\sigma} : \Omega \times [0, T] \to \mathbb{S}^3$ et la densité des débris d'usure $\zeta : \Gamma_C \times [0, T] \to \mathbb{R}$, tels que

$$\boldsymbol{\sigma} = \mathcal{A}\big(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})\big) + \mathcal{G}\big(\boldsymbol{\varepsilon}(\boldsymbol{u})\big) \quad \text{dans } \Omega \times (0, T) \tag{1}$$

$$\mathrm{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{dans } \Omega \times (0, T) \tag{2}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{sur } \Gamma_D \times (0, T) \tag{3}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_N \quad \text{sur } \Gamma_N \times (0, T) \tag{4}$$

$$-\sigma_\nu = p_\nu(u_\nu - \eta\zeta\,\chi_{[D_w]} - g) \quad \text{sur } \Gamma_C \times (0, T) \tag{5}$$

$$\|\boldsymbol{\sigma}_\tau\| \leqslant \mu|\sigma_\nu|; \quad \boldsymbol{\sigma}_\tau = -\mu|\sigma_\nu|\frac{\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*}{\|\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*\|} \quad \text{if } \dot{\boldsymbol{u}}_\tau \neq \boldsymbol{v}^* \quad \text{sur } \Gamma_C \times (0, T) \tag{6}$$

$$\dot{\zeta} - \mathrm{div}(k\nabla\zeta) = k\beta\mu p_\nu R^*\big(\|\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*\|\big)\chi_{[D_w]} \quad \text{sur } \Gamma_C \times (0, T) \tag{7}$$

$$\zeta = 0 \quad \text{sur } \gamma \times (0, T) \tag{8}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \zeta(0) = \zeta_0 \quad \text{dans } \Omega \tag{9}$$

Ici et partout dans ce travail le point au dessus représente la dérivée par rapport au temps, $\|\boldsymbol{v}\|$ est la norme du vecteur $\boldsymbol{v}$ et les indices $\nu$ et $\tau$ indiquent les composantes normales et tangentielles des tenseurs et des vecteurs. L'Éq. (1) est la loi de comportement où $\boldsymbol{\varepsilon}(\boldsymbol{u})$ dénote le tenseur des déformations linéarisé, (2) représente l'équation d'équilibre, (3) et (4) sont respectivement les conditions aux limites de déplacement-traction et (5) et (6) représentent respectivement les conditions de contact avec compliance normale et frottement. Précisons que le coefficient de frottement dépend de la solution puisqu'on suppose $\mu = \mu(\zeta, \|\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*\|)$. L'Éq. (7) représente l'équation de diffusion des débris d'usure, accompagnée de la condition aux limites (8), alors que (9) sont les conditions initiales.

L'étude du problème $P$ est motivée par des possibles applications en orthopédie et notamment dans l'étude du comportement mécanique des prothèses ; en effet, l'usure des surfaces de contact diminue la durée de vie d'une prothèse, voir par exemple [1,2].

Dans ce travail nous décrivons en détails le modèle mécanique ci-dessous, nous précisons des hypothèses sur les données puis, en utilisant la formule de Green, nous obtenons une formulation variationnelle du problème. Le résultat central est présenté dans le Théorème 3.1 ; il montre que, sous une hypothèse de petitesse, le problème mécanique $P$ admet une solution faible, unique. La démonstration est basée sur des arguments d'équations paraboliques, d'inéquations variationnelles elliptiques et de point fixe.

## 1. The model

We model the process in which a viscoelastic body, that is acted upon by volume forces and surface tractions, is in frictional contact with a moving foundation and, as a result, a part of its surface wears out. The wear particles are assumed to remain and diffuse on the potential contact surface. Such situations arise, among others, in orthopedic biomechanics of joint prostheses after arthoplasty. Since friction and wear debris influence the quality and long term performance of artificial joints and implants, they need to be taken into account when modelling these processes, see, e.g., [1,2] and the references therein.

Let $\Omega \subset \mathbb{R}^3$ denote the domain occupied by the body and $\Gamma$ the boundary of $\Omega$, which is assumed to be Lipschitz, and is divided into three disjoint measurable parts $\Gamma_D$, $\Gamma_N$ and $\Gamma_C$, such that $\text{meas}(\Gamma_D) > 0$. The body is clamped on $\Gamma_D$, prescribed surface tractions $\boldsymbol{f}_N$ act on $\Gamma_N$ and volume forces $\boldsymbol{f}_0$ act on $\Omega$. A gap $g$ exists between the contact surface $\Gamma_C$ and the foundation, and is measured along the outward normal $\boldsymbol{\nu}$. We assume that the coordinate system is such that $\Gamma_C$ occupies a regular domain in the $x_3 = 0$ plane, the foundation is planar and is moving with velocity $\boldsymbol{v}^*$ in the plane $x_3 = -g \leqslant 0$. Notice that our results below still hold in the case of an arbitrary foundation provided that the gap function is positive and belongs to $L^2(\Gamma_C)$. We assume that $\Gamma_C$ is divided into two subdomains $D_d$ and $D_w$ by a smooth curve $\gamma^*$, and wear takes place only on the part $D_w$, while the diffusion of the wear particles takes place in the whole of $\Gamma_C$. The boundary $\gamma = \partial\Gamma_C$ of $\Gamma_C$ is assumed Lipschitz and is composed of two parts $\gamma_d$ and $\gamma_w$. Thus, $\partial D_w = \gamma_w \cup \gamma^*$ and $\partial D_d = \gamma_d \cup \gamma^*$. The setting is depicted in Fig. 1.

In what follows $\mathbb{S}^3$ will represent the space of second order symmetric tensors on $\mathbb{R}^3$ while "$\cdot$" and $\|\cdot\|$ will denote the inner product and the Euclidean norm on $\mathbb{R}^3$ and $\mathbb{S}^3$, respectively. We denote by $\boldsymbol{u}$ the displacement vector, $\boldsymbol{\sigma}$ the stress field and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ the linearized strain tensor. Below $u_\nu$ and $\boldsymbol{u}_\tau$ represent the *normal* and *tangential* displacements, respectively, while $\sigma_\nu$ and $\boldsymbol{\sigma}_\tau$ represent the *normal* and *tangential* stresses, respectively; $[0, T]$ denotes the time interval of interest, $T > 0$, and a dot above a variable represents the time derivative.

The wear of the surface is described by the *wear function* $w = w(\boldsymbol{x}, t)$ which is defined on $D_w$ and the diffusion of the wear debris by the *wear particle surface density function* $\zeta = \zeta(\boldsymbol{x}, t)$ which is defined on $\Gamma_C$. Here $\boldsymbol{x} = (x_1, x_2, 0)$, since $\Gamma_C$ belongs to the plane $Ox_1x_2$. The wear function $w$ measures the volume density of material removed per unit surface area, see, e.g., [3] and references therein. We assume that $\zeta = \beta w$ in $D_w$, where $\beta$ is a conversion factor from wear depth (mm) to wear particles surface density, which is assumed to be a positive constant. This assumption simplifies the model, since $w = \eta\zeta$ in $D_w$, for $\eta = 1/\beta$. Next, we extend $w$ by zero to the whole of $\Gamma_C$, and thus, $w = \eta\zeta\,\chi_{[D_w]}$ on $\Gamma_C \times (0, T)$, where $\chi_{[D_w]}$ is the characteristic function of $D_w$.

The diffusion of the particles is described by the nonlinear evolutionary equation,

$$\dot{\zeta} - \operatorname{div}(k\,\nabla\zeta) = \kappa\beta\|\boldsymbol{\sigma}_\tau\|R^*\big(\|\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*\|\big)\chi_{[D_w]} \quad \text{in } \Gamma_C \times (0, T) \tag{10}$$

Here $k$ denotes the wear particle diffusion coefficient, $\kappa$ is the wear rate constant, and $R^*: \mathbb{R}_+ \to \mathbb{R}_+$ is the truncation operator: $R^*(r) = r$ if $r \leqslant R$, $R^*(r) = R$ if $r > R$, $R$ being a given positive constant. We need this operator in order to avoid some mathematical difficulties, however, from the physical point of view the use of $R^*$ is not restrictive since, in practice, the slip velocity is bounded and no smallness assumption will be made on $R$. We use $\chi_{[D_w]}$ on the right-hand side of (10) since the particles are produced only in $D_w$, and the rate of production is multiplied by $\beta$. Notice that below the wear particle diffusion coefficient $k$ is assumed to be given, however, a more

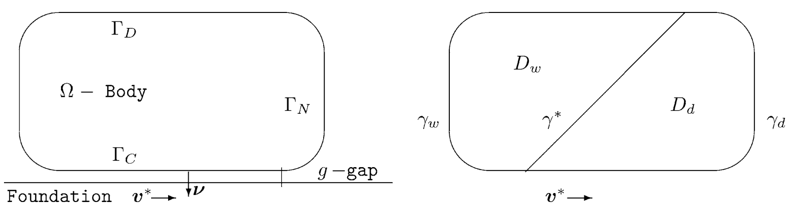


Fig. 1. The setting (left); contact surface $\Gamma_C$; wear is produced in $D_w$ (right).

Fig. 1. Le modèle (à gauche) – surface de contact $\Gamma_C$ ; l'usure est produite en $D_w$ (à droite).

complicated version of the model may be obtained by taking $k = k(\zeta)$. We note that without diffusion, condition (10) reduces to that of [3] for small to medium contact stresses.

Next, we describe the process of frictional contact on the surface $\Gamma_C$. We use a version of the *normal compliance* condition to model the contact (see, e.g., [4–6] and references therein). Since the process involves the wear of the contacting surfaces we need to take into account the change in the geometry by replacing the gap $g$ with $g + w$. Therefore,

$$-\sigma_\nu = p_\nu(u_\nu - \eta\zeta\,\chi_{[D_w]} - g) \quad \text{on } \Gamma_C \times (0,T) \tag{11}$$

The precise assumptions on $p_\nu$ will be given below. The friction law is

$$\|\boldsymbol{\sigma}_\tau\| \leqslant \mu|\sigma_\nu|; \quad \text{if } \dot{\boldsymbol{u}}_\tau \neq \boldsymbol{v}^* \text{ then } \quad \boldsymbol{\sigma}_\tau = -\mu|\sigma_\nu|\frac{\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*}{\|\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*\|} \quad \text{on } \Gamma_C \times (0,T) \tag{12}$$

Here, $\mu = \mu(\zeta, \|\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*\|)$ is the coefficient of friction which is assumed to depend on the density of the wear particles and on the slip rate.

To conclude, keeping in mind (10), (11) and (12), the classical formulation of the problem of *frictional contact of a viscoelastic body with wear diffusion* is as follows.

**Problem *P*.** Find a displacement field $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^3$, a stress field $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^3$ and a surface particle density field $\zeta : \Gamma_C \times [0,T] \to \mathbb{R}$, such that

$$\boldsymbol{\sigma} = \mathcal{A}\big(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})\big) + \mathcal{G}\big(\boldsymbol{\varepsilon}(\boldsymbol{u})\big) \quad \text{in } \Omega \times (0,T) \tag{13}$$

$$\operatorname{Div}\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in } \Omega \times (0,T) \tag{14}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_D \times (0,T) \tag{15}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_N \quad \text{on } \Gamma_N \times (0,T) \tag{16}$$

$$-\sigma_\nu = p_\nu(u_\nu - \eta\zeta\,\chi_{[D_w]} - g) \quad \text{on } \Gamma_C \times (0,T) \tag{17}$$

$$\|\boldsymbol{\sigma}_\tau\| \leqslant \mu|\sigma_\nu|; \quad \boldsymbol{\sigma}_\tau = -\mu|\sigma_\nu|\frac{\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*}{\|\dot{u}_\tau - \boldsymbol{v}^*\|} \quad \text{if } \dot{\boldsymbol{u}}_\tau \neq \boldsymbol{v}^* \quad \text{on } \Gamma_C \times (0,T) \tag{18}$$

$$\dot{\zeta} - \operatorname{div}(k\nabla\zeta) = k\beta\mu p_\nu R^*\big(\|\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*\|\big)\chi_{[D_w]} \quad \text{on } \Gamma_C \times (0,T) \tag{19}$$

$$\zeta = 0 \quad \text{on } \gamma \times (0,T) \tag{20}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \zeta(0) = \zeta_0 \quad \text{in } \Omega \tag{21}$$

Here, (13) represents the viscoelastic constitutive law of the material in which $\mathcal{A}$ and $\mathcal{G}$ are given nonlinear constitutive functions which may depend on $\boldsymbol{x} \in \Omega$ (the body may be nonhomogenuous); (14) represents the equation of equilibrium, since the process is assumed quasistatic; (15) and (16) are the displacement and traction boundary conditions, and (20) is an absorbing boundary condition, since once a wear particle reaches the boundary $\gamma = \partial\Gamma_C$ it dissapears; finally, (21) represent the initial conditions in which $\boldsymbol{u}_0$ and $\zeta_0$ are given.

## 2. Variational formulation

To obtain a variational formulation for problem $P$ we use the standard notation for $L^p$, $C$, $C^1$ and Sobolev spaces associated with the domains $\Omega \subset \mathbb{R}^3$ and $\Gamma_C \subset \mathbb{R}^2$. Moreover, for the stress field we use the space $Q = \{\boldsymbol{\tau} = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega),\ 1 \leqslant i, j \leqslant 3\} = L^2(\Omega)^{3\times3}_s$, endowed with the canonical inner product $(\cdot,\cdot)_Q$, and for the displacement field we use the space $V = \{\boldsymbol{v} \in H^1(\Omega)^3 \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D\}$. It follows from Korn's inequality that $V$ is a real Hilbert space endowed with the inner product $(\boldsymbol{u},\boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q$. For an element $\boldsymbol{v} \in V$ we denote by $\boldsymbol{v}$ its trace on $\Gamma$ and by $v_\nu = \boldsymbol{v}\cdot\boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu\boldsymbol{\nu}$ its *normal* and *tangential* components on the boundary.

For the surface particle density function we use the space $H_0^1(\Gamma_C)$. We denote by $H^{-1}(\Gamma_C)$ the dual of $H_0^1(\Gamma_C)$, and $\langle\cdot,\cdot\rangle$ represents the duality pairing between $H^{-1}(\Gamma_C)$ and $H_0^1(\Gamma_C)$.

To study the mechanical problem $P$ we make the following assumptions on the data.

The *viscosity operator* $\mathcal{A}:\Omega\times\mathbb{S}^3\to\mathbb{S}^3$ satisfies: there exist two constants $L_{\mathcal{A}}>0$ and $m_{\mathcal{A}}>0$ such that

$$\begin{aligned}
&\text{(a)}\quad \big\|\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\big\|\leqslant L_{\mathcal{A}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|,\quad \forall\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^3,\ \text{a.e. } \boldsymbol{x}\in\Omega\\
&\text{(b)}\quad \big(\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\big)\cdot(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)\geqslant m_{\mathcal{A}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|^2,\quad \forall\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^3,\ \text{a.e. } \boldsymbol{x}\in\Omega\\
&\text{(c)}\quad \boldsymbol{x}\mapsto\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon})\ \text{is Lebesgue measurable on } \Omega,\quad \forall\boldsymbol{\varepsilon}\in\mathbb{S}^3\\
&\text{(d)}\quad \boldsymbol{x}\mapsto\mathcal{A}(\boldsymbol{x},\boldsymbol{0})\in Q
\end{aligned}\tag{22}$$

The *elasticity operator* $\mathcal{G}:\Omega\times\mathbb{S}^3\to\mathbb{S}^3$ satisfies: there exists a constant $L_{\mathcal{G}}>0$ such that

$$\begin{aligned}
&\text{(a)}\quad \big\|\mathcal{G}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{G}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\big\|\leqslant L_{\mathcal{G}}\ \|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|,\quad \forall\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^3,\ \text{a.e. } \boldsymbol{x}\in\Omega\\
&\text{(b)}\quad \boldsymbol{x}\mapsto\mathcal{G}(\boldsymbol{x},\boldsymbol{\varepsilon})\ \text{is Lebesgue measurable on } \Omega,\ \forall\boldsymbol{\varepsilon}\in\mathbb{S}^3\\
&\text{(c)}\quad \boldsymbol{x}\mapsto\mathcal{G}(\boldsymbol{x},\boldsymbol{0})\in Q
\end{aligned}\tag{23}$$

The *normal compliance function* $p_\nu:\Gamma_C\times\mathbb{R}\to\mathbb{R}_+$ satisfies: there exist two constants $L_\nu>0$ and $p_\nu^*>0$ such that

$$\begin{aligned}
&\text{(a)}\quad \big|p_\nu(\boldsymbol{x},u_1)-p_\nu(\boldsymbol{x},u_2)\big|\leqslant L_\nu|u_1-u_2|,\quad \forall u_1,u_2\in\mathbb{R},\ \text{a.e. } \boldsymbol{x}\in\Gamma_C\\
&\text{(b)}\quad \boldsymbol{x}\mapsto p_\nu(\boldsymbol{x},u)\ \text{is Lebesgue measurable on } \Gamma_C,\ \forall u\in\mathbb{R}\\
&\text{(c)}\quad \boldsymbol{x}\mapsto p_\nu(\boldsymbol{x},u)=0\quad \text{for } u\leqslant 0,\ \text{a.e. } \boldsymbol{x}\in\Gamma_C\\
&\text{(d)}\quad p_\nu^*(\boldsymbol{x},u)\leqslant p_\nu^*\quad \forall u\in\mathbb{R},\ \text{a.e. } \boldsymbol{x}\in\Gamma_C
\end{aligned}\tag{24}$$

The *coefficient of friction* $\mu:\Gamma_C\times\mathbb{R}\to\mathbb{R}_+$ satisfies: there exist two constants $L_\mu>0$ and $\mu^*>0$ such that

$$\begin{aligned}
&\text{(a)}\quad \big|\mu(\boldsymbol{x},a_1,b_1)-\mu(\boldsymbol{x},a_2,b_2)\big|\leqslant L_\mu\big(|a_1-a_2|+|b_1-b_2|\big),\quad \forall a_1,a_2,b_1,b_2\in\mathbb{R},\ \text{a.e. } \boldsymbol{x}\in\Gamma_C\\
&\text{(b)}\quad \boldsymbol{x}\mapsto\mu(\boldsymbol{x},a,b)\ \text{is Lebesgue measurable on } \Gamma_C,\quad \forall a,b\in\mathbb{R}\\
&\text{(c)}\quad \mu(\boldsymbol{x},a,b)\leqslant\mu^*,\quad \forall a,b\in\mathbb{R},\ \text{a.e. } \boldsymbol{x}\in\Gamma_C
\end{aligned}\tag{25}$$

The *forces, tractions, particle diffusion coefficient, wear rate constant* and the *initial data* satisfy:

$$\begin{aligned}
&\text{(a)}\quad \boldsymbol{f}_0\in C\big([0,T];L^2(\Omega)^3\big),\quad \boldsymbol{f}_N\in C\big([0,T];L^2(\Gamma_N)^d\big)\\
&\text{(b)}\quad k\in L^\infty(\Gamma_C),\quad k\geqslant k^*>0\ \text{a.e. on } \Gamma_C\\
&\text{(c)}\quad \kappa\in L^\infty(\Gamma_{D_w}),\quad \kappa\geqslant 0\ \text{a.e. on } \Gamma_{D_w}\\
&\text{(d)}\quad \boldsymbol{u}_0\in V,\quad \zeta_0\in L^2(\Gamma_C)
\end{aligned}\tag{26}$$

Next, we define the function $\boldsymbol{f}:[0,T]\to V$, the functional $j:L^2(\Gamma_C)\times V^3\to\mathbb{R}$, the bilinear form $a:H_0^1(\Gamma_C)\times H_0^1(\Gamma_C)\to\mathbb{R}$ and the operator $F:H_0^1(\Gamma_C)\times V^3\to H^{-1}(\Gamma_C)$ by

$$\begin{aligned}
&\big(\boldsymbol{f}(t),\boldsymbol{v}\big)_V=\int_\Omega\boldsymbol{f}_0(t)\cdot\boldsymbol{v}\,\mathrm{d}x+\int_{\Gamma_N}\boldsymbol{f}_N(t)\cdot\boldsymbol{v}\,\mathrm{d}S\\
&j(\zeta,\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})=\int_{\Gamma_C}p_\nu(u_\nu-\eta\zeta\chi_{[D_w]}-g)w_\nu\,\mathrm{d}S+\int_{\Gamma_C}\mu\big(\zeta,\|\boldsymbol{v}_\tau-\boldsymbol{v}^*\|\big)p_\nu(u_\nu-\eta\zeta\chi_{[D_w]}-g)\|\boldsymbol{w}_\tau-\boldsymbol{v}^*\|\,\mathrm{d}S\\
&a(\zeta,\xi)=\int_{\Gamma_C}k\,\nabla\zeta\cdot\nabla\xi\,\mathrm{d}S\\
&\big\langle F(\zeta,\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}),\xi\big\rangle=\int_{D_w}\beta\kappa\mu\big(\zeta,\|\boldsymbol{v}_\tau-\boldsymbol{v}^*\|\big)p_\nu(u_\nu-\eta\zeta-g)R^*\big(\|\boldsymbol{w}_\tau-\boldsymbol{v}^*\|\big)\xi\,\mathrm{d}S
\end{aligned}$$

for all $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in V$, $\zeta,\xi\in H_0^1(\Gamma_C)$ and $t\in[0,T]$.

Using Green's formula leads to the following variational formulation of problem $P$.

**Problem $P_V$.** Find a displacement field $\boldsymbol{u}:[0,T]\to V$ and a surface particle density field $\zeta:[0,T]\to H_0^1(\Gamma_C)$ such that

$$\begin{aligned}&\big(\mathcal{A}\big(\boldsymbol{\varepsilon}\big(\dot{\boldsymbol{u}}(t)\big)\big), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}\big(\dot{\boldsymbol{u}}(t)\big)\big)_Q + \big(\mathcal{G}\big(\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big)\big), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}\big(\dot{\boldsymbol{u}}(t)\big)\big)_Q \\ &\quad + j\big(\zeta(t), \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t), \boldsymbol{v}\big) - j\big(\zeta(t), \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t), \dot{\boldsymbol{u}}(t)\big) \geqslant \big(\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t)\big)_V \quad \forall \boldsymbol{v} \in V,\ t \in [0, T] \end{aligned} \tag{27}$$

$$\big\langle \dot{\zeta}(t), \xi \big\rangle + a\big(\zeta(t), \xi\big) = \big\langle F\big(\zeta(t), \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t), \dot{\boldsymbol{u}}(t)\big), \xi \big\rangle \quad \forall \xi \in H_0^1(\Gamma_C),\ \text{a.e. } t \in (0, T) \tag{28}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \qquad \zeta(0) = \zeta_0 \tag{29}$$

## 3. An existence and uniqueness result

Our main result, concerning the well-posedness of problem $P_V$, is the following.

**Theorem 3.1.** *Assume that* (22)–(26) *hold. Then, there exists a constant* $c_0 > 0$ *which depends on* $\Omega$, $\Gamma_D$, $\Gamma_C$, $m_{\mathcal{A}}$, $L_\nu$, $L_\mu$, $\|\kappa\|_{L^\infty(D_w)}$, $\beta$ *and* $R$ *such that, if* $p_\nu^* < c_0$ *and* $\mu^* < c_0$*, there exists a unique solution of problem* $P_V$*. Moreover, the solution satisfies*

$$\boldsymbol{u} \in C^1\big([0, T]; V\big), \quad \zeta \in L^2\big(0, T; H_0^1(\Gamma_C)\big) \cap C\big([0, T]; L^2(\Gamma_C)\big), \quad \dot{\zeta} \in L^2\big(0, T; H^{-1}(\Gamma_C)\big) \tag{30}$$

**Proof.** The proof of the theorem was based on arguments for evolutionary equations, time-dependent elliptic variational inequalities and fixed points. Full details can be found in [7].

Let now $\{\boldsymbol{u}, \zeta\}$ denote a solution of problem $P_V$ and let $\boldsymbol{\sigma}$ be the stress field given by (1). Using (22) and (23) it follows that $\boldsymbol{\sigma} \in C([0, T]; Q)$.

A triplet of functions $\{\boldsymbol{u}, \boldsymbol{\sigma}, \zeta\}$ which satisfies (13), (27)–(29) is called a *weak solution* of the mechanical problem $P$. We conclude from the theorem that, if the normal compliance function $p_\nu$ and the coefficient of friction $\mu$ are sufficiently small, then problem $P$ has a unique weak solution.

The following features make the variational problem $P_V$ a difficult mathematical problem: (a) The strong nonlinearities in the functional $j$ and the operator $F$. (b) The dependence of the nondifferentiable functional $j$ on the solution $\{\boldsymbol{u}, \zeta\}$ and on $\dot{\boldsymbol{u}}$. (c) The dependence of $F$ on $\{\boldsymbol{u}, \zeta\}$ and on $\dot{\boldsymbol{u}}$. (d) The strong coupling between (27) and (28). These reasons lead to an interesting but difficult mathematical model. The case when there is no diffusion of the wear particles was investigated in [8] where the existence of a unique weak solution has been proved without any smallness assumptions.

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