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A model for frictional slip in woven fabrics

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Abstract

A continuum model for frictional slip of the yarns of a plain-weave fabric is presented. The model is based on the assumption that the weave is composed of two families of continuously distributed yarns constrained at all times to occupy a common evolving surface in three-dimensional space. The two families may slide relative to one another on the surface, subject to their respective equations of motion, fiber constitutive equations, and frictional slip rules. The theory is intended for the quantitative analysis of deformation, slip and energy dissipation during a ballistic impact event. ***To cite this article: B. Nadler, D.J. Steigmann, C. R. Mecanique 331 (2003).***

Résumé

Modèle de glissement avec frottement pour textiles ondulés. Un modèle continu de frottement avec glissement de fibres pour armure textile est présenté. Le modéle repose sur l'hypothése que l'armure est composée de deux familles de fibres contraintes d'occuper la même surface évoluant dans un espace à trois dimensions. Les deux familles peuvent glisser relativement l'une par rapport à l'autre sur cette surface, tout en étant soumises à leur équation de mouvement respective, aux équations constitutives des fibres et à leur loi de glissement. La théorie a pour but de présenter l'analyse quantitative de la déformation, du glissement et de l'énergie dissipée lors d'un impact balistique. ***Pour citer cet article : B. Nadler, D.J. Steigmann, C. R. Mecanique 331 (2003).***

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1. Introduction

Slipping of the yarns of a fabric is usually regarded as undesirable in conventional textile applications. In fabric shielding, however, slipping is unavoidable and plays a significant role in the dynamics of a ballistic impact event [1]. In the present work we incorporate the effects of slipping with friction in a theory for the dynamic response of fabric. We consider a macroscopic formulation suitable for structural dynamics analysis in which the

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small-scale interplay between yarns of the weave is not accounted for explicitly. Instead, such effects are accounted for indirectly though constitutive equations which can be extracted from available empirical data [2]. Kuznetsov [3] considered certain restricted modes of frictionless yarn sliding for equilibrium problems, but the theory proposed here is apparently the first to incorporate the mechanics of general frictional slipping. The model is intended for the prediction of the response of a single thin sheet of woven fabric. Accordingly, it is based on a membrane-type assumption in which flexural resistance is neglected.

Numerical implementation will be discussed elsewhere.

2. Dynamics of a surface regarded as a single family of continuously distributed fibers

The woven fabric is supposed to consist of two families of continuously distributed yarns that are orthogonal in a specified reference plane κ. In the present section we discuss the basic equations for a single family. The behavior of two interacting families is described in Section 3.

The motion is described by the map

$$\mathbf{x} = \boldsymbol{\chi}_\kappa(\mathbf{X}, t) \tag{1}$$

where t is the time, $\mathbf{X}$ is the position of a material point on the plane κ, $\mathbf{x}$ is the position of the same point on a surface ω at time t, and $\boldsymbol{\chi}_\kappa$ is the map from κ to ω. Let $\{\mathbf{L}, \mathbf{M}, \mathbf{k}\}$ be a fixed positively-oriented orthonormal basis with $\kappa' = \mathrm{Span}\{\mathbf{L}, \mathbf{M}\}$, where κ' is the translation space of κ. Then the gradient of $\boldsymbol{\chi}_\kappa$ with respect to $\mathbf{X}$ is [4]

$$\mathbf{F} = \lambda \mathbf{l} \otimes \mathbf{L} + \mu \mathbf{m} \otimes \mathbf{M} \tag{2}$$

where $\mathbf{l}, \mathbf{m}$ are unit vectors tangent to ω at $\mathbf{x}$ and

$$\lambda = |\mathbf{FL}|, \qquad \mu = |\mathbf{FM}| \tag{3}$$

are the stretches of material curves aligned with $\mathbf{L}$ and $\mathbf{M}$, respectively, on κ. The deformation gradient maps elements of κ' to elements of $T_{\omega(\mathbf{X},t)}$, the tangent space to ω at the material point $\mathbf{X}$. The orientation of the surface is given by the unit-normal field

$$\mathbf{n}(\mathbf{X}, t) = \mathbf{FL} \times \mathbf{FM}/|\mathbf{FL} \times \mathbf{FM}| \tag{4}$$

where, on the right-hand side, the vectors $\mathbf{L}$ and $\mathbf{M}$ may be replaced by an arbitrary pair, $\mathbf{i}$ and $\mathbf{j}$, say, such that $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is a positive orthonormal basis.

We assume $\mathbf{L}$ to be the direction field of a yarn. The plane κ is formed by the set of distinct lines parallel to $\mathbf{L}$, each of which is regarded as a yarn. Yarn stretch is given by the function $\lambda(\mathbf{X}, t)$ and the orientation after deformation of the yarn passing through $\mathbf{X}$ in κ is given by $\mathbf{l}(\mathbf{X}, t)$. The functions μ and $\mathbf{m}$ may be used to determine the extent of bunching or spreading of yarns in the course of deformation.

Let $\mathbf{X}(S)$ be the arclength parametrization of a material curve γ on κ. The traction – or force per unit length – on the material lying to the left of γ transmitted by yarns intersecting γ is

$$\mathbf{p} = \mathbf{P}\boldsymbol{\nu} \tag{5}$$

where $\boldsymbol{\nu} = \mathbf{k} \times \mathbf{X}'(S)$ is the rightward unit normal to γ when traversed in the sense of increasing S, and

$$\mathbf{P} = f\,\mathbf{l} \otimes \mathbf{L} \tag{6}$$

is the Piola stress. Thus,

$$\mathbf{p} = f\,\mathbf{l}(\mathbf{L} \cdot \boldsymbol{\nu}) \tag{7}$$

The force transmitted along the yarns per unit transverse length is the traction across their orthogonal trajectories. For these we have $\boldsymbol{\nu} = \mathbf{L}$ and

$$\mathbf{p} = f\,\mathbf{l} \tag{8}$$

Thus, the scalar field $f(\mathbf{X}, t)$ represents the force transmitted by the yarns per unit reference length. The actual yarn force is f/n, where n is the number of yarns per unit length.

If γ is a piecewise-smooth closed curve bounding a simply-connected part of κ, then the integral of $\mathbf{p}$ around it combines with the surface integral of any distributed force to balance the rate of change of the linear momentum of the enclosed material. For smooth fields, the local form of this balance law is

$$\operatorname{Div}\mathbf{P} + J\mathbf{f} = \rho\dot{\mathbf{v}} \tag{9}$$

where Div is the (two-dimensional) divergence operator on κ, $\mathbf{f}$ is the distributed force per unit area of the *current* surface ω, ρ is the mass per unit area of κ, $\mathbf{v}$ $(=\dot{\mathbf{x}})$ is the particle velocity, the superposed dot is used to denote the material time derivative $(\partial/\partial t$ at fixed $\mathbf{X})$, and

$$J = (\det\mathbf{C})^{1/2} \tag{10}$$

is the local ratio of current to reference surface areas, where

$$\mathbf{C} = \mathbf{F}^t\mathbf{F} \tag{11}$$

is the (positive semi-definite) right Cauchy–Green deformation tensor.

A power identity follows by forming the scalar product of (9) with $\mathbf{v}$ and integrating the result over an arbitrary simply-connected subregion π of the plane κ. Thus,

$$\int\limits_{\partial\pi} \mathbf{p}\cdot\mathbf{v}\,\mathrm{d}S + \int\limits_{\pi} J\mathbf{f}\cdot\mathbf{v}\,\mathrm{d}A = \int\limits_{\pi} \frac{1}{2}\rho\big(|\mathbf{v}|^2\big)^{\cdot}\,\mathrm{d}A + \int\limits_{\pi} \mathbf{P}\cdot\dot{\mathbf{F}}\,\mathrm{d}A \tag{12}$$

The distributed force $\mathbf{f}$ is assumed to arise entirely from contact of the two interacting surfaces. It is convenient to decompose it into parts normal and tangential to T_ω:

$$\mathbf{f} = p\mathbf{n} + \boldsymbol{\tau} \tag{13}$$

where $p \in \mathbb{R}$ and $\boldsymbol{\tau} \in T_\omega$. Then from (9),

$$Jp = \mathbf{n}\cdot(\rho\dot{\mathbf{v}} - \operatorname{Div}\mathbf{P}) \tag{14}$$

and

$$J\boldsymbol{\tau} = \mathbb{P}(\rho\dot{\mathbf{v}} - \operatorname{Div}\mathbf{P}) \tag{15}$$

where

$$\mathbb{P} = \mathbf{I} - \mathbf{n}\otimes\mathbf{n} \tag{16}$$

is the projection onto T_ω and $\mathbf{I}$ is the identity for three-space.

In the following sections we adapt concepts from nonlinear contact mechanics [5–8] to the description of interacting surfaces.

3. Interaction of two congruent surfaces

We assume the weave to consist of two interacting material surfaces $\kappa^{(\alpha)}$; $\alpha = 1, 2$, each of the type described in the foregoing. Although it is not essential to do so, for simplicity we assume both to occupy a common reference plane in which their respective yarns are orthogonal. The material surfaces are then required to be congruent and similarly oriented in all configurations, while their respective yarns may undergo shearing and relative sliding. Here and henceforth we use Greek superscripts enclosed in braces to identify variables associated with a material surface. We adopt the notation used in [5]. When distinct superscripts of this kind are used in an expression, the intention is that they assume distinct numerical values. Expressions of the latter type relate variables pertaining to both surfaces together, and so describe some aspect of the interaction.

All of the equations of the previous section apply to each surface separately. For example, the motion of material surface $\kappa^{(\alpha)}$ is

$$\mathbf{x}^{(\alpha)} = \boldsymbol{\chi}_{\kappa^{(\alpha)}}(\mathbf{X}^{(\alpha)}, t) \tag{17}$$

To describe surface interactions, we consider a particular point on the current surface ω with position $\mathbf{x}$ which at time t is common to both material surfaces. The corresponding material points on the reference surfaces satisfy the relation

$$\boldsymbol{\chi}_{\kappa^{(\alpha)}}(\mathbf{X}^{(\alpha)}, t) = \boldsymbol{\chi}_{\kappa^{(\beta)}}(\mathbf{X}^{(\beta)}, t) \tag{18}$$

which is assumed to possess a smooth solution of the form

$$\mathbf{X}^{(\beta)} = \check{\mathbf{X}}^{(\beta)}(\mathbf{X}^{(\alpha)}, t) \tag{19}$$

This furnishes a relationship between the material points on the two surfaces which interact at position $\mathbf{x}$ at time t. Following [5], we say that such points are *associated*. Then, if $F^{(\alpha)}(\mathbf{X}^{(\alpha)}, t)$ is a function defined on $\kappa^{(\alpha)}$, its counterpart at the point $\mathbf{X}^{(\beta)} \in \kappa^{(\beta)}$ with which $\mathbf{X}^{(\alpha)} \in \kappa^{(\alpha)}$ is currently associated has the value

$$\check{F}^{(\beta)} \doteq F^{(\beta)}(\check{\mathbf{X}}^{(\beta)}, t) \tag{20}$$

The difference between $F^{(\beta)}$ and $F^{(\alpha)}$ at associated points may be specified as a function defined on $\kappa^{(\alpha)}$. Thus,

$$G^{(\alpha)}(\mathbf{X}^{(\alpha)}, t) \doteq \check{F}^{(\beta)} - F^{(\alpha)} = F^{(\beta)}(\check{\mathbf{X}}^{(\beta)}, t) - F^{(\alpha)}(\mathbf{X}^{(\alpha)}, t) \tag{21}$$

The material derivative relative to $\kappa^{(\alpha)}$ is

$$\dot{G}^{(\alpha)} \doteq \partial G^{(\alpha)}/\partial t|_{\mathbf{X}^{(\alpha)}} = \partial F^{(\beta)}/\partial t + [\check{\mathbf{X}}^{(\beta)}]^{\cdot} \cdot \nabla^{(\beta)} F^{(\beta)} - \dot{F}^{(\alpha)} \tag{22}$$

where $\nabla^{(\beta)}$ is the gradient with respect to $\mathbf{X}^{(\beta)}$.

If material points of the two surfaces are associated at time t, then by definition the current position of a point on $\kappa^{(\beta)}$ coincides with that of its associated point on $\kappa^{(\alpha)}$. We express this as $[\mathbf{x}]^{(\alpha)} = \mathbf{0}$, where

$$[\mathbf{x}]^{(\alpha)} \doteq \check{\mathbf{x}}^{(\beta)} - \mathbf{x}^{(\alpha)} \tag{23}$$

Suppose that an interval of time exists in which (18) is valid. This means that for each t in this interval, there exists a material point of $\kappa^{(\beta)}$ which is associated with a fixed point of $\kappa^{(\alpha)}$. It follows that $[\mathbf{x}]^{(\alpha)} \equiv \mathbf{0}$ at the considered point $\mathbf{X}^{(\alpha)} \in \kappa^{(\alpha)}$, and time differentiation there yields

$$[\mathbf{v}]^{(\alpha)} + \check{\mathbf{F}}^{(\beta)}[\check{\mathbf{X}}^{(\beta)}]^{\cdot} = \mathbf{0} \tag{24}$$

where

$$[\mathbf{v}]^{(\alpha)} = \check{\mathbf{v}}^{(\beta)} - \mathbf{v}^{(\alpha)} \tag{25}$$

is the velocity, relative to $\kappa^{(\alpha)}$, of the material point on $\kappa^{(\beta)}$ currently associated with the point $\mathbf{X}^{(\alpha)}$,

$$\check{\mathbf{F}}^{(\beta)} = \mathbf{F}(\check{\mathbf{X}}^{(\beta)}, t) \doteq \nabla^{(\beta)} \boldsymbol{\chi}_{\kappa^{(\beta)}}|_{\check{\mathbf{X}}^{(\beta)}} \tag{26}$$

is the deformation gradient at the associated point, and

$$[\check{\mathbf{X}}^{(\beta)}]^{\cdot} = \partial \check{\mathbf{X}}^{(\beta)}/\partial t|_{\mathbf{X}^{(\alpha)}} \tag{27}$$

is the velocity of the associated point on $\kappa^{(\beta)}$.

If (18) holds on an open patch of $\kappa^{(\alpha)}$, then (19) holds on an associated patch of $\kappa^{(\beta)}$, and differentiation with respect to $\mathbf{X}^{(\alpha)}$ gives

$$\mathbf{F}^{(\alpha)} = \check{\mathbf{F}}^{(\beta)}\check{\mathbf{H}} \tag{28}$$

where $\check{\mathbf{H}}$ is the gradient of (19). It follows from this that the tangent planes $T_\omega^{(\alpha)}$ and $\check{T}_\omega^{(\beta)}$ coincide at the material point $\mathbf{X}^{(\alpha)}$, and from (24) that $[\mathbf{v}]^{(\alpha)} \in T_\omega^{(\alpha)}$. We orient them similarly, so that

$$\mathbf{n}^{(\alpha)} = \check{\mathbf{n}}^{(\beta)} \quad \text{and} \quad \mathbb{P}^{(\alpha)} = \check{\mathbb{P}}^{(\beta)} \tag{29}$$

Relative slip of the material surfaces occurs when $[\mathbf{v}]^{(\alpha)} \neq \mathbf{0}$. Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be a fixed orthonormal basis for the reference plane, and let

$$\mathbf{a}_\mu = \check{\mathbf{F}}^{(\beta)}\mathbf{e}_\mu \tag{30}$$

Then $T_\omega^{(\alpha)} = \mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$. Let $\mathbf{a}^\mu$ be the duals to the $\mathbf{a}_\mu$ on $T_\omega^{(\alpha)}$. From (24) it follows that

$$[\check{\mathbf{X}}^{(\beta)}]^{\cdot} = -(\mathbf{e}_\mu \otimes \mathbf{a}^\mu)[\mathbf{v}]^{(\alpha)} \tag{31}$$

The evolution of the associated point is thus determined by its relative slip, which in turn is given by a constitutive equation to be discussed in the following section.

The equation of motion for $\kappa^{(\alpha)}$ is

$$\mathrm{Div}^{(\alpha)}\,\mathbf{P}^{(\alpha)} + J^{(\alpha)}\mathbf{f}^{(\alpha)} = \rho^{(\alpha)}\mathbf{a}^{(\alpha)} \tag{32}$$

where $\mathbf{a}^{(\alpha)} = \dot{\mathbf{v}}^{(\alpha)}$ is the acceleration. Let $s^{(\alpha)}$ be the image of an arbitrary subregion $\pi^{(\alpha)}$ of $\kappa^{(\alpha)}$ at time t under the map (17). We assume that the net interaction force, acting on the two surfaces together, vanishes:

$$\int_{s^{(\alpha)}} (\mathbf{f}^{(\alpha)} + \check{\mathbf{f}}^{(\beta)})\,\mathrm{d}a = \mathbf{0} \tag{33}$$

where $\check{\mathbf{f}}^{(\beta)}$ is the interaction force acting on the associated subregion of $\kappa^{(\beta)}$. The arbitrariness of $s^{(\alpha)}$ yields

$$\check{\mathbf{f}}^{(\beta)} = -\mathbf{f}^{(\alpha)} \tag{34}$$

Substituting this into the equation of motion for $\kappa^{(\beta)}$ yields the motion of $\check{\mathbf{X}}^{(\beta)}$ in terms of the motion of $\mathbf{X}^{(\alpha)}$:

$$\check{\rho}^{(\beta)}\check{\mathbf{a}}^{(\beta)} = \mathrm{Div}^{(\beta)}\,\check{\mathbf{P}}^{(\beta)} - [\check{J}^{(\beta)}/J^{(\alpha)}](\rho^{(\alpha)}\mathbf{a}^{(\alpha)} - \mathrm{Div}^{(\alpha)}\,\mathbf{P}^{(\alpha)}) \tag{35}$$

The normal and tangential components of the interaction force on $\kappa^{(\alpha)}$, which figure in the constitutive equation for slip, are given by (14) and (15), respectively. Thus,

$$J^{(\alpha)}p^{(\alpha)} = \mathbf{n}^{(\alpha)} \cdot (\rho^{(\alpha)}\mathbf{a}^{(\alpha)} - \mathrm{Div}\,\mathbf{P}^{(\alpha)}) \tag{36}$$

and

$$J^{(\alpha)}\boldsymbol{\tau}^{(\alpha)} = \mathbb{P}^{(\alpha)}(\rho^{(\alpha)}\mathbf{a}^{(\alpha)} - \mathrm{Div}\,\mathbf{P}^{(\alpha)}) \tag{37}$$

According to (13) and (34), their counterparts at an associated point of $\kappa^{(\beta)}$ are

$$\check{p}^{(\beta)} = -p^{(\alpha)} \quad \text{and} \quad \check{\boldsymbol{\tau}}^{(\beta)} = -\boldsymbol{\tau}^{(\alpha)} \tag{38}$$

A mechanical power identity for the composite surface is obtained by adding the expression (12) for $\pi^{(\alpha)}$ to its counterpart for the set of associated points on $\kappa^{(\beta)}$. In the resulting equation, the power of the interaction forces may be reduced, with the aid of (13) and (34), to the form

$$P^{(\alpha)} = -\int_{s^{(\alpha)}} \mathbf{f}^{(\alpha)} \cdot [\mathbf{v}]^{(\alpha)}\,\mathrm{d}a = -\int_{\bar{s}^{(\alpha)}} \boldsymbol{\tau}^{(\alpha)} \cdot [\mathbf{v}]^{(\alpha)}\,\mathrm{d}a \tag{39}$$

where we have used $[\mathbf{v}]^{(\alpha)} \in T_\omega^{(\alpha)}$ in the second equality and $\bar{s}^{(\alpha)} \subset s^{(\alpha)}$ is the region of non-zero slip on the current surface. To ensure that slip is dissipative we impose the requirement

$$\boldsymbol{\tau}^{(\alpha)} \cdot [\mathbf{v}]^{(\alpha)} \geqslant 0 \tag{40}$$

Equality holds in the case of frictionless slip. For slip with friction we assume that

$$\boldsymbol{\tau}^{(\alpha)} \cdot [\mathbf{v}]^{(\alpha)} > 0 \quad \text{for all nonzero } [\mathbf{v}]^{(\alpha)} \tag{41}$$

4. Constitutive equations

4.1. Yarn response

Biaxial dead-load equilibrium experiments are widely used to characterize the response of woven fabric [9]. In a typical experiment, a cross-shaped sample of fabric with edges parallel to the warp and weft directions is placed horizontally in a loading device, and the variation of extension with applied force in the interior of a horizontal test section (where the deformation is deemed to be approximately homogeneous) is recorded for various fixed values of the transverse force. Sliding is minimal in such experiments and the resistance of the weave to local shearing is usually deemed to be negligible in the presence of substantial direct forces along the warp and weft. Thus, in principle, experiments yield data directly in the form

$$f^{(\alpha)} = g^{(\alpha)}(\lambda^{(\alpha)}, \check{f}^{(\beta)}) \tag{42}$$

where $g^{(\alpha)}$ is a constitutive function for the relevant yarn family.

It is conceivable that rate effects may have a non-trivial effect on fabric response in the applications envisaged but for definiteness we do not include them either in the foregoing constitutive equations or in those for friction.

The nature of the experiments means that real data are restricted to tensile forces. In principle, then, to characterize the response in compression, a non-equilibrium experiment is required since the fabric may be expected to buckle into a non-homogeneous deformation mode if maintained in equilibrium. The data would then cease to reflect *material* properties *per se*. It is to be expected that true constitutive data in the presence of compressive stresses would thus be very difficult to obtain. Indeed, we are not aware of any such experiments. However, we believe the idealization

$$g^{(\alpha)}(\cdot, \check{f}^{(\beta)}) \to \pm\infty \quad \text{as } \lambda^{(\alpha)} \to 0, \infty, \text{ respectively} \tag{43}$$

to be consistent with physically realistic behavior.

A model for wide-mesh networks, which are characterized by the absence of any 'Poisson effect', may be obtained by specializing $g^{(\alpha)}$ to depend only on $\lambda^{(\alpha)}$. However, this decoupling is not observed in the biaxial response of woven fabrics. This is due to the fact that a yarn of the weave describes a curve in space that oscillates about a mean curve. The local curvature associated with this oscillation is known as the *crimp* of the yarn. Roughly, the stretching of one family is accompanied by straightening, or decrimping, of that family. If the orthogonal family is unstressed, its crimp is in turn increased to a degree that depends on the extent to which the first family is stretched. The end-to-end length of a fiber of the orthogonal family is thereby reduced. A stress is then required to restore the orthogonal family to its original length, or to some other length. By contrast, this mechanism is absent in networks in which the local interaction between fiber families occurs only at nodes where the fibers are tied together.

Generally, the stress required to extend a yarn family is relatively low for highly crimped yarns due to their flexibility and to the predominant role played by yarn bending. The stress response stiffens dramatically during decrimping since strain, rather than flexure, then plays the major role in accommodating the overall deformation. These effects are modified to some degree by the deformation and stress in the orthogonal yarn family. Micromechanical analyses of the crimping/decrimping mechanism may be found in [10] and [11]. In the present theory, it is modelled by replacing the actual fibers by their projections onto the tangent plane of a mean surface. Thus, the small-scale structure of the actual fabric is replaced by a continuum with an appropriate constitutive response.

4.2. Friction

The experimental record for friction in fabrics is not definitive. Some experimenters argue in support of Coulomb friction [12,13] while others favor a modification in which the norm of the friction force is related to the contact pressure by a power-law expression [14]. We adopt a framework encompassing both possibilities.

Consideration of the oscillatory local structure of the yarns of a weave suggests that the pressure acting on a given member of the pair of interacting surfaces changes sign as the yarns are traversed. This is due to the use of (4) to calculate the orientation field in (13). For this reason we assume the interaction pressure to manifest itself through

$$p \doteq |p^{(\alpha)}| = |\check{p}^{(\beta)}| \tag{44}$$

in the constitutive equation for slip with friction.

A framework for rate-independent frictional response has been discussed in [8]. We adapt it here to describe inter-yarn, or intra-weave, friction. Thus, in the case of Coulomb friction, we have

$$\boldsymbol{\tau}^{(\alpha)} = p\mathbf{r}^{(\alpha)}(\mathbf{u}^{(\alpha)}, \mathbf{A}, \mathbf{B}), \quad \text{where } \mathbf{u}^{(\alpha)} = [\mathbf{v}]^{(\alpha)} / \big|[\mathbf{v}]^{(\alpha)}\big| \tag{45}$$

provided that $p \neq 0$ and $[\mathbf{v}]^{(\alpha)} \neq \mathbf{0}$, where $\mathbf{r}^{(\alpha)}$ is a constitutive function, and

$$\mathbf{A} = \mathbf{l}^{(\alpha)} \otimes \mathbf{l}^{(\alpha)}, \qquad \mathbf{B} = \check{\mathbf{l}}^{(\beta)} \otimes \check{\mathbf{l}}^{(\beta)} \tag{46}$$

are *structural tensors* which characterize the anisotropy of the weave. Using results from representation theory, He and Curnier [8] derived the general form of the function $\mathbf{r}^{(\alpha)}$ compatible with objectivity. Thus,

$$\mathbf{r}^{(\alpha)}(\mathbf{u}^{(\alpha)}, \mathbf{A}, \mathbf{B}) = \mathbf{H}\mathbf{u}^{(\alpha)} \tag{47}$$

where

$$\mathbf{H} = \alpha(I, J)\mathbf{I} + \beta(I, J)\mathbf{A} + \gamma(I, J)\mathbf{B} \tag{48}$$

where α, β, γ are constitutive functions to be determined by experiment, and where

$$I = \mathbf{u}^{(\alpha)} \cdot \mathbf{A}\mathbf{u}^{(\alpha)}, \qquad J = \mathbf{u}^{(\alpha)} \cdot \mathbf{B}\mathbf{u}^{(\alpha)} \tag{49}$$

We note that $\mathbf{H}$ depends on $\mathbf{u}^{(\alpha)}$. The dissipation inequality (41) is equivalent to

$$\mathbf{u}^{(\alpha)} \cdot \mathbf{H}\mathbf{u}^{(\alpha)} > 0 \quad \text{for all unit } \mathbf{u}^{(\alpha)} \tag{50}$$

To obtain an adjustment of the power-law type one may replace p in (45) by $p_0(p/p_1)^x$ where p_0 and p_1 are parameters with dimensions of pressure and x is a positive constant.

In the classical Coulomb law the ratio of the norm of the shear stress to the pressure is given by a fixed coefficient of friction. The analogue in the present problem is obtained by requiring that α, β, γ assume constant values. Inequality (50) then requires that $\mathbf{H}$ be non-singular, so that (45) may be inverted to yield the *slip rule*

$$\mathbf{u}^{(\alpha)} = p^{-1}\mathbf{H}^{-1}\boldsymbol{\tau}^{(\alpha)} \tag{51}$$

and the associated *slip criterion*

$$\boldsymbol{\tau}^{(\alpha)} \cdot \mathbf{H}^{-2}\boldsymbol{\tau}^{(\alpha)} = p^2 \tag{52}$$

The interacting surfaces are deemed to be in a state of *stick* defined by $[\mathbf{v}]^{(\alpha)} = \mathbf{0}$ if

$$\boldsymbol{\tau}^{(\alpha)} \cdot \mathbf{H}^{-2}\boldsymbol{\tau}^{(\alpha)} < p^2 \tag{53}$$

Additional models for anisotropic friction which may be relevant here are discussed in the fundamental work of He and Curnier [8]. Further, wear of the yarns is known to be pronounced in some circumstances [13] but this is not considered here as it seems unlikely to have a significant effect on the dynamics of a single impact event, provided the fabric is not worn prior to impact.

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