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Asymptotic decomposition of a singular perturbation problem with unbounded energy

Franck Fontvieille [a], Gregory P. Panasenko [b], Jérôme Pousin [a]

[a] MAPLY, UMR CNRS 5585, INSA de Lyon, Bât. Léonard de Vinci, 21, avenue Jean Capelle, 69621 Villeurbanne, France

[b] Equipe d'analyse numérique, UPRES EA 3058, Université de Saint-Etienne, 23, rue Paul Michelon, 42023 Saint-Etienne, France

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Abstract We consider a singular perturbation with unbounded energy. We propose here an effective method of finite element computation, fit for accounting for the linear behavior of the solution. The Hilbert space of the variational formulation, $H_0^2(0,1)$, is replaced by a simpler subspace containing an asymptotic solution of the initial problem. Error estimates are derived by eliminating some degrees of freedom and a numerical experiment is developped. ***To cite this article: F. Fontvieille et al., C. R. Mecanique 330 (2002) 507–512.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

computational solid mechanics / asymptotic decomposition / finite element / singular perturbation / error estimates

Décomposition asymptotique d'une perturbation singulière d'énergie sans limite

Résumé Nous considérons un problème de perturbation singulière sans limite dans l'espace d'énergie finie. Nous proposons une méthode de résolution numérique par éléments finis adaptés au comportement linéaire de la solution. L'espace de Hilbert de la formulation variationnelle, $H_0^2(0,1)$, est remplacé par un sous-espace plus simple qui contient une solution asymptotique du problème initial. Des estimations d'erreur sont obtenues en éliminant des degrés de liberté et des essais numériques sont présentés. ***Pour citer cet article : F. Fontvieille et al., C. R. Mecanique 330 (2002) 507–512.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

mécanique des solides numérique / décomposition asymptotique / éléments finis / perturbation singulière / estimation d'erreur

Version française abrégée

La méthode asymptotique de décomposition partielle de domaine (MAPDD) présentée dans [1,2] est considérée ici sous forme variationnelle pour traiter un problème monodimensionnel de perturbation singulière d'ordre quatre dépendant d'un petit paramètre ε proposé dans [3] et [4] comme un problème modèle issu de la modélisation des coques lié à la « sensitivité ». La discrétisation de ce problème demande

E-mail addresses: franck.fontvieille@insa-lyon.fr (F. Fontvieille); panasenko@anum.univ-st-etienne.fr (G.P. Panasenko); pousin@insa-lyon.fr (J. Pousin).

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un grand nombre de points, ceci d'autant plus si ε est petit, d'où l'intérêt d'une méthode permettant de réduire le nombre de noeuds du maillage.

L'étude asymptotique de ce problème montre que la solution est très proche d'une fonction affine par morceaux en dehors d'un petit voisinage de certains « points singuliers ». La méthode de décomposition asymptotique partielle de domaine consiste à remplacer l'espace classique de Hilbert H associé au problème par un sous-espace H_{dec} qui tient compte du comportement asymptotique de la solution. Cet espace dépend d'un petit paramètre $\delta(\varepsilon) > \varepsilon$. Nous définissons des « macro-éléments » dont la mesure du support est comparable à celle du domaine tout entier en dehors des couches limites de taille δ (voir définition (1)). Pour le problème traité, on montre que pour tout $L \in \mathbb{N}$, il existe $K \in \mathbb{R}$, tel que, en choisissant $\delta = K\varepsilon|\ln(\varepsilon)|$, on commet une erreur d'ordre ε^L entre la solution dans H et la solution dans l'espace décomposé H_{dec}. Dans la résolution numérique par éléments finis, nous utilisons des éléments finis de Hermite de classe $\mathcal{C}^1$ (voir par exemple [5]), combinés à des fonctions de base globales particulières où certains degrés de liberté sont éliminés, définies sur des domaines macroscopiques, qui permettent de résoudre le problème approché dans H_{dec} (voir Fig. 5). L'ordre de convergence obtenu entre la solution approchée $u^h_{\varepsilon,\mathrm{dec}}$ et la solution exacte est le même que pour une résolution par éléments finis d'Hermite classiques uniquement (voir Théorème 2.1 et la Fig. 2), mais le nombre de noeuds utilisés dans la discrétisation est très inférieur dès que ε devient petit (voir Tableau 1). La Fig. 3 illustre le comportement des solutions obtenues avec notre méthode.

Introduction

The variational version of the method of asymptotic partial decomposition of domain (MAPDD), presented in [1,2] is applied here to the following fourth order differential equation, for $\varepsilon > 0$, for $f \in H^{-2}(]0,1[)$, find u_ε such that:

$$\begin{cases} -u_\varepsilon'' + \varepsilon^2 u_\varepsilon'''' = f \\ u_\varepsilon(0) = u_\varepsilon(1) = 0; \quad u_\varepsilon'(0) = u_\varepsilon'(1) = 0 \end{cases} \tag{$\mathcal{P}_1$}$$

The variational formulation of this problem is: *find* $u_\varepsilon \in H_0^2(0,1)$ *such that*:

$$B^\varepsilon(u_\varepsilon, v) = \int_0^1 u_\varepsilon' v' \,\mathrm{d}x + \varepsilon^2 \int_0^1 u_\varepsilon'' v'' \,\mathrm{d}x = (f, v) \quad \forall v \in H_0^2(0,1) \tag{$\mathcal{P}_1'$}$$

Replacing ε by zero in ($\mathcal{P}_1$), we obtain the 'limit problem':

$$\begin{cases} -u'' = f \\ u(0) = u(1) = 0 \end{cases} \tag{$\mathcal{P}_2$}$$

Problem ($\mathcal{P}_1$) was introducted in [3] and [4] as a model problem describing some mathematical difficulties in shell theory; f was taken in the form of the derivative of Dirac distribution centered in 1/2, i.e., $(f, v) = -v'(1/2)$. Problem ($\mathcal{P}_1'$) is well posed in $H_0^2(0,1)$ due to the H^2-coercivity of the form B^ε consequence of the Poincaré inequality, while the 'limit problem' ($\mathcal{P}_2$) is ill posed in $H_0^1(0,1)$ for $f \notin H^{-1}(0,1)$ and consequently u_ε does not tend to u in $H_0^1(0,1)$ in the strong sense. In [3] the numerical method uses an adapted mesh, it seems however that the number of nodes in the adapted mesh depends on ε and it increases when ε tends to zero, moreso, the process of construction of the adapted mesh is iterative and so we should add the nodes of meshes for all previous iterations; this emphasises the interest of a numerical method allowing to reduce the number of nodes of the mesh.

Here below, we propose the method of asymptotic partial decomposition of domain for problem ($\mathcal{P}_1'$). The asymptotic analysis of this problem shows that the solution is close to a piecewise affine function outside of some small neighborhoods of the 'singular points' 0, 1, and 1/2. Consequently, we pass from ($\mathcal{P}_1'$) to the partially decomposed problem:

Let $\delta > 0$ be a positive small parameter, $\varepsilon < \delta < 1$. Denote $I_\delta =]\delta, \frac{1}{2} - \frac{\delta}{2}[\cup]\frac{1}{2} + \frac{\delta}{2}, 1 - \delta[$, and define

$$H_{\mathrm{dec}} = \left\{\varphi : \varphi \in H_0^2(0,1) \text{ and } \varphi|_{I_\delta} \in \mathbb{P}_1\left(\overline{I_\delta}\right)\right\}$$

Here (as well as further below) $\mathbb{P}_n(I)$ stands for the space of polynomials of degree less or equal to n defined on I. We also denote for the whole paper, for m integer, $\|\cdot\|_{m,]0,1[}$ as the usual norm and $|\cdot|_{m,]0,1[}$ as the usual semi-norm on $H^m(0,1)$.

H_{dec} is a Hilbert space. Indeed it is closed in $H_0^2(0,1)$ supplied by the ordinary norm. Hence, according to the Lax–Milgram lemma, for all $f \in H^{-2}(0,1)$, the problem $B^\varepsilon(u_{\varepsilon,\mathrm{dec}}, v) = \langle f, v\rangle$ $\forall v \in H_{\mathrm{dec}}$ has a unique solution $u_{\varepsilon,\mathrm{dec}}$ in H_{dec}.

1. Approximation of the problem stated in H_0^2 by a problem in H_{dec}

We first build a formal asymptotic expansion $u_\varepsilon^{(\infty)}$ of u_ε; then we modify it to obtain an asymptotic approximation of $u_{\varepsilon,\mathrm{dec}}$ in H_{dec}.

LEMMA 1.1. – *A formal asymptotic expansion of the solution of problem* $(\mathcal{P}_1)$ *is*:

$$u_\varepsilon^{(\infty)}(x) \sim \sum_{j=0}^{\infty} \varepsilon^j \left[u_j(x) + u_j^{Cl_0}\left(\frac{x}{\varepsilon}\right) + u_j^{Cl_{1/2}}\left(\frac{x-1/2}{\varepsilon}\right) + u_j^{Cl_1}\left(\frac{x-1}{\varepsilon}\right)\right]$$

where, for all points different of $1/2$, *we have*:

$$\begin{cases} -u_0'' = 0 & \text{with convention the } u_j = 0 \\ u_j'' = u_{j-2}'''' & \text{if } j < 0 \end{cases}$$

and boundary and jump (*in* $x = 1/2$, *noted* $[\cdot]$) *conditions*:

$$\begin{cases} u_j(0) = -u_{j-1}'(0), \\ u_j(1) = +u_{j-1}'(1), \end{cases} \qquad \begin{cases} [u_0] = -1, \quad [u_0'] = 0 \\ [u_1] = 0, \quad [u_1'] = 0 \\ [u_j] = [u_{j-2}''] \\ [u_j'] = [u_{j-2}'''] \quad \text{for } j \geqslant 2 \end{cases}$$

The correctors for boundary and internal layers are given by:

$$\begin{cases} u_j^{Cl_0}(\xi) = +u_{j-1}'(0)\,\mathrm{e}^{-\xi}, \\ u_j^{Cl_1}(\xi) = -u_{j-1}'(1)\,\mathrm{e}^{\xi}, \end{cases} \qquad u_j^{Cl_{1/2}}(\xi) = \begin{cases} \dfrac{[u_{j-1}'] + [u_j]}{2}\,\mathrm{e}^{\xi} & \text{if } \xi < 0 \\ \dfrac{[u_{j-1}'] - [u_j]}{2}\,\mathrm{e}^{-\xi} & \text{if } \xi > 0 \end{cases}$$

Then, we follow the principle exposed in [2], and we get:

THEOREM 1.2. – *For all* $L > 0$, *there exists* $K > 0$ *such that, if* $\delta = K\varepsilon|\ln\varepsilon|$, *then*:

$$\|u_\varepsilon - u_{\varepsilon,\mathrm{dec}}\|_{2,]0,1[} = \mathcal{O}\left(\varepsilon^L\right)$$

2. Finite element solution

Now we are going to build an affine equivalent finite element family to solve the partially decomposed problem. This family will contain some basic functions which are linear on some large intervals (i.e. linear super-elements) as well as standard Hermite elements.

Let δ be given by Theorem 1.2. Let M be a given positive integer. Consider the mesh of $M+2$ nodes of the segment $[0,1]$ such that the extremities of the intervals $[\delta, (1-\delta)/2]$ and $[(1+\delta)/2, 1-\delta]$ are consecutive nodes of the mesh. Taking in consideration the symmetry of the problem with respect to the point $x = 1/2$, we will simplify the presentation of the construction of finite element family considering only one linear super-element whose support of linear part is the interval $[a,b]$, as in Fig. 1.

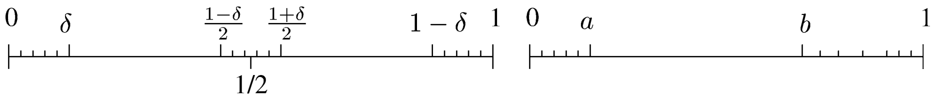

Figure 1. Mesh.

Figure 1. *Maillage.*

So, we consider the approximation of the space $H_{\text{dec}} = \{v \in H_0^2(0,1) : v|_{[a,b]} \in \mathbb{P}_1(a,b)\}$. Let x_i, $i \in \{0,\dots,M+1\}$, be the nodes of the mesh; denote $K_i = [x_i, x_{i+1}]$, $i \in \{0,\dots,M\}$. In particular, we get $K_N = [x_N, x_{N+1}] = [a,b]$.

Notice that the space dimension is equal to one and $H_{\text{dec}} \subset \mathcal{C}^1(0,1)$. That is why we consider Hermite elements of class $\mathcal{C}^1$ (see, for example, [5], p. 80), defined by the following relations:

$$\widehat{K} = [0,1]; \quad \widehat{\Sigma} = \{\hat{\phi}_{i,\hat{K}},\ 1 \leqslant i \leqslant 4\} = \{v \mapsto v(0),\ v \mapsto v'(0),\ v \mapsto v(1),\ v \mapsto v'(1)\}; \quad P_{\hat{K}} = \mathbb{P}_3(0,1)$$

According to notations of [5], we note the local projector $\Pi_{\hat{K}}$, and $p_{i,\hat{K}}$ the local basis functions associated to the local degrees of freedom $\hat{\phi}_{i,\hat{K}}$, for $1 \leqslant i \leqslant 4$. The linearity requirement for functions $v \in H_{\text{dec}}$ means that $v'(a) = v'(b) = \frac{v(b)-v(a)}{b-a}$. It means that the values of the derivatives $v'(a)$ and $v'(b)$ can be eliminated for K_N. Denote $|K_i| = |x_{i+1} - x_i|$, and define $h = \max_{K_i \neq [a,b]} |K_i|$, the approximated space $H_{\text{dec},h}$ is defined by the relation:

$$H_{\text{dec},h} = \{v \in H_{\text{dec}} : v|_{K_i} \in \mathbb{P}_3(K_i),\ 0 \leqslant i \leqslant M\} \tag{1}$$

Let us prove that the space $H_{\text{dec},h}$ is correctly defined. For $1 \leqslant i < N$, and $N+1 < i \leqslant M$, denote w_{2i} and w_{2i+1}, the global basis functions associated respectively to global degrees of freedom $\Phi_{h,2i} : v \mapsto v(x_i)$ and $\Phi_{h,2i+1} : v \mapsto v'(x_i)$ (for which x_i is neither a nor b). We have that w_{2i} and w_{2i+1} belong to the class $\mathcal{C}^1(K_{i-1} \cup K_i)$, and, $w_{2i}|_{K_{i-1}} = p_{3,K_{i-1}}$, $w_{2i}|_{K_i} = p_{1,K_i}$, $w_{2i+1}|_{K_{i-1}} = p_{4,K_{i-1}}$, $w_{2i+1}|_{K_i} = p_{2,K_i}$. The global projector Π_h is defined as

$$\Pi_h(v) = \sum_{x_i : x_i \notin \{a,b\}} \left[v(x_i) w_{2i} + v'(x_i) w_{2i+1}\right] + v(a) w_a + v(b) w_b$$

w_a and w_b (cf. Fig. 5) can be defined as the functions satisfying the following conditions:

The support of w_a and w_b is $K_{N-1} \cup K_N \cup K_{N+1}$, these functions are $\mathcal{C}^1$.

$w_a|_{K_{N-1}} \in \mathbb{P}_3(K_{N-1})$, $w_a|_{K_N} \in \mathbb{P}_1(K_N)$, $w_a|_{K_{N+1}} \in \mathbb{P}_3(K_{N+1})$, and $w_a(x_{N-1}) = w_a'(x_{N-1}) = w_a(x_{N+1}) = w_a'(x_{N+1}) = 0$, the same assertions are all valid for w_b.

$$w_a(a) = 1, \quad w_a(b) = 0, \quad w_b(a) = 0, \quad w_b(b) = 1$$

Π_h can be defined as the projector associated to an affine family of finite elements belonging to the class $\mathcal{C}^1$ with the condition $v'(a) = v'(b) = (v(b) - v(a))/(b-a)$, this, because we get $(\Pi_h v)|_K = \Pi_K v$, $\forall K$ in the mesh.

Now we can formulate the theorem on the error estimate:

THEOREM 2.1. – *Let $2 < M$, $\bigcup_{i=0}^{M} K_i$ be a mesh of $[0,1]$, such that the closure of every connected component of I_δ (see introduction) is an element of the mesh. Let k and m be two integers, such that $m \geqslant 0$, $k \geqslant 1$, $m \leqslant k+1 \leqslant 4$. Let $f \in H^{k-3}(0,1)$, and $u^h_{\varepsilon,\text{dec}}$ be the solution of the following discrete problem*:

Find $u^h_{\varepsilon,\text{dec}} \in H_{\text{dec},h}$ such that: $B^\varepsilon(u^h_{\varepsilon,\text{dec}}, v_h) = (f, v_h)$ for all $v_h \in H_{\text{dec},h}$.

Then, there exists a constant C independent of h, such that the estimate holds

$$\left\| u_{\varepsilon,\text{dec}} - u^h_{\varepsilon,\text{dec}} \right\|_{m,]0,1[} \leqslant C h^{k+1-m} |u_{\varepsilon,\text{dec}}|_{k+1,]0,1[}$$

Draft of the proof. – The classical results on interpolation are valid for all K such that $K \neq K_N$ (see [5]). For K_N, every $v \in H_{\text{dec}}(]0,1[)$ satisfies the following property: $\Pi_h v|_{K_N} = v|_{K_N}$.

3. Numerical results

In Fig. 2 we present the L^1 norm of the error as a function of the mesh size. Accordingly to the estimate given in Theorem 2.1 we have a fourth order convergence rate. The figure corresponds to computation with the MAPDD and with classical Hermite finite elements. Fig. 3 depictes the obtained solutions. We also notice that if δ is not too large, then the solution calculated with the MAPDD seems to be less numericaly oscillating for 'large' mesh step size than classical method (cf. Fig. 4). With regard to the number of mesh nodes, in classical Hermite we need a step size of the same order than ε, so a number of nodes similar to $1/\varepsilon$. For the MAPDD, we need a step size similar to ε, but the number of nodes needed is of the order of δ/ε, say, $\mathcal{O}(|\ln\varepsilon|)$, as we can check in Table 1. We notice with last data that the factor K of $\delta = K\varepsilon|\ln\varepsilon|$ can be of order 10.

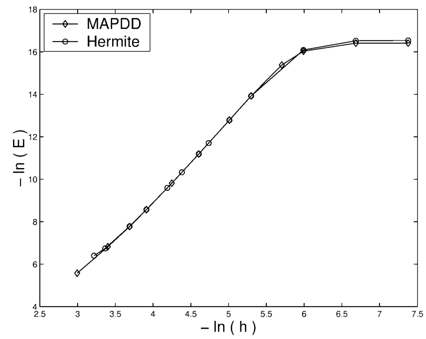

Figure 2. Convergence rates for $(\mathcal{P}_1)$ problem.

Figure 2. *Courbes de convergence pour le problème* $(\mathcal{P}_1)$ $(\varepsilon = 10^{-2},\ \delta = 0{,}2)$.

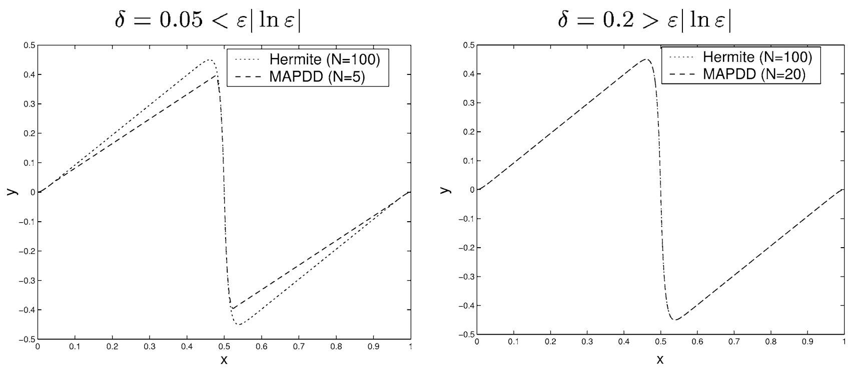

Figure 3. Solutions of problem $(\mathcal{P}_1)$ calculated with MAPDD and Hermite.

Figure 3. *Solutions du problème* $(\mathcal{P}_1)$ *calculées avec la MAPDD et Hermite* $(\varepsilon = 10^{-2})$.

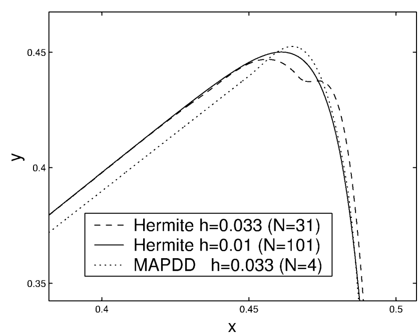

Figure 4. Stability comparison between MAPDD and classical Hermite FEM.

Figure 4. *Comparaison de stabilité entre la MAPDD et Hermite classique.*

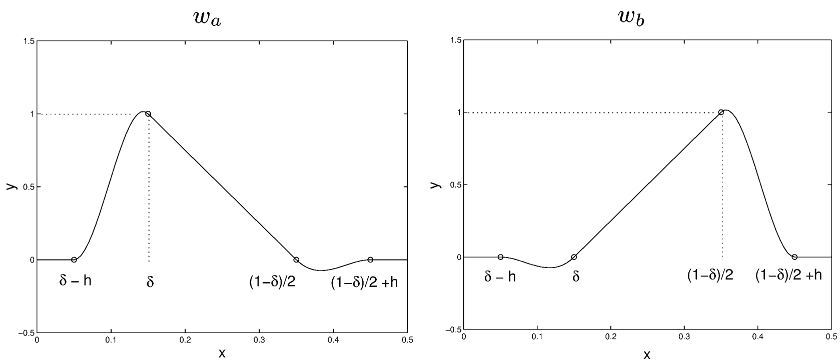

Figure 5. 'Super-element' map.

Figure 5. *Fonction de base « de super-élément ».*

Table 1. Comparison between δ and ε.

Tableau 1. *Relations entre δ et ε.*

ε	$\varepsilon\|\ln\varepsilon\|$	δ	h	N MAPDD	N classical Hermite	$\|u_\varepsilon^h - u_{\varepsilon,\mathrm{dec}}^h\|_{L^1}$
10^{-2}	5×10^{-2}	0.2	10^{-2}	63	101	10^{-5}
10^{-3}	7×10^{-3}	0.02	10^{-3}	63	1001	2×10^{-5}
″	″	0.05	″	153	″	$1{,}2\times10^{-11}$
10^{-4}	9×10^{-4}	0.01	2×10^{-4}	153	5001	9×10^{-8}
″	″	0.005	″	78	″	9×10^{-10}

Remark on numerical discrepancy: if $\|\!|\cdot|\!\|_{2,\varepsilon}$ denotes the induced norm by the bilinear form B_ε, and if $f \in L^2(0,1)$, then we have:

$$\left|\!\left|\!\left| u_{\varepsilon,\mathrm{dec}} - u_{\varepsilon,\mathrm{dec}}^h \right|\!\right|\!\right|_{2,\varepsilon} \leqslant C\frac{h^2}{\varepsilon}\|f\|_{0,]0,1[}$$

Moreover, for a fourth order problem, the Hermite finite element leads to a large condition number which affects the rate of convergence for small mesh size.

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