Piston Stokes flow in a semi-infinite channel

Anthony M.J. Davis

Department of Mathematics, University of Alabama, Tuscaloosa, AL 35487, USA

Received 29 April 2002; accepted 6 May 2002

Note presented by Keith Moffatt.

Abstract The piston flow is bounded by rigid walls at $y = \pm 1$, $x > 0$ and generated by the uniform translation of the end wall $x = 0$. After Katopodes, Davis and Stone [3] constructed a solution in terms of biorthogonal eigenfunctions, Meleshko and Krasnopolskaya [1] used a variation of an asymptotic technique developed by Meleshko and Gomilko [2] to examine the pointwise convergence of the non-orthogonal series. However, they overlooked the nonuniqueness of their solution and the consequent solvability condition which is shown here to necessitate a minor modification without significant harm to their contribution. ***To cite this article: A.M.J. Davis, C. R. Mecanique 330 (2002) 457–459.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

computational fluid mechanics / piston Stokes flow / solvability condition

L'écoulement de Stokes autour d'un piston dans un conduit semi-infini

Résumé L'écoulement autour d'un piston est limité par les parois rigides situées en $y = \pm 1$, $x > 0$, et engendré par une translation uniforme de la paroi extrême $x = 0$. Suite à la construction d'une solution de ce problème en termes des fonctions propres bi-orthogonales par Katopodes, Davis et Stone [3], Meleshko et Krasnopolskaya [1] ont examiné la convergence en un point de la série non-orthogonale en utilisant une variante de la technique asymptotique développée par Melesko et Gomilko [2]. Cependant, ils n'ont pas remarqué la non-unicité de leur solution et par conséquent, une condition de solvabilité supplémentaire que nous montrons ici, et qui oblige à introduire une modification mineure dans leurs résultats, qui ne compromet pas la validité générale de leur contribution. ***Pour citer cet article : A.M.J. Davis, C. R. Mecanique 330 (2002) 457–459.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS–

mécanique des fluides numérique / écoulement de Stokes / condition de solvabilité

1. Nonuniqueness and solvability

The alternative solution to this biharmonic problem in the semi-infinite strip, $x > 0$, $|y| \leqslant 1$, given by [1], modified from [2]:

$$\Psi = \frac{1}{2}\left(3y - y^3\right) + \sum_{k=1}^{\infty} \frac{(-1)^k}{\beta_k}\left(\frac{6}{\beta_k^2} + xY_k\right) \mathrm{e}^{-\beta_k x} \sin\beta_k y + \frac{1}{\pi}\int_0^{\infty} X(s)\left(\frac{\sinh sy}{\sinh s} - y\frac{\cosh sy}{\cosh s}\right)\frac{\sin sx}{s}\,\mathrm{d}s, \quad \beta_k = k\pi \tag{1}$$

E-mail address: adavis@euler.math.ua.edu (A.M.J. Davis).

© 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS. Tous droits réservés
S1631-0721(02)01493-6/FLA

includes both a Fourier series in y and a Fourier transform in x. Consequently, the application of the no-slip boundary conditions

$$\Psi = y, \quad \frac{\partial\Psi}{\partial x} = 0 \quad \text{at } x = 0, \qquad \Psi = \pm 1, \quad \frac{\partial\Psi}{\partial y} = 0 \quad \text{at } y = \pm 1$$

yields the following infinite system of linear integro-algebraic equations for the unknown sequence $\{Y_k\}$ and function $X(s)$:

$$\begin{aligned} &Y_k - \frac{4\beta_k^2}{\pi}\int_0^\infty X(s)\frac{s\tanh s}{(s^2+\beta_k^2)^2}\,\mathrm{d}s = \frac{6}{\beta_k} \quad (k \geqslant 1) \\ &X(s)\left(1 - \frac{2s}{\sinh 2s}\right) - \sum_{k=1}^\infty Y_k \frac{4s^2\beta_k}{(s^2+\beta_k^2)^2} = 2 - \sum_{k=1}^\infty \frac{12}{s^2+\beta_k^2} \quad (0 < s < \infty) \end{aligned} \tag{2}$$

On the right-hand side of (2),

$$\sum_{k=1}^\infty \frac{2s}{s^2+\beta_k^2} = \coth s - \frac{1}{s} \tag{3}$$

in which it may be noted that, although individual terms $\to 0$ as $s \to \infty$, the left-hand side is a Riemann sum for $(2/\pi)\int_0^\infty \frac{\mathrm{d}x}{1+x^2}$, which $\to 1$, in agreement with the right-hand side. It may also be observed, with the aid of the summation

$$\sum_{k=1}^\infty \frac{4s\beta_k^2}{(s^2+\beta_k^2)^2} = \frac{\sinh 2s - 2s}{2\sinh^2 s} \tag{4}$$

that $\{Y_k = \beta_k;\ k \geqslant 1\}$, $X(s) = s\coth s$ is a homogeneous solution of (2). Uniqueness is recovered by requiring a bounded sequence but, according to the Fredholm alternative theory, there is a solvability condition to be constructed and satisfied.

First, the convergence is accelerated but with a degree of flexibility not considered by [1]. Write

$$X(s) = X_0 + \frac{x(s) - (1-C)\frac{3}{s}(\coth s - \frac{1}{s})}{1 - 2s/(\sinh 2s)}, \qquad Y_k = Y_0 + (1+C)\frac{3}{\beta_k} + y_k \tag{5}$$

with the constants X_0, Y_0 determined, as in [1], by

$$X_0\frac{2}{\pi} - Y_0 = 0, \qquad 2 - X_0 + Y_0\frac{2}{\pi} = 0 \tag{6}$$

The substitution of (5) into (2) then yields

$$\begin{aligned} &y_k - \frac{4\beta_k^2}{\pi}\int_0^\infty \frac{x(s)}{(s^2+\beta_k^2)^2}\frac{2s\sinh^2 s}{\sinh 2s - 2s}\,\mathrm{d}s = f_k \quad (k \geqslant 1) \\ &x(s) - \sum_{k=1}^\infty y_k \frac{4s^2\beta_k}{(s^2+\beta_k^2)^2} = g(s) \quad (0 < s < \infty) \end{aligned} \tag{7}$$

with the constants eliminated according to (6) and

$$\begin{aligned} &f_k = \frac{12\beta_k^2}{\pi}(1-C)\int_0^\infty \left[1 - \frac{s - \tanh s}{s(1 - 2s/\sinh 2s)}\right]\frac{\mathrm{d}s}{(s^2+\beta_k^2)^2} - X_0\frac{4\beta_k^2}{\pi}\int_0^\infty \frac{s(1-\tanh s)}{(s^2+\beta_k^2)^2}\,\mathrm{d}s \\ &g(s) = X_0\frac{2s}{\sinh 2s} + Y_0\left[\sum_{k=1}^\infty \frac{4s^2\beta_k}{(s^2+\beta_k^2)^2} - \frac{2}{\pi}\right] - 3(1+C)\frac{\mathrm{d}}{\mathrm{d}s}\left(\coth s - \frac{1}{s}\right) \end{aligned} \tag{8}$$

after use of (3). Note that (6)–(8) imply that $g(0) = 1 - C = x(0)$ and hence the numerator in (5) cancels the double zero in the denominator. This also ensures the absence in (1) of pole contributions at $s = 0$, which would modify the forcing term. Evidently,

$$\{y_k = \beta_k;\, k \geqslant 1\}, \qquad x(s) = \frac{s(\sinh 2s - 2s)}{2\sinh^2 s}$$

is a homogeneous solution of (7) but uniqueness is now assured.

Next, the summability of $\{y_k\}$ and the integrability of $x(s)$ allows the derivation, aided by (4), of the solvability condition

$$\sum_{k=1}^{\infty} f_k + \frac{1}{\pi}\int_0^{\infty} g(s)\,\mathrm{d}s = 0 \tag{9}$$

The integral of the term multiplying Y_0 in (8) can be evaluated without introducing the trigamma function. Thus

$$\sum_{k=1}^{\infty}\frac{4t^2k}{(t^2+k^2)^2} - 2 = 2t\sum_{k=1}^{\infty}\int_0^{\infty} u\,\mathrm{e}^{-ku}\sin tu\,\mathrm{d}u - 2$$

$$= 2t\int_0^{\infty}\frac{u\,\mathrm{e}^{-u}}{1-\mathrm{e}^{-u}}\sin tu\,\mathrm{d}u - 2 = 2\int_0^{\infty}\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{u\,\mathrm{e}^{-u}}{1-\mathrm{e}^{-u}}\right)\cos tu\,\mathrm{d}u$$

which yields

$$\frac{1}{\pi}\int_0^{\infty}\left[\sum_{k=1}^{\infty}\frac{4s^2\beta_k}{(s^2+\beta_k^2)^2} - \frac{2}{\pi}\right]\mathrm{d}s = \lim_{u\to 0}\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{u\,\mathrm{e}^{-u}}{1-\mathrm{e}^{-u}}\right) = -\frac{1}{2}$$

Thus the solvability condition (9) reduces, on substitution of (8), to

$$0 = (1-C)\frac{3}{\pi}\int_0^{\infty}\left(\frac{1}{s^2} - \frac{1}{\sinh^2 s}\right)\mathrm{d}s + \frac{X_0}{\pi} - \frac{Y_0}{2} - (1+C)\frac{3}{\pi}\left[\coth s - \frac{1}{s}\right]_0^{\infty} = -\frac{6C}{\pi}$$

which is satisfied when $C = 0$, in contrast to the value $C = 1$ used by [1]. The cancellation of X_0, Y_0 is expected because the shifts cannot influence the solvability of the system. Thus, from (5), the correct asymptotic behaviour is obtained by writing

$$X(s) = \frac{2\pi^2}{\pi^2-4} + \frac{x(s) - \frac{3}{s}(\coth s - \frac{1}{s})}{1 - 2s/(\sinh 2s)}, \qquad Y_k = \frac{4\pi}{\pi^2-4} + \frac{3}{\beta_k} + y_k \tag{10}$$

The ‘shift’ functions in (10) should be regarded as optimal choices because only their leading terms at infinity are determined by the solvability condition. For example, the same value of C is obtained if the denominator $(1 - 2s/\sinh 2s)$ is omitted from (5) but subsequent substitution into the solution (1) shows that the zeros of $\cosh s$ yield a different Fourier series that is presumably, but not obviously, cancelled from the total solution. Preferable are the poles associated with Moffatt vortices and the eigenfunction expansion used by [3].

References

[1] V.V. Meleshko, T.S. Krasnopolskaya, Piston Stokes flow in a semi-infinite channel, C. R. Acad. Sci. Paris, Série IIb 329 (2001) 1–6.

[2] V.V. Meleshko, A.M. Gomilko, Infinite systems for a biharmonic problem in a rectangle, Proc. Roy Soc. London A 453 (1997) 2139–2160.

[3] F.V. Katopodes, A.M.J. Davis, H.A. Stone, Piston flow in a two-dimensional channel, Phys. Fluids 12 (2000) 1240–1243.