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Thin electromagnetic waveguides

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Abstract We study the propagation of electromagnetic waves in a guide the section of which is a thin annulus. Owing to the presence of a small parameter, explicite approximations of the eigenmodes can be computed. ***To cite this article: N. Turbé, C. R. Mecanique 330 (2002) 391–396.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

acoustics / waves / vibrations

Guides électromagnétiques minces

Résumé On étudie la propagation d'ondes dans un guide électromagnétique dont la section est une couronne mince. Grâce à la présence d'un petit paramètre, des approximations des modes propres sont proposées. ***Pour citer cet article : N. Turbé, C. R. Mecanique 330 (2002) 391–396.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Pour un guide fermé, les modes propres s'obtiennent par calculs numériques. Dans cette note, on étudie la propagation d'une onde dans un guide électromagnétique dont la section Ω est une couronne mince. En supposant le milieu homogène et l'épaisseur de la couronne constante, on peut expliciter des développements asymptotiques.

La propagation d'une onde électromagnétique est régie par les équations de Maxwell [1]. Après changement de fonctions [2] et élimination du champ magnétique, on montre que la connaissance du champ électrique résulte de la résolution d'un problème de Dirichlet pour la composante longitudinale e_3 et d'un problème de Neumann pour la composante transversale e (inconnues à valeurs réelles) :

$$\Delta e_3 + \left(\omega^2\varepsilon\mu - \beta^2\right)e_3 = 0 \quad \text{dans } \Omega, \quad e_3 = 0 \quad \text{sur } \partial\Omega$$
$$\operatorname{rot}\operatorname{rot} e - \left(\omega^2\varepsilon\mu - \beta^2\right)e = \beta \operatorname{grad} e_3 \quad \text{dans } \Omega, \quad e \wedge \vec{N} = 0 \quad \text{sur } \partial\Omega$$

ω est la pulsation et β le nombre d'onde dans la direction x_3. La constante diélectrique ε et la perméabilité magnétique μ du milieu sont des constantes réelles, positives. $\vec{N}$ désigne un vecteur normal à $\partial\Omega$. La condition aux limites sur e est associée à l'opérateur rot rot défini par : $\operatorname{rot}\operatorname{rot} e = (e_{2,12} - e_{1,22}, e_{1,21} - e_{2,11})$.

L'épaisseur ηl de la couronne est supposée constante, petite d'ordre η.

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La frontière étant de classe C^1, on introduit des coordonnées adimensionnées (ζ_1, ζ_2), associées à la base locale $(\vec{n}, \vec{\tau})$, liée à l'un des bords de la frontière $\partial\Omega$. Les éléments propres du problème sont cherchés sous forme de développements en puissances de η. Par identification, dans les équations, des termes de puissances successives, on obtient les résultats suivants :

- Onde magnétique transverse

La courbe de dispersion et le champ électrique longitudinal ont pour approximations :

$$\omega^2\varepsilon\mu = \beta^2 + \eta^{-2}l^{-2}\left(\alpha_0^2 + 2\eta^2\alpha_0\alpha_2\right), \quad w_0 = C_0\sin(\alpha_0\zeta_1)$$

où $\alpha_0 = \alpha_0^{(n)} = n\pi$ $(n \in \mathbf{N}^*)$ et $(C_0(\zeta_2), \alpha_2)$ sont éléments propres d'une équation de Hill.

- Onde électrique transverse

La courbe de dispersion et le champ magnétique longitudinal ont pour approximations :

$$\omega^2\varepsilon\mu = \beta^2 + \eta^{-2}l^{-2}\left(a_0^2 + 2\eta^2 a_0 a_2\right), \quad p_0 = D_0\cos(a_0\zeta_1)$$

où $a_0 = a_0^{(n)} = n\pi (n \in \mathbf{N}^*)$ et $(D_0(\zeta_2), a_2)$ sont éléments propres d'une équation de Hill.

La justification des approximations est obtenue à partir d'une forme variationnelle du problème. Elle permet l'application du théorème de Kato de perturbations de valeurs spectrales.

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1. Introduction

For closed waveguides, frequencies and guided modes are determined by computational methods. However, when one size of the waveguide is small compared to others, asymptotic methods can lead quickly to approximations. We study here the propagation in a guide the section of which is a thin annulus. Some assumptions allow explicit calculations: the electromagnetic material is homogeneous, the distance between the two curves of the boundary is constant.

2. Statement of the problem

Let us consider an electromagnetic cylindrical waveguide $\Omega \times \mathbf{R}$. The bounded section has a boundary $\partial\Omega$, constituted by two curves, Γ_0 and Γ_1. The distance between them is small compared to the dimensions of the section Ω.

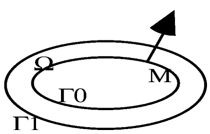

Figure 1. The problem.
Figure 1. *Le problème.*

Along $\Gamma_0 \times \mathbf{R}$ and $\Gamma_1 \times \mathbf{R}$, the waveguide is in contact with a perfect infinite conductor.

Let (x_1, x_2, x_3) represents the cartesian coordinates of a point of $\mathbf{R}^3$. We denote by $x = (x_1, x_2)$ the generic point of $\mathbf{R}^2$. An electromagnetic wave $(\vec{E}(x)\exp\mathrm{i}(\omega t - \beta x_3), \vec{H}(x)\exp\mathrm{i}(\omega t - \beta x_3))$ propagates in the guide. The real positive given parameter β is the wave number in the x_3-direction. Maxwell's harmonic equations [1] govern the propagation:

$$\mathrm{i}\omega\varepsilon\vec{E} - \mathrm{Rot}_\beta\,\vec{H} = 0, \quad \mathrm{i}\omega\mu\vec{H} + \mathrm{Rot}_\beta\,\vec{E} = 0 \quad \text{in } \Omega$$

$$\vec{E} \wedge \vec{N} = 0, \quad \vec{H} \cdot \vec{N} = 0 \quad \text{on } \partial\Omega$$

$\vec{N}$ is a normal vector to $\partial\Omega$ and $\mathrm{Rot}_\beta \vec{u}$ is the vector with the following components: $(u_{3,2} + \mathrm{i}\beta u_2, -u_{3,1} - \mathrm{i}\beta u_1, u_{2,1} - u_{1,2})$ (the notation $u_{i,j} = \partial u_i / \partial x_j$ is used). We assume that the dielectric constant ε and the magnetic permeability μ are real, positive constants.

We introduce real valued unknown fields [2]: $e_1 = E_1$, $e_2 = E_2$, $e_3 = \mathrm{i}E_3$, $h_1 = H_1$, $h_2 = H_2$, $h_3 = -\mathrm{i}H_3$. The equations now can be written:

$$\omega\varepsilon\vec{e} = \mathrm{rot}_{-\beta}\,\vec{h}, \qquad \omega\mu\vec{h} = \mathrm{rot}_\beta\,\vec{e},$$

where $\mathrm{rot}_\beta\,\vec{u}$ is the vector with components: $(u_{3,2} - \beta u_2, -u_{3,1} + \beta u_1, u_{2,1} - u_{1,2})$.

Eliminating the magnetic field $\vec{h}$, the problem in the electric field $\vec{e}$ follows:

$$\mathrm{rot}_{-\beta}(\mathrm{rot}_\beta \vec{e}) = \omega^2\varepsilon\mu\vec{e} \quad \text{in } \Omega \tag{1}$$

$$\vec{e} \wedge \vec{N} = 0 \quad \text{on } \partial\Omega \tag{2}$$

For β given wave number, there exists an infinite countable sequence of eigenfrequencies.

Let us denote by e_3 and e the longitudinal and transversal components, respectively, of the electric field $\vec{e}$. From (1), the relation $\mathrm{div}\, e = \beta e_3$ results, with $\mathrm{div}\, e = e_{1,1} + e_{2,2}$, and problem (1), (2) leads to the e_3-Dirichlet problem and the e-Neumann problem:

$$\Delta e_3 + \left(\omega^2\varepsilon\mu - \beta^2\right)e_3 = 0 \qquad \text{in } \Omega \tag{3}$$

$$e_3 = 0 \qquad \text{on } \partial\Omega \tag{4}$$

$$\mathrm{rot}\,\mathrm{rot}\, e - \left(\omega^2\varepsilon\mu - \beta^2\right)e = \beta\,\mathrm{grad}\, e_3 \quad \text{in } \Omega \tag{5}$$

$$e \wedge \vec{N} = 0 \qquad \text{on } \partial\Omega \tag{6}$$

This last boundary condition is associated with the rot rot operator which is defined by: $\mathrm{rot}\,\mathrm{rot}\, e = (e_{2,12} - e_{1,22}, e_{1,21} - e_{2,11})$.

Related to the distance between Γ_0 and Γ_1, a small parameter η allows an approximation of the eigenfrequencies and the eigenfunctions.

3. Approximations for a C^1 boundary

We introduce local coordinates. We denote by ξ_2 the arc length parameter along the boundary Γ_0. A point P is located in Ω, from a point M on Γ_0, by $M\vec{P} = \xi_1\vec{n}$, where $\vec{n} = \vec{n}(\xi_2)$ is the unitary normal to Γ_0 in M, oriented to the interior of Ω.

Let $\vec{\tau}$ be the tangential vector to Γ_0 in M, $\vec{\tau} = \vec{x}_3 \wedge \vec{n}$. From $O\vec{P} = O\vec{M} + \xi_1\vec{n}$, we have

$$\mathrm{d}\vec{P} = \vec{\tau}s\,\mathrm{d}\xi_2 + \vec{n}\,\mathrm{d}\xi_1 \tag{7}$$

with $s = 1 + \xi_1 R^{-1}$, $R(\xi_2)$ radius of curvature of Γ_0 in M.

Owing to relation (7), we can express $\mathrm{grad}\, p$ and Δp ($p(\xi_1, \xi_2)$ scalar function):

$$\mathrm{grad}\, p = \frac{\partial p}{\partial \xi_1}\vec{n} + \frac{1}{s}\frac{\partial p}{\partial \xi_2}\vec{\tau}, \qquad \Delta p = \frac{\partial^2 p}{\partial \xi_1^2} + \frac{1}{s}\frac{\partial}{\partial \xi_2}\left(\frac{1}{s}\frac{\partial p}{\partial \xi_2}\right) + \frac{1}{sR}\frac{\partial p}{\partial \xi_1}$$

Let l be a characteristic length of Γ_0. We define the dimensionless coordinates of P: $\xi_1 = \eta l \zeta_1, \xi_2 = l\zeta_2$, with $\eta \ll 1$ and $\zeta_1 = \mathrm{O}(1)$, $\zeta_2 = \mathrm{O}(1)$.

We assume that the equation of Γ_0 (resp. Γ_1) is $\zeta_1 = 0$ (resp. $\zeta_1 = 1$).

- Transverse magnetic (TM) wave: $e_3 \neq 0$, $h_3 = 0$.

With the new variables, the longitudinal electric field problem is written as follows:

$$\frac{\partial^2 e_3}{\partial \zeta_1^2} + \frac{\eta}{s\rho}\frac{\partial e_3}{\partial \zeta_1} + \frac{\eta^2}{s^2}\frac{\partial^2 e_3}{\partial \zeta_2^2} + \frac{\eta^3 \zeta_1 \rho'}{s^3 \rho^2}\frac{\partial e_3}{\partial \zeta_2} + \eta^2 l^2\left(\omega^2 \varepsilon \mu - \beta^2\right) e_3 = 0 \quad \text{in } \Omega \tag{8}$$

$$e_3 = 0 \quad \text{on } \partial\Omega \tag{9}$$

where $R = l\rho$, $s = 1 + \eta\rho^{-1}\zeta_1$ and $\rho' = \mathrm{d}\rho/\mathrm{d}\zeta_2$.

The presence of the small parameter η leads to search expansions of the solution:

$$e_3 = w_0 + \eta w_1 + \eta^2 w_2 + \cdots, \qquad \eta l\sqrt{\omega^2\varepsilon\mu - \beta^2} = \alpha = \alpha_0 + \eta\alpha_1 + \eta^2\alpha_2 + \cdots$$

We identify in (8), (9) the terms η^0, η^1 and η^2, assuming that ρ is not small, for instance O(1) compared to η. We obtain:

$$w_0 = C_0 \sin(\alpha_0\zeta_1), \quad \alpha_0 = \alpha_0^{(n)} = n\pi \quad \left(n \in \mathbf{N}^*\right)$$

$$w_1 = C_1 \sin(\alpha_0\zeta_1) - \frac{C_0}{2\rho}\zeta_1 \sin(\alpha_0\zeta_1), \quad \alpha_1 = 0$$

$$w_2 = C_2 \sin(\alpha_0\zeta_1) - \frac{C_1}{2\rho}\zeta_1 \sin(\alpha_0\zeta_1) + \frac{3C_0}{8\rho^2}\zeta_1^2 \sin(\alpha_0\zeta_1)$$

C_0, C_1, C_2 depend on ζ_2. The term α_2 and the function $C_0(\zeta_2)$ result from the boundary condition on $\zeta_1 = 1$ for w_2. This condition gives the following Hill's equation

$$C_0'' + \left(\frac{1}{4\rho^2} + 2\alpha_0\alpha_2\right) C_0 = 0$$

This equation has periodic solutions for an infinite countable sequence of the parameter α_2 [3]. So, we obtain the leading term of the longitudinal field and the approximation of the dispersion relation: $\omega^2\varepsilon\mu = \beta^2 + \eta^{-2}l^{-2}(\alpha_0^2 + 2\eta^2\alpha_0\alpha_2)$.

As for the transverse field, from $h_3 = 0$ we get $\operatorname{rot}\operatorname{rot} e = 0$. Thus, from (5), we obtain the approximation of e on the local basis. With $e = u\vec{n} + v\vec{\tau}$, the leading term of each component is: $u_1 = -\beta l \alpha_0^{-1} C_0 \cos(\alpha_0\zeta_1)$ and $v_2 = -\beta l \alpha_0^{-2} C_0' \sin(\alpha_0\zeta_1)$.

For a circular section, we can compare some eigenvalues. The characteristic length of Γ_0 is the radius $\rho = 1$ and the approximation is

$$\eta^{-1}\alpha_{\text{app}} = \eta^{-1} n\pi + \frac{\eta}{2n\pi}\left(m^2 - \frac{1}{4}\right)$$

Some computated eigenvalues $\eta^{-1}\alpha$ and approximations $\eta^{-1}\alpha_{\text{app}}$ are given for $\eta = 0.067$ in Table 1.

A good agreement between them is observed.

- Transverse electric (TE) wave: $e_3 = 0$, $h_3 \neq 0$.

In this case, $\operatorname{div} e = 0$ and the Hodge's decomposition [1] of e is: $e_1 = p_{,2}$, $e_2 = -p_{,1}$.

The unknown scalar function p can be determined from the Neumann problem:

$$\Delta p + \left(\omega^2\varepsilon\mu - \beta^2\right) p = 0 \quad \text{in } \Omega$$

$$\frac{\partial p}{\partial n} = 0 \quad \text{on } \partial\Omega$$

In a similar way as for the TM wave (with different boundary conditions), we search, for solution of this eigenvalue problem, expansions:

$$p = p_0 + \eta p_1 + \eta^2 p_2 + \cdots, \qquad \eta l\sqrt{\omega^2\varepsilon\mu - \beta^2} = a_0 + \eta a_1 + \eta^2 a_2 + \cdots$$

Table 1. Computed eigenvalues and approximations for TM wave.
Tableau 1. *Valeurs propres et approximations (onde magnétique transverse).*

	$\eta^{-1}\alpha$	$\eta^{-1}\alpha_{\text{app}}$	$\eta^{-1}\alpha$	$\eta^{-1}\alpha_{\text{app}}$
$m=0$	46.8869	46.8867	93.7776	93.7775
$m=1$	46.8969	46.8974	93.7826	93.7828
$m=2$	46.9269	46.9294	93.7976	93.7988
	$n=1$	$n=1$	$n=2$	$n=2$

Table 2. Computed eigenvalues and approximations for TE wave.
Tableau 2. *Valeurs propres et approximations (onde électrique transverse).*

	$\eta^{-1}\alpha$	$\eta^{-1}\alpha_{\text{app}}$	$\eta^{-1}\alpha$	$\eta^{-1}\alpha_{\text{app}}$
$m=0$	46.8969	46.8974	93.7826	93.7828
$m=1$	46.9067	46.9081	93.7875	93.7882
$m=2$	46.9369	46.9401	93.8026	93.8042
	$n=1$	$n=1$	$n=2$	$n=2$

We obtain: $p_0 = D_0\cos(a_0\zeta_1)$ with $a_0 = a_0^{(n)} = n\pi$ $(n \in \mathbf{N}^*)$, and then $a_1 = 0$. The term a_2 and the function $D_0(\zeta_2)$ result from the boundary condition on $\zeta_1 = 1$ for p_2. This condition gives the Hill equation:

$$D_0'' + \left(2a_0a_2 - \frac{3}{4\rho^2}\right)D_0 = 0$$

We get then the leading term of each component on the local basis of the transversal field: $v_0 = l^{-1}D_0a_0\sin(a_0\zeta_1)$, $u_1 = l^{-1}D_0'\cos(a_0\zeta_1)$. An approximation of the dispersion relation is, in this case: $\omega^2\varepsilon\mu = \beta^2 + \eta^{-2}l^{-2}(a_0^2 + 2\eta^2a_0a_2)$ (eigenvalues of multiplicity at least two).

For a circular section, with $\rho = 1$, some computated eigenvalues $\eta^{-1}\alpha$ and approximations,

$$\eta^{-1}\alpha_{\text{app}} = \eta^{-1}n\pi + \frac{\eta}{2n\pi}\left(m^2 + \frac{3}{4}\right)$$

are compared for $\eta = 0.067$ in Table 2.

- Transverse electric magnetic (TEM) wave $e_3 = 0 = h_3$.

In this case, $\omega^2\varepsilon\mu = \beta^2$ and the leading term of e is $u_0\vec{n}$, u_0 constant.

We can mention that for TM and TE waves, the cut-off frequencies are large of order η^{-1}.

4. Justification of the approximations

For the TM wave, let us consider the e_3-eigenvalue problem (8), (9), denoted by P_η. Using Green's formula, from (3), (4), one has for P_η:

$$e_3 \in H_0^1(\Omega) \quad \text{and} \quad a_\eta(e_3,\varphi) = \alpha^2(e_3,\varphi)_\eta \quad \forall\varphi \in H_0^1(\Omega) \tag{10}$$

with

$$a_\eta(e_3,\varphi)=\int_\Omega\left(\frac{\partial e_3}{\partial\zeta_1}\frac{\partial\varphi}{\partial\zeta_1}+\frac{\eta^2}{s^2}\frac{\partial e_3}{\partial\zeta_2}\frac{\partial\varphi}{\partial\zeta_2}\right)s\,\mathrm{d}\zeta_1\,\mathrm{d}\zeta_2\quad\text{and}\quad(e_3,\varphi)_\eta=\int_\Omega e_3\varphi s\,\mathrm{d}\zeta_1\,\mathrm{d}\zeta_2$$

As Ω can be identified with $[0,1]\times[0,T]$ (T such that lT is the length of the interior closed curve Γ_0), in fact the test function φ belongs to $H^1([0,1]\times[0,T])$ and satisfies: $\varphi(0,\zeta_2)=\varphi(1,\zeta_2)=0$ and $\varphi(\zeta_1,0)=\varphi(\zeta_1,T)$.

According to the previous results, we make a change of eigenfunction and test function:

$$e_3(\zeta_1,\zeta_2)=s^{-1/2}f(\zeta_1,\zeta_2)\quad\text{and}\quad\varphi(\zeta_1,\zeta_2)=s^{-1/2}\psi(\zeta_1,\zeta_2)$$

We obtain for P_η:

$$f\in(\Omega)\quad\text{and}\quad b_\eta(f,\psi)=\alpha^2(f,\psi)\quad\forall\psi\in H_0^1(\Omega)\tag{11}$$

with

$$\begin{aligned}b_\eta(f,\psi)=\int_\Omega\Bigg[&\frac{\partial f}{\partial\zeta_1}\frac{\partial\psi}{\partial\zeta_1}-\frac{\eta}{2s\rho}\left(f\frac{\partial\psi}{\partial\zeta_1}+\psi\frac{\partial f}{\partial\zeta_1}\right)+\frac{\eta^2}{4s^2\rho^2}f\psi+\frac{\eta^2}{s^2}\frac{\partial f}{\partial\zeta_2}\frac{\partial\psi}{\partial\zeta_2}\\&-\frac{\eta^3\zeta_1\rho'}{2s^3\rho^2}\left(f\frac{\partial\psi}{\partial\zeta_2}+\psi\frac{\partial f}{\partial\zeta_2}\right)+\frac{\eta^4\zeta_1^2\rho'^2}{4s^4\rho^4}f\psi\Bigg]\mathrm{d}\zeta_1\,\mathrm{d}\zeta_2\quad\text{and}\quad(f,\psi)=\int_\Omega f\psi\,\mathrm{d}\zeta_1\,\mathrm{d}\zeta_2\end{aligned}$$

From the bilinear form $b_\eta(f,\psi)$ in (11), we retain only the terms of order η^0, η^1 and η^2. After some integrations by parts, these terms lead to the expression:

$$\tilde b_\eta(f,\psi)=\int_\Omega\left(\frac{\partial f}{\partial\zeta_1}\frac{\partial\psi}{\partial\zeta_1}+\eta^2\frac{\partial f}{\partial\zeta_2}\frac{\partial\psi}{\partial\zeta_2}-\frac{\eta^2}{4\rho^2}f\psi\right)\mathrm{d}\zeta_1\,\mathrm{d}\zeta_2$$

And problem P_η appears to be a perturbation of the eigen problem:

$$\tilde e\in H_0^1(\Omega)\quad\text{and}\quad\tilde b_\eta(\tilde e,\psi)=\gamma^2(\tilde e,\psi)\quad\forall\psi\in H_0^1(\Omega)\tag{12}$$

We check that the function $\tilde e(\zeta_1,\zeta_2)=C_0(\zeta_2)\sin(\alpha_0\zeta_1)$, $\alpha_0=\alpha_0^{(n)}=n\pi$ and C_0 solution of the Hill equation of Section 3, is an eigenfunction of problem (12), associated with the eigenvalue $\gamma^2=\alpha_0^2+2\eta^2\alpha_0\alpha_2$. Kato's theorem [4] gives a justification of the approximations of Section 3.

For the TE wave, the same approach leads to the justification.

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