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Continuum modeling for the modulated vibration modes of large repetitive structures

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Abstract By homogenization theory, one can predict the vibrations of long repetitive structures in the low frequency range. Beyond this range, many modes have a modulated shape. Based on a multiple scale analysis, a continuum model is presented, that is able to account for this class of modes. This model involves a real coefficient that can be computed from the finite element resolution of problems defined on a few basic cells. An application in 2D elasticity is presented. ***To cite this article: E.M. Daya et al., C. R. Mecanique 330 (2002) 333–338.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

computational solid mechanics / solids and structures

Modèle continu pour les modes de vibrations modulés des longues structures répétitives

Résumé Grâce à la théorie de l'homogénéisation, on peut prédire les basses fréquences de vibrations des structures longues et répétitives. Pour des fréquences moyennes, beaucoup de modes ont une forme modulée. Nous présentons ici un modèle continu qui permet de prendre en compte cette classe de modes, grâce à la méthode des échelles multiples. Ce modèle dépend d'un paramètre réel qu'on peut calculer en résolvant par éléments finis des problèmes définis sur quelques cellules de base. Une application est présentée dans le cas de l'élasticité 2D. ***Pour citer cet article : E.M. Daya et al., C. R. Mecanique 330 (2002) 333–338.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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1. Introduction

Large structures exhibiting a repetitive form are used in many domains, as aerospace industry. Generally, the eigenmodes of these structures can appear as overall modes or modulated ones. For instance, let us consider a structure as the one pictured in Fig. 1. If the displacement is locked at one or several points of each basic cell, only modulated modes exist, sometimes together with a few localized modes [1]. On the contrary if all the basic cells have stress free boundaries except the first and the last one, the smallest eigenfrequencies correspond to overall modes, also called beam modes. In these two cases, most of the eigenfrequencies are closely located in well separated bands, see Fig. 2. Typical shapes for the modulated modes are presented in Fig. 3: they appear as slow modulations of a periodic mode. The latter property

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suggests that a continuum model can be established to describe this type of modes, by using a classical multiple scale analysis.

The classical homogenization theory [2] can be applied to get a good approximation of overall modes. This approach is to replace the actual repetitive structure by a substitute beam model that is equivalent to the original in some sense, by considering the constitutive relation, the strain energy and/or kinetic energy. Thus, they are many ways to deduce such an equivalent continuum beam see for instance [3–6]. All these theories are more or less based on Bernoulli kinematical assumptions. However, the latter theories are not valid in the case where the repetitive structure presents modulated modes. Indeed, these modes involves local deformations, that are not accounted by Bernoulli kinematics.

It is possible to built up another continuum modeling for this class of modes, as established in [7] in the case of simple periodic beam. In this paper, we apply the same ideas when the starting model is two-dimensional elasticity. The local deformations modes and the coefficient of the equivalent continuum model will be computed by the finite element method.

2. Two-scale analysis

2.1. Basic expansions

Let us consider an elastic repetitive structure, as pictured in Fig. 1. It is defined by the repetition of two-dimensional cells in x-direction. The corresponding classical free vibration problem can be written as follows:

$$\sigma_{xx,x} + \sigma_{xy,y} + \rho\lambda u = 0, \quad \sigma_{xy,x} + \sigma_{yy,y} + \rho\lambda v = 0 \tag{1}$$

$$Eu_{,x} - (\sigma_{xx} - \nu\sigma_{yy}) = 0, \quad Ev_{,y} - (\sigma_{yy} - \nu\sigma_{xx}) = 0, \quad E(u_{,y} + v_{,x}) - (1+\nu)\sigma_{xy} = 0 \tag{2}$$

where $\lambda = \omega^2$ is the square of the natural frequency. The total length in the x-direction is denoted by L and the basic cell length is denoted by l_x. $N = L/l_x$ is the number of cells.

We apply a two-scale expansion method to analyze the vibration problem (1), (2). The small parameter is defined by $\eta = l_x/L$. As usual within the multiple scale expansion method, the unknowns $U = (u, v, \sigma_{\alpha\beta})$ are assumed to be functions of three independent variables, i.e., the rapid variables x, y and the slow variable X. The definition of this X and the expansions rules are classical:

$$X = \eta x, \quad \partial_x \to \partial_x + \eta\partial_X \tag{3}$$

$$U(X, x, y) = \sum_{i=0}^{\infty} \eta^i U_i(X, x, y), \quad \lambda = \sum_{i=0}^{\infty} \eta^i \lambda_i \tag{4}$$

The unknowns $U = (u, v, \sigma_{\alpha\beta})$ and λ are expanded into powers of η. Each mode U is assumed to be "locally" periodic, i.e., periodic with respect to the rapid variable x. Inserting Eqs. (3), (4) into (1), (2), we find equations at the three first orders 1, η, η^2. By combining the constitutive law and the equilibrium equation, we get the following displacement problems for the displacement $\mathbf{u}_i = (u_i, v_i)$:

$$L_0\mathbf{u}_0 + \rho\lambda_0\mathbf{u}_0 = 0 \tag{5}$$

$$L_0\mathbf{u}_1 + \rho\lambda_0\mathbf{u}_1 = -\partial_X L_1\mathbf{u}_0 - \rho\lambda_1\mathbf{u}_0 \tag{6}$$

$$L_0\mathbf{u}_2 + \rho\lambda_0\mathbf{u}_2 = -\partial_X L_1\mathbf{u}_1 - \partial_X^2 L_2\mathbf{u}_0 - \rho\lambda_1\mathbf{u}_1 - \rho\lambda_2\mathbf{u}_0 \tag{7}$$

$$L_0 = \frac{E}{1-\nu^2}\begin{bmatrix} \partial_x^2 + (1-\nu)\partial_y^2 & \partial_x\partial_y \\ \partial_x\partial_y & \partial_y^2 + (1-\nu)\partial_x^2 \end{bmatrix}, \quad L_1 = \frac{E}{1-\nu^2}\begin{bmatrix} 2\partial_x & \partial_y \\ (1-\nu)\partial_y & 2(1-\nu)\partial_x \end{bmatrix},$$

$$L_2 = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & 0 \\ 0 & 1-\nu \end{bmatrix}$$

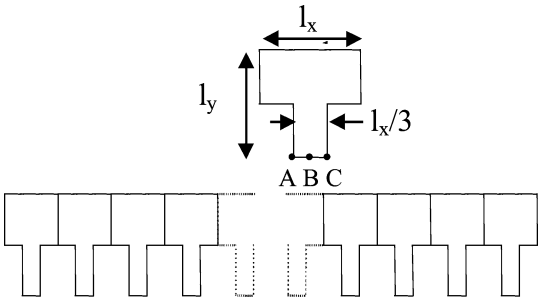

Figure 1. The considered repetitive structure and the basic cell.

Each Eq. (5), (6) or (7) will be considered as an elastic system posed on a period. The period consists of a few basic cells. The latter equations have also to be completed, first by the usual periodicity along the ends of the period, second by classical boundary conditions along the remainder of the cells. Here, we consider the following conditions:

$$u(A) = u(B) = v(B) = u(C) = 0 \tag{8}$$

and stress free conditions otherwise. With such restraints on each cell, the beam modes do not exist and the present analysis can hold for any frequency rather than the standard homogenization theory [7].

2.2. Deducing the continuum model

The general solution of Eqs. (5) and (6) can be written in the following form:

$$\mathbf{u}_0(x, X, y) = A_0(X)\mathbf{w}_0(x, y), \qquad \mathbf{u}_1(x, X, y) = A_1(X)\mathbf{w}_0(x, y) + A_0'\mathbf{w}_1(x, y) \tag{9}$$

where $\mathbf{w}_0$ and $\mathbf{w}_1$ solve the following equations to be completed by periodicity conditions and boundary conditions on the cell:

$$L_0\mathbf{w}_0 + \rho\lambda_0\mathbf{w}_0 = 0, \qquad L_0\mathbf{w}_1 + \rho\lambda_0\mathbf{w}_1 = -L_1\mathbf{w}_0 - \rho\lambda_1\mathbf{w}_0 \tag{10}$$

At this stage, the amplitudes $A_0(X)$ and $A_1(X)$ in (9) are arbitrary functions of the slow variable X, because only the rapid variables x and y appear in the differential operator $L_0 + \rho\lambda_0 I$. This operator is singular. Hence, the nonhomogeneous equations (6) and (7) have a solution if and only if the right-hand sides F of these equations satisfy the following solvability condition:

$$\langle F, \mathbf{w}_0\rangle = \iint_{\text{period}} F(X, x, y)\mathbf{w}_0(x, y)\,\mathrm{d}x\,\mathrm{d}y = 0 \tag{11}$$

By using the solvability condition for the problems at the second order (6) and at the third order (7), we obtain, first $\lambda_1 = 0$ and second a differential equation to be satisfied by the envelope:

$$CA_0'' + \lambda_2 A_0 = 0 \tag{12a}$$

$$C = \frac{\langle L_1\mathbf{w}_1 + L_2\mathbf{w}_0, \mathbf{w}_0\rangle}{\rho\langle \mathbf{w}_0, \mathbf{w}_0\rangle} \tag{12b}$$

So the simple differential equation (12a) is the sought continuum model. It is an eigenvalue problem for the amplitude $A_0(X)$, the eigenvalue being λ_2. To get the real constant C, it is necessary to compute the local deformation modes $\mathbf{w}_0$ and $\mathbf{w}_1$. The corresponding equations (10) are discretized by standard finite elements.

2.3. What boundary conditions for the continuum model?

It is not an easy task to deduce boundary conditions for the amplitude equation (12a). Often, there exist localized modes, that are not in agreement with the assumptions of the multiple scale method [1]. A correct treatment of boundary conditions should include a boundary layer analysis [7,1], that should be intricate in 2D elasticity. That is why we shall limit ourselves to a simple study of boundary conditions, and to

an application where this simple analysis is more or less valid. As for the boundary condition associated with the 2D model (1), (2), we assume here that the deflections at the ends $v(0, y)$ and $v(L, y)$ are zero. If the boundary layers are disregarded, the expansions (4) would be valid up to these ends. As explained for instance in [7], this yields boundary conditions for the amplitude. Let us underline that the latter boundary condition is not the same for all the periodic modes. The so deduced boundary conditions for the amplitude depend on the properties of the periodic mode $\mathbf{w}_0(x, y)$, that can be identical to zero along the ends $x = 0$, $x = L$ (case 2) or not (case 1):

$$A_0(0) = A_0(L\eta) = 0 \quad \text{in case 1}, \qquad A_0'(0) = A_0'(L\eta) = 0 \quad \text{in case 2} \tag{13}$$

The constant C is deduced from formulae (12b) after computing $\mathbf{w}_0$ and $\mathbf{w}_1$. Thus the amplitude equation (12a) is solved analytically on account of the boundary conditions. This leads to an approximation of the spectrum close to λ_0:

$$\begin{aligned} \lambda(n) &= \lambda_0 + \frac{Cn^2\pi^2}{L^2} + \theta\left(\eta^3\right), \quad n = 1, 2, \ldots \quad \text{in case 1} \\ \lambda(n) &= \lambda_0 + \frac{Cn^2\pi^2}{L^2} + \theta\left(\eta^3\right), \quad n = 0, 1, \ldots \quad \text{in case 2} \end{aligned} \tag{14}$$

Note that in this way, we have generated an infinite number of approximated eigenvalues λ. The range of validity of this approximation is now discussed.

3. Numerical results

Consider the structure defined in Fig. 1. The material data are $E = 2.1 \cdot 10^{11}$, $\nu = 0.3$, $\rho = 7800$. The boundary conditions along the cells have been presented previously, see (8), and those at the ends of the whole structure are:

$$v(0, y) = v(L, y) = 0 \tag{15}$$

The structure and the basic cell have been meshed by four node quadrilateral elements. The whole structure with 20 cells has been split into 320 elements, which corresponds to 966 d.o.f. For the basic cell, only 54 d.o.f. are needed. The obtained eigenfrequencies $\omega/2\pi$ are reported in Fig. 2. They are closely located in well separated packets. The first mode and the last mode of the first packet are plotted on Fig. 3. The first mode appears as a slow modulation of a periodic one. The last mode is exactly periodic. Note that the corresponding periodic modes are different: at the beginning of the packet, the period is $2l_x$ and only l_x at the end. These periodic modes have been obtained by the eigenvalue problem (5) set on two basic cells. The so computed eigenfrequencies $\lambda_{\text{begin}} = 133.03$ and $\lambda_{\text{end}} = 146.65$ are quite the same as the limits of the first packet.

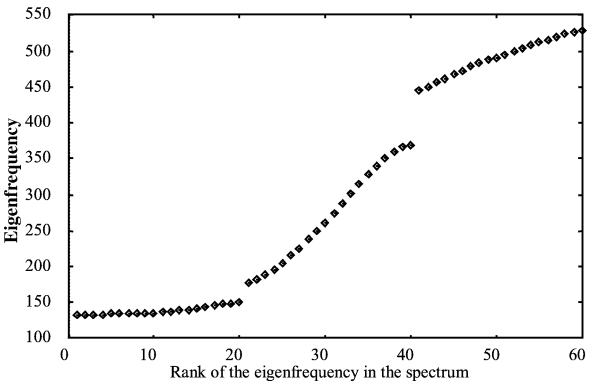

Figure 2. The three first packets of eigenfrequencies obtained by direct simulation for $N = 20$, $l_x = 6$ and $l_y = 1$.

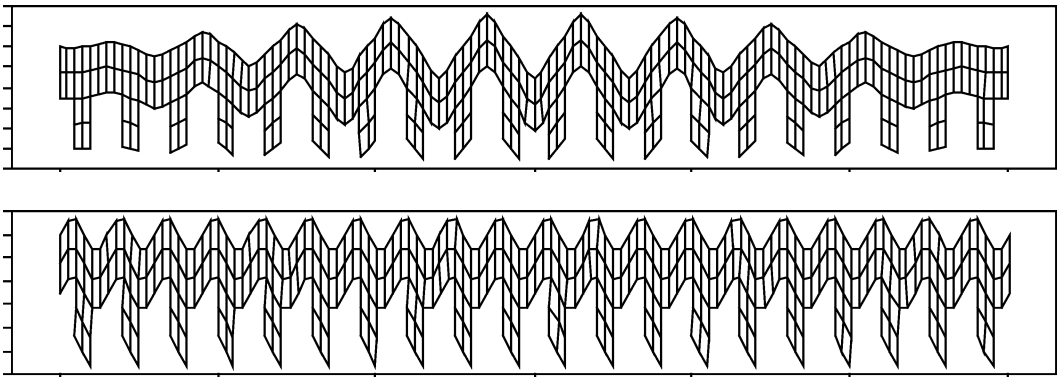

Figure 3. The first and the 20th eigenmodes when $N = 20$, $l_x = 6$ and $l_y = 1$.

Table 1. The first eight eigenfrequencies of the first packet when $N = 20$, $l_x = 6$ and $l_y = 1$.

Mode number	1	2	3	4	5	6	7	8
Proposed method	133.08	133.22	133.46	133.80	134.23	134.75	135.37	136.07
Direct computation	133.05	133.10	133.20	133.33	133.52	133.76	134.07	134.45

Table 2. The last eight eigenfrequencies of the first packet when $N = 20$, $l_x = 6$ and $l_y = 1$.

Mode number	13	14	15	16	17	18	19	20
Proposed method	123.40	130.88	136.89	141.61	145.18	147.68	149.16	149.65
Direct computation	138.28	139.61	141.19	143.03	145.09	147.19	148.93	149.65

Table 3. The first eight eigenfrequencies of the first packet when $N = 60$, $l_x = 6$ and $l_y = 1$.

Mode number	1	2	3	4	5	6	7	8
Proposed method	133.03	133.05	133.08	133.11	133.16	133.22	133.29	133.37
Direct computation	133.03	133.04	133.05	133.06	133.08	133.10	133.13	133.16

Table 4. The last eight eigenfrequencies of the first packet when $N = 60$, $l_x = 6$ and $l_y = 1$.

Mode number	53	54	55	56	57	58	59	60
Proposed method	146.96	147.68	148.29	148.78	149.16	149.43	149.60	149.65
Direct computation	146.50	147.19	147.84	148.43	148.93	149.32	149.57	149.65

In Tables 1, 2 we present some eigenfrequencies at the beginning and at the end of the first packet. The smallest frequencies are obtained from (13) with $\lambda_0 = \lambda_{\text{begin}}$, the constant $C = 737811.25$ being computed from the first $2l_x$-periodic mode $\mathbf{w}_0$ and the boundary conditions of case 1. As for the last frequencies, we get $C = -842415.46$ by starting from $\lambda_0 = \lambda_{\text{end}}$, $\mathbf{w}_0$ being the second $2l_x$-periodic mode (that is the first l_x-periodic mode). Because the vertical component of $\mathbf{w}_0$ is zero at the end of the cell, the boundary conditions are those of case 2. From the presented numerical results, it appears that the approximations are valid near the ends of a packet, but not at the center. We also observed that the first term of the expansions (4) yields good approximations of the exact modes.

With 60 cells, the results are reported in Tables 3, 4. For this case, the asymptotic method is much more accurate. Hence as expected, the larger the number of cells is, the more efficient the asymptotic method is.

4. Conclusions

A specific homogenized method has been developed to describe the modulated vibration modes of large repetitive structures. So we have deduced an amplitude equation, that is similar to those from bifurcation theory. In the present version of the theory, it is not possible to describe the whole packet. To achieve this goal, it is necessary to account for the interaction between two periodic modes, a first approach being presented in [1]. Likely, the approximated analysis would be better with a more accurate treatment of the boundary conditions.

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