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Stress equivalence principle for saturated porous media

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Abstract The stress equivalence principle for saturated porous media is studied in the plastic domain using a homogenization approach. The skeleton is composed of a micro-isotropic and micro-homogeneous material. The stress localization law in saturated porous media is first obtained. This makes it possible to define an appropriate effective stress tensor in the sense of the stress equivalence principle. The form of the effective stress tensor is examined for two particular yield functions of skeleton material. ***To cite this article: D. Lydzba, J.-F. Shao, C. R. Mecanique 330 (2002) 297–303.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

granular media / porous media / effective stress / poroplasticity / homogenization

Principe d'équivalence en contrainte pour milieux poreux saturés

Résumé Le principe d'équivalence en contrainte pour milieux poreux saturés est étudié dans le domaine plastique en utilisant une approche d'homogénéisation. Le squelette est composé d'un matériau micro-isotrope et micro-homogène. La loi de localisation des contraintes dans le milieu poreux saturé est d'abord déterminée. Celle-ci permet de définir une contrainte effective appropriée dans le sens du principe d'équivalence en contrainte. La forme du tenseur des contraintes effectives est étudiée pour deux fonctions de charge particulières du matériau squelette. ***Pour citer cet article : D. Lydzba, J.-F. Shao, C. R. Mecanique 330 (2002) 297–303.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

milieux granulaires / milieux poreux / contrainte effective / poroplasticité / homogénéisation

1. Introduction

The concept of effective stress provides a possibility to extend the constitutive equations and complementary plastic laws of dry material to saturated porous media by using the strain and stress equivalence principles [1]. The validity of the strain equivalence principle in the elastic domain has been confirmed from theoretical point of view ([2,3], among others). The validity of the effective stress concept in the inelastic range is still an open problem, particularly for cohesive materials like rocks and concrete.

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Recently, based on a strong assumption on the plastic change of porosity, a plastic effective stress has been proposed for the plastic modeling of porous media [4]. The validity of the assumption used needs, however, to be checked. More rigorous results, based on a homogenization approach, have also been obtained [5–7]. Particularly, it has been shown in [5,6], that the failure surface of a saturated porous medium can be expressed by that for dried material replacing the usual stress by the effective one. The appropriate form of the effective stress depends on the kind of failure criterion used at micro-level. The authors in [7] have found similar results concerning the initial yield surface. These studies were limited to two particular states of plastic domain. The present paper proposes a generalization of the work in [5,7] to the full plastic hardening domain. The porous medium is assumed to be composed of a micro-homogeneous and micro-isotropic skeleton material. A homogenization approach for periodic media [8] is used.

2. Stress localization law

2.1. Dry material

For the porous medium without any liquid in the pores, a local boundary value problem with a prescribed distribution of plastic strain inside the unit cell can be written [8]:

$$\begin{cases} \sigma^s_{ij,i} = 0 & \text{in } V_s \\ \sigma^s_{ij} n_i = 0 & \text{on } \Gamma \\ \sigma^s_{ij}(y) = C_{ijkh}\left[e_{kh}(u) - \varepsilon^p_{kh}\right] & \text{in } V_s \\ u_i(y) = E_{ij} y_j + u^*_i(y) & \text{in } V_s \\ f\left(\sigma^s_{ij}\right) \leqslant 0, \quad u^*_i(y)\ Y\text{-periodic}, \quad \sigma^s_{ij} n_i \text{ antiperiodic} & \end{cases} \tag{1}$$

where $\sigma^s_{ij}(y)$ are components of the solid micro-stress tensor, $u_i(y)$ – the displacement field, $u^*_i(y)$ – the fluctuating part of the displacement field, E_{ij} – the solid macro-strain tensor (treated as given), $e_{ij}(u)$ – the solid micro-strain tensor and $\varepsilon^p_{ij}(y)$ – the plastic micro-strain tensor (considered as given), C_{ijkl} – the elastic stiffness tensor of the skeleton material. The function $f(\sigma^s_{ij}) \leqslant 0$ represents the plastic yield criterion (at microscopic level). V_s is the volume of the solid material within the unit cell, Γ – the phase separation surface.

Introducing a plastic macro-strain tensor E^p_{ij} and a self-equilibrated residual stress field $\sigma^{\text{res}}_{ij}(y)$ (the stress field under a null macro-stress), the above local problem can be transformed to:

$$\sigma^s_{ij}(y) - \sigma^{\text{res}}_{ij}(y) = \left[C_{ijkh} + C_{ijlm} e_{lm}\left(\xi^{kh}\right)\right]\left(E_{kh} - E^p_{kh}\right) \tag{2}$$

and, finally, to the stress localization law [8]:

$$\sigma^s_{ij}(y) - \sigma^{\text{res}}_{ij}(y) = L_{ijkh}(y)\Sigma^s_{kh}, \quad L_{ijkh}(y) = \left[C_{ijpq} + C_{ijlm} e_{lm}\left(\xi^{pq}\right)\right] S^{\text{hom}}_{pqkh} \tag{3}$$

The tensor $\Sigma^s_{ij}\left(= \frac{1}{\|V\|}\int_{V_s} \sigma^s_{ij}(y)\,\mathrm{d}y\right)$ is the solid macro-stress tensor, S^{hom}_{pqkh} is the overall elastic compliance tensor, $L_{ijkh}(y)$ are components of the stress localization operator and $\xi^{kh}_i(y)$ are components of the fluctuating part of the displacement field at $\{\varepsilon^p_{ij}(y) = 0,\ E_{ij} = \delta_{ik}\delta_{jh}\}$, δ_{ij} is the Kronecker symbol and $\|V\|$ denotes a measure of the unit cell volume.

The plastic macro-strain tensor E^p_{ij} and residual stress field $\sigma^{\text{res}}_{ij}(y)$ are linear functionals of the plastic micro-strain field $\varepsilon^p_{ij}(y)$ and are determined by the following formulae [8]:

$$E^p_{ij} = \frac{1}{\|V\|}\int_{V_s} L_{khij}(y)\varepsilon^p_{kh}(y)\,\mathrm{d}y \tag{4}$$

$$\sigma^{\text{res}}_{ij}(y) = -\int_{V_s} R_{ijkh}(y, y')\varepsilon^p_{kh}(y')\,\mathrm{d}y' \tag{5}$$

where $R_{ijkh}(y, y')$ depends only on the geometry of microstructure within the unit cell.

2.2. Fully saturated material

For the porous material saturated with a liquid pressure p and with a prescribed distribution of plastic strain $\bar{\varepsilon}^p_{ij}(y)$, a local boundary value problem can be written:

$$\begin{cases} \sigma^s_{ij,i} = 0 & \text{in } V_s \\ \sigma^s_{ij} n_i = -p\delta_{ij} n_i & \text{on } \Gamma \\ \sigma^s_{ij}(y) = C_{ijkh}\left[e_{kh}(\bar{u}) - \bar{\varepsilon}^p_{kh}\right] & \text{in } V_s \\ \bar{u}_i(y) = \overline{E}_{ij} y_j + \bar{u}^*_i(y) & \text{in } V_s \\ f\left(\sigma^s_{ij}\right) \leqslant 0, \quad \bar{u}^*_i(y)\ Y\text{-periodic}, \quad \sigma^s_{ij} n_i \text{ antiperiodic} \end{cases} \tag{6}$$

As for the dry material, the above local problem can be transformed to:

$$\sigma^s_{ij}(y) - \bar{\sigma}^{\text{res}}_{ij}(y) = \left[C_{ijkh} + C_{ijlm} e_{lm}\left(\xi^{kh}\right)\right]\left[\overline{E}_{kh} - \overline{E}^p_{kh}\right] + C_{ijkh} e_{kh}(\eta) p \tag{7}$$

where: $\overline{E}^p_{ij}$ and $\bar{\sigma}^{\text{res}}_{ij}(y)$ are again the linear funtionals (4) and (5) of the plastic micro-strain field $\bar{\varepsilon}^p_{ij}(y)$, $\eta_i(y)$ are components of the fluctuating part of the displacement field at $\{\bar{\varepsilon}^p_{ij}(y) = 0,\ \overline{E}_{ij} = 0,\ p = 1\}$ [2].

The assumption of the micro-homogeneity and micro-isotropy of the skeleton material implies [7]:

$$\eta_i(y) = \frac{\xi^{kh}_i(y)\delta_{kh}}{3K_s} \tag{8}$$

and therefore, Eq. (7) can be rewritten as:

$$\sigma^s_{ij}(y) - \bar{\sigma}^{\text{res}}_{ij}(y) + p\delta_{ij} = \left[C_{ijkh} + C_{ijlm} e_{lm}\left(\xi^{kh}\right)\right]\left[\overline{E}_{kh} - \overline{E}^p_{kh} + \frac{p\delta_{kh}}{3K_s}\right] \tag{9}$$

where K_s is a bulk modulus of the skeleton material.

The above equation, after the volume averaging, results in the stress localization law for the saturated material:

$$\sigma^s_{ij}(y) - \bar{\sigma}^{\text{res}}_{ij}(y) + p\delta_{ij} = L_{ijkh}(y)\Sigma^{eT}_{kh} \tag{10}$$

where $\Sigma^{eT}_{ij}\ \left(= \frac{1}{\|V\|}\int_{V_s}(\sigma^s_{ij}(y) + p\delta_{ij})\,\mathrm{d}y\right)$ is the so-called Terzaghi's effective stress tensor.

3. Form of the effective stress

Following [8], the closure of the elastic domain in the macro-stress space, at prescribed distribution of residual stress field, can be expressed as:

- dry material

$$\mathbf{E}_{\text{D}}\left(\left\{\sigma^{\text{res}}_{ij}\right\}\right) = \left\{\Sigma^s_{ij} \mid f\left(L_{ijkh}(y)\Sigma^s_{kh} + \sigma^{\text{res}}_{ij}(y)\right) \leqslant 0\ \forall y \in V_s\right\} \tag{11}$$

- saturated material

$$\mathbf{E}_{\text{S}}\left(p, \left\{\bar{\sigma}^{\text{res}}_{ij}\right\}\right) = \left\{\Sigma^{eT}_{ij} \mid f\left(L_{ijkh}(y)\Sigma^{eT}_{kh} - p\delta_{ij} + \bar{\sigma}^{\text{res}}_{ij}(y)\right) \leqslant 0\ \forall y \in V_s\right\} \tag{12}$$

The set $\mathbf{E}_{\text{S}}(p; \{\bar{\sigma}^{\text{res}}_{ij}\})$ denotes, at prescribed residual stress field and value of p, the closure of the elastic domain in the Terzaghi's effective stress space.

It is clear from the above definitions that determination of the elastic domain, for both cases, requires the knowledge of the whole field of the residual stress $\{\sigma^{\text{res}}_{ij}(y)\}$ – for the dry material and $\{\bar{\sigma}^{\text{res}}_{ij}(y)_s\}$ – for the saturated material. In practice, however, some approximate models as work – or strain-hardening laws are used. For instance, assuming the strain-hardening rule to be valid for the dry material, the closure of the elastic domain is then approximated as:

$$\mathbf{E}_{\text{D}}\left(E^p_{ij}\right) = \left\{\Sigma^s_{ij} \mid F_D\left(\Sigma^s_{ij}; \chi\left(E^p_{ij}\right)\right) \leqslant 0\right\} \tag{13}$$

where $F_D(\Sigma^s_{ij}; \chi(E^p_{ij}))$ represents the macroscopic loading function for the dry material.

In the following, the concept of effective stress is examined by considering two classic plastic yield criteria for the skeleton material.

3.1. The Von-Mises criterion

This criterion is pressure insensitive. Therefore, defining for the saturated material an equivalent stress as:

$$\sigma_{ij}^{eq}(y) = \sigma_{ij}^{s}(y) + p\delta_{ij} \tag{14}$$

one gets the following identities:

$$(\forall p)\begin{cases} \sigma_{ij}^{eq}(y) = C_{ijkh}\left(e_{kh}^{eq}(\tilde{u}) - \bar{\varepsilon}_{kh}^{p}\right) \quad \text{with } e_{kh}^{eq}(\tilde{u}) = e_{kh}(\bar{u}) + \dfrac{p\delta_{kh}}{3K_s} \\ f\left(\sigma_{ij}^{s}\right) = f\left(\sigma_{ij}^{eq}\right), \quad \dfrac{\partial f(\sigma_{ij}^{s})}{\partial\sigma_{kh}^{s}} = \dfrac{\partial f(\sigma_{ij}^{eq})}{\partial\sigma_{kh}^{eq}} \end{cases} \tag{15}$$

The property (15b) together with the definition of the elastic domain (12) imply:

$$\mathbf{E}_{\mathrm{S}}\left(p, \{\bar{\sigma}_{ij}^{\mathrm{res}}\}\right) = \left\{\Sigma_{ij}^{eT} \mid f\left(L_{ijkh}(y)\Sigma_{kh}^{eT} + \bar{\sigma}_{ij}^{\mathrm{res}}(y)\right) \leqslant 0 \ \forall y \in V_s\right\} \tag{16}$$

The identities (15) enable also to transform the local problem (Eqs. (6)) to the equivalent one:

$$\begin{cases} \sigma_{ij,i}^{eq} = 0 & \text{in } V_s \\ \sigma_{ij}^{eq} n_i = 0 & \text{on } \Gamma \\ \sigma_{ij}^{eq}(y) = C_{ijkh}\left[e_{kh}^{eq}(\tilde{u}) - \bar{\varepsilon}_{kh}^{p}\right] & \text{in } V_s \\ \tilde{u}_i(y) = E_{ij}^{eq} y_j + u_i^{*}(y) \quad \text{with } E_{ij}^{eq} = \overline{E}_{ij} + \dfrac{p\delta_{ij}}{3K_s} & \text{in } V_s \\ f\left(\sigma_{ij}^{eq}\right) \leqslant 0, \quad u_i^{*}(y)\ Y\text{-periodic}, \quad \sigma_{ij}^{eq} n_i \text{ antiperiodic} & \end{cases} \tag{17}$$

Furthermore, for the plastic deformation of the skeleton material governed by an associative plastic flow rule, the properties (15b) lead to:

$$\dot{\bar{\varepsilon}}_{ij}^{p}(y) = \begin{cases} \dfrac{(\partial f/\partial\sigma_{kl}^{eq})C_{klmn}\dot{e}_{mn}^{eq}(y)}{(\partial f/\partial\sigma_{kl}^{eq})C_{klmn}(\partial f/\partial\sigma_{mn}^{eq})}\dfrac{\partial f}{\partial\sigma_{ij}^{eq}} & \text{for } f\left(\sigma_{ij}^{eq}(y)\right) = 0 \wedge \dot{f}\left(\sigma_{ij}^{eq}(y)\right) = 0 \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

The system of Eqs. (17), (18) represents the local elasto-plastic problem for the saturated material expressed by the equivalent fields introduced. It is clear that the corresponding local problem for the dry material is of the same form, except it contains the solid's strain and stress fields instead of the equivalent ones.

Let us now consider the saturated porous material, initially free of the plastic micro-strain field, subjected to a history $\{\overline{E}_{ij}(t), p(t)\}$. Let the pair:

$$\left\{\overline{E}_{ij}(t), p(t)\right\} \mapsto \left\{e_{ij}\left(\bar{u}(y,t)\right), \bar{\varepsilon}_{ij}^{p}(y,t)\right\} \tag{19}$$

to characterize a solution of the local elasto-plastic problem for the saturated material, i.e., the loading history and corresponding micro-strain fields induced in the unit cell. According to the system obtained (Eqs. (17), (18)), the above pair is equivalent to:

$$\left\{E_{ij}^{eq}(t)\right\} \mapsto \left\{e_{ij}^{eq}\left(\tilde{u}(y,t)\right), \bar{\varepsilon}_{ij}^{p}(y,t)\right\} \tag{20}$$

or:

$$\left\{E_{ij}(t) = E_{ij}^{eq}(t)\right\} \mapsto \left\{e_{ij}\left(u(y,t)\right) = e_{ij}^{eq}\left(\tilde{u}(y,t)\right), \varepsilon_{ij}^{p}(y,t) = \bar{\varepsilon}_{ij}^{p}(y,t)\right\} \tag{21}$$

The latter relation is a solution of the local problem for the dry material. It indicates that the solution for the saturated material can be recovered from the solution of the local problem for the dry material by imposing the history $\{E_{ij}(t)\} = \{\overline{E}_{ij}(t) + (p(t)/(3K_s))\delta_{ij}\}$. For these corresponding local problems, as a consequence of the functional relations (4), (5) and the constitutive equations, one gets also:

$$\begin{cases} \sigma_{ij}^{s}(y,t) = \sigma_{ij}^{eq}(y,t);\ \sigma_{ij}^{\text{res}}(y,t) = \bar{\sigma}_{ij}^{\text{res}}(y,t) & \forall y \in V_s \\ E_{ij}^{p}(t) = \overline{E}_{ij}^{p}(t); \quad \Sigma_{ij}^{s}(t) = \Sigma_{ij}^{eT}(t) \end{cases} \quad \text{for } E_{ij}(t) = \overline{E}_{ij}(t) + \frac{p(t)}{3K_s}\delta_{ij} \tag{22}$$

where the variables on the left side of Eqs. (22) correspond to the dry material whereas on the right side to the saturated material.

Now, comparing the definitions of the elastic domains (11) and (16), it follows immediately from (22) that:

$$\mathbf{E}_{\mathrm{S}}\big(p, \{\bar{\sigma}_{ij}^{\text{res}}\}\big) = \mathbf{E}_{\mathrm{D}}\big(\{\bar{\sigma}_{ij}^{\text{res}}\}\big) \tag{23}$$

which clearly indicates that the Terzaghi's effective stress fulfills the stress equivalence principle for the material considered. It is obvious that this statement can be extended also for porous materials composed of a uniform material obeying any arbitrary pressure insensitive yield criterion with an associated or a non-associative plastic flow rule. For the case of non-associative plasticity, the plastic potential has to be, however, of the form:

$$\tilde{g}(\sigma_{ij}) = aI_1 + g(J_2, J_3) \tag{24}$$

where I_1 is the first invariant of stress tensor, J_3 is the third invariant of deviatoric stress tensor, $g(J_2, J_3)$ is an arbitrary function of J_2 and J_3, 'a' is a parameter (could be $a = 0$). Such the form of the plastic potential allows to express the rate of the plastic micro-strain using the equivalent fields introduced above and the formula is of the same form as for the dry material. Therefore, the relations (22) as well as the identity (23) are still hold true.

Let us return to approximate models. If, for example, a description for the dry material uses, in a plastic range, the loading function (13) and a plastic flow rule as:

$$\dot{E}_{ij}^{p} = \lambda \frac{\partial G(\Sigma_{kh}^{s})}{\partial \Sigma_{ij}^{s}} \tag{25}$$

therefore, it follows from (22) that the model can be also successfully adopted for a description of the saturated material, using the Terzaghi's effective stress, i.e.:

$$\mathbf{E}_{\mathrm{S}}\big(p, \overline{E}_{ij}^{p}\big) = \big\{\Sigma_{ij}^{eT} \mid F_D\big(\Sigma_{ij}^{eT}; \chi\,(\overline{E}_{ij}^{p})\big) \leqslant 0\big\} \tag{26}$$

$$\dot{\overline{E}}_{ij}^{p} = \lambda \frac{\partial G(\Sigma_{kh}^{eT})}{\partial \Sigma_{ij}^{eT}} \tag{27}$$

The above relations clearly indicate that the Terzaghi's effective stress fulfills, in the plastic range, the stress as well as the strain equivalence principle for saturated media.

3.2. The Coulomb–Mohr criterion

The criterion is a pressure-sensitive. In contrast to the former case, we do not attempt to validate or not the concept of effective stress for any arbitrary history $\{\overline{E}_{ij}(t), p(t)\}$. The effective stress concept has been, however, confirmed for a particular history corresponding to a so-called drained condition, i.e., the history $\{\overline{E}_{ij}(t),\ p(t) = \text{const.}\}$. In the following, this case is only considered. The equivalent stress has now the following form:

$$\sigma_{ij}^{eq} = \frac{\sigma_{ij}^{s} + p\delta_{ij}}{1 + p\,\mathrm{tg}\,\varphi/c} \tag{28}$$

which fulfills the identity:

$$f\left(\sigma_{ij}^{s}\right)=\left(1+\frac{p\operatorname{tg}\varphi}{c}\right)f\left(\sigma_{ij}^{eq}\right) \tag{29}$$

where: φ represents a friction angle ($\varphi \geqslant 0$), c is the internal cohesion of the skeleton material ($c > 0$ is assumed). This equivalent stress was proposed in [5] for the study of failure criterion of saturated porous media.

Since $p \geqslant 0$, Eq. (29) implies:

$$f\left(\sigma_{ij}^{s}\right)\leqslant 0 \quad \text{if } f\left(\sigma_{ij}^{eq}\right)\leqslant 0; \qquad \frac{\partial f(\sigma_{ij}^{s})}{\partial\sigma_{kh}}=\frac{\partial f(\sigma_{ij}^{eq})}{\partial\sigma_{kh}^{eq}} \tag{30}$$

The relations (30) enable to transform the local problem (Eqs. (6)) to the equivalent one described by the system (17). Eqs. (17c) and (17d) involve now, however, the new equivalent variables, i.e.:

$$\begin{cases}\sigma_{ij}^{eq}(y)=C_{ijkh}\left(e_{kh}^{eq}(\tilde{u})-\dfrac{\varepsilon_{kh}^{p}}{1+p\operatorname{tg}\varphi/c}\right) & \text{with } e_{kh}^{eq}(\tilde{u})=\dfrac{e_{kh}(\bar{u})+p\delta_{kh}/(3K_s)}{1+p\operatorname{tg}\varphi/c}\\ \tilde{u}_i(y)=E_{ij}^{eq}y_j+u_i^{*}(y) & \text{with } E_{ij}^{eq}=\dfrac{\overline{E}_{ij}+p\delta_{ij}/(3K_s)}{1+p\operatorname{tg}\varphi/c}\end{cases} \tag{31}$$

Furthermore, for a given pore pressure ($p(t)=\text{const.}$) and the associated plastic flow rule, a rate of the plastic micro-strain can be presented as:

$$\frac{\dot{\bar{\varepsilon}}_{ij}^{p}(y)}{1+p\operatorname{tg}\varphi/c}=\begin{cases}\dfrac{(\partial f/\partial\sigma_{kl}^{eq})C_{klmn}\dot{e}_{mn}^{eq}(y)}{(\partial f/\partial\sigma_{kl}^{eq})C_{klmn}(\partial f/\partial\sigma_{mn}^{eq})}\dfrac{\partial f}{\partial\sigma_{ij}^{eq}} & \text{for } f\left(\sigma_{ij}^{eq}(y)\right)=0\wedge\dot{f}\left(\sigma_{ij}^{eq}(y)\right)=0\\ 0 & \text{otherwise}\end{cases} \tag{32}$$

Again, as for the Von-Mises criterion, the local elasto-plastic problem for the saturated material, expressed by the equivalent fields, is of the same form as for the dry material. Therefore, if the pair:

$$\left\{\overline{E}_{ij}(t),\,p(t)=\text{const.}\right\}\mapsto\left\{e_{ij}\left(\bar{u}(y,t)\right),\bar{\varepsilon}_{ij}^{p}(y,t)\right\} \tag{33}$$

characterize a solution of the local elasto-plastic problem for the saturated medium then one gets for the dry medium:

$$\left\{E_{ij}(t)=E_{ij}^{eq}(t)\right\}\mapsto\left\{e_{ij}\left(u(y,t)\right)=e_{ij}^{eq}\left(\tilde{u}(y,t)\right),\varepsilon_{ij}^{p}(y,t)=\frac{\bar{\varepsilon}_{ij}^{p}(y,t)}{1+p\operatorname{tg}\varphi/c}\right\} \tag{34}$$

For these corresponding local problems, the relation (34) implies:

$$\begin{cases}\sigma_{ij}^{s}(y,t)=\sigma_{ij}^{eq}(y,t); \quad \sigma_{ij}^{\text{res}}(y,t)=\dfrac{\bar{\sigma}_{ij}^{\text{res}}(y,t)}{1+p\operatorname{tg}\varphi/c}, \quad \forall y\in V_s\\ E_{ij}^{p}(t)=\dfrac{\overline{E}_{ij}^{p}(t)}{1+p\operatorname{tg}\varphi/c}; \quad \Sigma_{ij}^{s}(t)=\Sigma_{ij}^{eq}(t)\end{cases} \tag{35}$$

where the variables on the left side correspond to the dry material whereas on the right side to the saturated material.

The volume average of the equivalent micro-stress tensor (28) gives the macroscopic equivalent stress tensor :

$$\Sigma_{ij}^{eq}=\frac{\Sigma_{ij}^{eT}}{1+p\operatorname{tg}\varphi/c}=\frac{1}{\|V\|}\int_{V_s}\sigma_{ij}^{eq}(y)\,\mathrm{d}y \tag{36}$$

Accordingly, the closure of the elastic domain for the saturated material can be expressed as:

$$\mathbf{E}^{eq}\left(p,\left\{\bar{\sigma}_{ij}^{\text{res}}\right\}\right)\left\{\Sigma_{ij}^{eq}\;\middle|\;f\left(L_{ijkh}(y)\Sigma_{kh}^{eq}+\frac{\bar{\sigma}_{ij}^{\text{res}}(y)}{1+p\operatorname{tg}\varphi/c}\right)\leqslant 0,\ \forall y\in V_s\right\} \tag{37}$$

which together with the definition of the elastic domain (11) and the relations (35) result in:

$$\mathbf{E}^{eq}\left(p, \{\bar{\sigma}_{ij}^{\mathrm{res}}\}\right) = \mathbf{E}_{\mathrm{D}}\left(\left\{\frac{\bar{\sigma}_{ij}^{\mathrm{res}}}{1 + p\,\mathrm{tg}\,\varphi/c}\right\}\right) \tag{38}$$

The above result clearly indicates, that the equivalent macro-stress defined by (36) fulfills the stress equivalence principle, at the drained condition. It is clear that this statement is also valid for a non-associative plastic flow rule described by the potential (24) with a function $g(J_2, J_3)$ being a homogeneous function of deviatoric stress of the degree one.

According the approximate model described by (13) and (25), it follows from (35) that the model can be also successfully adopted, using the equivalent macro-stress tensor introduced, for the saturated material subjected to the drained condition, i.e.:

$$\mathbf{E}^{eq}\left(p, \overline{E}_{ij}^{p}\right) = \left\{\Sigma_{ij}^{eq} \;\middle|\; F_D\left(\Sigma_{ij}^{eq}; \chi\left(\frac{\overline{E}_{ij}^{p}}{1 + p\,\mathrm{tg}\,\varphi/c}\right)\right) \leqslant 0\right\} \tag{39}$$

$$\frac{\dot{\overline{E}}_{ij}^{p}}{1 + p\,\mathrm{tg}\,\varphi/c} = \lambda \frac{\partial G(\Sigma_{kh}^{eq})}{\partial \Sigma_{ij}^{eq}} \tag{40}$$

4. Conclusion

The validity of the effective stress concept has been investigated for saturated porous media composed of micro-homogeneous and micro-isotropic skeleton material. Two kinds of materials have been considered, respectively obeying a pressure independent (Von-Mises for instance) or a pressure dependent (Coulomb–Mohr) yield criterion at the local level. For the first type of material, it has been proved that the Terzaghi's effective stress fulfills, in the plastic range, the stress as well as the strain equivalence principles. For the second type of material, the effective stress tensor proposed in [5] for failure condition is generalized to plastic hardening range. Its validity has been proved for the drained condition.

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