C. R. Mecanique 330 (2002) 279–283

Nonuniqueness of solutions to differential equations for boundary-layer approximations in porous media

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Received 22 October 2001; accepted after revision 4 March 2002

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Abstract The free convection, along a vertical flat plate embedded in a porous medium, can be described in terms of solutions to $f''' + \frac{\alpha+1}{2} f f'' - \alpha f'^2 = 0$, for all $t \in (0, +\infty)$. The purpose of this Note is to study the nonuniqueness of solutions to this problem, with the initial conditions, $f(0) = a \in \mathbb{R}$ and $f'(0) \in \{0, 1\}$, where $\alpha \in (-\frac{1}{3}, 0)$. No assumption at infinity is imposed. We show that this problem has an infinite number of unbounded global solutions. Moreover, we prove that the first and the second derivative of solutions tend to 0 as t approaches infinity. ***To cite this article: M. Guedda, C. R. Mecanique 330 (2002) 279–283.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

porous media / boundary layer / existence and nonuniqueness

Non unicité de solutions des équations différentielles pour un problème de couches limites en milieux poreux

Résumé La modélisation d'un phénomène de convection naturelle dans un milieu poreux, occupant un domaine non borné, nous conduit à l'équation différentielle $f''' + \frac{\alpha+1}{2} f f'' - \alpha f'^2 = 0$, dans $(0, +\infty)$. Nous montrons que, pour $\alpha \in (-\frac{1}{3}, 0)$, cette équation avec les conditions initiales $f(0) = a \in \mathbb{R}$, $f'(0) = 0$ ou 1, admet une infinité de solutions globales, dont les dérivées d'ordre un et deux convergent vers 0 à l'infini. ***Pour citer cet article : M. Guedda, C. R. Mecanique 330 (2002) 279–283.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

milieu poreux / couche limite / existence et non unicité

1. Introduction

The free convection, along a vertical flat plate embedded in a porous medium, can be described in terms of solutions to the differential equation [1–3]

$$f''' + \frac{\alpha+1}{2} f f'' - \alpha f'^2 = 0 \tag{1.1}$$

posed in the semi-infinite interval $(0, +\infty)$, with the boundary conditions

$$f(0) = 0, \qquad f'(0) = 1 \tag{1.2}$$

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S1631-0721(02)01458-4/FLA

and

$$\lim_{t\to+\infty} f'(t) = 0 \tag{1.3}$$

Here the plate is impermeable and its temperature is assumed to be a power function of the height, i.e., $T(x,y)|_{y=0} = T_\infty + Ax^\alpha$, where $A > 0$, $\alpha \in \mathbb{R}$ and T_∞ is the temperature far from the plate which is the same at time $t' < 0$. The x-axis being parallel to the direction of gravity but directed upwards. For a physical consideration $-\frac{1}{3} \leqslant \alpha \leqslant 1$ [1,2]. Eq. (1.1) with suitable boundary conditions appears also in the context of boundary layer flow on stretching permeable surfaces with mass transfer parameter $a \neq 0$ [4,5]. In the last situation condition (1.2) takes the form

$$f(0) = a, \qquad f'(0) = 1 \tag{1.4}$$

The real a is referred also to as the suction/injection parameter. The case $a > 0$ corresponds to the suction and $a < 0$ to the injection of the fluid. A complete physical derivation of (1.1)–(1.4) is given in [6,2,4,5,3].

Results concerning problem (1.1)–(1.3) can be found in [1] in which the numerical solution has been performed in the case where $-\frac{1}{3} < \alpha < 0$. For the case $\alpha > -\frac{1}{2}$ numerical investigations are in [7] and [2]. The mathematical analysis is also considered in [2]. The authors showed the nonexistence of solutions to (1.1)–(1.3) if $\alpha < -\frac{1}{2}$ satisfying

$$\lim_{t\to\infty} f' f^2(t) = 0 \tag{1.5}$$

A similar argument is used in [4] to prove the nonexistence of nontrivial solutions to (1.1), (1.3) satisfying

$$f(0) = 0, \qquad f'(0) = 0 \tag{1.6}$$

where $\alpha \neq -\frac{1}{2}$.

Very recently Belhachmi, Brighi and Taous [3,8] showed the nonexistence of solutions to (1.1)–(1.3) for $\alpha \leqslant -\frac{1}{2}$ without condition (1.5). Among other results they proved that this problem has an infinite number of solutions when $\alpha = -\frac{1}{3}$ whereas uniqueness holds for $0 \leqslant \alpha \leqslant \frac{1}{3}$.

The remainder of this Note is to give negative answer to the open questions raised in [3,8] on the uniqueness of solution to (1.1)–(1.3) and of solutions to problem (1.1)–(1.3) supplemented with the condition

$$0 \leqslant f'(t) \leqslant 1, \quad \forall\, t \geqslant 0 \tag{1.7}$$

where $-\frac{1}{3} < \alpha < 0$. We shall show, in particular, that condition (1.2) is sufficient to provide an unbounded physical solution to (1.1)–(1.3); that is a solution satisfying (1.7).

For a mathematically consideration, we analyze Eq. (1.1) with more general initial conditions. It is proposed here to show, for $-\frac{1}{3} < \alpha < 0$, that Eq. (1.1) subject to the conditions

$$f(0) = a, \qquad f'(0) = \varepsilon \tag{1.8}$$

in which $\varepsilon \in \{0, 1\}$ and a is a real fixed, has an infinite number of unbounded solutions such that (1.3) holds, at least one of which satisfies (1.7) if $\varepsilon = 1$ and $a \geqslant 0$.

Remark 1.1. – As it is noticed by Brighi [9] a more general condition $f'(0) = b \geqslant 0$ can be transformed to $f'(0) = 0, 1$ by introducing a new function $kf(kt)$ which solves (1.1).

2. The result

As announced before we consider Eq. (1.1) with a more general initial condition (1.8). By using the shooting argument our main result can be formulated by

THEOREM 2.1. – *Let $a \in \mathbb{R}$, $\varepsilon = 0, 1$ and $\alpha \in (-\frac{1}{3}, 0)$. For any $\gamma > -\frac{1+\alpha}{2}a\varepsilon$, there exists a unique global solution, f_γ, to* (1.1), (1.8) *such that $f_\gamma''(0) = \gamma$. The function $f_\gamma(t)$ goes to infinity with t, and the first and the second derivative of f_γ tend to 0 as t approaches infinity.*

Moreover if (in addition) $\gamma \leqslant 0$, $\varepsilon = 1$ we have

$$0 \leqslant f_\gamma'(t) \leqslant 1 \tag{2.1}$$

Remark 2.1. – The condition $\gamma > -\frac{1+\alpha}{2}a\varepsilon$ can be replaced by $\gamma \geqslant -\frac{1+\alpha}{2}a\varepsilon$ if $(a, \varepsilon, \gamma) \neq (0, 0, 0)$.

Remark 2.2. – In [3] the authors showed the existence of $\gamma_\star < 0$ such that problem (1.1)–(1.3), (1.7) has a solution, $f_\star$, satisfying $f_\star''(0) = \gamma_\star$, for $\alpha \geqslant -\frac{1}{3}$. Moreover $0 \leqslant f_\star(t) \leqslant 2/\sqrt{\alpha+1}$. This solution is different from f_γ. Therefore problem (1.1)–(1.3), has at least two physical solutions if $-\frac{1}{3} < \alpha < 0$.

3. Proof

Let $-\frac{1}{3} < \alpha < 0$, a be a real and $\varepsilon = 0, 1$. We consider the initial value problem,

$$\begin{cases} f''' + \dfrac{\alpha+1}{2} f f'' - \alpha f'^2 = 0 \\ f(0) = a, \quad f'(0) = \varepsilon, \quad f''(0) = \gamma \end{cases} \tag{3.1}$$

The real γ is regarded as the shooting parameter. For every $\gamma \in \mathbb{R}$ problem (3.1) has a unique local solution f_γ defined on $(0, T_\gamma)$, $T_\gamma \leqslant +\infty$. This solution is of class C^3 on $[0, T_\gamma)$, in fact $f_\gamma \in C^\infty$. To begin with, suppose that a is nonnegative. We shall show that, for any $\gamma > -\frac{1+\alpha}{2}a\varepsilon$, f_γ is global and condition (1.3) is satisfied. The proof will proceed via a series of lemmas. Below we give an identity which is obtained by a simple integration of $(3.1)_1$ over $(0, t)$ [3] and will be useful in the proofs.

$$f_\gamma''(t) + \frac{1+\alpha}{2} f_\gamma'(t) f_\gamma(t) = \gamma + \frac{1+\alpha}{2}a\varepsilon + \frac{3\alpha+1}{2}\int_0^t f_\gamma'^2(s)\,\mathrm{d}s, \quad \forall t < T_\gamma \tag{3.2}$$

This propriety indicates that f_γ cannot have a local maximum. Let us note that if $T_\gamma < +\infty$, then $\lim_{t\uparrow T_\gamma} |f_\gamma(t)| + |f_\gamma'(t)| + |f_\gamma''(t)| = +\infty$. In fact the existence time T_γ is characterized by

LEMMA 3.1. – *If $T_\gamma < +\infty$, then $\lim_{t\uparrow T_\gamma} |f_\gamma(t)| = +\infty$.*

The proof is similar as in [10].

LEMMA 3.2. – *If $a \geqslant 0$ then $f_\gamma' > 0$, $f_\gamma > 0$ on $(0, T_\gamma)$ and $T_\gamma = +\infty$; that is f_γ is global. Moreover, f_γ' and f_γ'' are bounded.*

Proof. – It is not difficult to see that $f_\gamma' > 0$, $f_\gamma > 0$ on $(0, T_\gamma)$. To demonstrate that $T_\gamma = +\infty$ we consider a Lyapunov function for f_γ

$$E(t) = \frac{1}{2}\big(f_\gamma''(t)\big)^2 - \frac{\alpha}{3}\big(f_\gamma'(t)\big)^3$$

which satisfies

$$E'(t) = -\frac{1+\alpha}{2} f_\gamma \big(f_\gamma''\big)^2 \leqslant 0$$

thanks to $(3.1)_1$. Therefore E is bounded and then f_γ'' and f_γ' are bounded, since $\alpha < 0$. This in turn implies that if $T_\gamma < \infty$ the function f_γ is bounded which is absurd. □

LEMMA 3.3. – *If $a \geqslant 0$ then $f_\gamma(t)$ tends to infinity with t, $f_\gamma'(t)$ and $f_\gamma''(t)$ tend to 0 as $t \to \infty$.*

Proof. – From $(3.1)_1$ it follows that f_γ' is monotone on $(t_1, +\infty)$, t_1 large enough. Since f_γ' is bounded there exists $l \in \mathbb{R}_+$ such that $\lim_{t\to+\infty} f_\gamma'(t) = l$. This implies in particular the existence of a sequence (t_n)

tending to $+\infty$ with n such that $\lim_{n\to+\infty} f''_\gamma(t_n)=0$ and then $\lim_{t\to+\infty} f''_\gamma(t)=0$, by using the function E.

Next we suppose that f_γ is bounded, therefore $l=0$. Subsequently

$$0=\gamma+\frac{1+\alpha}{2}a\varepsilon+\frac{3\alpha+1}{2}\int_0^{+\infty} f'_\gamma(t)^2\,\mathrm{d}t$$

This is impossible if $\alpha>-\frac{1}{3}$. Therefore f_γ is unbounded and then $\lim_{t\to+\infty} f_\gamma(t)=+\infty$. It remains to prove that $l=0$ and f_γ, $\gamma\leqslant 0$, satisfies (2.1). Suppose that $l>0$. Together with $(3.1)_1$ we get, as t approaches infinity

$$f''_\gamma(t)=-\frac{1+\alpha}{2}l^2t+\frac{3\alpha+1}{2}l^2t+\mathrm{o}(t)$$
$$f''_\gamma(t)=\alpha l^2t+\mathrm{o}(t)$$

This is only possible if $\alpha=0$. Then $l=0$.

To show that f_γ satisfies (2.1) for $-\frac{1+\alpha}{2}a\leqslant\gamma\leqslant 0$, $\varepsilon=1$ we prove that $f''_\gamma(t)\leqslant 0$ for any $t\geqslant 0$ and then $0\leqslant f'_\gamma(t)\leqslant 1$. This ends the proof in the case where $a\geqslant 0$. □

Next we suppose that $a<0$. The first simple consequence is that $f_\gamma(t)<0$ and $f'_\gamma(t)>0$ for small $t>0$. Since f_γ cannot have a local maximum, we have two possibilities either $f_\gamma(t)$ vanishes at a some point and remains positive after this point or $f_\gamma(t)<0$ for all $t>0$. In fact we have

LEMMA 3.4. – *Assume $a<0$ and $\gamma>-\frac{1+\alpha}{2}a\varepsilon$. Then f_γ has exactly one zero, say t_0, tends to $+\infty$, and the functions f'_γ, f''_γ converge to 0 as $t\to+\infty$.*

Proof. – Assume on the contrary that $f_\gamma(t)<0$ for all $t\in(0,T_\gamma)$. Since $f'_\gamma>0$ then $T_\gamma=+\infty$ and in view of (3.2) we have $f''_\gamma>0$. Therefore, $\lim_{t\to+\infty} f_\gamma(t)\in(a,0]$ and $\lim_{t\to+\infty} f'_\gamma(t)=0$. This is absurd since f'_γ is positive and increasing function. To finish the proofs of Lemma 3.4 and Theorem 2.1 we note that

$$f_\gamma(t_0)=0,\qquad f''_\gamma(t_0)>0$$

Then the new function $h(t)=f_\gamma(t+t_0)$ satisfies Eq. $(3.1)_1$ and

$$h(0)\geqslant 0,\qquad h''(0)>-\frac{1+\alpha}{2}h(0)h'(0)$$

Therefore we use Lemmas 3.2 and 3.3 to conclude. □

Remark 3.1. – Since γ is arbitrary we deduce that problem (1.1)–(1.3) has an infinite number of solutions and it has a solution satisfying (1.7). These solutions are unbounded. This gives an answer to the open questions of [3].

We finish this Note by two results obtained in [11].

PROPOSITION 3.1. – *Let $\varepsilon=0,1$. For any $\gamma\geqslant-\frac{1+\alpha}{2}a\varepsilon$, $\gamma^2+\varepsilon^2+a^2\neq 0$, we have*

$$\lim_{t\to+\infty} f_\gamma(t)f''_\gamma(t)=\lim_{t\to+\infty} f'''_\gamma(t)=0$$

and

$$\lim_{t\to+\infty} f_\gamma(t)f'_\gamma(t)=+\infty$$

Remark 3.2. – The situation is different if $\alpha\in(-\frac{1}{2},-\frac{1}{3})$. In [3] it is shown that any possible solution, f, to (1.1)–(1.3) with $\alpha\in(-\frac{1}{2},-\frac{1}{3})$ satisfies $\lim_{t\to+\infty} ff'(t)=0$.

The following theorem shows that problem (1.1), (1.4), (1.7) has a solution for any $\alpha \in (-\frac{1}{2}, -\frac{1}{3})$ provided that $a \geqslant \sqrt{1/(1+\alpha)}$ [11].

THEOREM 3.1. – *For any* $-\frac{1}{2} < \alpha < 0$ *and any* $a \geqslant \sqrt{1/(1+\alpha)}$*, the problem*

$$\begin{cases} f''' + \dfrac{\alpha+1}{2} f f'' - \alpha f'^2 = 0 \\ f(0) = a, \quad f'(0) = 1, \quad f'(+\infty) = 0 \\ 0 \leqslant f'(t) \leqslant 1 \end{cases}$$

has at least one unbounded solution satisfying $\lim_{t\to+\infty} f_\gamma f_\gamma''(t) = 0$, $\lim_{t\to\infty} f_\gamma^2 f_\gamma'(t) = +\infty$.

Acknowledgements. The author would like to thank B. Brighi for stimulating discussions. This work was partially supported by Direction des Affaires Internationales (UPJV) Amiens, France.

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