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Effective equations describing the flow of a viscous incompressible fluid through a long elastic tube

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Abstract We study the flow of a viscous incompressible fluid through a long and narrow elastic tube whose walls are modeled by the Navier equations for a curved, linearly elastic membrane. The flow is governed by a given small time dependent pressure drop between the inlet and the outlet boundary, giving rise to creeping flow modeled by the Stokes equations. By employing asymptotic analysis in thin, elastic, domains we obtain the reduced equations which correspond to a Biot type viscoelastic equation for the effective pressure and the effective displacement. The approximation is rigorously justified by obtaining the error estimates for the velocity, pressure and displacement. Applications of the model problem include blood flow in small arteries. We recover the well-known Law of Laplace and provide a new, improved model when shear modulus of the vessel wall is not negligible. ***To cite this article: S. Čanić, A. Mikelić, C. R. Mecanique 330 (2002) 661–666.***

fluid mechanics / creeping blood flow / viscoelasticity

Equations efficaces décrivant l'écoulement d'un fluide visqueux incompressible dans un tuyau long et élastique

Résumé Nous considérons l'écoulement d'un fluide incompressible visqueux à travers un tuyau long et de faible épaisseur, ayant la paroi latérale obéissant aux équations de Navier pour une membrane courbe élastique linéaire. L'écoulement est régi par une petite chute de pression entre l'entrée et la sortie du tuyau et on a un écoulement lent décrit par les équations de Stokes. En utilisant la méthode des développements asymptotiques, avec la réduction dimensionnelle dans la partie mince, nous obtenons les équations limites. Elles correspondent aux équations viscoélastiques de Biot pour la pression efficace et les déplacements efficaces. L'approximation est justifiée rigoureusement en obtenant une estimation d'erreur pour la vitesse, la pression et les déplacements « dilatés ». Des applications incluent l'écoulement sanguin dans des arterioles. Nous retrouvons la bien connue Loi de Laplace et donnons un nouveau modèle amélioré lorsque le module de cisaillement de la paroi n'est pas négligeable. ***Pour citer cet article : S. Čanić, A. Mikelić, C. R. Mecanique 330 (2002) 661–666.***

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Nous considérons l'écoulement quasi-statique axisymétrique d'un fluide Newtonien incompressible à travers un cylindre droit, à petite épaisseur,

$$\Omega_\varepsilon = \left\{x \in \mathbb{R}^3;\, x = (r\cos\vartheta, r\sin\vartheta, z),\ r < \varepsilon R,\ 0 < z < L\right\} \tag{1}$$

La vitesse du fluide v^ε et la pression p^ε satisfont les équations de Stokes dans des coordonnées cylindriques

$$-\mu\left(\frac{\partial^2 v_r^\varepsilon}{\partial r^2} + \frac{\partial^2 v_r^\varepsilon}{\partial z^2} + \frac{1}{r}\frac{\partial v_r^\varepsilon}{\partial r} - \frac{v_r^\varepsilon}{r^2}\right) + \frac{\partial p^\varepsilon}{\partial r} = 0 \quad \text{dans } \Omega_\varepsilon \times \mathbb{R}_+ \tag{2}$$

$$-\mu\left(\frac{\partial^2 v_z^\varepsilon}{\partial r^2} + \frac{\partial^2 v_z^\varepsilon}{\partial z^2} + \frac{1}{r}\frac{\partial v_z^\varepsilon}{\partial r}\right) + \frac{\partial p^\varepsilon}{\partial z} = 0 \quad \text{et} \quad \frac{\partial v_r^\varepsilon}{\partial r} + \frac{\partial v_z^\varepsilon}{\partial z} + \frac{v_r^\varepsilon}{r} = 0 \quad \text{dans } \Omega_\varepsilon \times \mathbb{R}_+ \tag{3}$$

Nous supposons que la paroi latérale du cylindre $\Sigma_\varepsilon = \{r = \varepsilon R\} \times (0, L)$ est élastique et que sa déformation est décrite par les équations de Navier

$$F_r = \frac{h(\varepsilon)E(\varepsilon)}{1-\sigma^2}\left(\frac{\sigma}{\varepsilon R}\frac{\partial s^\varepsilon}{\partial z} + \frac{\eta^\varepsilon}{\varepsilon^2 R^2}\right) - h(\varepsilon)G(\varepsilon)k(\varepsilon)\frac{\partial^2 \eta^\varepsilon}{\partial z^2} + \rho_w h(\varepsilon)\frac{\partial^2 \eta^\varepsilon}{\partial t^2} \tag{4}$$

$$F_z = -\frac{h(\varepsilon)E(\varepsilon)}{1-\sigma^2}\left(\frac{\partial^2 s^\varepsilon}{\partial z^2} + \frac{\sigma}{\varepsilon R}\frac{\partial \eta^\varepsilon}{\partial z}\right) + \rho_w h(\varepsilon)\frac{\partial^2 s^\varepsilon}{\partial t^2} \tag{5}$$

Dans (4) et (5), η^ε est le déplacement radial et s^ε le déplacement longitudinale dans les coordonnées de Lagrange (voir Fig. 1), $h = h(\varepsilon)$ est l'épaisseur de la membrane, ρ_w sa densité, $E = E(\varepsilon)$ le module d'Young, $0 < \sigma < 1$ le coefficient de Poisson, $G = G(\varepsilon)$ le module de cisaillement et $k = k(\varepsilon)$ le facteur de correction de Timoshenko (voir [1]). F_r et F_z sont les composantes radiale et longitudinale de la force exercée par le fluide sur la paroi et données par

$$F_r\vec{e}_r + F_z\vec{e}_z = \left(p^\varepsilon I - 2\mu D(v^\varepsilon)\right)\vec{e}_r \quad \text{sur } \Sigma_\varepsilon \times \mathbb{R}_+ \tag{6}$$

où $D(v^\varepsilon)$ est le tenseur des vitesses de déformation. La vitesse du fluide v^ε est liée avec la vitesse de la paroi par

$$v_r^\varepsilon = \frac{\partial \eta^\varepsilon}{\partial t} \quad \text{et} \quad v_z^\varepsilon = \frac{\partial s^\varepsilon}{\partial t} \quad \text{sur } \Sigma_\varepsilon \times \mathbb{R}_+ \tag{7}$$

Nous ajoutons au système (1)–(7) les conditions aux limites (13)–(16).

On va obtenir les équations asymptotiques qui décrivent l'écoulement à l'échelle du temps dominante et aussi les oscillations de la membrane induites par le réponse du matériau élastique. Nous posons $\tilde{t} = \omega^\varepsilon t$ avec $\omega^\varepsilon = \varepsilon^2/\mu$. Ce choix conduit à l'égalité d'énergie (17), qui nous permet d'obtenir les estimations a priori pour la solution. Soient $E_0 = \lim_{\varepsilon\to 0} h(\varepsilon)E(\varepsilon)/\varepsilon$ et $G_0 = \lim_{\varepsilon\to 0} G(\varepsilon)h(\varepsilon)k(\varepsilon)\varepsilon$. Nous utilisons l'approche de Ciarlet et al. (cf. [3]) ou on remplace les inconnues par leurs « dilatées » $v(\varepsilon)(z, r, t) = v^\varepsilon(z, \varepsilon r, t)$ et $p(\varepsilon)(z, r, t) = p^\varepsilon(z, \varepsilon r, t)$. Ω_ε devient le domaine « non-mince » $\Omega = \Omega_1$. La vitesse et la pression « dilatées » satisfont les estimations a priori (19)–(21). Ensuite nous cherchons les équations efficaces en utilisant les développements asymptotiques suivantes

$$v(\varepsilon)(z, r, t) = \frac{\varepsilon^2}{\mu}\sum_{i\geqslant 0}\varepsilon^i v^i(z, r, t), \qquad p(\varepsilon)(z, r, t) = p^0(z, t) + \sum_{i\geqslant 1}\varepsilon^i p^i(z, r, t) \tag{8}$$

$$\eta^\varepsilon(z, t) = \varepsilon\sum_{i\geqslant 0}\varepsilon^i \eta^i(z, t), \qquad s^\varepsilon(z, t) = \sum_{i\geqslant 0}\varepsilon^i s^i(z, t) \tag{9}$$

qui nous donnent en Tableau 1 le problème parabolique pour la pression efficace $p = p^0$, et les expressions suivants pour les autres inconnus

$$\frac{E_0\eta^0}{R(1-\sigma^2)}=\frac{R}{2-\sigma}\left(-\frac{1}{2}p+\frac{h(\varepsilon)E(\varepsilon)R}{\varepsilon}\frac{1}{8(1-\sigma^2)}\frac{\partial^2}{\partial z^2}\int_0^t p\right) \tag{10}$$

$$\frac{\partial s^0}{\partial z}=\frac{R}{2-\sigma}\left(-\frac{R\sigma}{8}\frac{\partial^2}{\partial z^2}\int_0^t p+\frac{1-\sigma^2}{E_0}p\right) \tag{11}$$

$$v_z^0(z,r,t)=\frac{r^2-R^2}{4}\frac{\partial p}{\partial z}(z,t)+\frac{\partial s}{\partial t}(z,t),\qquad v_r^0=\frac{R}{r}\frac{\partial \eta}{\partial t}+\frac{(R^2-r^2)^2}{4r}\frac{\partial^2 p}{\partial z^2}-\frac{R^2-r^2}{2}\frac{\partial^2 s}{\partial z\partial t} \tag{12}$$

Notons que pour $G_0=0$, on obtient la loi (22), liant la pression efficace p et le déplacement radial efficace η. Pour $\sigma=1/2$ (22) est exactement la Loi de Laplace (voir [4]).

Nous justifions le problème efficace par une estimation d'erreur. Soit $\{\beta^{bl},\pi^{bl}\}$ la solution pour les équation de la couche limite (23) et (24). Alors on a

THÉORÈME 0.1. – *Soient* $\gamma(\varepsilon)=p(\varepsilon)-p-\pi^\varepsilon$, $w(\varepsilon)=\frac{\mu}{\varepsilon^2}v(\varepsilon)-v_z^0\vec{e}_z-\varepsilon v_r^0\vec{e}_r+\beta^\varepsilon$, $s(\varepsilon)=s^\varepsilon-s^0$ *et* $\eta(\varepsilon)=\eta^\varepsilon-\varepsilon\eta^0$. *Alors on a*

$$\|w(\varepsilon)\|_{L^2(0,T;W)}+\|\gamma(\varepsilon)\|_{L^2(\Omega\times(0,T))}\leqslant C\varepsilon^{3/2}$$

$$\sup_{0\leqslant t\leqslant T}\left\{\sqrt{G_0}\left\|\frac{\partial\eta(\varepsilon)}{\varepsilon\partial z}(t)\right\|_{L^2(0,L)}+\sqrt{E_0}\left\|\frac{\eta(\varepsilon)}{\varepsilon}(t)\right\|_{L^2(0,L)}+\left\|\frac{\partial s(\varepsilon)}{\partial z}(t)\right\|_{L^2(0,L)}\right\}\leqslant C\varepsilon^{3/2}$$

1. Statement of the problem

We study the quasi-static axisymmetric flow of a Newtonian incompressible fluid in a thin right cylinder Ω_ε, given by (1), whose radius is small with respect to its length Fluid velocity v^ε and pressure p^ε satisfy Stokes equations in Eulerian cylindrical coordinates (2) and (3). We assume that the lateral wall of the cylinder $\Sigma_\varepsilon=\{r=\varepsilon R\}\times(0,L)$ is elastic and that it satisfies the Navier equations (4) and (5). We denote by η^ε the radial and by s^ε the longitudinal displacement from the reference state, see Fig. 1, $e=h(\varepsilon)$ is the membrane thickness, ρ_w the wall volumetric mass, $E=E(\varepsilon)$ is the Young modulus, $0<\sigma<1$ the Poisson ratio, $G=G(\varepsilon)$ is the shear modulus and $k=k(\varepsilon)$ is the Timoshenko shear correction factor (see [1]). The radial and the longitudinal component of the forces exerted by the fluid on the lateral wall, F_r and F_z, and given by (6), where $D(v^\varepsilon)$ is the rate of the strain tensor. Fluid velocity v^ε is coupled with the velocity of the wall through (7). We note that the true coupling between the Eulerian quantity v^ε and the Lagrangian quantities $(\eta^\varepsilon,s^\varepsilon)$ is given by $v^\varepsilon(x+(\eta^\varepsilon,s^\varepsilon),t)=(\partial\eta^\varepsilon/\partial t,\partial s^\varepsilon/\partial t)(x,t)$. After linearization, compatible with linear elasticity and creeping flow, we approximate the interface by its initial position and the interface condition for the velocity by (7).

Initially, the cylinder is filled with fluid at velocity zero, and the entire structure is in an equilibrium at some reference pressure P_0,

$$\eta^\varepsilon=s^\varepsilon=\frac{\partial\eta^\varepsilon}{\partial t}=\frac{\partial s^\varepsilon}{\partial t}=0\quad\text{on }\Sigma_\varepsilon\times\{0\} \tag{13}$$

The following boundary conditions are motivated by the fact that only the pressure field is correctly observed at the outlet boundary:

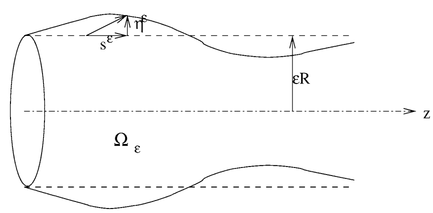

Figure 1. Wall displacement.

Figure 1. *Déformation de la paroi latérale.*

$$v_r^\varepsilon = 0 \quad \text{and} \quad p^\varepsilon = 0 \quad \text{on } \big(\partial\Omega_\varepsilon \cap \{z=0\}\big) \times \mathbb{R}_+ \tag{14}$$

$$v_r^\varepsilon = 0 \quad \text{and} \quad p^\varepsilon = A(t) \quad \text{on } \big(\partial\Omega_\varepsilon \cap \{z=L\}\big) \times \mathbb{R}_+ \tag{15}$$

$$\frac{\partial s^\varepsilon}{\partial z} = \eta^\varepsilon = 0 \quad \text{for } z=0, \quad s^\varepsilon = \eta^\varepsilon = 0 \quad \text{for } z=L \text{ and } \forall t \in \mathbb{R}_+ \tag{16}$$

The time-dependent pressure drop $A(t)$ between the inlet and the outlet boundary drives the flow. We are assuming that the pressure drop is small compared to the reference pressure P_0 so that the acceleration of the fluid is negligible compared to the effects of the fluid viscosity. Blood flow in small arteries satisfies these assumptions [2].

2. Main results

We derive the asymptotic equations that describe: (a) the flow occurring at the leading order time scale and (b) the oscillations of the membrane caused by a response of the elastic material. Since they occur at different time scales we introduce the scaling $\tilde{t} = \omega^\varepsilon t$. Energy estimate (17) below implies that $\omega^\varepsilon = \varepsilon^2/\mu$. This choice of ω^ε balances the term involving the time derivative of the velocity and the inertia term. It needs to be compatible with the oscillations of the pressure drop $A(t)$, which is related to the heart pulsation, blood viscosity and the size of the vessel via the so called Womersley number α [1]. The results that follow are all given in terms of the rescaled time $\tilde{t}$ where the wiggle sign is dropped for notational simplicity.

Energy estimate. – We first state the energy equality

$$\begin{aligned}
&\omega^\varepsilon h(\varepsilon)\varepsilon R \frac{\mathrm{d}}{2\,\mathrm{d}t}\bigg\{(\omega^\varepsilon)^2\rho_w \int_0^L \bigg(\bigg|\frac{\partial\eta^\varepsilon}{\partial t}\bigg|^2 + \bigg|\frac{\partial s^\varepsilon}{\partial t}\bigg|^2\bigg)\mathrm{d}z + G(\varepsilon)k(\varepsilon)\int_0^L \bigg|\frac{\partial\eta^\varepsilon}{\partial z}\bigg|^2 \mathrm{d}z \\
&\quad + \frac{E(\varepsilon)}{1-\sigma^2}\bigg(\sigma\int_0^L \bigg(\frac{\eta^\varepsilon}{\varepsilon R} - \frac{\partial s^\varepsilon}{\partial z}\bigg)^2 + (1-\sigma)\int_0^L \bigg(\bigg|\frac{\eta^\varepsilon}{\varepsilon R}\bigg|^2 + \bigg|\frac{\partial s^\varepsilon}{\partial z}\bigg|^2\bigg)\bigg)\bigg\} \\
&\quad + 2\mu\int_{\Omega_\varepsilon(t)} \big|D\big(v^\varepsilon(t)\big)\big|^2 r\,\mathrm{d}r\,\mathrm{d}z = -\int_0^{\varepsilon R} A(t)v_z^\varepsilon(t,r,L)r\,\mathrm{d}r
\end{aligned} \tag{17}$$

with $v_r^\varepsilon = \omega^\varepsilon\partial\eta^\varepsilon/\partial t$ and $v_z^\varepsilon = \omega^\varepsilon\partial s^\varepsilon/\partial t$ on $\Sigma_\varepsilon \times (0,T)$, which provides the following a priori estimate

PROPOSITION 2.1. – *Solution* $(v_r^\varepsilon, v_z^\varepsilon, \eta^\varepsilon, s^\varepsilon)$ *satisfies the following a priori estimate.*

$$\begin{aligned}
&\omega^\varepsilon \frac{h(\varepsilon)E(\varepsilon)}{4\varepsilon R(1+\sigma)}\bigg\{\int_0^L \big|\eta^\varepsilon(t)\big|^2\mathrm{d}z + \varepsilon^2R^2\bigg\|\frac{\partial s^\varepsilon}{\partial z}(t)\bigg\|_{L^2}^2\bigg\} + \mu\int_0^t\int_{\Omega_\varepsilon}\big|D(v^\varepsilon)\big|^2 r\,\mathrm{d}r\,\mathrm{d}z \\
&\quad + \omega^\varepsilon G(\varepsilon)k(\varepsilon)\varepsilon R\int_0^L \bigg|\frac{\partial\eta^\varepsilon(t)}{\partial z}\bigg|^2\mathrm{d}z \leqslant C(R,L)\frac{\varepsilon^4}{\mu}\int_0^t \big\{\big|A(\tau)\big|^2 + \big|\partial_\tau A(\tau)\big|^2\big\}\,\mathrm{d}\tau
\end{aligned} \tag{18}$$

A priori solution estimates follow from here. We use the approach introduced by Ciarlet et al. (see, e.g., [3] and the references therein) and consider the scaled velocity $v(\varepsilon)(z,r,t) = v^\varepsilon(z,\varepsilon r,t)$ and pressure $p(\varepsilon)(z,r,t) = p^\varepsilon(z,\varepsilon r,t)$ defined on the scaled, fixed domain $\Omega = \Omega_1$. Let $E_0 = \lim_{\varepsilon\to 0} h(\varepsilon)E(\varepsilon)/\varepsilon$ and $G_0 = \lim_{\varepsilon\to 0} G(\varepsilon)h(\varepsilon)k(\varepsilon)\varepsilon$. Then (18) leads to the following a priori estimates for the scaled unknowns.

COROLLARY 2.1. – *Scaled solution* $(v(\varepsilon), p(\varepsilon), \eta^\varepsilon, s^\varepsilon)$ *satisfies the a priori estimates*

$$\bigg\|\frac{v(\varepsilon)_r}{r}\bigg\|_{L^2(\Omega\times(0,T))} + \bigg\|\frac{\partial v(\varepsilon)_z}{\partial r} + \varepsilon\frac{\partial v(\varepsilon)_r}{\partial z}\bigg\|_{L^2(\Omega\times(0,T))} + \bigg\|\frac{\partial v(\varepsilon)_r}{\partial r}\bigg\|_{L^2(\Omega\times(0,T))} \leqslant C\frac{\varepsilon^2}{\mu}\|A\|_{\mathcal{V}} \tag{19}$$

$$\frac{\mu}{\varepsilon}\bigg\|\frac{\partial v(\varepsilon)_z}{\partial z}\bigg\|_{L^2(\Omega\times(0,T))} + \big\|p(\varepsilon)\big\|_{L^2(\Omega\times(0,T))} + \frac{1}{\varepsilon}\bigg\|\frac{\partial}{\partial r}p(\varepsilon)\bigg\|_{L^2(0,T;H^{-1}(\Omega))}^2 \leqslant C\|A\|_{\mathcal{V}} \tag{20}$$

$$\frac{1}{\varepsilon^2}\left\|\eta^\varepsilon(t)\right\|^2_{L^2(0,L)}+\left\|\frac{\partial s^\varepsilon}{\partial z}(t)\right\|^2_{L^2(0,L)}+\frac{G_0}{E_0}\frac{1}{\varepsilon^2}\left\|\frac{\partial \eta^\varepsilon(t)}{\partial z}\right\|^2_{L^2(0,L)}\leqslant CE_0\|A\|^2_{\mathcal{V}} \tag{21}$$

where $\|A\|^2_{\mathcal{V}}=\int_0^T\{|A(\tau)|^2+|\partial_\tau A(\tau)|^2\}\,\mathrm{d}\tau$.

These a priori estimates are insufficient to control the velocity field correctly. Further calculations imply

$$\left\|\frac{\partial v(\varepsilon)_z}{\partial r}\right\|_{L^2(\Omega\times(0,T))}+\varepsilon\left\|\frac{\partial v(\varepsilon)_r}{\partial z}\right\|_{L^2(\Omega\times(0,T))}\leqslant\frac{\varepsilon^2}{\mu}\left(\|A\|_{\mathcal{V}}+\varepsilon\left\|\frac{\partial s^\varepsilon}{\partial t}\right\|_{L^2((0,L)\times(0,T))}\right)$$

We observe that the above estimates also hold for the time derivatives of the unknowns, but with A replaced by $\partial_t A$, assuming the compatibility condition $A(0)=0$.

Asymptotic expansions. – The above a priori estimates provide the multi-scale asymptotic expansions listed in (8) and (9).

The reduced problem. – By plugging the multi-scale expansions into the scaled Stokes equations and by taking into account the boundary conditions which are the Navier equations for the membrane, we obtain the initial-boundary value problem (see Table 1) for the effective pressure $p=p^0$,

Table 1. The initial boundary value problem.

Tableau 1. *Le problème parabolique.*

Small G_0	Non-negligible G_0
$\left(\frac{5}{2}-2\sigma\right)\frac{\partial p}{\partial t}=\frac{E_0R}{8}\frac{\partial^2 p}{\partial z^2}$	$\frac{\partial}{\partial t}\left\{\left(\frac{5}{2}-2\sigma\right)p-\left(1-\sigma^2\right)\frac{G_0R^2}{2E_0}\frac{\partial^2 p}{\partial z^2}\right\}=\frac{\partial^2}{\partial z^2}\left\{\frac{E_0R}{8}p-\frac{G_0R^3}{8}\frac{\partial^2 p}{\partial z^2}\right\}$
$p(0,t)=p(z,0)=0;\quad p(L,t)=A(t)$	$p(0,t)=0;\quad p(L,t)=A(t)$
$p(z,0)=0$	$\frac{\partial^2 p}{\partial z^2}(0,t)=0;\quad \frac{E_0R}{8(1-\sigma^2)}\frac{\partial^2 p}{\partial z^2}(L,t)=\frac{1}{2}\frac{\mathrm{d}A}{\mathrm{d}t}$

and formulas (10)–(12) for the other effective unknowns. Denote by $\mathcal{P}$ the initial value problem for p coupled with (10)–(12). In the case when the shear stress terms are negligible we find that pressure is directly related to the radial displacement via

$$p=\frac{E_0\eta}{(1-\sigma/2)R^2}=\left(\lim_{\varepsilon\to 0}\frac{h(\varepsilon)E(\varepsilon)}{(1-\sigma/2)\varepsilon R}\right)\left(1-\sqrt{\frac{\mathcal{A}(0)}{\mathcal{A}}}+\mathcal{O}\left(\frac{\eta}{R}\right)\right) \tag{22}$$

where $\mathcal{A}$ denotes the cross-sectional area of the vessel. For $\sigma=1/2$ (incompressible materials) our calculation implies $s=0$ and Eq. (22) reduces to the Law of Laplace [4] typically used in literature [5, 2,1] to model vessel walls via the independent ring model. We note that if the inertia terms have been taken into account to model the flow by the Navier–Stokes equations, the reduced equation for the pressure (or effective displacement) would have been hyperbolic. Creeping flow leads to a parabolic problem for the effective pressure, shown above in $\mathcal{P}$.

Convergence result. – As in the study of the Stokes and the Navier–Stokes equations in thin, rigid, cylinders (see [6] and subsequent works by numerous authors), the a priori estimates and uniqueness for problem $\mathcal{P}$ allow us to justify the effective model by obtaining the weak limit as $\varepsilon\to 0$.

THEOREM 2.2. – *Suppose for simplicity that* $G_0>0$ *and let* $W=\{\varphi\in L^2(\Omega)\mid \frac{\partial\varphi}{\partial r}\in L^2(\Omega)\}$. *Then the sequence* $(\mu/\varepsilon^2 v(\varepsilon)_z,\,p(\varepsilon),\,\eta^\varepsilon/\varepsilon,\,s^\varepsilon)$ *converges weakly in* $L^2(0,T;W\times L^2(\Omega)\times H^1(0,L)^2)$ *to a unique solution of the initial value problem* $\mathcal{P}$. *Furthermore,* $\frac{\mu}{\varepsilon^2}v_r(\varepsilon)\rightharpoonup 0$ *weakly in* $L^2(0,T;W)$.

Error estimates. – Since weak convergence does not provide enough information about the accuracy of the approximation of the true solution by the solution of the reduced problem $\mathcal{P}$, we find the order of approximation by calculating the error estimates of the difference between the two solutions. We found that the reduced problem gives rise to a boundary layer in the velocity and pressure at the outlet boundary $z = L$. We construct the boundary layer explicitly by considering the following abstract problem on a semi-infinite rigid-wall cylinder $Z^- = S \times \mathbb{R}_-$, where $S = \{r < R\} \times \{y_3 = 0\}$:

$$-\Delta\beta^{bl} + \nabla\pi^{bl} = 0 \quad \text{and} \quad \operatorname{div}\beta^{bl} = 0 \quad \text{in } Z^- \tag{23}$$

$$\beta_r^{bl} = v_r^0(t, r, L), \quad -2\frac{\partial\beta_z^{bl}}{\partial y_3} + \pi^{bl} = -2\frac{\partial v_z^0}{\partial z}(t, r, L) \quad \text{on } y_3 = 0 \quad \text{and} \quad \beta^{bl} = 0 \quad \text{on } \partial S \times \mathbb{R}_- \tag{24}$$

The results from [7] give the existence of a unique smooth velocity and pressure fields, exponentially decreasing when $|y_3| \to \infty$. We now define the boundary layer velocity and the boundary layer pressure to be $\beta^\varepsilon(t, r, z) = \varepsilon\beta^{bl}(t, r, \frac{z-L}{\varepsilon})$ and $\pi^\varepsilon(t, r, z) = \varepsilon^2\pi^{bl}(t, r, \frac{z-L}{\varepsilon})$, respectively. These functions are exponentially small away from the outlet boundary. Assuming that the solution of problem $\mathcal{P}$ is sufficiently smooth and that the convergence toward G_0 and E_0 is sufficiently fast, we obtain

THEOREM 2.3. – *Let* $\gamma(\varepsilon) = p(\varepsilon) - p - \pi^\varepsilon$, $w(\varepsilon) = \frac{\mu}{\varepsilon^2}v(\varepsilon) - v_z^0\vec{e}_z - \varepsilon v_r^0\vec{e}_r + \beta^\varepsilon$, $s(\varepsilon) = s^\varepsilon - s^0$ *and* $\eta(\varepsilon) = \eta^\varepsilon - \varepsilon\eta^0$ *. Then we have*

$$\left\|w(\varepsilon)\right\|_{L^2(0,T;W)} + \left\|\gamma(\varepsilon)\right\|_{L^2(\Omega\times(0,T))} \leqslant C\varepsilon^{3/2} \tag{25}$$

$$\sup_{0\leqslant t\leqslant T}\left\{\sqrt{G_0}\left\|\frac{\partial\eta(\varepsilon)}{\varepsilon\partial z}(t)\right\|_{L^2(0,L)} + \sqrt{E_0}\left\|\frac{\eta(\varepsilon)}{\varepsilon}(t)\right\|_{L^2(0,L)} + \left\|\frac{\partial s(\varepsilon)}{\partial z}(t)\right\|_{L^2(0,L)}\right\} \leqslant C\varepsilon^{3/2} \tag{26}$$

Therefore, we have shown that the error between the solution of the full set of axisymmetric Stokes equations coupled with the Navier equations for the elastic tube wall, and the solution of the reduced problem $\mathcal{P}$, modified by the boundary layer velocity and pressure, is of order $\varepsilon^{3/2}$, where ε is the ratio between the radius and the length of the elastic tube. We point out that in the interior of the domain, away from the outlet boundary, boundary layer has negligible effect and the accuracy of the approximation of the solution to the original, non-reduced problem, by the solution of problem $\mathcal{P}$ away from the outlet boundary is ε^2.

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