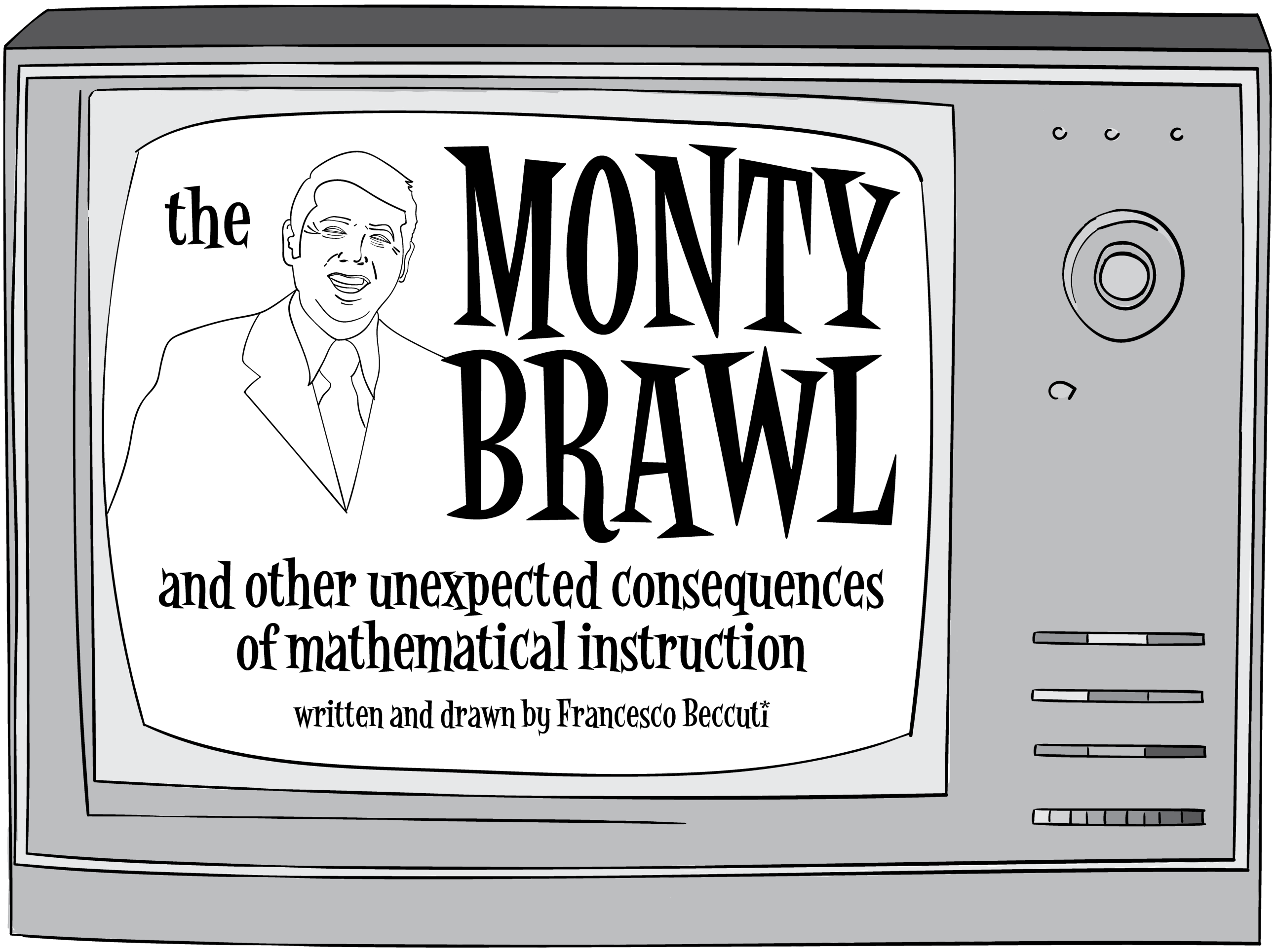
The Mathematics Enthusiast

Manuscript 1632

The Monty Brawl and other unexpected consequences of mathematical instruction

Francesco Beccuti



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The Mathematics Enthusiast, ISSN 1551-3440, vol. 21, nos. 1&2, pp. 377-404

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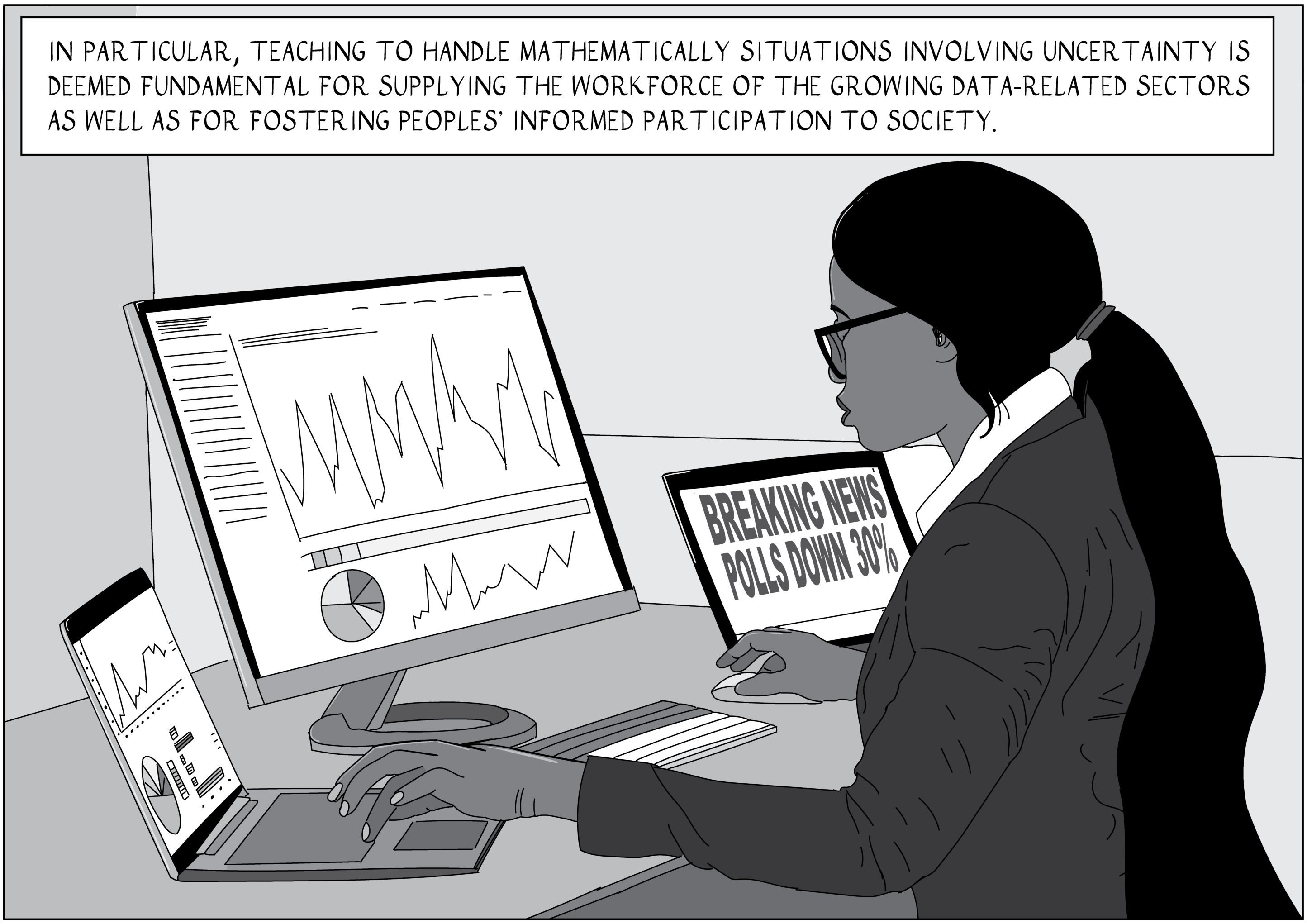
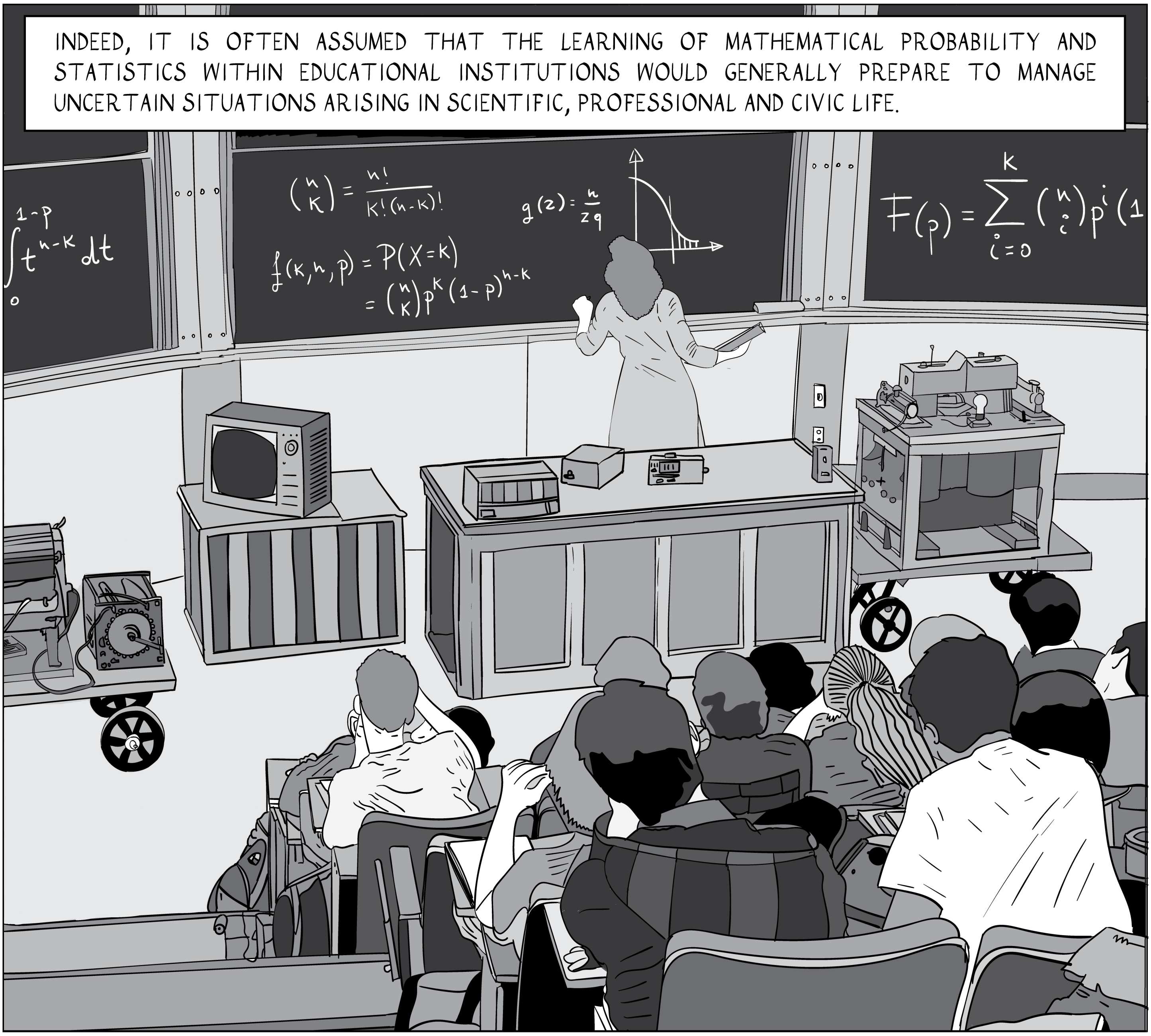
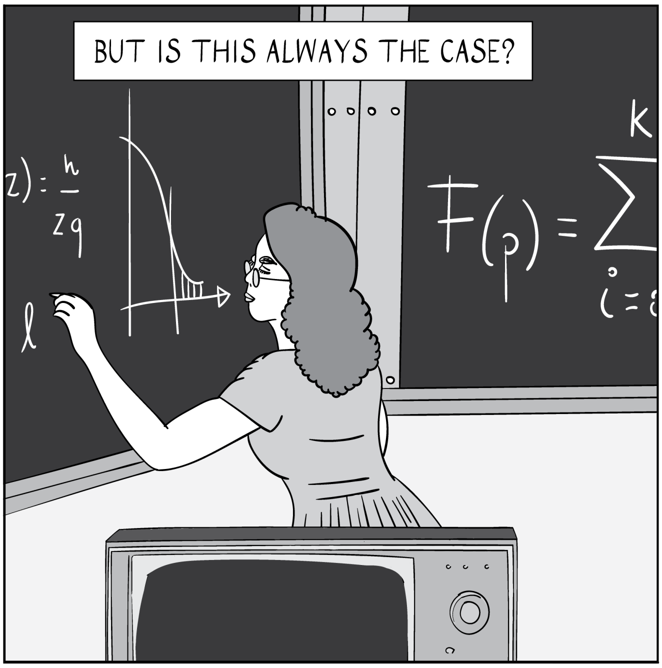
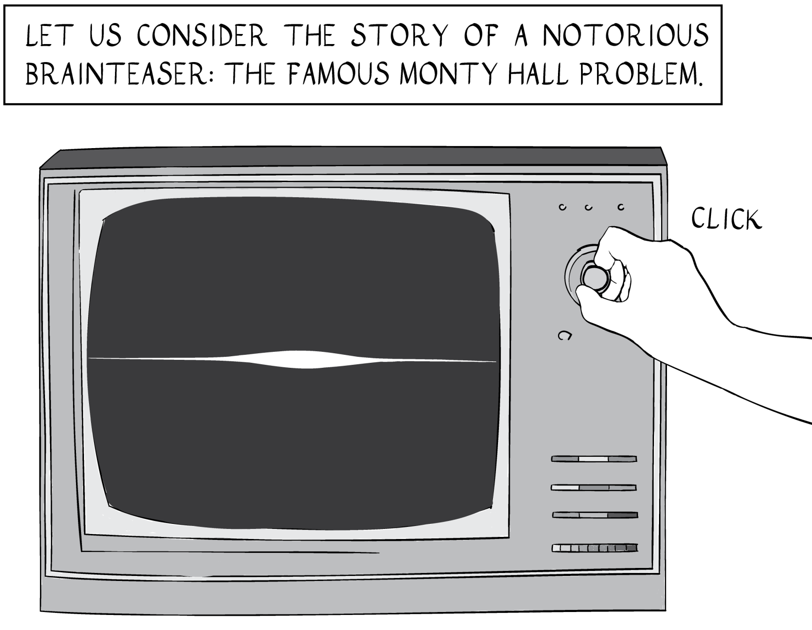
THE DEVELOPMENT AND DISSEMINATION OF MATHEMATICAL KNOWLEDGE AND MATHEMATICAL COMPETENCES ARE DEEMED TO BE VITAL FOR THE ADVANCEMENT OF A PROSPEROUS TECHNO-SCIENTIFIC SOCIETY

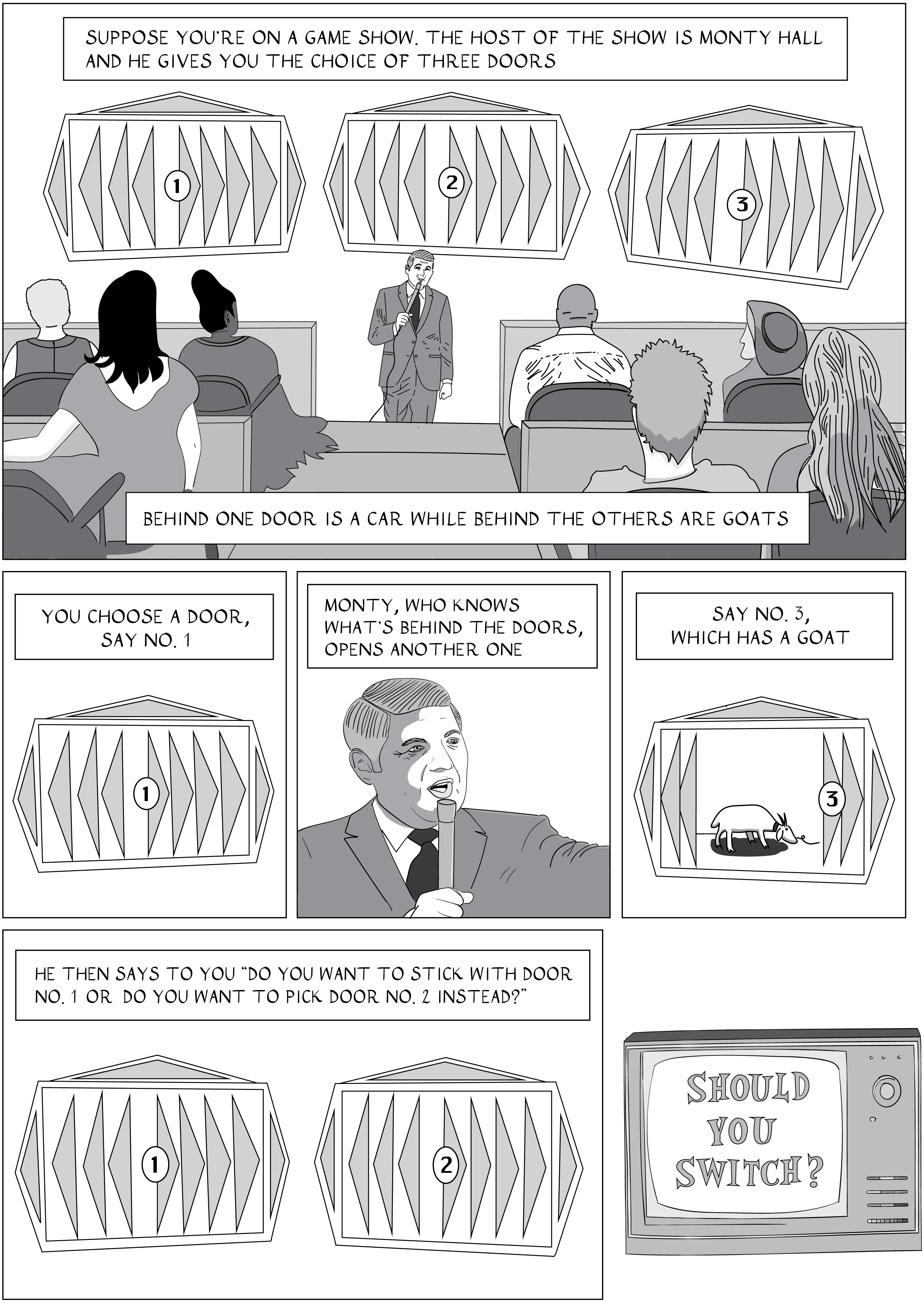

LEARNING MATHEMATICS IS CONSIDERED IMPORTANT FOR ACCESSING AND PERFORMING JOBS RELATED TO THE FUNCTIONING OF VARIOUS SECTORS OF SCIENCE AND INDUSTRY.


IN PARTICULAR, TEACHING TO HANDLE MATHEMATICALLY SITUATIONS INVOLVING UNCERTAINTY IS DEEMED FUNDAMENTAL FOR SUPPLYING THE WORKFORCE OF THE GROWING DATA-RELATED SECTORS AS WELL AS FOR FOSTERING PEOPLES' INFORMED PARTICIPATION TO SOCIETY.
BREAKING NEWS
POLLS DOWN 30%

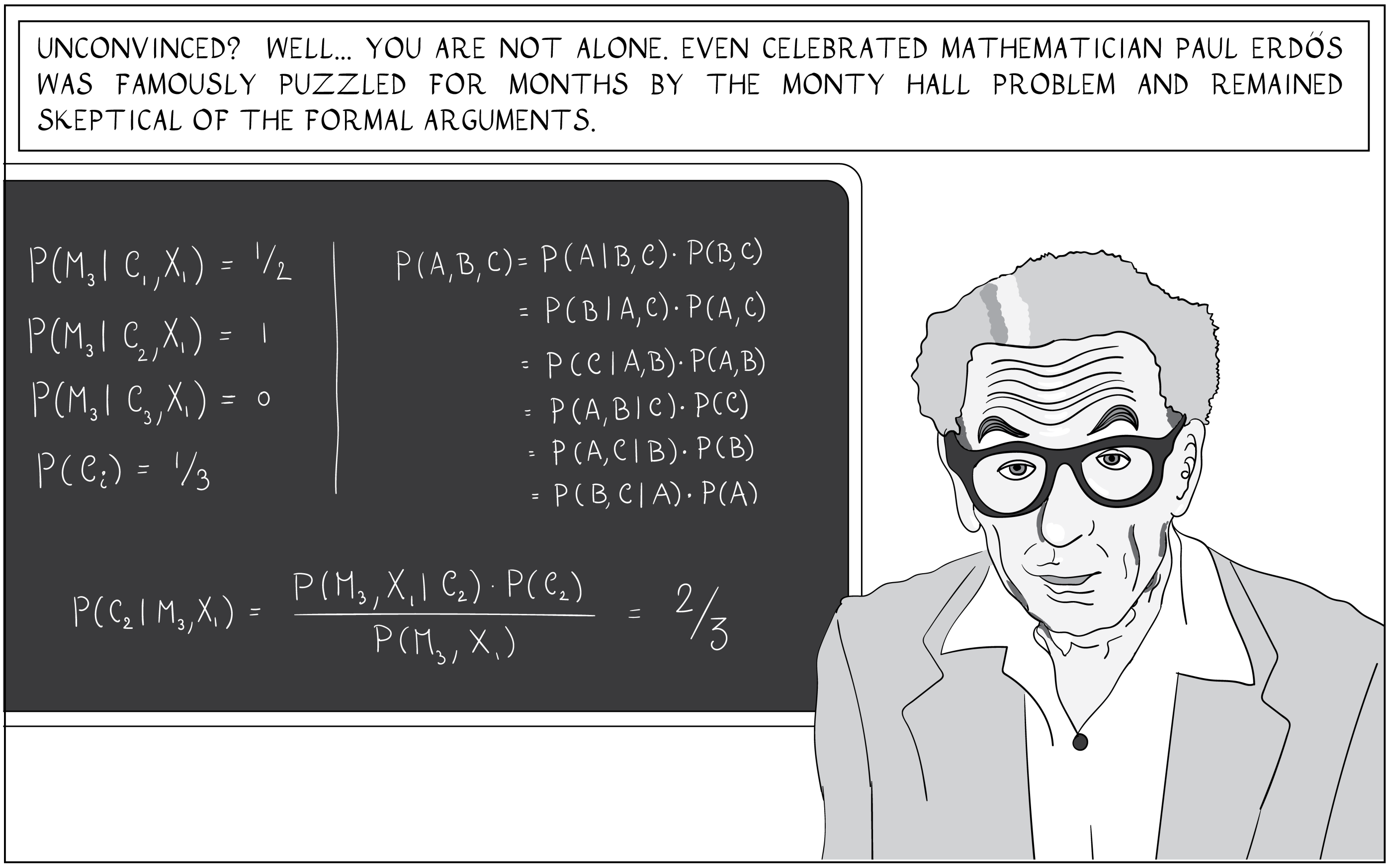

INDEED, IT IS OFTEN ASSUMED THAT THE LEARNING OF MATHEMATICAL PROBABILITY AND STATISTICS WITHIN EDUCATIONAL INSTITUTIONS WOULD GENERALLY PREPARE TO MANAGE UNCERTAIN SITUATIONS ARISING IN SCIENTIFIC, PROFESSIONAL AND CIVIC LIFE.

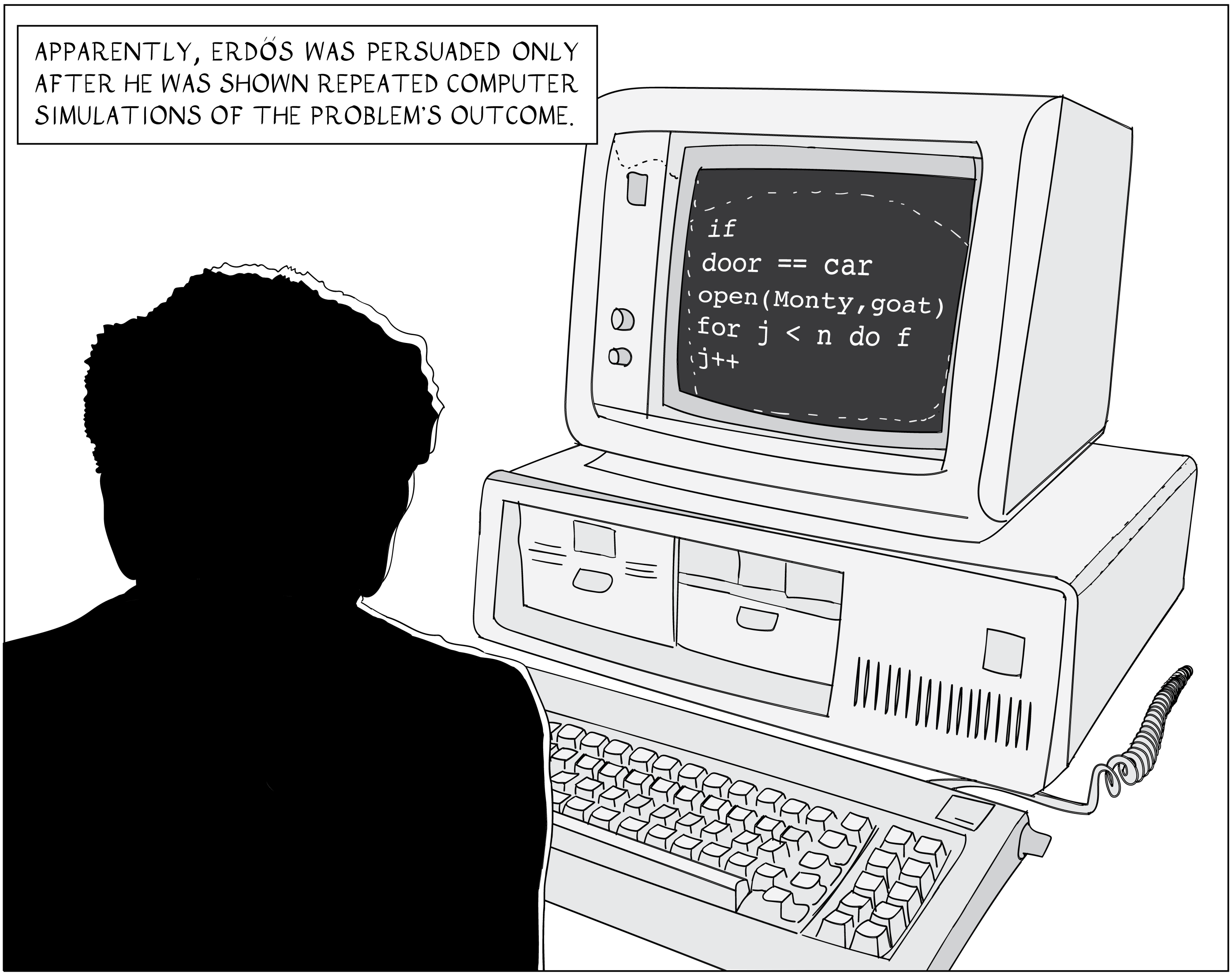

BUT IS THIS ALWAYS THE CASE?


LET US CONSIDER THE STORY OF A NOTORIOUS BRAINTEASER: THE FAMOUS MONTY HALL PROBLEM.
CLICK


SUPPOSE YOU'RE ON A GAME SHOW. THE HOST OF THE SHOW IS MONTY HALL AND HE GIVES YOU THE CHOICE OF THREE DOORS
1
2
3
BEHIND ONE DOOR IS A CAR WHILE BEHIND THE OTHERS ARE GOATS
YOU CHOOSE A DOOR, SAY NO. 1
1
MONTY, WHO KNOWS WHAT'S BEHIND THE DOORS, OPENS ANOTHER ONE
SAY NO. 3, WHICH HAS A GOAT
3
HE THEN SAYS TO YOU "DO YOU WANT TO STICK WITH DOOR NO. 1 OR DO YOU WANT TO PICK DOOR NO. 2 INSTEAD?"
1
2
SHOULD YOU SWITCH?

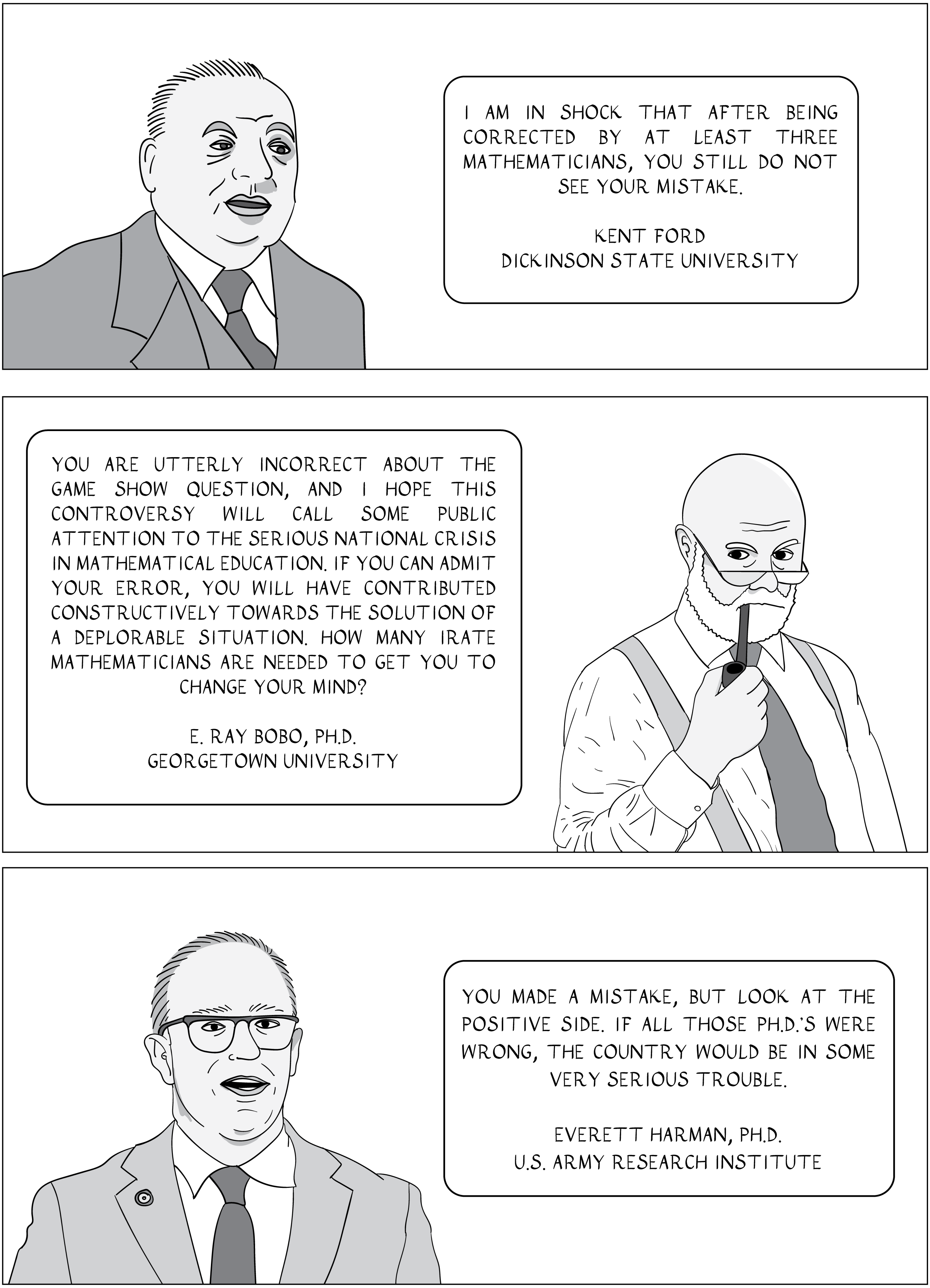

SO, SHOULD YOU SWITCH?
IT TURNS OUT THAT MOST PEOPLE TEND TO STICK WITH THEIR ORIGINAL CHOICE RATHER THAN PICKING THE REMAINING DOOR.
THIS IS THE CASE EVEN WHEN IT IS MADE EXPLICIT THAT THE HOST OF THE SHOW IS NOT TRYING TO TRICK THE PLAYER AND THAT HE WOULD OFFER A SWITCH BOTH IF THE PLAYER INITIALLY CHOSE A WINNING DOOR AND IF SHE CHOSE A LOSING DOOR.
INDEED, AT FIRST GLANCE IT SEEMS THAT, AFTER MONTY OPENS A DOOR, THERE IS NO ADVANTAGE IN CHANGING ONE'S ORIGINAL CHOICE.
HOWEVER, SWITCHING IS THE BEST OPTION IF ONE AIMS TO WIN THE CAR. THIS IS BECAUSE, IF YOU SWITCH, THEN YOU WIN THE CAR IF AND ONLY IF YOUR FIRST CHOICE WAS A DOOR HIDING A GOAT. CONVERSELY, IF YOU STICK WITH YOUR INITIAL CHOICE, YOU WIN IF AND ONLY IF THE LATTER WAS A DOOR HIDING THE CAR ALREADY. BUT IT IS MORE LIKELY THAT YOUR INITIAL CHOICE WAS FOR A DOOR HIDING A GOAT AND NOT THE CAR.
THUS, PROVIDED THAT MONTY OFFERS YOU THE OPTION TO SWITCH REGARDLESS OF YOUR INITIAL CHOICE, THEN SWITCHING ALLOWS YOU TO WIN THE CAR TWO TIMES OUT OF THREE. THIS ARGUMENT IS TIPICALLY FORMALIZED IN MATHEMATICAL TEXTBOOKS BY ARGUING FROM USUAL DEFINITIONS AND RULES OF PROBABILITY THEORY.


UNCONVINCED? WELL... YOU ARE NOT ALONE. EVEN CELEBRATED MATHEMATICIAN PAUL ERDŐS WAS FAMOUSLY PUZZLED FOR MONTHS BY THE MONTY HALL PROBLEM AND REMAINED SKEPTICAL OF THE FORMAL ARGUMENTS.
$P(M_3 | C_1, X_1) = 1/2$
$P(M_3 | C_2, X_1) = 1$
$P(M_3 | C_3, X_1) = 0$
$P(C_i) = 1/3$
$P(A,B,C) = P(A|B,C) \cdot P(B,C)$
$= P(B|A,C) \cdot P(A,C)$
$= P(C|A,B) \cdot P(A,B)$
$= P(A,B|C) \cdot P(C)$
$= P(A,C|B) \cdot P(B)$
$= P(B,C|A) \cdot P(A)$
$P(C_2 | M_3, X_1) = \frac{P(M_3, X_1 | C_2) \cdot P(C_2)}{P(M_3, X_1)} = 2/3$


APPARENTLY, ERDŐS WAS PERSUADED ONLY AFTER HE WAS SHOWN REPEATED COMPUTER SIMULATIONS OF THE PROBLEM'S OUTCOME.
if
door == car
open(Monty,goat)
for j < n do f
j++

Note: The images accompanying the written words of mathematicians involved in the Monty Brawl are products of my imagination. They are intended for sole purposes of storytelling and are not meant to bear any resemblance to the physical appearance of the writers nor to depict them in a negative light.


LET ME EXPLAIN. IF ONE DOOR IS SHOWN TO BE A LOSER, THAT INFORMATION CHANGES THE PROBABILITY OF EITHER REMAINING CHOICE – NEITHER OF WHICH HAS ANY REASON TO BE MORE LIKELY – TO 1/2. AS A PROFESSIONAL MATHEMATICIAN, I'M VERY CONCERNED WITH THE GENERAL PUBLIC'S LACK OF MATHEMATICAL SKILLS. PLEASE HELP BY CONFESSING YOUR ERROR AND IN THE FUTURE BEING MORE CAREFUL.
ROBERT SACHS, PH.D.
GEORGE MASON UNIVERSITY
MAY I SUGGEST THAT YOU OBTAIN AND REFER TO A STANDARD TEXTBOOK ON PROBABILITY BEFORE YOU TRY TO ANSWER A QUESTION OF THIS TYPE AGAIN?
CHARLES REID, PH.D.
UNIVERSITY OF FLORIDA
YOUR ANSWER TO THE QUESTION IS IN ERROR. BUT IF IT IS ANY CONSOLATION, MANY OF MY ACADEMIC COLLEAGUES HAVE ALSO BEEN STUMPED BY THIS PROBLEM.
BARRY PASTERNACK, PH.D.
CALIFORNIA FACULTY ASSOCIATION


I AM IN SHOCK THAT AFTER BEING CORRECTED BY AT LEAST THREE MATHEMATICIANS, YOU STILL DO NOT SEE YOUR MISTAKE.
KENT FORD
DICKINSON STATE UNIVERSITY
YOU ARE UTTERLY INCORRECT ABOUT THE GAME SHOW QUESTION, AND I HOPE THIS CONTROVERSY WILL CALL SOME PUBLIC ATTENTION TO THE SERIOUS NATIONAL CRISIS IN MATHEMATICAL EDUCATION. IF YOU CAN ADMIT YOUR ERROR, YOU WILL HAVE CONTRIBUTED CONSTRUCTIVELY TOWARDS THE SOLUTION OF A DEPLORABLE SITUATION. HOW MANY IRATE MATHEMATICIANS ARE NEEDED TO GET YOU TO CHANGE YOUR MIND?
E. RAY BOBO, PH.D.
GEORGETOWN UNIVERSITY
YOU MADE A MISTAKE, BUT LOOK AT THE POSITIVE SIDE. IF ALL THOSE PH.D.'S WERE WRONG, THE COUNTRY WOULD BE IN SOME VERY SERIOUS TROUBLE.
EVERETT HARMAN, PH.D.
U.S. ARMY RESEARCH INSTITUTE


OF COURSE NOT ALL MATHEMATICIANS WROTE THE SAME.
YOU ARE INDEED CORRECT. MY COLLEAGUES AT WORK HAD A BALL WITH THIS PROBLEM, AND I DARE SAY THAT MOST OF THEM, INCLUDING ME AT FIRST, THOUGHT YOU WERE WRONG!
SETH KALSON, PH.D.
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
I PUT MY SOLUTION OF THE PROBLEM ON THE BULLETIN BOARD IN THE PHYSICS DEPARTMENT OFFICE AT THE NAVAL ACADEMY, FOLLOWING IT WITH A DECLARATION THAT YOU WERE RIGHT. ALL MORNING I TOOK A LOT OF CRITICISM AND ABUSE FROM MY COLLEAGUES, BUT BY LATE IN THE AFTERNOON MOST OF THEM CAME AROUND. I EVEN WON A FREE DINNER FROM ONE OVERCONFIDENT PROFESSOR.
EUGENE MOSCA, PH.D.
U.S. NAVAL ACADEMY, ANNAPOLIS, MARYLAND
AND THE MONTY BRAWL INITIATED BY MARILYN PROMPTED MANY SCHOOL TEACHERS TO TRY OUT THE EXPERIMENT BY ENGAGING THEIR STUDENTS WITH CUPS AND PENNIES.
OUR CLASS, WITH UNBRIDLED ENTHUSIASM, IS PROUD TO ANNOUNCE THAT OUR DATA SUPPORT YOUR POSITION. THANK YOU SO MUCH FOR YOUR FAITH IN AMERICA'S EDUCATORS TO SOLVE THIS.
JACKIE CHARLES,
HENRY GRADY ELEMENTARY, TAMPA, FLORIDA

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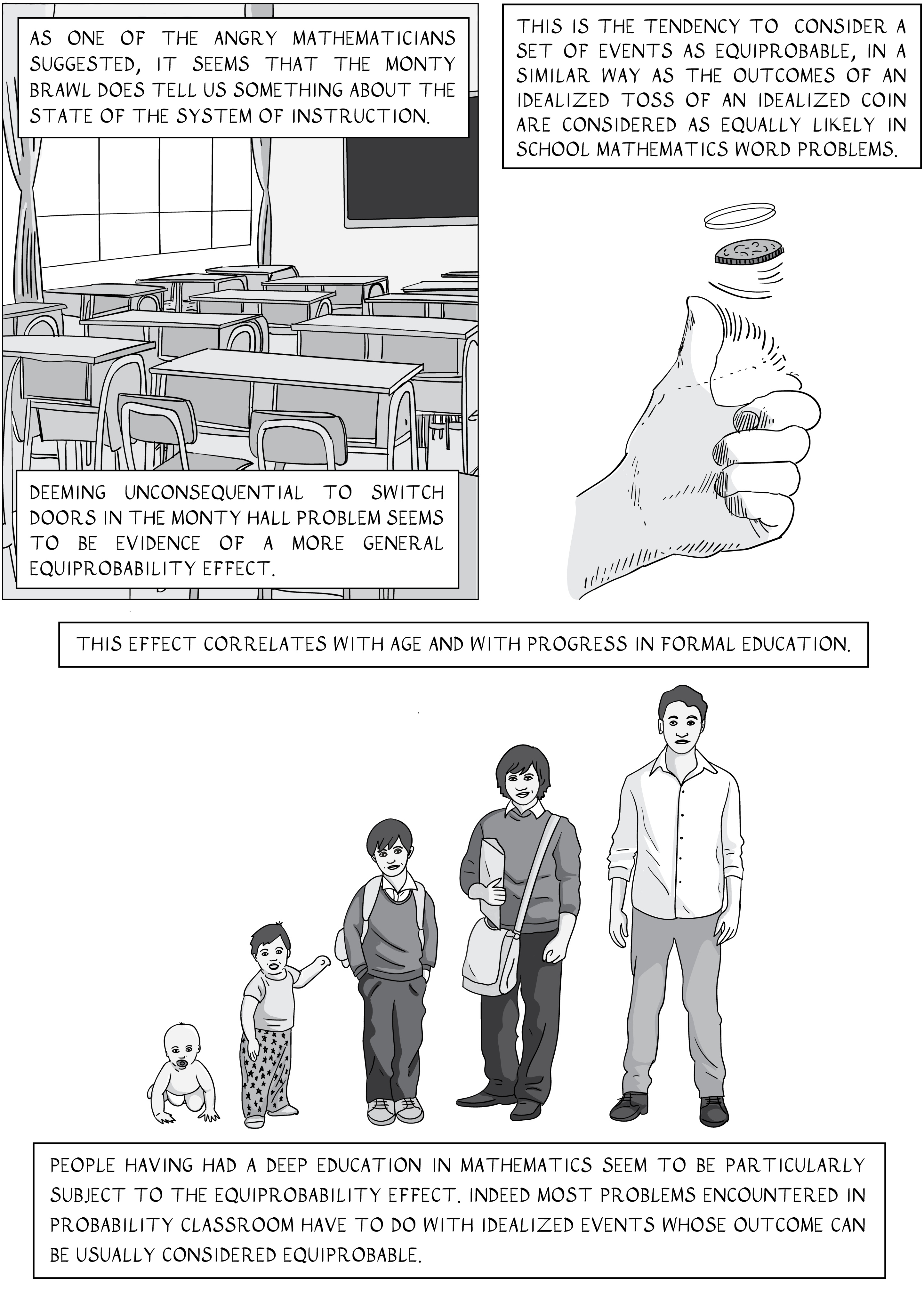
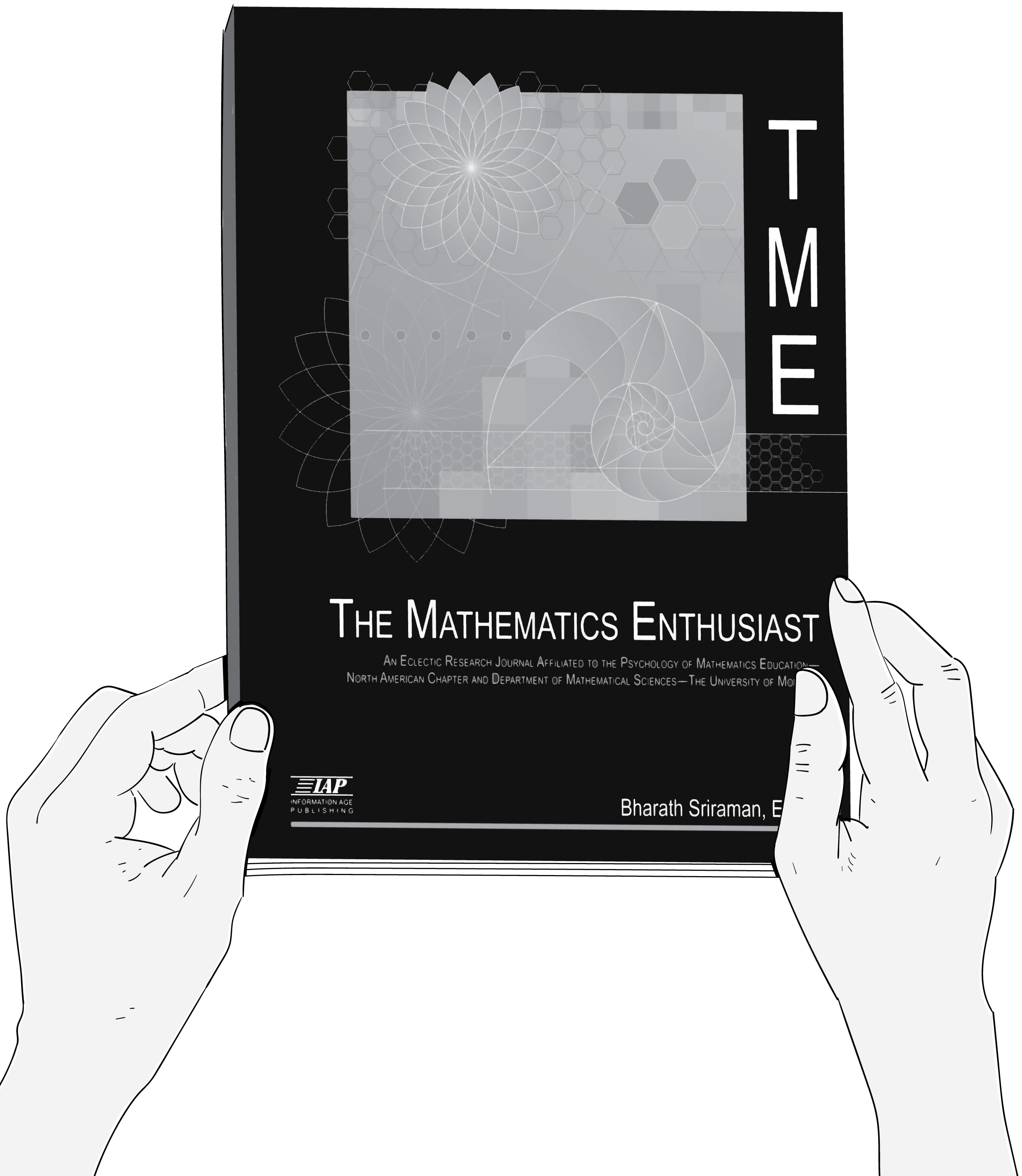
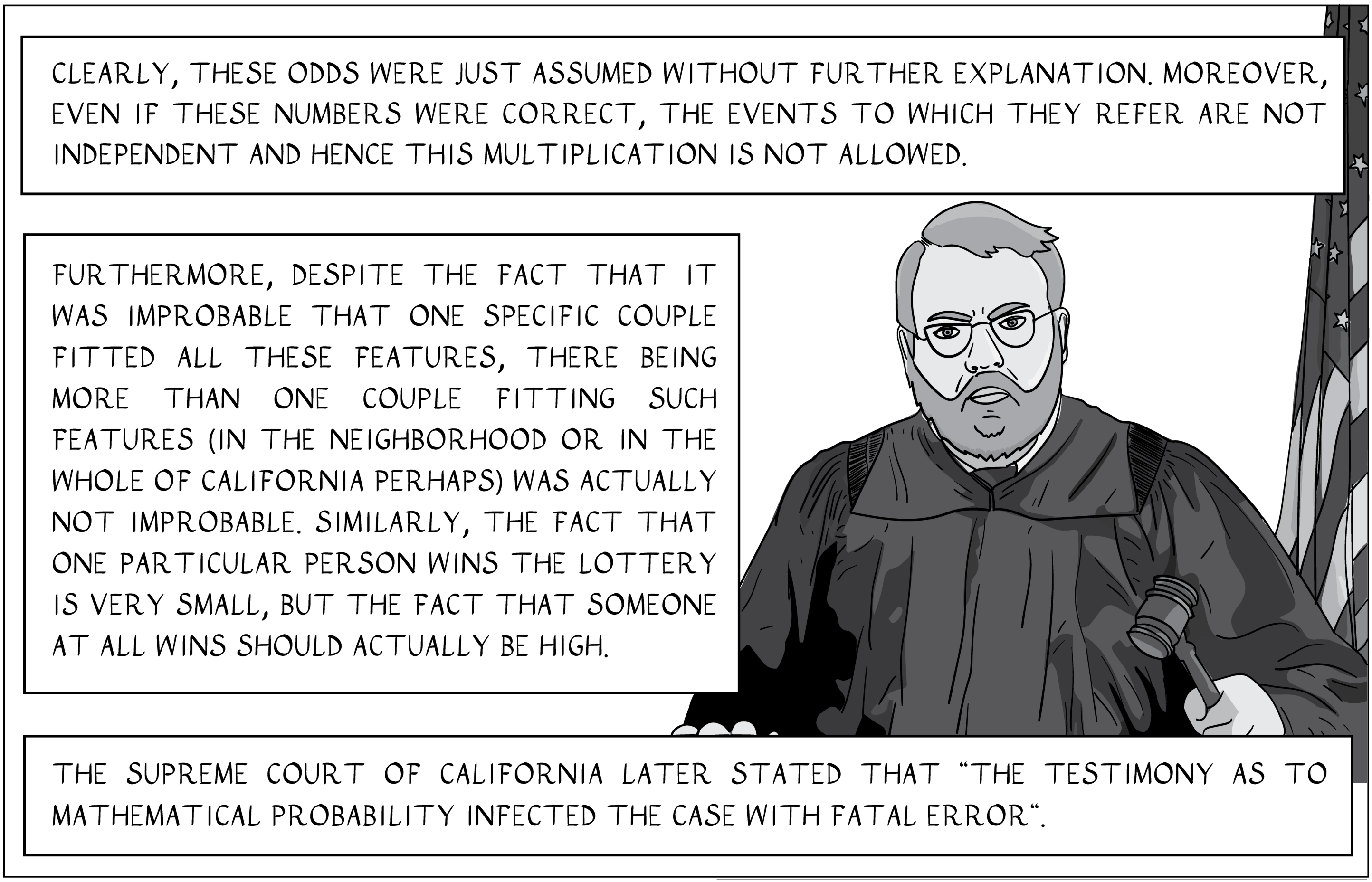


IF THE MONTY HALL PROBLEM IS ONLY AN ABSTRACT PUZZLE, OTHER MORE CONCRETE CASES IN WHICH PEOPLE AND EXPERTS WERE LED ASTRAY AS A CONSEQUENCE OF THEIR EDUCATION IN PROBABILITY AND STATISTICS WERE DISCUSSED IN THIS JOURNAL.



The case of Malcom and Janet Collins



The case of Sally Clark

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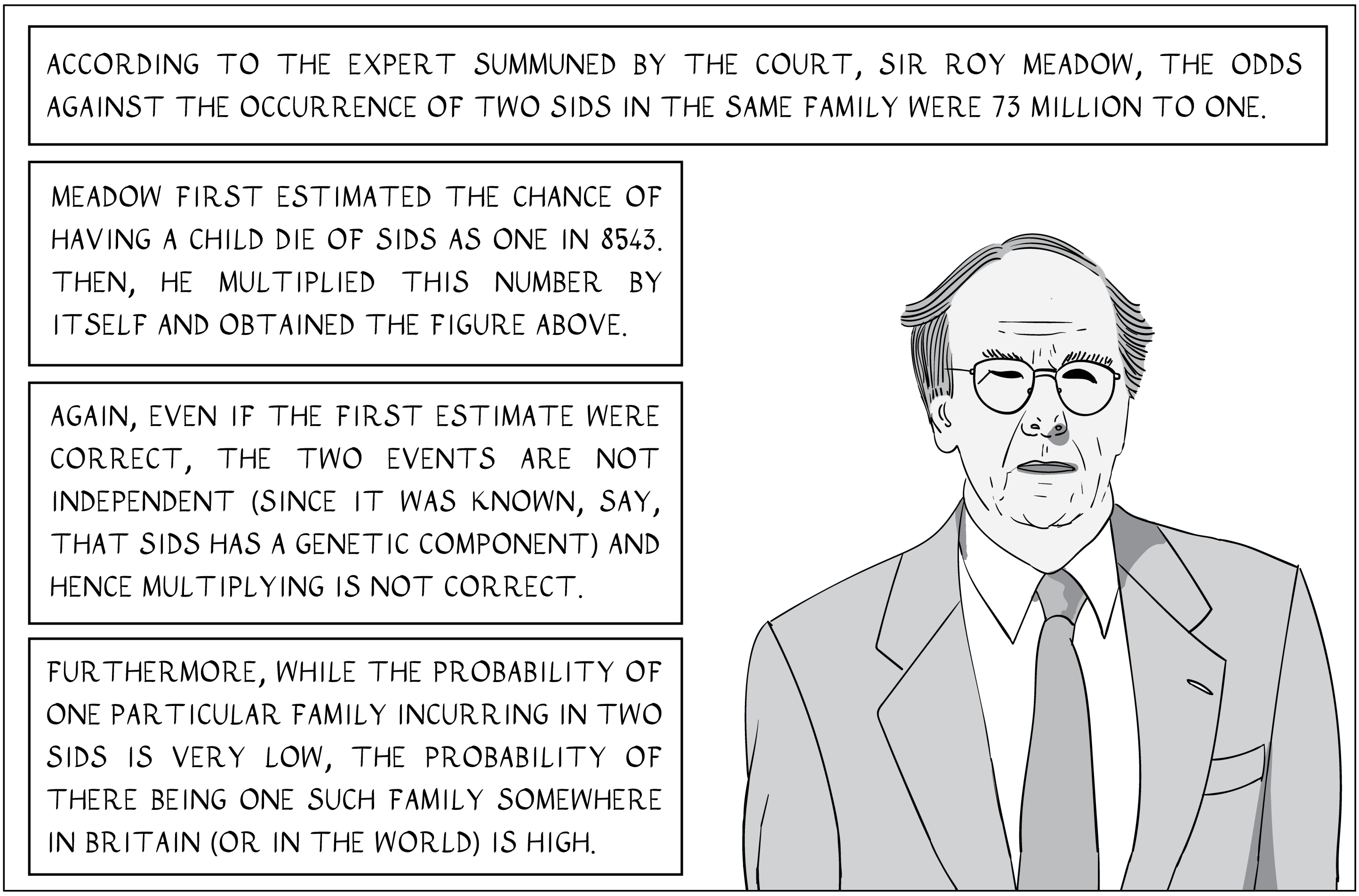
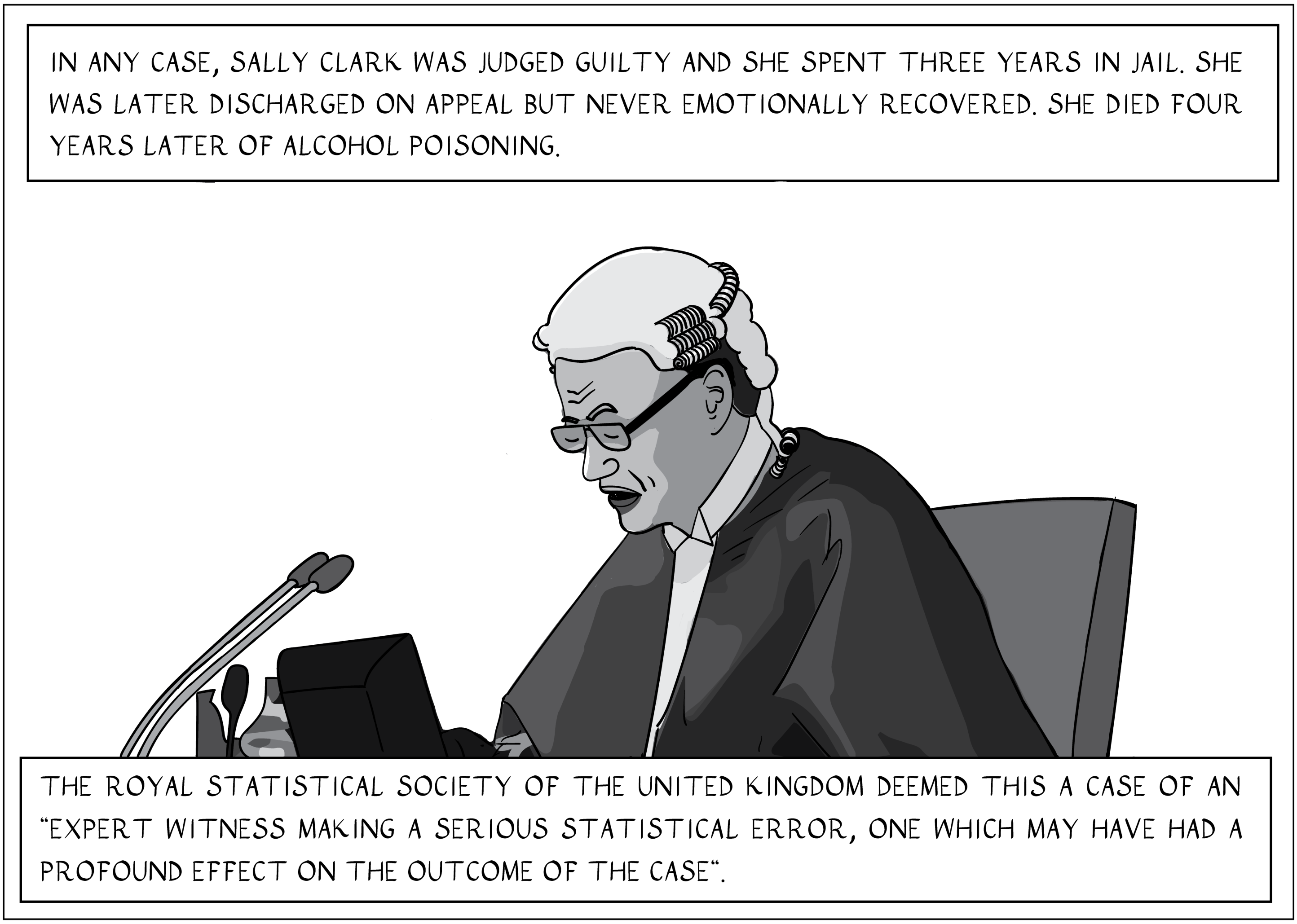
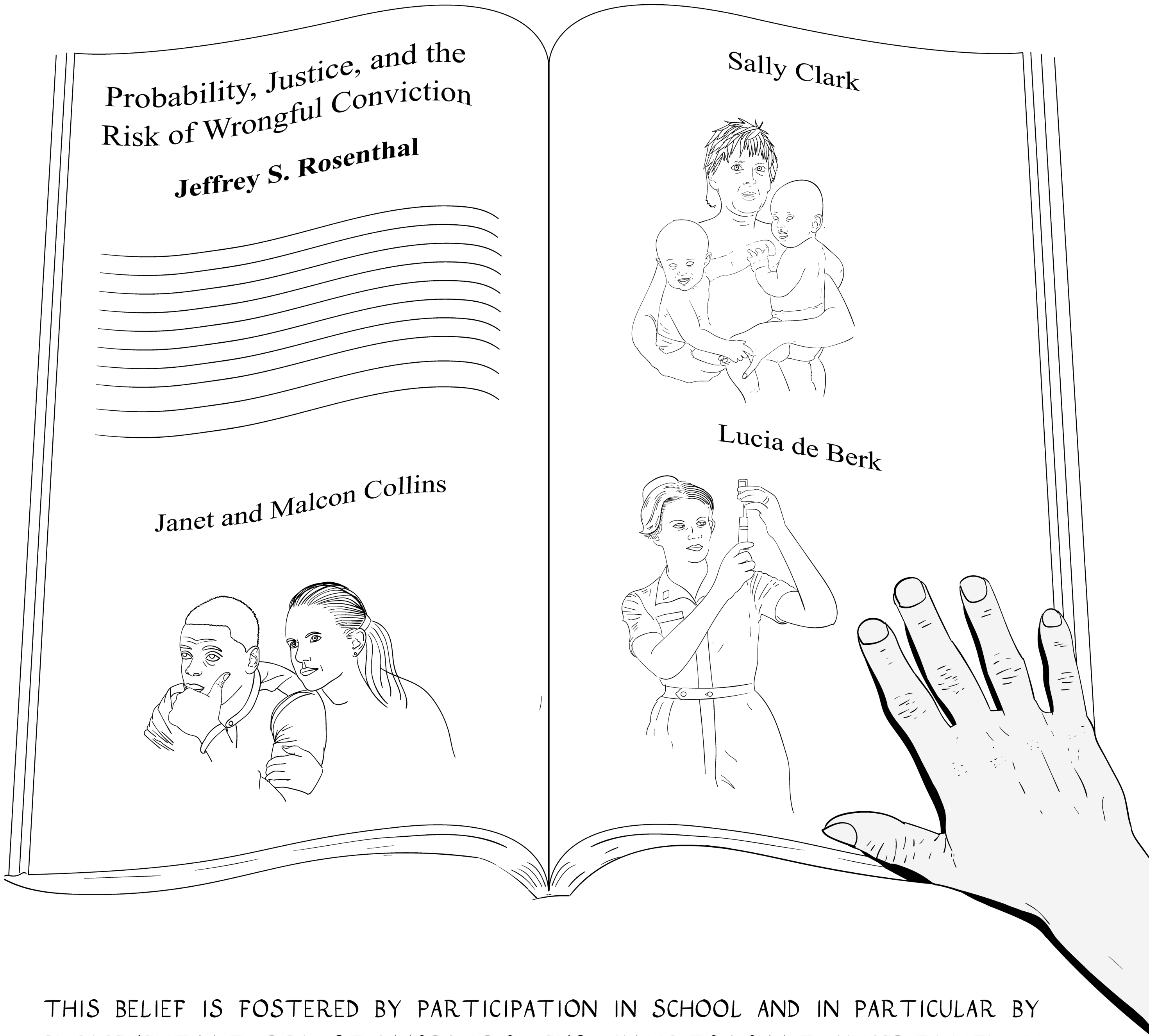




The case of Lucia de Berk

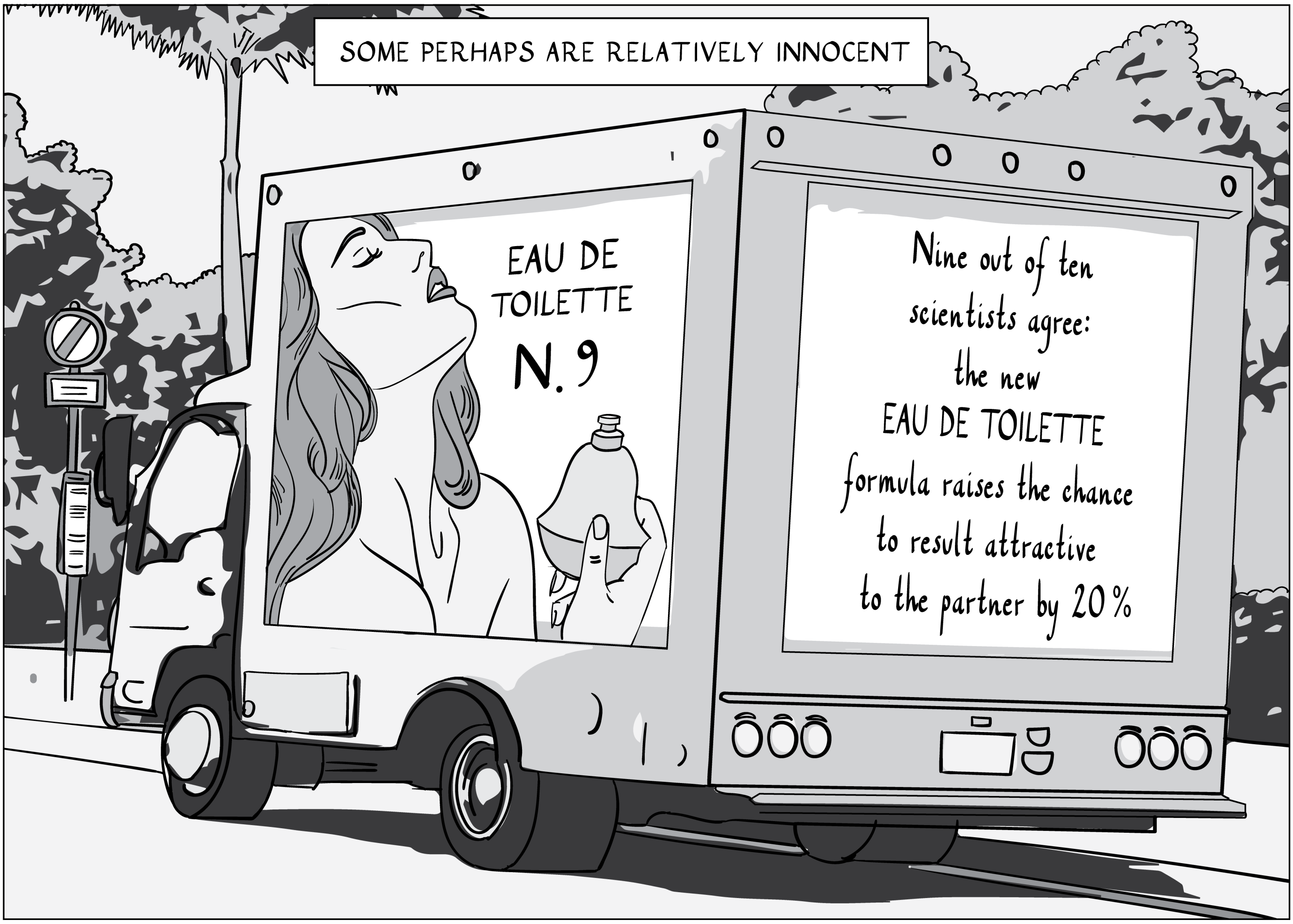
OF COURSE, NOTHING PREVENTS THE FACT THAT LUCIA DE BERK, SALLY CLARK OR THE COLLINSES WERE ACTUALLY GUILTY. NEVERTHELESS, IT SURELY WAS A MISTAKE TO CONDEMN THEM SOLELY OR PRIMARILY ON THE BASIS OF THE PROBABILISTIC ARGUMENTS PUT FORWARD BY THEIR PROSECUTORS. IN ALL THREE CASES WHAT HAPPENED CAN BE INTERPRETED AS A STEMMING FROM THE BELIEF THAT THESE CASES COULD BE RESOLVED BY MEANS OF NUMBERS AND OF MATHEMATICAL ARGUMENTS PROVIDED BY MATHEMATICAL EXPERTS.



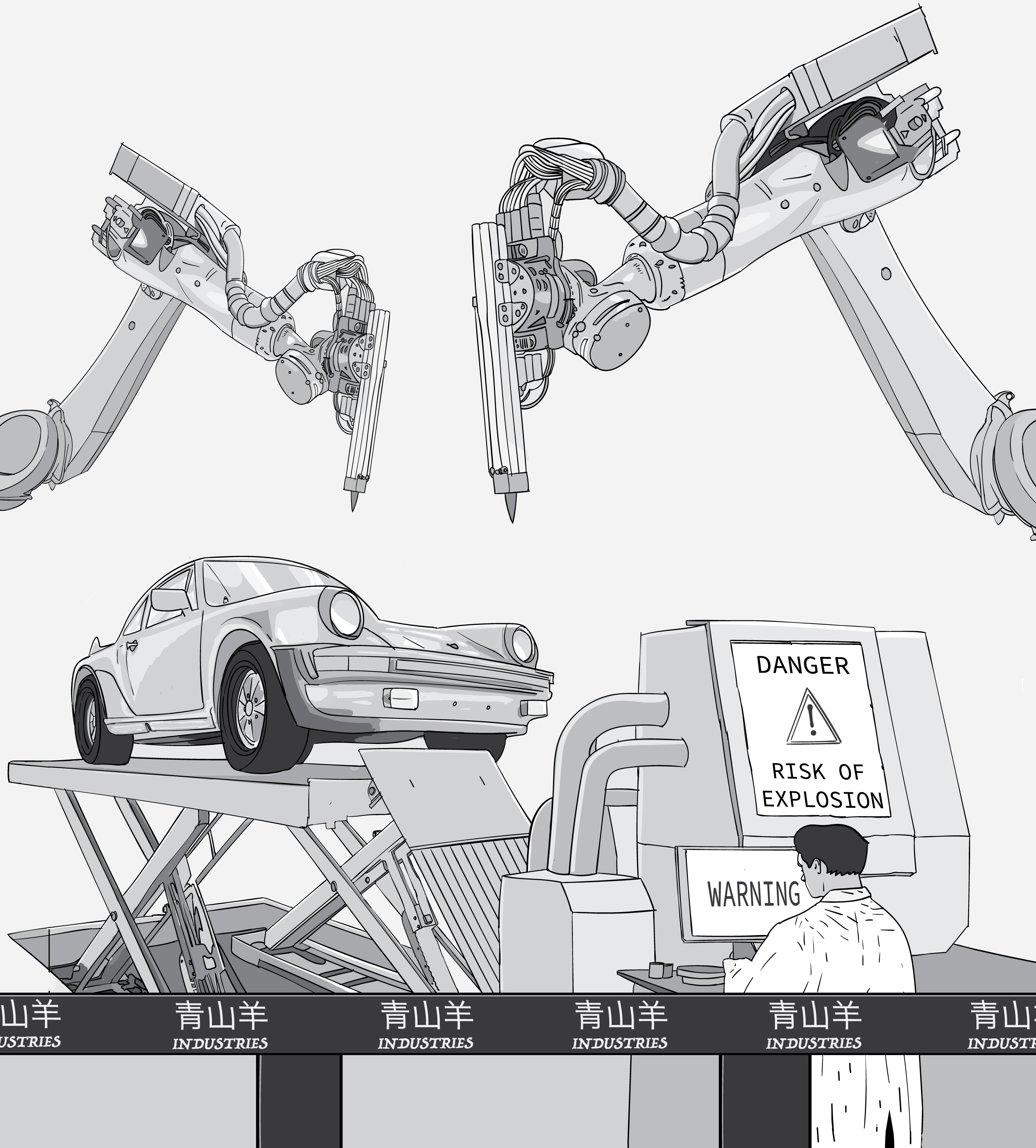
THIS BELIEF IS FOSTERED BY PARTICIPATION IN SCHOOL AND IN PARTICULAR BY ENGAGEMENT WITH REALISTIC WORD PROBLEMS HAVING TO DO WITH UNCERTAINTY. IN SUCH PROBLEMS EVERY EVENT CAN BE ASSIGNED A PROBABILITY, OFTEN EVENTS ARE INDEPENDENT TO ONE ANOTHER AND MATHEMATICAL CALCULATIONS ALWAYS LEAD TO THE ONLY CORRECT ANSWER.


BUT SO WHAT?
MATHEMATICAL INSTRUCTION DEALING WITH PROBLEMS OF UNCERTAINTY HAS SOMETIMES UNWANTED EFFECTS...
BOTH IN THE CONTEXT OF PUZZLES, TESTS OR PROBLEM SOLVING,
6.
7.
8.
9.
10.
11.
12.
13.
AS WELL AS IN MORE IMPACTFUL WORLDLY SCENARIOS
GUILTY
with 95%
chance

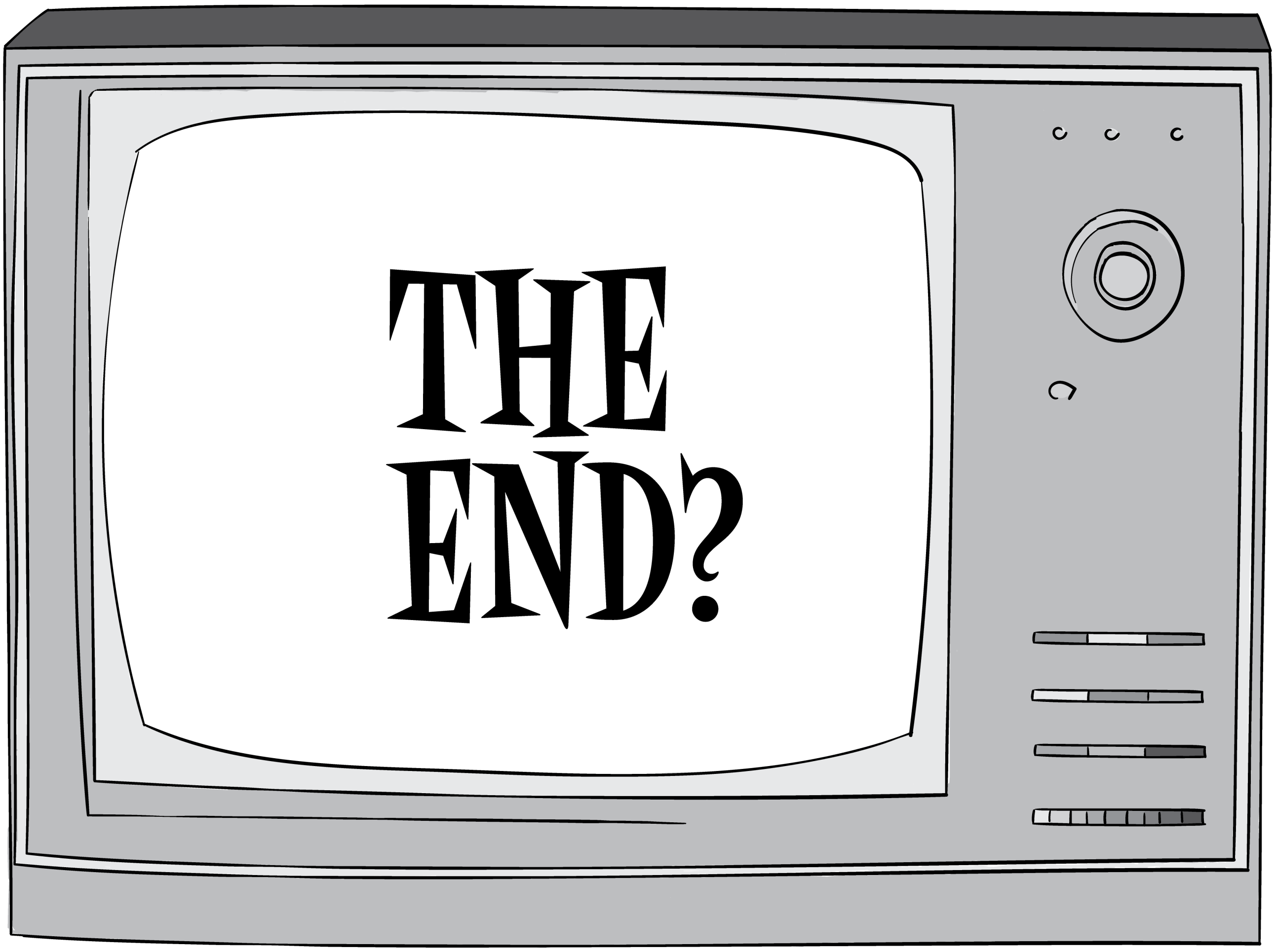
OTHER TYPICAL SITUATIONS COME TO MIND WHEN ONE REFLECTS ON THE MEDIATED RELATIONSHIP BETWEEN EXPERTS AND THE PUBLIC INVOLVING THE USAGE OF CONCEPTS AND ARGUMENTS FROM STATISTICS AND PROBABILITY ACQUIRED IN SCHOOL.



TEACHING TO TREAT SITUATIONS INVOLVING UNCERTAINTY WITH FORMAL MATHEMATICAL TOOLS IS IMPORTANT FOR THE DEVELOPMENT AND THE MAINTAINANCE OF A TECHNICAL, SCIENTIFIC AND INDUSTRIAL SOCIETY.



HOWEVER, IT SEEMS THAT FORMAL INSTRUCTION IN PROBABILITY AND STATISTICS CAN SOMETIMES LEAD TO MISTAKE, TO BIASED JUDGEMENT AS WELL AS, IN A FEW CASES, EVEN TO PERSONAL OR SOCIAL DISASTER.


THE
END?

ACKNOWLEDGMENTS

I wish to thank Melissa Andrade-Molina for having been, with her own work, the main inspiration for commencing this project, as well as for the comments and suggestions she kindly provided at a later stage. Furthermore, I wish to thank Jordan Hill and Veronica Perez for having motivated me to complete this work as well as for their corrections on a previous version of this article. A final thank to Francesco M. Saettone and Alberto Botto Poala for reviewing a last version of this article.

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APPENDIX

Following Gill (2011), a simple but rigorous solution to the Monty Hall problem relies on finding the unconditional probability of winning by switching. This requires the assumption that the probability of initially choosing a winning door is 1/3, which is justified by the configuration of the problem itself. Indeed, the fact that there are two goats and one car and that each of these three is hidden behind one of three identical doors (and that no other information is given) justifies the assumption that the initial subjective probability of choosing a door hiding the car is 1/3, while the initial subjective probability of choosing a door hiding a goat is 2/3. From the statement of the problem, we know that Monty opens a door which is not the one hiding the car and which is not the player's initial choice. Thus, it follows trivially that the player wins by switching if and only if she initially chose a losing door. Therefore, 2/3 is the subjective probability of winning by switching. This also simply follows by enumeration of all the possible cases, as summarized in the following table.

player's initial choice	position of the car	door opened by Monty	player stays	player switches
1	1	2 or 3	WIN	LOSE
1	2	3	LOSE	WIN
1	3	2	LOSE	WIN
2	1	3	LOSE	WIN
2	2	1 or 3	WIN	LOSE
2	3	1	LOSE	WIN
3	1	2	LOSE	WIN
3	2	1	LOSE	WIN
3	3	1 or 2	WIN	LOSE

According to Gill, this simple argument succeeds in convincing everyone except writers of basic textbooks in probability and statistics. Indeed, in these, the solution of the Monty Hall problem is often given in terms of conditional probability, usually as an exercise on the application of Bayes' theorem and by employing further assumptions (i.e., that the car was hidden at random behind the doors and that, when Monty has a choice between opening two doors, he opens one at random). These assumptions are nonetheless justifiable by the undeterminateness of the formulation of the problem and arguably correspond better to a frequentist philosophy of probability. Thus, such assumptions were explicitly programmed in the code reproduced in the next page. This is a code written in the language Python which simulates one million rounds of the Monty Hall problem and counts how many times the player wins by switching at all rounds.

```
import random

NumRounds = 1000000
wins = 0
random.seed()
switch = True

for i in range(NumRounds) :

    doors = ["goat", "goat", "car"]
    random.shuffle(doors)
    player_choice = random.randrange(3)

    if random.choice([True, False]):
        for j, content in enumerate(doors):
            if j != player_choice and content == "goat":
                monty_opens = j
                break
    else:
        for j, content in reversed(list(enumerate(doors))):
            if j != player_choice and content == "goat":
                monty_opens = j
                break

    if switch:
        for j, content in enumerate(doors):
            if j != player_choice and j != monty_opens:
                switch_to = j
    else:
        switch_to = player_choice

    if doors[switch_to] == "car":
        wins += 1

print("\n\n We simulated ", NumRounds, "rounds of the game \
and the player won", wins, "times.")
```

The following is an example of the program's output, which can help us in corroborating our intuition that switching is the optimal strategy for the Monty Hall problem.

```
We simulated 1000000 rounds of the game and the player won 666784 times.
```

Notice that, by changing the value of `NumRounds` to any integer one can virtually simulate any number of rounds (provided one has enough computational power at disposal). Furthermore, changing the variable `switch` to `False` would simulate the situation in which the player always stays with her initial choice of door.