The Chain Rule Does Not Have to be a Pain Rule

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Abstract: Relational and instrumental understanding of the Chain Rule can help teachers provide students a deeper and more meaningful calculus experience.
Keywords: Instrumental understanding; relational understanding; the Chain Rule

Introduction

Teaching and learning the Chain Rule needs a makeover. Too often, high school and undergraduate students are taught the Chain Rule in the same old ways. Students often become overwhelmed and confused by dry textbooks and the basic explanations of the Chain Rule, because they are often asked to solely memorize the rule. These basic memorizations must be changed due to the fact that students are lacking a deeper understanding of the Chain Rule. The following excerpts are the memories of pre-service teachers' experiences learning the Chain Rule:

> "I recall the first time I ever learned about the Chain Rule. My high school calculus teacher asked us to find the derivative of a function comprised of one function inside of another function. We had all class to find the solution and we used the entire hour! Most of us never figured out how to find the derivative. The next day, he showed us what he called a 'shortcut' to solve these types of problems. I found the Chain Rule so incredibly confusing that I spent many sessions with my tutor trying to grasp an understanding of the concept." (Rose)

> "The first time I encountered the Chain Rule was in my AP calculus course in high school. My teacher took a fairly straightforward approach by introducing the idea of the Chain Rule and then immediately giving us the formula. We did a couple of examples of this formula and then were just expected to understand what it was actually doing. I remember spending countless hours on homework and scouring the internet to find resources to help teach myself the material. I never understood or saw the greater picture

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> of what the Chain Rule meant until I reached college and took calculus as a freshman." (Mary)
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> "I first learned the Chain Rule in my AP calculus class in high school, where it was presented to me as a mere formula. I remember copying down the rule taken from the textbook for our class notes. To explain the formula, my teacher just described it as the derivative of the outer function keeping the inner function the same and multiplying it by the derivative of the inner function. At first, it was difficult for me to understand when to apply the rule because there is also the Quotient and Product Rule. I did not know where the Chain Rule was derived, its meaning, when to apply it, or what it looked like visually. I had a superficial view of the Chain Rule and tied my own semantic meaning to it." (Teresa)
>
> "My first experience with the Chain Rule came my senior year in high school in precalculus, and it was hardly talked about, merely mentioned. My first application of the Chain Rule came in college, and I found it to be rather difficult to understand. I was able to apply it after practice, but I struggled to do it accurately for more complex problems. Throughout the lessons, we never talked about why we use the Chain Rule and what is really going on behind the formula given. Today, I still struggle with the rules around the chain rule, and I feel like it is due to the lack of understanding around the operation inside, the derivative of the outer function, keeping the inner function and multiplying by the inner function's derivative." (Joseph)

When asked to describe their previous learning experiences with the Chain Rule, we found that the above are typical comments from pre-service secondary mathematics teachers (PSMTs). Merely memorizing mathematical facts might prohibit students from seeing underlying concepts or understanding any ideas connected to procedures. Given that PSMTs will teach mathematics to our next generation, it is extremely important for instructors to consider how to balance concepts and procedures when teaching undergraduate mathematics to PSMTs. Moreover, learning mathematics is not just about getting answers to questions, but rather it is about developing insight into relationships and structures, and that mathematical understanding emerges through explanation, exploration, and sense-making instead of merely memorizing a procedure (National Council of Teacher of Mathematics, 2000; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). In this article, we will describe three activities focused on developing students' conceptual understanding of and procedural fluency with the Chain Rule.

2 Instrumental vs. Relational Understanding

Developing procedural fluency and conceptual understanding is key in the world of mathematics. If one does not understand how to complete the math problem at hand, one cannot be successful in its solution. However, understanding can be separated into two different meanings, instrumental understanding and relational understanding. In an overly simplified definition, instrumental understanding is the idea of “rules without reasons” (Skemp, 1976, p. 89). More often than not, students receive this type of learning and understanding within their classrooms. This type of understanding tells students that they only need to be able to repeat a procedure over and over again without understanding the deeper concepts. The less common style of understanding is relational. Relational understanding is far more complex than the former instrumental understanding. This type of understanding requires one to know what to do and why they are doing it (Skemp, 1976). In order to have relational understanding, an individual must understand the underlying rules of the operations and be able to make mathematical connections to previous knowledge. If instrumental understanding is so popular among math teachers, should it not be the most advantageous? While this style of understanding is easier to comprehend, more immediate, and often results in the correct answer quicker; it cannot provide a full grasp of the mathematical concepts (Skemp, 1976). In the same vein, if relational understanding is typically unused, should it not be ignored? Relational understanding is more adaptable to new tasks, easier to remember, an excellent goal, and more organic to teach (Skemp, 1976). However, when used in conjunction, the two styles of understanding can form a more full vision of the mathematical concepts.

When discussing relational and instrumental understanding, it is easy to put the two schools of thought against each other in an either/or type relationship. This type of relationship hinders the full potential of utilizing both styles of understanding. When used together, they can assist in creating a more effective, efficient learning environment for students. Ultimately, both instrumental and relational understanding are pertinent in a student’s overall learning. The concepts of instrumental and relational understanding must work hand in hand, as one should not exist without the other, to ensure students receive the richest learning environment possible. In regards to the Chain Rule, many students have been taught solely by instrumental understanding. This is enough for students to squeak by and complete their tests in a relatively successful manner. However, as emphasized in the excerpts from pre-service teachers, it is not a full or enjoyable experience. These students are seeking further knowledge outside of their course that otherwise could be sufficiently provided through more teaching activities that foster relational understanding within a calculus class. Since math teachers should have strong knowledge in their content area, we find it important to discuss a general background of the Chain Rule. This refresher will be vital in providing a foundation to teach the Chain Rule activities detailed in this article.

3 Chain Rule Background

Before learning about why the Chain Rule works, we should establish an understanding of what the Chain Rule is. The Chain Rule is applied to composition functions of the form $f(g(x))$,where we take the derivative of the outside function keeping the inside function the same, producing $f'(g(x))$.Then we would take the derivative of the inside function, $g'(x)$,and multiply it by our previous step, giving us a final product of $f'(g(x)) * g'(x)$. Some students may find it beneficial to identify the two functions as an "inner" and an "outer" function. Incorporating instrumental and relational understanding into chain rule activities can be quite the challenge! To alleviate some concerns, we have included a variety of activities that range from beginner to advanced chain rule knowledge.

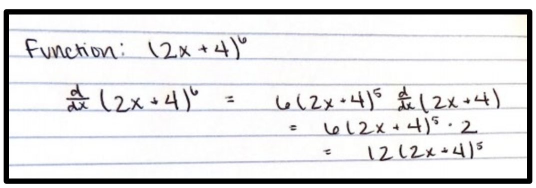


Figure 1: Sample student's work of Chain Rule utilization

4 Chain Rule "M&M's"

When students are first learning what the Chain Rule means, they often cannot grasp what they are actually accomplishing. Students cannot conceptualize the different parts of the Chain Rule and can greatly benefit from a hands-on activity to see the application in real-time. This is where the "M&M" activity can become useful! This activity utilizes the structure of regular "M&M's" and peanut "M&M's" to help students connect their understanding of this candy to how the Chain Rule functions (National Math and Science Initiative, 2014). Let's consider a standard function as $f(x)$.A regular "M&M" has a candy outer layer with a chocolate center. That candy outer layer can be seen as the outer part of the function, or the $f(\ )$piece. The chocolate center can be seen as the inside piece of the function or the'x'that we see within the parenthesis. A peanut "M&M" has a candy outer layer, a chocolate inner layer, and a peanut in the very center. This differs slightly from a standard "M&M", so now let's consider a standard function of □(□(□)). Once again, the candy outer layer is the outer part of the function, or the □( ).The chocolate is now considered the □( ) piece since it wraps around the peanut. Lastly, the peanut center of the "M&M" is considered the '□' of the function that we find within the □( ) function.

I. "Plain" type				
Function $f(u)$	**Rewrite (if necessary)**	**Candy shell** $f(\ )$	**Chocolate** u	**Derivative** $f'(u)\cdot u'$
$(2x+4)^6$				
$\sqrt{x^3-9}$	$(x^3-9)^{1/2}$	$u^{1/2}$	x^3-9	$\frac{1}{2}(x^3-9)^{-1/2}\cdot 3x^2=\frac{3x^2}{2\sqrt{x^3-9}}$
$\frac{1}{(3x-1)^2}$				
$7(3x+1)^2$				
$\frac{4}{\sqrt[3]{2x}}$				
$\frac{2}{3}(5x-3)^{\frac{3}{5}}$				
$\frac{5}{x-4}$				
$\frac{3}{2(4x^2-9)}$				

Figure 2: Chain Rule "M&M's" part one (National Math and Science Initiative, 2014, p. 58)

II. "Peanut" type					
Function $f(g(u))$	**Rewrite (if necessary)**	**Candy shell** $f(\ )$	**Chocolate** $g(\ )$	**Peanut** u	**Derivative** $f'(g(u))\cdot g'(u)\cdot u'$
$\frac{1}{x^2+3x-1}$					
$\sqrt{\frac{5}{1-x^2}}$					
$\cos^2(4x)$	$(\cos 4x)^2$	u^2	$\cos u$	$4x$	$2(\cos 4x)(-\sin 4x)\cdot 4=-8\sin 4x\cos 4x=-4\sin 8x$
$\sqrt[5]{3x^3+4x}$					
$\sin^3(\pi x)$					
$\sqrt[4]{\tan\left(\frac{x}{2}\right)}$					
$\sin\left(\ln\sqrt{2x}\right)$					

Figure 3: Chain Rule "M&M's" part two (National Math and Science Initiative, 2014, p. 59)

This "M&M" activity can be implemented relatively easily in any classroom. Depending on class time, this could be used as a single-day or two-day activity. The first day would be working with regular "M&M's" and the second day would be working with the peanut "M&M's". Students should be placed in small groups of four people or less to be the most efficient. In order to achieve equity with students, each student in the group should be assigned a certain amount of problems to work on. After each student completes their own set of problems, they should share with their group mates and explain how they completed their work as well as their thought processes.

As mentioned near the beginning of this article, teachers need to ensure their students have both instrumental and relational understanding of the topics being taught. Since this activity is primarily instrumental in nature, it should be used in combination with the below tasks to further relational understanding. However, this specific task focuses on instrumental understanding to help drive home the "how" for students. The instrumental understanding comes from the connection between the structures of regular "M&M's" and peanut "M&M's" to the structure of the Chain Rule. Students are able to use their prior knowledge of the "M&M's" to understand the inner workings of a complicated equation. Additionally, students use their prior knowledge of functions and composition functions to understand the Chain Rule pieces. A further instrumental aspect comes in with the process of filling out the table. For example, students are using the functions given to break down each part and practice each part to fundamentally understand the process of completing the Chain Rule.

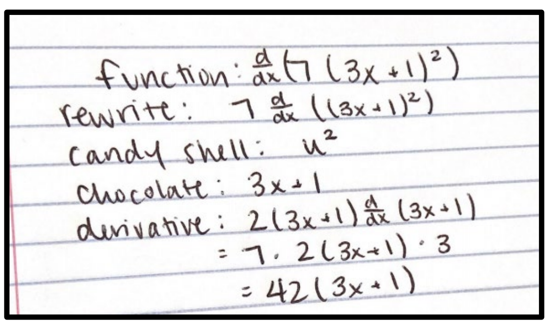
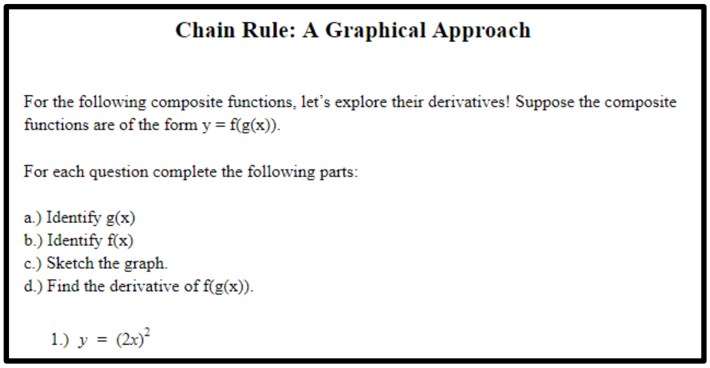


Figure 4: Sample student's work of the Chain Rule "M&M's"

5 Graphing with the Chain Rule

When initially learning the Chain Rule, students are not visualizing what is happening while taking the derivative of the "inner" and "outer" parts of a composite function. This was evident in the excerpts from PSMT's experiences when learning the Chain Rule. In this GeoGebra activity, students will deepen their understanding of the Chain Rule by exploring how it works through a graphical lens. This is beneficial for challenging students to investigate the relationship between □(□), □(□), □′(□),and □′(□(□))while representing these functions on a graph. In this activity, students are given various composite functions of the form $\square = \square(\square(\square))$,where $\square(\square)$is the “inner” function and $\square(\square)$is the “outer” function. Students will have to identify the two functions provided for each Chain Rule problem. They will type $\square(\square)$and$\square(\square)$into separate boxes within the GeoGebra application, where each function is portrayed in a different color. Additionally, the derivative of $\square = \square(\square(\square))$is represented graphically from the input values. Students are able to click and drag the sliders to explore the graph at different points. At the bottom corner of the application, students can click on the checkbox “Show Tangents” to see the tangent lines of both functions at various points. The point $'\square'$ may be slid to a specific point on the x-axis causing the tangent lines denoted as m1 and m2 to change. These tangent lines, m1 and m2, are expressed as $\square'(\square)$and $\square'(\square(\square))$,respectively. This activity is excellent for teachers with access to one-to-one devices to implement in their classroom. Students will click on the GeoGebra link and follow the worksheet provided by the teacher as a guide at their own pace. The worksheet starts with simple functions and gradually increases in complexity to functions of higher orders, logarithmic, exponential, and trigonometric functions. In order to best explore the new concepts, students will work independently for this intermediate activity. Following their online explorations, students will collaborate as a class to discuss their findings.

This activity promotes and incorporates instrumental and relational understanding for the students. The relational understanding involved in the activity is found when students apply their previously learned knowledge of how to use the Chain Rule and how to graph the function $\square(\square)$, and the derivative, $\square'(\square)$.Students are tasked with building on this idea to see the relationships between composite functions, $\square(\square(\square))$,and their derivatives, $\square'(\square(\square)) * \square'(\square)$,discovering the Chain Rule. The instrumental understanding can be seen through students applying what they have learned from graphing composite functions. An example within this activity may be the assigned problem on the worksheet, challenging students to apply previously and recently acquired knowledge by identifying the functions $\square(\square)$and $\square(\square)$,finding the derivative of $\square(\square(\square))$, and constructing the graphs.



Chain Rule: A Graphical Approach

For the following composite functions, let's explore their derivatives! Suppose the composite functions are of the form y = f(g(x)).

For each question complete the following parts:

a.) Identify g(x)
b.) Identify f(x)
c.) Sketch the graph.
d.) Find the derivative of f(g(x)).

1.) $y = (2x)^2$

Figure 5: A graphical approach expectations for the Chain Rule

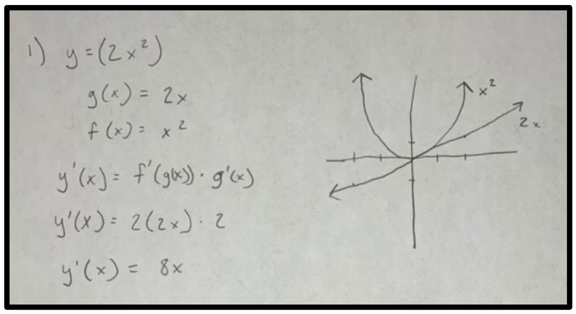


Figure 6: Sample student's Chain Rule work using a graphical approach

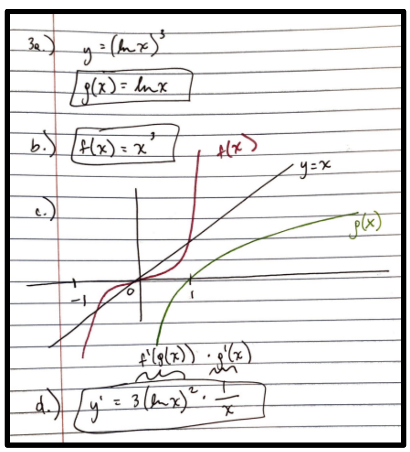


Figure 7: Sample student's Chain Rule work using a graphical approach

6 Taking it Further

Once students have grasped the basic concepts of the Chain Rule and applied those concepts to graphing, it can be very easy to end the activities there. However, there are benefits to taking the learning one step further by combining all the learned skills. In this activity, students will be partnered up in sets of two. Each student will receive their own activity sheet where they will develop and derive their own composite function. This will be an excellent time to remind students what a composite function is and why the Chain Rule is key to their derivation. After developing their function, students will trade their papers with their partners. Now, each student will graph their partner's composite function and derivative. This allows students to practice the graphing techniques learned previously and develop their own visual representations of how the Chain Rule is functioning. Students will trade their papers back to their rightful owners and engage in a productive dialogue about each graph and its relationship to the original function.

This activity incorporates both an instrumental and relational understanding of the Chain Rule. Students have the ability to apply the rules learned in class to a problem in order to reap a correct solution. This application involves instrumental understanding. Beyond the instrumental, students are incorporating more relational understanding by developing their own composite function and engaging in productive dialogue regarding the functions' derivation and graph.

Figure 8: Function frenzy expectations

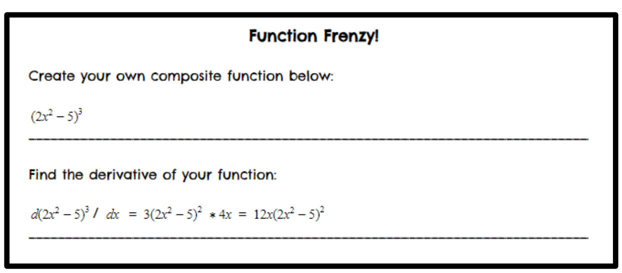


Figure 9: Sample student's work of the function frenzy

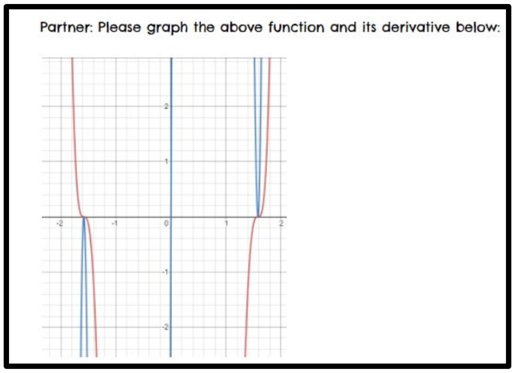
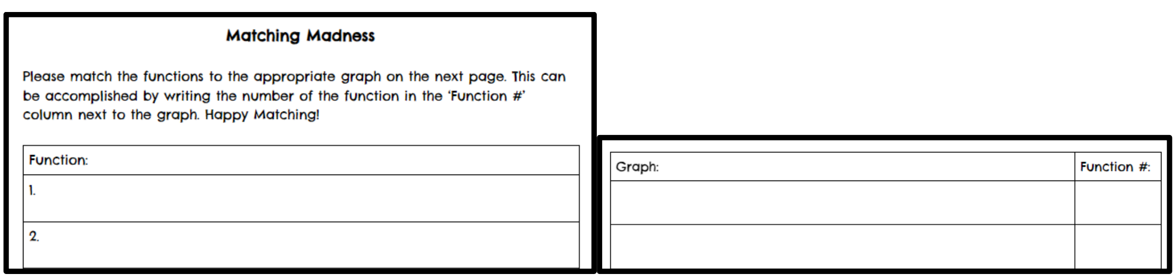


Figure 10: Sample student's work of the function frenzy

Following the completion of the 'Function Frenzy!' activity, the math teacher may compile the students' functions and graphs into the below document, 'Matching Madness'. This custom worksheet will provide students the opportunity to match the relationships of functions and graphs. This initial task may be done alone, or with partners. Once students have completed matching, they may partner up and explain their process of thinking to their partner. In general, this activity is highly instrumental as students are constantly repeating the same task over and over by applying similar rules and notions. However, this task also engages relational understanding through dialogue about their findings and the general background that all functions and graphs were student-developed. As a duo, these activities fully incorporate the skills learned in both the "M&M" activity as well as the graphing activity. Students have the opportunity to combine their derivation and graphing skills, practice constructive mathematical arguments, and further their understanding of the Chain Rule.



Matching Madness

Please match the functions to the appropriate graph on the next page. This can be accomplished by writing the number of the function in the 'Function #' column next to the graph. Happy Matching!

Function:
1.
2.

Graph:	Function #:

Figure 11: Matching madness expectations

7 Conclusion

We asked some of the same PSMTs from the introduction to consider how the implementation of these activities may be positive or negative within their classrooms. The following excerpts are their responses:

> "As a future teacher who will more than likely have to teach calculus at some point in my career, I want to ensure that my students have a richer and fuller understanding of the Chain Rule. Since this is one of the largest components and the easiest to misunderstand, I think that the above-referenced activities can help bridge the misunderstanding gap. Each of these activities utilizes different methods to help cover various learning styles. We have found items that help hands-on learners, visual learners, auditory learners, and any combination of those! Students who have multiple resources are much more likely to find something that makes sense to them and then connect that understanding to other activities." (Mary)

> "Looking towards a future career as a teacher and reflecting on my experiences as a student, if given the opportunity to teach the Chain Rule, I would implement these tasks in my lesson planning. These activities give students a more diverse experience with a difficult topic within Calculus. With all the moving parts and remembering the order of derivatives and multiplication can trip students up, so covering these with unique and additional resources could be extremely beneficial to the students. I would implement these tasks since they have the opportunity to reach all types of learners, hands-on, auditory, and visual, and it can also be easily differentiated to reach all learners regardless of current levels of understanding." (Joseph)

> "Reflecting on these Chain Rule activities, I feel these would enrich students' knowledge in their calculus class. Many students simply learn to memorize the definition of the Chain Rule, viewing it as an abstract, isolated mathematical concept. To deepen student understanding, these activities help reveal and highlight the connection the Chain Rule has to composite functions, derivatives, graphing of functions and the relationship among these components. By framing the Chain Rule in a way that builds upon prior knowledge such as utilizing key concepts from algebra students feel more comfortable with the material. As a future educator, I would like to implement these activities because they complement one another, increasing in complexity. Initially, students dissect and interpret the Chain Rule, view it through a graphical lens, and lastly tie these two ideas together by constructing their own scenario. With this structure, they hit all learning modalities and challenge students not only to understand the definition of the Chain Rule but analyze it in different contexts." (Teresa)

It is evident that the opportunity for complex and engaging activities are of great interest to future mathematics educators and current mathematics students. The PSMTs' reflections indicate that the Chain Rule activities provide both instrumental and relational understanding throughout and challenge students to think beyond the formulas. Even though our activities target an undergraduate student in calculus, they may also be friendly for students in high school. To provide a relational and instrumental understanding of the Chain Rule, the three activities that we describe create opportunities for mathematics teachers to incorporate engaging, hands-on activities into their classrooms and to strengthen their students' overall positive experience with Chain Rule concepts.

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