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# The construction of figured numbers in GeoGebra software using algebraic properties

Francisco Evamar Barros

Francisco Regis Vieira Alves

Paula Maria Machado Cruz Catarino

Renata Teofilo de Sousa

**The construction of figured numbers in GeoGebra software using algebraic properties**

Francisco Evamar Barros[1]
Department of Mathematics, Federal Institute of Education,
Science and Technology of State of Ceará- IFCE
Francisco Régis Vieira Alves[2]
Department of Mathematics, Federal Institute of Education,
Science and Technology of State of Ceará- IFCE
Paula Maria Machado Cruz Catarino[3]
University of Trás-os-Montes and Alto Douro – UTAD
Renata Teófilo de Sousa[4]
Department of Mathematics, Federal Institute of Education,
Science and Technology of State of Ceará- IFCE

Abstract**:** Figured numbers are sequences of natural numbers that, when represented by dots, form geometric configurations. The immediate sequence of figured numbers presents in its configuration the equilateral triangles, being the basis for finding several others, as long as a relationship between them is discovered. This article shows the construction of polygonal, stellar and three-dimensional figured numbers that can be constructed using the GeoGebra software as a resource to visualize the properties that relate these numbers to the triangular numbers. It is a way of observing the relationship that the figures shown have to each other, how we can understand the structure of these sequences, when presented in the form of figures, and the possibility of building figures that represent some of the most known sequences within the number's figures such as 2D figured numbers such as square, pentagonal, centered triangular and star numbers within 3D figured numbers. Thus, this work presents such construction through the properties of pyramidal numbers, cubic numbers, octahedral numbers and two forms of construction of the stella octangula numbers, with emphasis on the use of triangular numbers or using the other sequences that depart from this one. Furthermore, the work shows the possibility of constructing other sequences as well as the discovery of possible properties that relate them.

*Key words***:** Figured Numbers, GeoGebra, Recurring Numerical Sequences.

[1] fco.evamar@gmail.com
[2] fregis@ifce.edu.br
[3] pcatarino23@gmail.com
[4] rtsnaty@gmail.com

# Introduction

The figured numbers are treated in the History of Mathematics books and studied since the Pythagorean times, aiming at a union between geometry and arithmetic (Alves, Borges Neto and Maia, 2012). These numbers were arranged in a geometrical arrangement by the Pythagoreans, who used stones to represent flat and even three-dimensional figures (Roque and Pitombeira, 2012). However, current studies describe figures that represent polytopes of any dimension, as described by Coxeter (1973) only as geometric entities and studied by authors of Number Theory, such as Conway and Guy (1996), Burton (1986) and Alves, Borges Neto and Maia (2012) as sequences of numbers. Each figure has its own growth generating a particular sequence. For example, an arrangement that forms a triangle generates a triangular number, and when we add points on this triangle to form a larger triangle, we are building the sequence of triangular numbers.

Within the study of figure numbers, it is important to visualize the arrangement of points in order to understand how the growth occurs and, at the same time, present the geometry and arithmetic behind the arrangement presented. In this article, the construction of some figures based on the construction of triangular numbers is presented, which proposes an arithmetic relationship between the presented numbers and the triangular numbers, the basis of all the constructions presented in this work.

To start, we will present the triangular numbers using the GeoGebra software, with their recurrence relation and general term, as shown in Figure 1.

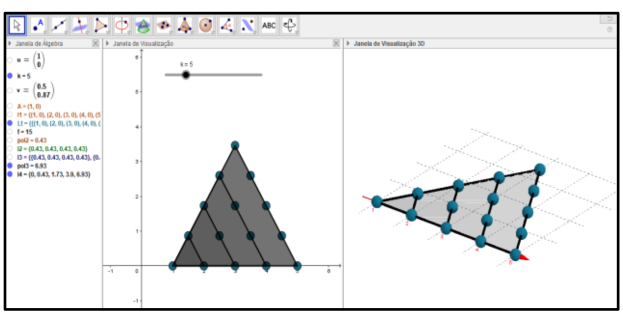

Figure 1. Fifth triangular number. Authors' elaboration.

In Figure 1, we have that the recurrence of the sequence can be seen in the part that represents the fifth triangular number. So, from the image, we see that to form the next triangular number, we need to add the previous triangular number to a natural number, which represents the order of the next number, that is:

$$T_{n+1} = T_n + (n+1), T_1 = 1$$

In this way, we can also determine the formula that represents the general term, that is, a way to find a triangular number without having to have its previous one. Also from Figure 1, we can see that a triangular number is formed by the sum of $n$ natural numbers. In this way, we have: $T_1 = 1, T_2 = 1+2, T_3 = 1+2+3, \ldots, T_n = 1+2+\cdots+n$ . This summation of natural numbers can be developed as follows:

$$\begin{aligned}
T_n &= 1+2+\cdots+(n-1)+n \\
T_n &= n+(n-1)+\cdots+2+1 \\
&\overline{\hphantom{= n+(n-1)+\cdots+2+1}}+ \\
2T_n &= \underbrace{(1+n)+(1+n)+\cdots+(1+n)}_{n \text{ vezes}} \\
T_n &= \frac{n(n+1)}{2}
\end{aligned}$$

This formula can be proved using induction, but we will not present it here because the focus of this work is to show how such properties can be observed with the help of GeoGebra.

For the construction to be possible, it is necessary to use commands such as “sequence” and “part of the list”, and, mainly, it is necessary to create a vector for the list to be translated. This vector must have a slope of 60°, as it is the value of the inside angle of an equilateral triangle, so we will form a regular triangle. This information is important because the vectors determine the type of triangle that will be formed with the points in the sequence. In this way, we can build other figurative numbers, such as square numbers, pentagonal numbers, among others, from a property of these numbers in relation to the triangular numbers.

For this, we present below the square and pentagonal numbers, their respective general term formulas and the way to construct other numbers from the use of the sequence of triangular numbers.

## Construction of flat figured numbers from triangular numbers

The square numbers form the sequence $(1, 4, 9, 16, \ldots)$ and we will call $Q_1 = 1, Q_2 = 4, Q_3 = 9, \ldots, Q_n = n^2$, (Alves, Borges Neto and Maia, 2012), which can be represented as shown in Figure 2:

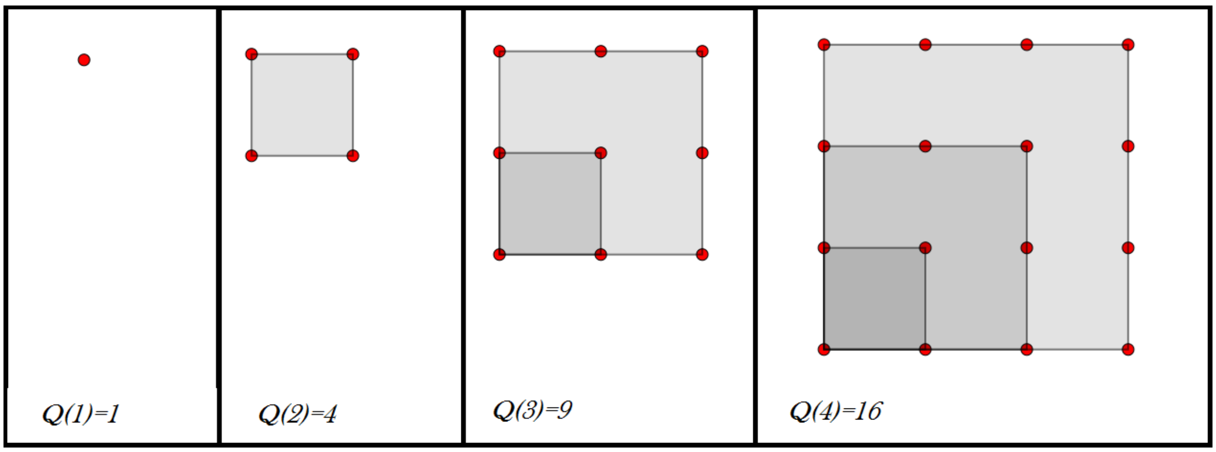


Figure 2. The first four square numbers. Authors' elaboration.

**Proposition 1:** Every square number is the sum of two consecutive triangular numbers.

**Proof:**

$$Q_n = n^2, = \frac{2n^2}{2}, = \frac{n^2 + n^2 + n - n}{2}, = \frac{n^2 - n + n^2 + n}{2}, = \frac{n(n+1) + (n-1)n}{2}, = T_n + T_{n-1}$$

In this way, we can form a square number using two triangular numbers. We just take the targeting vectors to form the square. That is, if the inside angle of a square is 90°, then we must have a vector of 45° (half of 90°) and form two sequences of triangular numbers. Using this argument, we create two angles, one 90° and one 135° (90° + 45°), as shown in Figure 3:

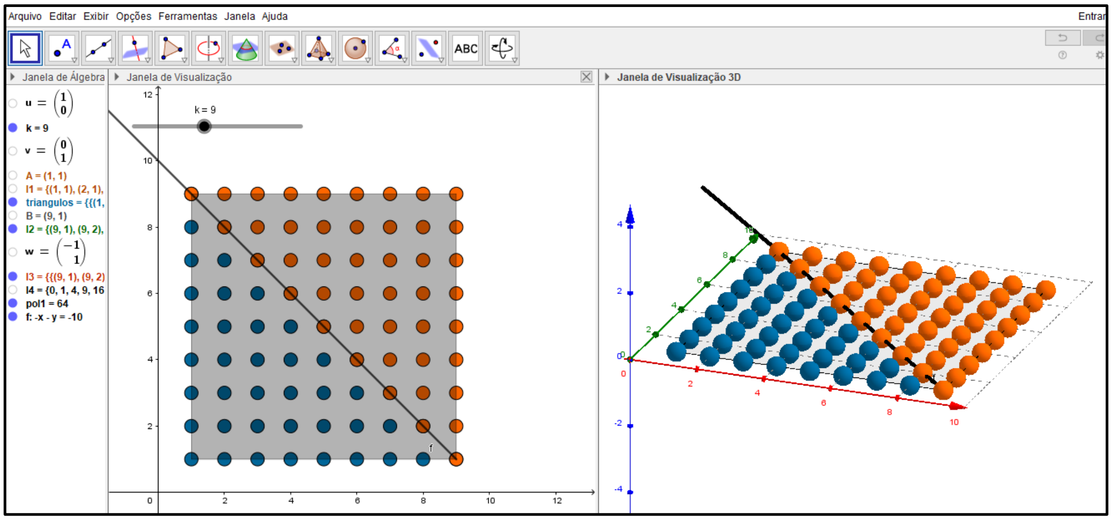


Figure 3. Ninth square number as the sum of two triangular numbers. Authors' elaboration.

From Figure 3 onwards, we notice two right triangles, which are triangular numbers, and a semi-right triangle, indicating that the property can be noticed in the figure in question.

The pentagonal numbers are the numbers that make up the sequence. $(1,5,12,22,...)$, where we define as $P_1=1, P_2=5, P_3=12, P_4=22$, so that $P_n=\frac{3n^2-n}{2}$ . This formula is already known from the writings of many authors such as Deza and Deza (2012) and Fossa (2018), who are authors who help in the theoretical foundation of this work.

**Proposition 2:** The $n-th$ pentagonal number is formed by the sum of three $(n-1)th$ triangular numbers plus $n$.

**Proof:**

$$3\cdot T_{n-1}+n=3\cdot\left(\frac{(n-1)n}{2}\right)+n=n\cdot\left(\frac{3(n-1)+2}{2}\right)=n\left(\frac{3n-1}{2}\right)=\frac{3n^2-n}{2}$$

For the construction of pentagonal numbers, we must create three lists of triangular numbers, plus a list of $n$ points. For that we need to know which vectors we should translate the triangle lists by. Note that in Figure 4 the pentagon angles are not calculated, but a pentagon is constructed and between its points are formed vectors through which the lists will be translated.

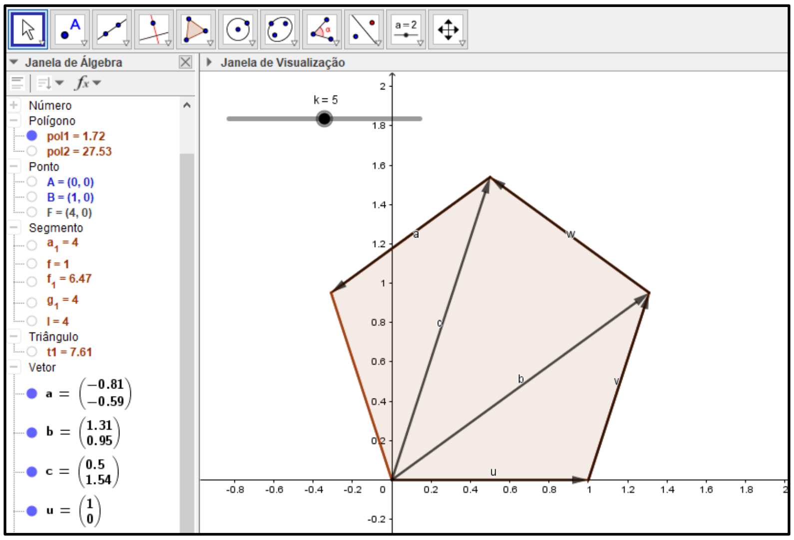


Figure 4. Polygon scheme of construction of pentagonal numbers. Authors' elaboration

In Figure 4 we have a pentagon with a 1u side (a unit of measure) and from this side we build vectors through which the lists of points or triangles will be translated, using the same commands used to build triangular numbers. Thus, in figure 5 a list of triangles is represented by vectors $u$ and $v$, in which the generated figure forms the first triangle inside a pentagon:

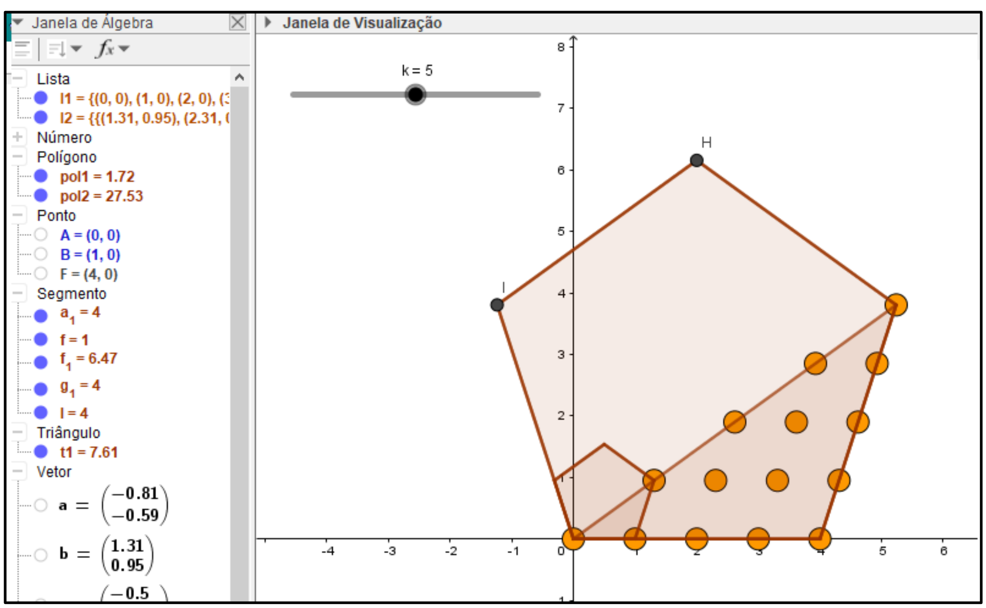


Figure 5. Construction of a list of points on the base polygon for pentagonal numbers. Authors' elaboration.

From the above and following the same procedure for the other lists, we can see that they will, in fact, form the pentagon and the sequence of pentagonal numbers.

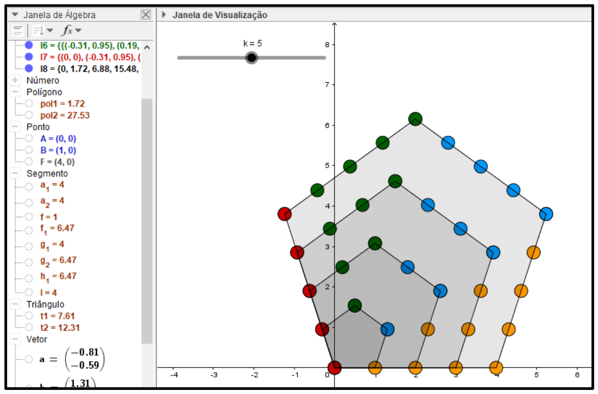


Figure 6. Fifth pentagonal number. Authors' elaboration.

Figure 6 presents proposition 2, with the triangular numbers and the line with $n$ points, differentiated by colors. The lines that cover each list of points define the succession of pentagonal numbers, according to the sequence presented above. In this way, we can also construct hexagonal, heptagonal, and so on, just defining vectors where the lists must translate.

For flat figure numbers, with different configurations from polygonal numbers, we must understand the structure of the figure formed by the given sequence. For example, centered polygonal numbers are formed by a central point and layers of points around it, forming polygons. The numbers that form these sequences are called centered triangular numbers, centered squares, etc. Figure 7 presents some of these numbers:

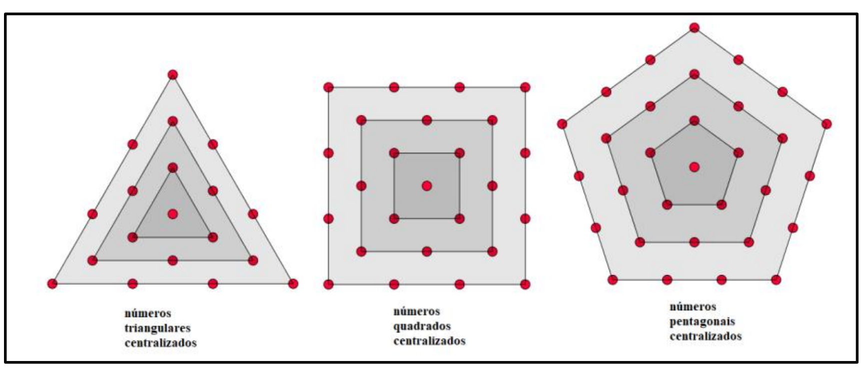


Figure 7. Example of centered polygonal numbers. Authors' elaboration.

When looking at Figure 7, it can be seen that these numbers bring some properties that relate them to the triangular numbers, so it is also possible to build these sequences with the support of the GeoGebra software. To do this, we present the triangular numbers and their general term formula.

We will define the sequence of centered triangular numbers, given by $(1,4,10,19,...)$ as $TC_1 = 1, TC_2 = 4, TC_3 = 10, ...$ and, we realized that: $TC_1 = 1,\ TC_2 = 1 + 3 \cdot 1 = 1 + 3T_1$ $TC_3 = 1 + 3 \cdot 1 + 3 \cdot 2 = 1 + 3 \cdot (1+2) = 1 + 3T_2$ . Thus $TC_n = 1 + 3T_{n-1}, TC_1 = 1$. This formula can be presented using GeoGebra from a central point, generating around it three lists of triangular numbers, where each list can start from its vertices. Figure 8 represents a centralized triangular number, built from this property.

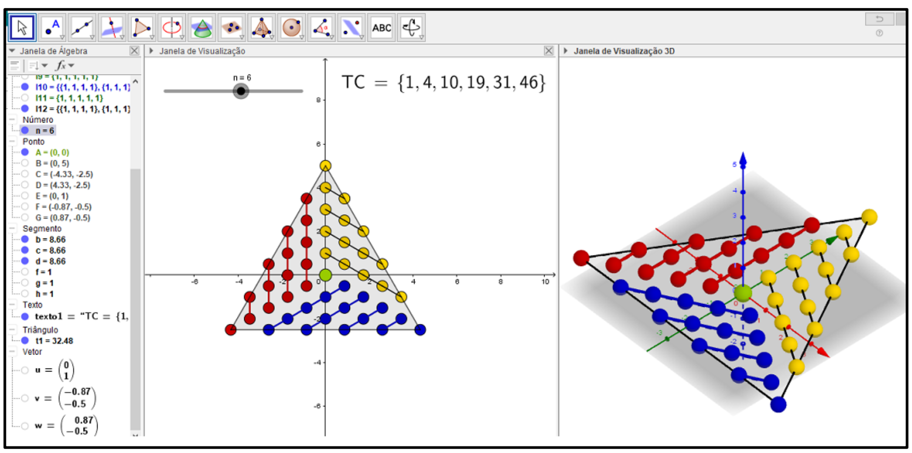


Figure 8. Sixth centralized triangular number. Author's elaboration.

In Figure 8 we can see that each list of triangular numbers started from each vertex of the triangle, that is, it is possible to use the same notion to build other centralized polygonal figured numbers. It is noteworthy that some extra features were used in this construction, such as segment lists, list with some terms of the sequence, varied colors, to improve the visualization. Thus, it is possible to observe the three lists of triangular numbers clearly. Remembering that, to build these numbers, the same lists of triangular numbers were used, translated by the vectors that start from the central point.

The next flat numbers that we can also build are called star numbers and are the same as the sequence of centered dodecagonal numbers, that is, their representation can either be constructed as a twelve-sided polygon with a centered point, or it can be presented as a six-pointed star, with a point also centered. The star numbers form the sequence $(1,13,37,73,121,...)$ and, we will call $E_1 = 1, E_2 = 13, E_3 = 37,...$ . By constructing the centralized triangular numbers, we can find a relationship that deduces the formula for the general term of the star numbers, in which it can be seen that: $E_2 = 13 = 1 + 12 = 1 + 12T_1$ , $E_3 = 37 = 1 + 36 = 1 + 12 \cdot 3 = 1 + 12T_2$ . This leads us to assume that $E_{n+1} = 1 + 12T_n$ , which is equivalent to the formula presented by Deza and Deza (2012) for the construction of this number. We will use a center point and six lists of square numbers. $Q_{n-1}$, added to a list of points such that $L_{n-1} = n - 1$. Thus, $E_n = 6Q_{n-1} + 6L_{n-1}$, i.e:

$$E_n = 1 + 6(Q_{n-1} + L_{n-1}) = 1 + 6(T_n + T_{n-1} + (n-1)) = 1 + 6(T_n + T_n) = 12T_n$$

Building the lists through the vectors, starting from the central point to the direction of each vertex of a regular hexagon and generating by these a list of square numbers, we have Figure 9:

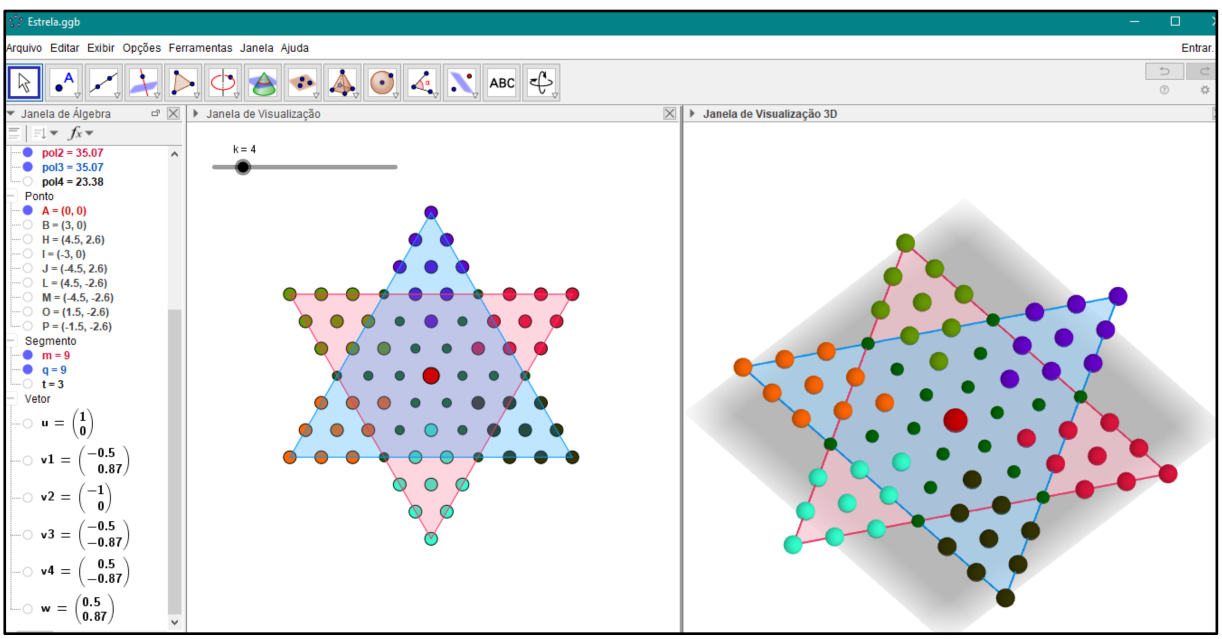


Figure 9. Fourth star number. Authors' elaboration.

We can see in Figure 9 that the lists of linear numbers are formed by smaller points. To improve the visualization, in Figure 10 we can see this configuration with an emphasis on the points, without the polygons.

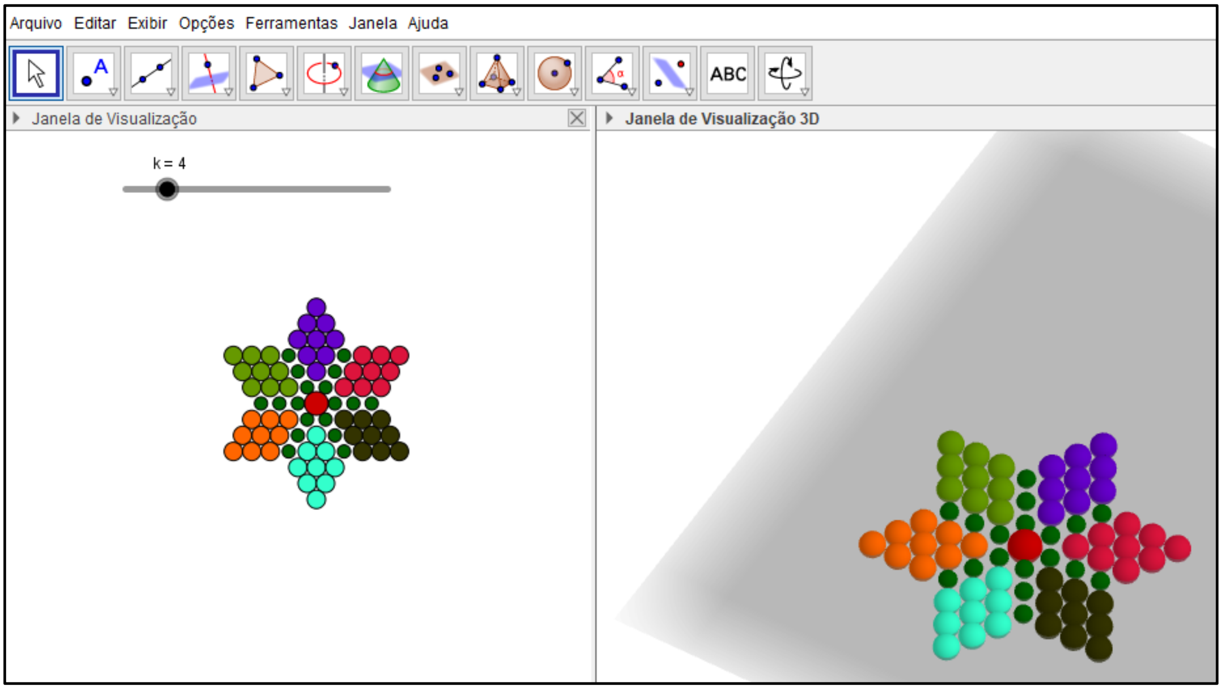


Figure 10. View fourth star number from your points in geogebra software. Authors' elaboration.

In figure 10 we can see a well-defined star, using the same configurations as in Figure 9, but without the need for polygons, without the axes and meshes, so that the figure is presented only with the arrangement of the points that form it.

In fact, in this section, the possibility of constructing different figures was shown, starting only from their relationship with the triangular numbers. In the next section, the configurations that form the three-dimensional figured numbers and their relationships with the triangular numbers are brought up, based on the observation of some important properties.

## Construction of three-dimensional figured numbers from triangular numbers

Three-dimensional, spatial, or polyhedral figured numbers are sequences of numbers that represent points that, distributed at equal distances, form polyhedra. To represent them we use the 3D GeoGebra window, as well as the relationships of these numbers with the triangular numbers. Among them, the first to be discussed are the tetrahedral numbers, formed by the sum of successive triangular numbers, that is, the $n-th$ tetrahedral number $T^3(n)$, is formed by the sum of the $n$ first triangular numbers $(T_n)$. Thus, we have: $T^3(1)=1, T^3(2)=1+3, T^3(3)=1+3+6, \ldots, T^3(n)=1+3+\cdots+\frac{n(n+1)}{2}$ with $n>0$, the formula of the general term for tetrahedral numbers is given by $T^3(n)=\frac{n(n+1)(n+2)}{6}$ with $T_1^3=1$ and $n\in\mathbb{N}$, (Alves, Borges Neto and Maia, 2012). When we handle the successive addition commands of the triangular numbers, through the "translate part of the list", using the triangular numbers as a list, we obtain Figure 11:

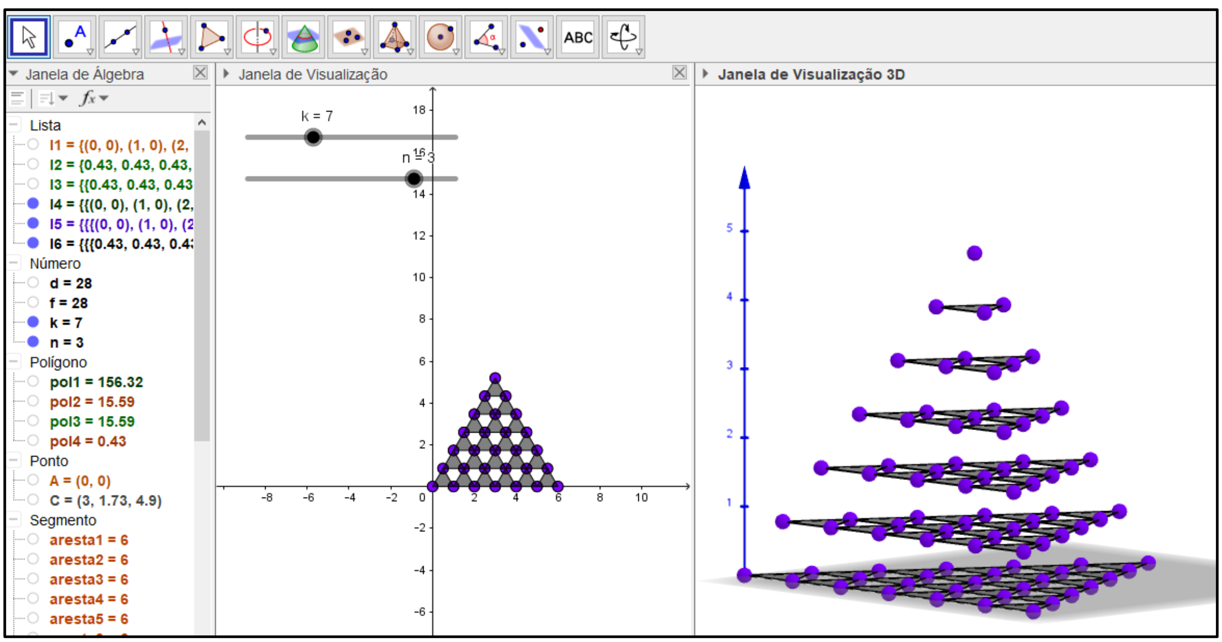


Figure 11. Seventh tetrahedral number. Author's elaboration.

Note that in Figure 11 we have a sum of $n$ triangular numbers, visualized in the 3D window, where the result is a pyramid of points that represent the tetrahedral numbers.

From assembling these numbers and using the same arguments, we can construct other pyramid numbers and find their general term formulas, as is the case for pentagonal square pyramid numbers. The first are the square pyramidal numbers that form the sequence $(1,5,14,30,...)$ and we will call $P_4^3(1)=1, P_4^3(2), P_4^3(3)=14,$ and so on. The formula for these numbers is given by $P_4^3(n)=\frac{n(n+1)(2n+1)}{6}$ with $P_4^3(n)=1$ (Conway and Guy, 1996, p. 57).

Using tetrahedral numbers with the support of GeoGebra software, we point out the following relationship:

$$P_4^3(n)=T^3(n)+T^3(n-1)$$

That is, from two consecutive tetrahedral numbers, to build the square pyramidal numbers, we then have the development:

$$\begin{aligned} P_4^3(n) &= T^3(n)+T^3(n-1) \\ &= \frac{n(n+1)(n+2)}{6}+\frac{(n-1)n(n+1)}{6} \\ &= \frac{\big(n(n+1)\big)\big(n+2+n-1\big)}{6} \\ &= n(n+1)(2n+1) \end{aligned}$$

Figure 12 shows how the construction actually features square pyramidal numbers:

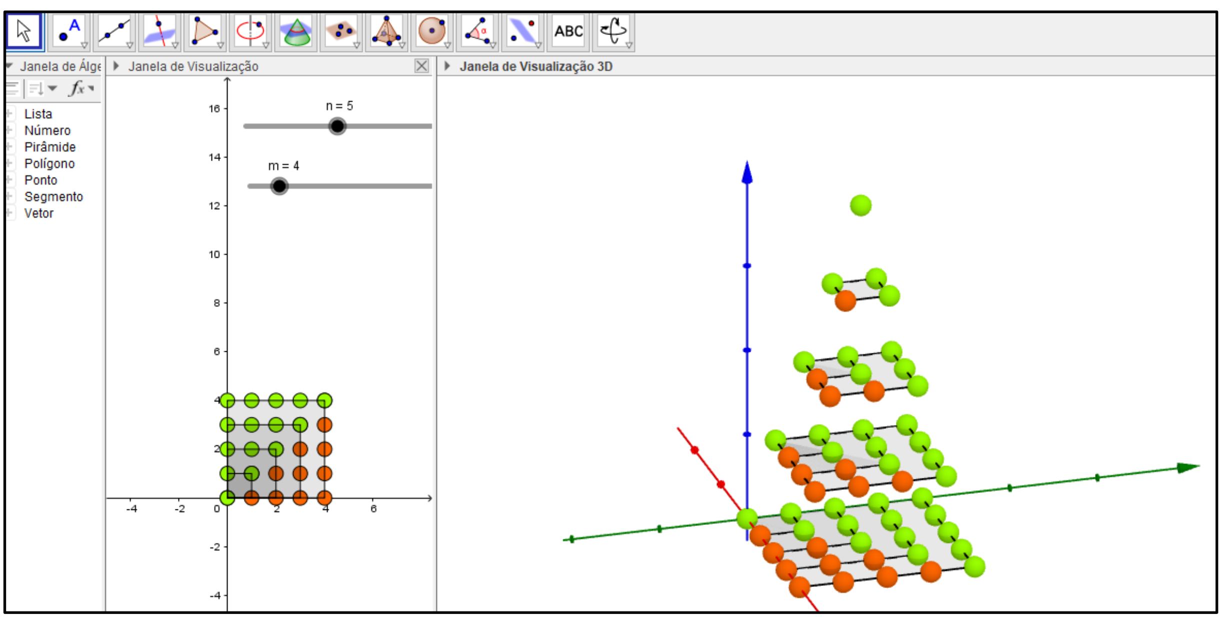


Figure 12. Fourth square pyramidal number highlighted at the base. Author's elaboration.

In Figure 12 it is possible to observe the two lists of tetrahedral numbers, with highlighted colors and united, forming a pyramid of points with a square base. Thus, from the figure and from the formula used, these numbers are the square pyramidal numbers.

Similarly, we will use three lists to form the pentagonal pyramid numbers. Since the base is formed by pentagonal numbers, discussed earlier, we can deduce its formula. According to Deza and Deza (2012), a pyramid number can be written according to the mathematical model:

$P_m^3(n) = \dfrac{n(n+1)((m-2)n-m+5)}{6}$ . Thus, a pentagonal pyramidal number has the formula for any term $P_5^3(n) = \dfrac{n(n+1)(3n)}{6}$ using tetrahedral numbers, so that $P_5^3(n) = T^3(n) + 2T^3(n-1)$ . In fact, we have to:

$$\begin{aligned} P_5^3(n) &= T^3(n) + 2T^3(n-1) \\ &= \frac{n(n+1)(n+2)}{6} + \frac{2(n-1)(n)(n+1)}{6} \\ &= \frac{\big(n(n+1)\big)\big(n+2+2n-2\big)}{6} \\ &= \frac{n(n+1)(3n)}{6} \end{aligned}$$

Figures 13 and 14 show the union of lists of tetrahedral numbers within a pentagonal base. The numbers formed, according to the mathematical models used, are pyramidal pentagonal numbers.

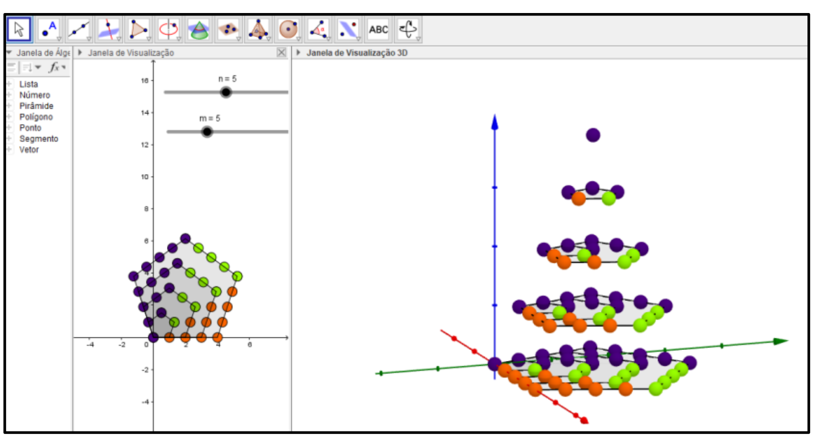


Figure 13. Fifth pentagonal pyramidal number, highlighted at the base. Authors' elaboration.

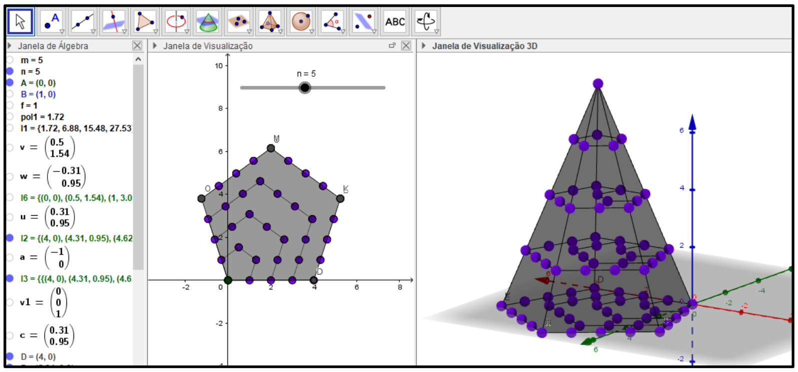


Figure 14. Another presentation of pentagonal pyramidal numbers in GeoGebra software. Authors' elaboration.

In Figures 13 and 14, the pyramids of pentagonal numbers, which form the pyramidal pentagonal numbers, can be observed in two ways: in the first, it is possible to observe the lists of tetrahedral numbers, seen in different colors; in the second, the lists are of the same color, as it presents the pyramid as a sum of $n-th$ pentagonal numbers.

Octahedral numbers, on the other hand, have a formula, according to Teo and Sloane (1985) and Alves and Barros (2019), $O(n)=\frac{n(2n^2+1)}{3}, O(1)=1$, and we can form them using geogebra software, handling two square-based pyramids. As we already have these numbers, we can form the following numbers from the following proposition: the $n-th$ octahedral number is formed by adding the $n-th$ square pyramidal number to the $(n-1)-th$ square pyramidal number. So, we have:

$$O(n)=\frac{(n-1)\cdot n\cdot(2n-1)}{6}+\frac{n\cdot(n+1)\cdot(2n+1)}{6}=\frac{n(2n^2-3n+1+2n^2+3n+1)}{6}=\frac{n(4n^2+2)}{6}$$

$$O(n)=\frac{n(2n^2+1)}{3}$$

From the square-based pyramid commands, previously used to build pyramid numbers, we can build octahedral numbers in GeoGebra, as shown in Figure 15:

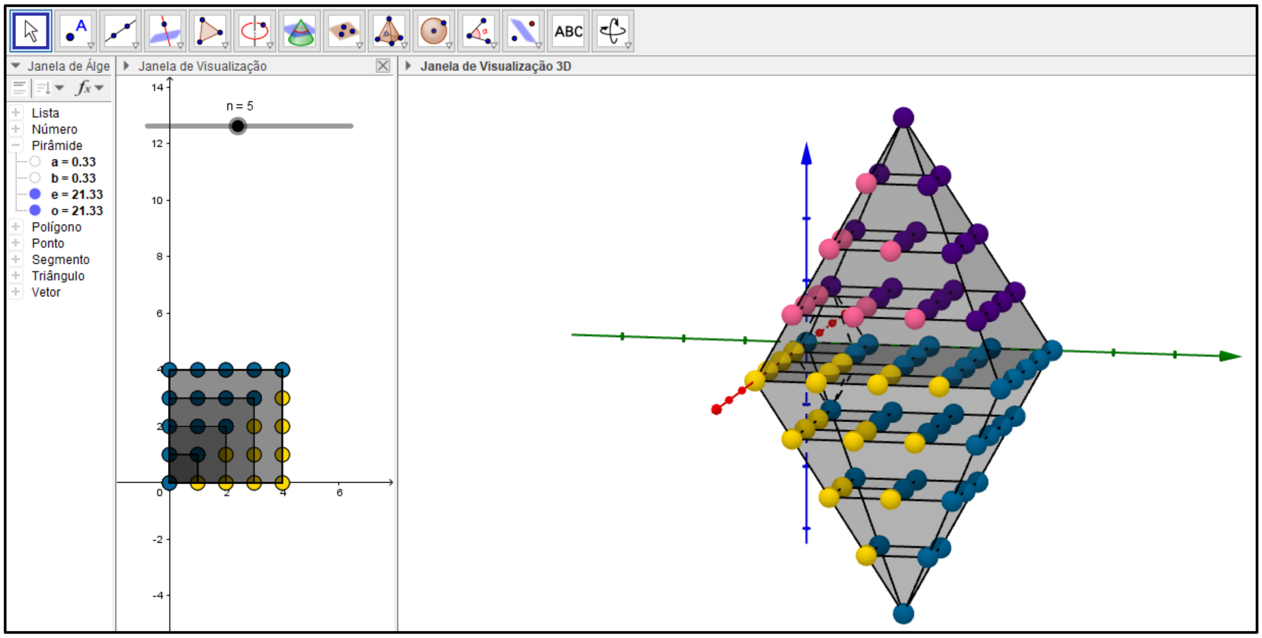


Figure 15. Fifth octahedral number. Authors' elaboration.

Figure 15 represents the 3D projection of the octahedral numbers as two constructions of inverted pyramidal numbers joined at the base so that the base is common to the two pyramids. In this way, we can have the two pyramids made up of square pyramid numbers in a row.

To construct cubic numbers from tetrahedrons, we can fill a cube with tetrahedrons so that no tetrahedron face overlaps the other in a neighboring tetrahedron. In this way, we can build six tetrahedrons inside a cube and, from that, assemble the cubic numbers and formulate a proposition. For this, we present the cubic numbers that form the sequence $(1,8,27,64,\ldots)$ and we will call

$C_1 = 1, C_2 = 8, C_3 = 27, \ldots, C_n = n^3$, where we have to:

**Proposition 4:** the $n - th$ cubic number is formed by adding the $n - th$, plus four times the $(n - 1) - th$, plus the $(n - 2) - th$ tetrahedral number.

**Proof**:

$$T_n^3 + 4T_{n-1}^3 + T_{n-2}^3 = \frac{n(n+1)(n+2)}{6} + \frac{4(n-1)(n)(n+1)}{6} + \frac{(n-2)(n-1)(n)}{6}$$

$$= \frac{n}{6}\big((n+1)(n+2) + 4(n-1)(n+1) + (n-2)(n-1)\big)$$

$$= \frac{n}{6}(n^2 + 3n + 2 + 4n^2 - 4 + n^2 - 3n + 2)$$

$$= \frac{n}{6}(6n^2) = n^3 = C_n$$

Figures 16 and 17 represent a cube formed with tetrahedral numbers. It is possible, by separating the colors of each point, to perceive pyramids of points in different directions, completing each part of the cube.

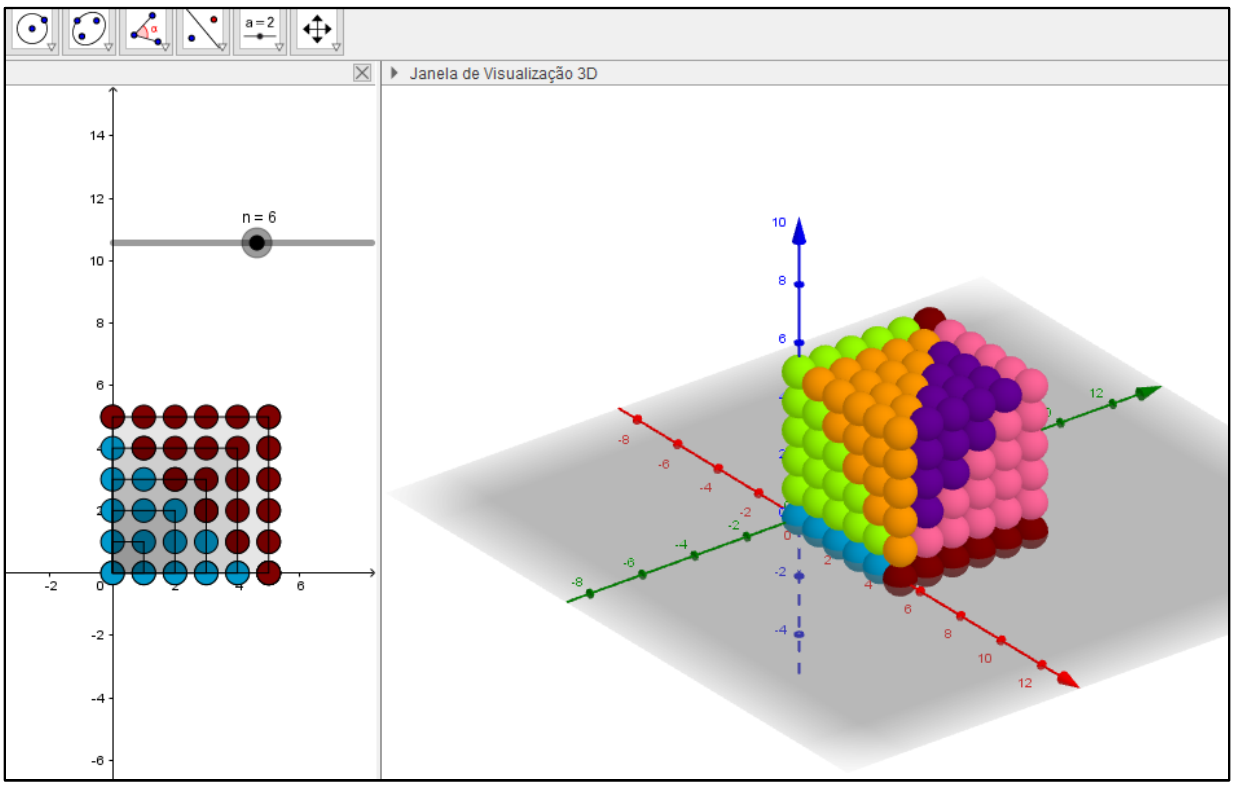


Figure 16. Fifth cubic number constructed from tetrahedral numbers. Authors' elaboration.

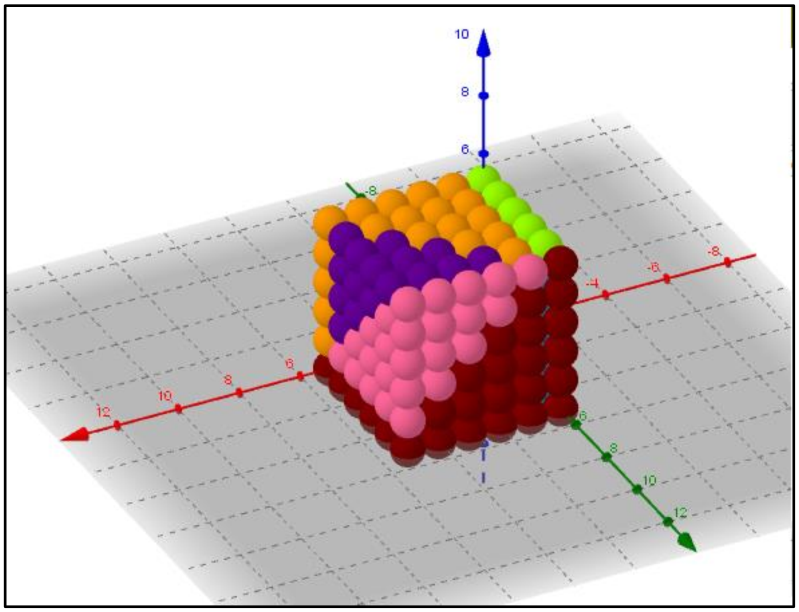

Figure 17. Close up view of a cubic number. Authors' elaboration.

The next figured number represents a starry polyhedron. This figure can be constructed from the intersection between two tetrahedrons. Or with an octahedron, where each triangular face is constructed from a tetrahedron with a point outside that octahedron. Figure 18 presents, in an analogous way, the three-dimensional representation of the star number, which can also be constructed using only tetrahedral numbers.

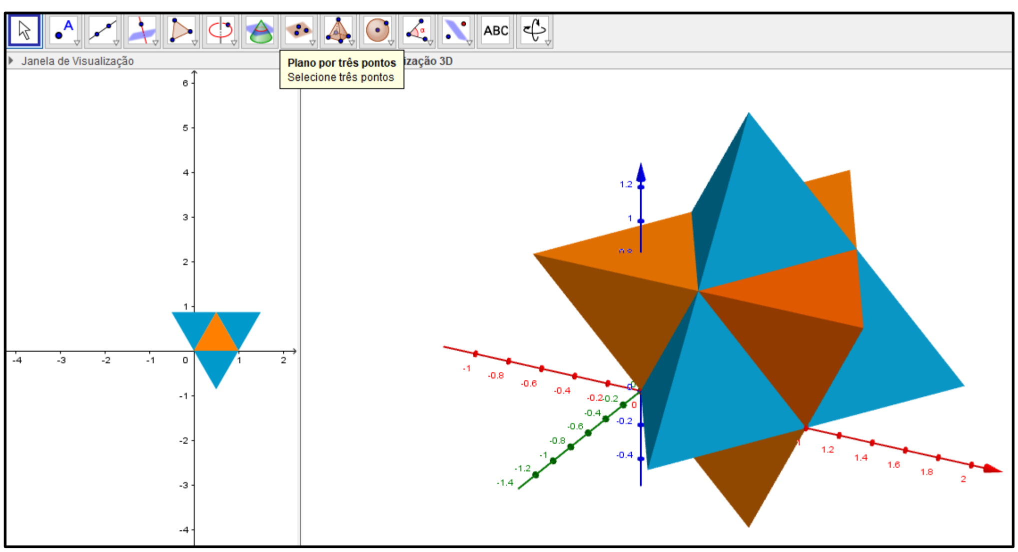


Figure 18. *Stella octangula* constructed with two tetrahedrons. Authors' elaboration.

Thus, Figure 18 explains the polyhedron on which this sequence called *stella octangula* numbers is based, in which two inverted tetrahedrons can be observed. The octahedron that focuses on the core of this figure can be filled in with octahedral numbers. From each face of the octahedron leave other smaller tetrahedrons, which generate the *stella octangula*. In this case, as we are talking about figured numbers, the numbers of the stella octangula or numbers of the eight-armed star, present as a general term formula $SO(n) = n(2n^2 - 1)$, according to Deza and Deza (2012).

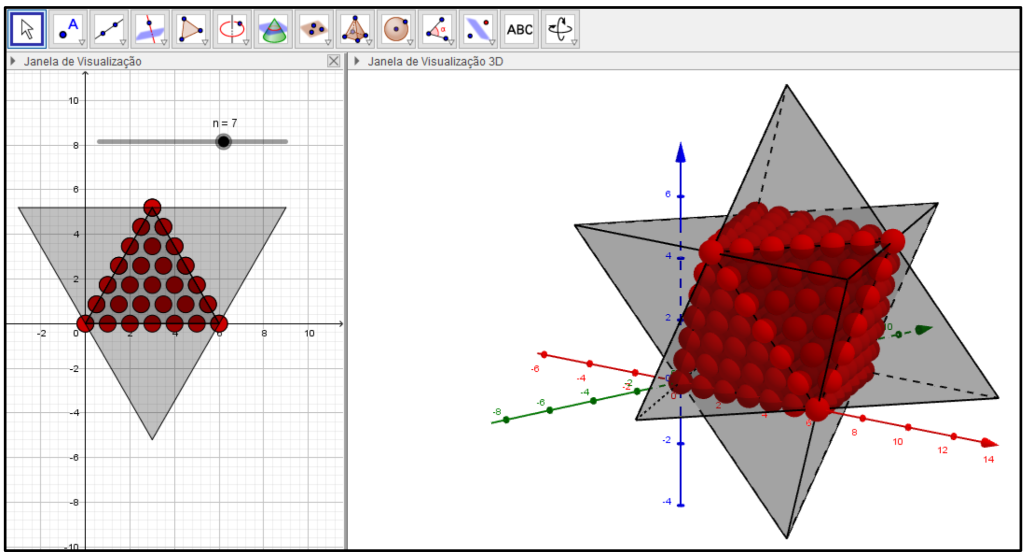


Figure 19. Octahedral numbers built into the nucleus of an octangula stella. Authors' elaboration.

Figure 19 shows the central octahedron, this observation being necessary, as, once the figure is completely constructed, it is difficult to perceive this polyhedron. From the construction using GeoGebra, we have two propositions about the *stella octangula* numbers:

**Proposition:** the $n - th$ number of the stella octangula is constituted by the sum of the $n - th$ octahedral number added to eight times the $(n - 1) - th$ tetrahedral number.

**Proof:**

$$
\begin{aligned}
SO(n) &= O(n) + 8T^3(n-1) \\
&= \frac{n(2n^2+1)}{3} + \frac{8(n-1)n(n+1)}{6} \\
&= \frac{2n(2n^2+1)}{6} + \frac{8(n-1)n(n+1)}{6} \\
&= \frac{2n\left(2n^2+1+4(n-1)(n+1)\right)}{6} \\
&= \frac{2n\left(2n^2+1+4n^2-4\right)}{6} \\
&= \frac{2n(6n^2-3)}{6} \\
&= n(2n^2-1)
\end{aligned}
$$

In Figure 20 we can see a construction of the *stella octangula* numbers:

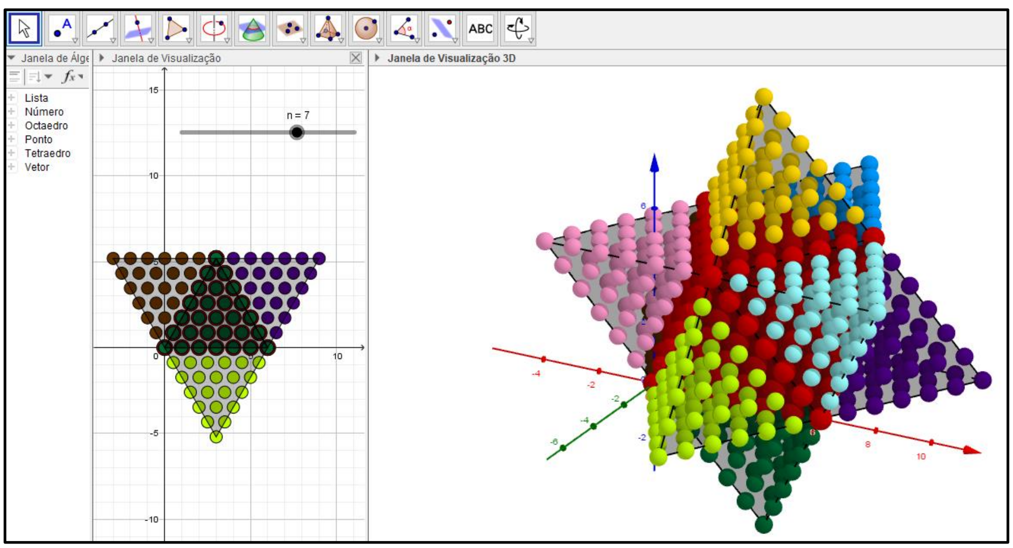


Figure 20. Seventh number of the stella octangula. Authors' elaboration.

Both Figures, Figure 19 and Figure 20, show the numbers of the *stella octangula* from two different angles so that its eight "arms" can be seen, highlighted in different colors. In the images it is possible to observe the 2D GeoGebra window and see that we can also form the numbers of the stella octangula using only tetrahedral numbers, with the first one being a unit smaller than twice the others. This is due to the fact that the central point, as illustrated in the figures, serves as a vertex for another tetrahedral number, while the $n-1$ points that divide it are the bases for the other tetrahedral numbers. This way we have to $SO(n)=T^3(2n+1)+4T^3(n+1)$, where we have to:

$$
\begin{aligned}
SO(n) &= T^3(2n-1)+4T^3(n-1)\\
&= \frac{(2n-1)(2n-1+1)(2n-1+2)}{6}+\frac{4(n-1)n(n+1)}{6}\\
&= \frac{(2n-1)(2n)(2n+1)+4(n-1)n(n+1)}{6}\\
&= \frac{2n\left(4n^2-1+2(n^2-1)\right)}{6}\\
&= \frac{2n\left(6n^2-3\right)}{6}\\
&= n\left(2n^2-1\right)
\end{aligned}
$$

Figures 21 and 22 show this construction with an emphasis on colors to differentiate the larger tetrahedron from the others, so we can easily observe the property used. Although you cannot visualize the colors, one of the tetrahedrons is based on visualization in two dimensions, that is, a triangle. Thus, the figures aim to present different rotations so that the visualization includes more points and that the eight "arms" of the *stella octangula* can be observed in the form of points, forming pyramids.

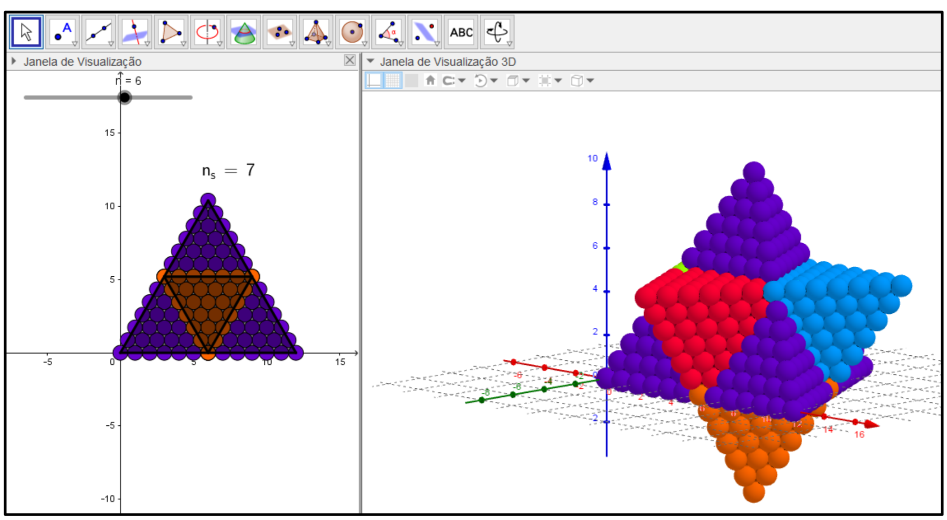


Figure 21. Another construction for the stella octangula numbers. Authors' elaboration.

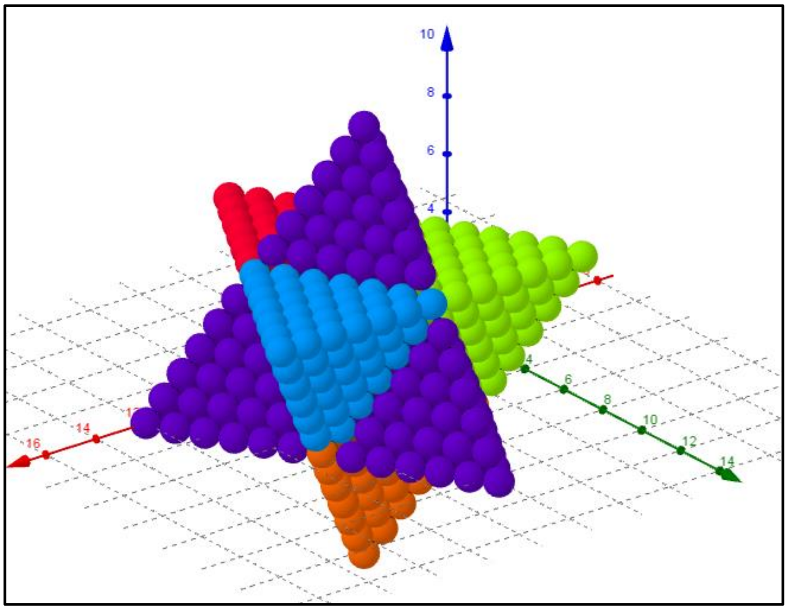

Figure 22. Enlarged view for the second construction of the *stella octangula* numbers. Authors' elaboration.

The figures presented in this section were obtained essentially based on tetrahedral numbers, and it is possible to present properties that relate them. In this way, we can form as many three-dimensional figured numbers as we wish. We also observe that these tetrahedral numbers are formed based on the triangular numbers, the latter being necessary for all the structures mentioned.

## Final considerations

This text presented the possibility of using the GeoGebra software as a resource to discover properties of several figured numbers, whether they are two-dimensional or three-dimensional. Thus, this work followed a path aimed at presenting relationships between triangular numbers and other figured numbers, showing that this sequence is fundamental for the construction of the others. Other possibilities can be listed, as long as you have the structure of the polytope in which the lists will be built and translated. Furthermore, there is the possibility of representing even 4D shapes or any sequence within the figures shown.

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