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Internal Yoneda Ext Groups, Central H-spaces, and Banded Types

Jarl Gunnar Taxerås Flaten, *Western University*

Supervisor: Christensen, John Daniel, *The University of Western Ontario*

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Abstract

We develop topics in synthetic homotopy theory using the language of homotopy type theory, and study their semantic counterparts in an ∞-topos. Specifically, we study Grothendieck categories and Yoneda Ext groups in this setting, as well as a novel class of *central* H-spaces along with their associated *bands*. The former are fundamental notions from homological algebra that support important computations in traditional homotopy theory. We develop these tools with the goal of supporting similar computations in our setting. In contrast, our results about central H-spaces and bands are new, even when interpreted into the ∞-topos of spaces.

In Chapter 2 we define and study Grothendieck categories in HoTT, which are abelian categories that satisfy additional axioms. The main subtlety in this development is the construction of left-exact coproducts (the AB4 axiom) from exactness of filtered colimits (AB5) in a cocomplete abelian category. In particular, it follows that coproducts of modules over a ring are left-exact, and this is one of our main results. Traditionally, it is easy to deduce AB4 from AB5 using that the finite subsets of a set X form a filtered category. However, the concept of "finite" bifurcates into multiple nonequivalent concepts in our constructive setting. Instead we consider lists of elements of X, inspired by Roswitha Harting's construction of internal coproducts of abelian groups in an elementary topos [Har82].

These results lay the foundation for our study of Yoneda Ext groups [Yon60, Mac63]. Chapter 3 describes our formalization of the higher Ext groups $\mathrm{Ext}^n_{\mathbb{Z}}(B, A)$ of abelian groups, and the expected (contravariant) long exact sequence. We give a novel proof of the usual six-term exact sequence from a fibre sequence of spaces of short exact sequences (Theorem 3.4.1). We also emphasize that our formalization can be adapted to modules over a ring, and that these higher Ext groups are interesting even for abelian groups. In Chapter 4 we further develop the theory of our Ext groups and relate their semantic counterparts to *sheaf Ext* [Gro57] in certain ∞-topos models. We also carry out a detailed study of *internal* injectivity and projectivity of modules an ∞-topos, and show that our Ext groups can be computed using resolutions of such in certain cases.

The final chapter 5 is mostly independent. In it, we study generalizations of Eilenberg–Mac Lane spaces called *central* H-spaces. Such H-spaces admit an astonishingly simple notion of torsor (independently studied in [Wär23]), namely that of a *banded type*. The type of such torsors form a delooping of a central H-space, analogously to how the type of torsors of a group G form a $\mathsf{K}(G, 1)$. Using centrality, we define a tensor product on banded types, producing an H-space structure which makes the type of torsors into a central H-space itself. Iterating this procedure, we obtain arbitrary deloopings of A (and also of pointed self-maps of A) which are in fact unique.

Keywords: homotopy type theory, homological algebra, Yoneda Ext, sheaf Ext, higher topos theory, internal logic, H-spaces, infinite loop spaces, formalization of mathematics.

Summary for Lay Audience

In homotopy theory, we study properties of shapes which are preserved by continuous deformations. Some of these shapes are easy to visualize, like circles, spheres, and tori. Others are higher-dimensional (even infinite-dimensional) and therefore impossible to visualize, but they are still important for various reasons. A basic homotopical property is how many connected components a shape has, or how many holes it has—but there are also much more sophisticated properties which give deep insight into the nature of a shape.

There are many kinds of shapes, and many ways of studying them. For instance, instead of taking a point as the basic building block of shapes, one can take another shape as the basic building block. Using the circle as the basic building block, drawing a circle produces a torus (since each "point" is replaced by a circle). In this thesis we study shapes from an *internal* perspective, which encompasses many ways of studying shapes. This approach can also be represented on a computer, meaning it can check that our proofs are correct.

This internal approach is presently being developed. We contribute to this endeavour by developing certain computational tools, called Ext groups, which are useful for computing properties of shapes. Roughly, Ext groups measure in how many ways two structures can be knitted together to form a new structure. (Such a "knitted structure" is called an *extension*, which is where the name Ext comes from.) In the context of shapes, they let us compute new properties from old ones. Ext groups play an important role in traditional computations, but have not been previously studied from the internal viewpoint.

We also introduce and study a new form of symmetry we call *centrality*. A shape is central whenever there is precisely one symmetry (of a certain kind) of the shape which sends its base point any other point. The circle, for example, is central because there is only one way of rotating the base point to any other point without going backwards or doing a full loop. Though a seemingly simple condition, centrality has surprising consequences.

Co-Authorship Statement

This thesis comprises both independent work and joint work with several others.

Chapters 2 and 3 are the respective papers [Fla23b, Fla23a] of which I am the sole author. The later paper is on the formalization of certain mathematics, and parts of this formalization were done in collaboration with Dan Christensen and Jacob Ender.

The material of Chapter 4 is joint with Dan Christensen and comprises the forthcoming paper [CF23]. I have written most of this chapter, with copious feedback from Dan. We have both made substantial theoretical contributions to this chapter, though the idea to study Yoneda Ext in homotopy type theory is due to Dan.

Chapter 5 is the paper [BCFR23] which is joint with Ulrik Buchholtz, Dan Christensen, and Egbert Rijke. Large parts of this paper were written by me, and I made substantial contributions to the main results of this paper, as well as to its conception.

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I thank Jacob Ender for contributions to the formalization of the Baer sum of Ext groups in Section 3.3.3, and the collaborators of the Coq-HoTT library [CH] for their careful review of, and contributions to, the various pull requests originating from both [Fla23a] (Chapter 3) and [BCFR23] (Chapter 5). Ali Caglayan deserves an additional thanks for his efforts in maintaining the Coq-HoTT library, and in particular for his rapid fixes to technical issues that invariably crop up.

I am grateful to Ulrik Buchholtz, David Jaz Myers, and Egbert Rijke for being such lively and stimulating collaborators, from whom I have learnt and gained a lot.

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Contents

Chapter 1

Introduction

This thesis contributes to the field of synthetic homotopy theory by investigating and developing certain fundamental topics from traditional homological algebra and homotopy theory in the language of homotopy type theory (HoTT). Our main motivation is two-fold: the language of HoTT may be interpreted into any ∞-topos, and, as such, it lets us prove theorems which hold in this generality. In addition, the formal rules of HoTT are amenable to representation on a computer, allowing us to formalize and verify these theorems using computers. Indeed, most of the results we present have already been formalized (see the formalization section below).

Chapters 2 through 4 concern the development of homological algebra in HoTT; more specifically, the theory of Grothendieck abelian categories, the theory of Ext groups and their long exact sequence, as well as related notions. Homological algebra plays a role in traditional homotopy theory similar to the one linear algebra plays in many other branches of mathematics—i.e., as a powerful computational tool. It was first developed in the generality used today by Grothendieck [Gro57], building on the work of many others (such as Buchsbaum [Buc55], Cartan and Eilenberg [CE56]), so as to allow for a systematic study of derived functors and sheaf cohomology in algebraic geometry. A tenet of Grothendieck's approach was to work with abelian categories that satisfy additional axioms, called the *AB-axioms*, which enable or facilitate development of the theory. Today we refer to abelian categories satisfying a selection of these axioms as *Grothendieck (abelian) categories*. In Chapter 2, we develop parts of the corresponding theory in our setting.

Since its introduction, homological algebra has become a standard tool in many branches of mathematics, and it plays a crucial role in many computations from traditional homotopy theory. Accordingly, we wish to build analogous tools so as to further develop synthetic homotopy theory. We will see that there are obstacles which hinder this development, such as *nonconstructive* aspects of the traditional approach, most notably the lack of enough injective and projective modules (or even abelian groups) in our setting.

One interesting nonconstructive aspect features in the construction of the coproduct $\bigoplus_{x:X} A_x$ of a family $(A_x)_{x:X}$ of abelian groups (or modules) indexed by a set X. Traditionally, we are used to there being a natural monomorphism $\bigoplus_{x:X} A_x \to \prod_{x:X} A_x$ from a coproduct to the corresponding product. In constructive settings, however, there is in general no such natural map, and possibly no nontrivial maps. An example where there is no nontrivial map is given by Harting [Har82] in the Sierpínski topos. Her insight was to consider the set $X :\equiv (2 \to 1)$

in this model, which (crucially) does not have *decidable equality*[1]. Indeed, for types X with decidable equality there is a natural monomorphism from the coproduct to the product. Despite these additional subtleties related to decidability, we construct a left-exact coproduct functor of modules in Chapter 2, inspired Harting's *internal coproduct* [Har82]. To do this, we replace a coproduct over X by a colimit over the type HX of (finite) lists of elements in X, which is a filtered category. By showing that the natural map $X \to HX$ is *final*, we realize a coproduct as a filtered colimit, which is left-exact. More generally, our construction shows that AB5 (filtered colimits are exact) implies AB4 (coproducts are left-exact) in any abelian category (Theorem 2.3.19).

We should explain what we mean by "model" in the previous paragraph, as Harting works in a 1-topos, which is not by itself a model of HoTT. These considerations also come up below. Any Grothendieck 1-topos $\mathcal{E}$ can be realized as the 0-truncated fragment of an ∞-topos $\mathcal{X}$, such as ∞-sheaves on a site presenting $\mathcal{E}$. Since $\mathcal{X}$ is a model of HoTT, this gives a formal connection between a (Grothendieck) 1-topos and HoTT. When we relate results in HoTT to counterparts in a 1-topos, it is through the ∞-topos $\mathcal{X}$. The situation for an *elementary* 1-topos is, at present, less clear. It is conjectured that any *elementary* ∞-topos [Ras22] is a model of HoTT. [KL18] Even if this is shown to be the case, this still does not completely clarify the relation to elementary 1-topos theory, as we do not know whether the latter can always be "embedded" into the former as in the Grothendieck case just described. Nevertheless, results about modules in an elementary 1-topos do inform and motivate the analogous development in HoTT, if only heuristically.

A substantial part of Chapter 4 is dedicated to studying injective and projective modules in models of HoTT. This study is important because it relates these fundamental notions in traditional mathematics to their counterparts in (models of) HoTT. In the case of projectives, it is well-known that sets themselves may not be projective in constructive settings, stemming from the lack of the axiom of choice. Accordingly, free abelian groups need not be projective (as modules), unless the generating set is. This phenomenon is also familiar to topos theorists: in the topos of G-sets (for a nontrivial discrete group G), the abelian group $\mathbb{Z}$ with trivial G-action is not projective. Indeed, the natural epimorphism $\mathbb{Z}G \to \mathbb{Z}$ from the group ring does not admit a section. However, $\mathbb{Z}$ is *internally* projective in this topos—and this topos generally admits enough *internally* projective modules, but not ordinary projectives. These observations are explained by the fact that the topos of G-sets satisfies the internal axiom of choice (IAC), but not the external version (see, e.g., [Joh77]). We give a sample computation of our Ext groups using an internally projective resolution in Proposition 4.3.12.

The case of injective modules is also interesting. Our study of these is facilitated by the existing literature on notions of injectivity of modules in an elementary topos. An important result—due to Harting for abelian group objects [Har83b], and Blechschmidt for module objects [Ble18]—is that various *internal* and *local* notions of injectivity coincide with ordinary injectivity in any localic topos, and that in general, ordinary injectivity implies internal injectivity. These results imply that any sheaf topos admits enough internal injectives (since it admits enough ordinary injectives), however this is not generally the case in the elementary setting. In fact, Blass constructed a model of ZF in which there are no nontrivial injective

[1] A type X has *decidable equality* if the type $\prod_{x,y:X}(x = y) \vee (x \neq y)$ holds. In particular, the law of the excluded middle implies that all sets have decidable equality.

abelian groups [Bla79], which gives rise to an elementary topos (with a natural numbers object) with the same property. Harting has further studied conditions for an elementary topos to have enough injective abelian groups [Har82]. Blass' model indicates to us that we should not be able to construct nontrivial injective modules in HoTT—though it is not a formal counter-model, as we explained above.

The aforementioned results of Harting and Blechschmidt which relate various notions of injectivity of modules in a topos rely on the fact that all of these notions are stable by base change [Har83b]. Harting's proof of this relies on her construction of the *internal coproduct* $\bigoplus_X$ of a family of abelian groups over an object X in a topos $\mathcal{E}$ [Har82], which yields a monomorphism-preserving left adjoint to base change of abelian group objects over X:

$$\bigoplus_X \; : \; \mathrm{Ab}_{\mathcal{E}/X} \leftrightarrows \mathrm{Ab}_{\mathcal{E}} \; : \; (-) \times X.$$

Here $\mathrm{Ab}_{\mathcal{E}}$ denotes the (abelian) category of abelian group objects in the elementary topos $\mathcal{E}$, and X-indexed families of such are abelian group objects in $\mathcal{E}/X$. A left adjoint between abelian categories which preserves monomorphisms has a right adjoint which preserves injective objects, as is easy to verify. Thus injectivity being stable by base change is an immediate consequence of $\bigoplus_X$ preserving monomorphisms. The proof that both *local* and *internal* notions of injectivity are stable as well is less direct, but also uses the internal coproduct.

By interpreting the notion of injectivity of modules from HoTT into an ∞-topos, we get another notion which we call *HoTT-injectivity*. In Chapter 4, we are interested in relating HoTT-injectivity to existing notions of injectivity (such as the ones mentioned above) in an ∞-topos. Though we also think it is a good custom to always relate semantic notions coming from HoTT to their traditional counterparts, our main motivation for doing this is to relate our Ext groups to *Ext sheaves* which are studied by algebraic geometers [Gro57].

As with any semantic notion, HoTT-injectivity is automatically stable by base change *in an ∞-topos*. Even though Harting showed that various notions of injectivity are stable by base change in a 1-topos, we do not get stability of the corresponding notions in an ∞-topos for free. This is because an ∞-topos has more slices than its 0-truncated fragment, since we can base change over objects which are not 0-truncated. The base change stability shown by Harting in a 1-topos only implies stability for base change over 0-truncated objects in an ambient ∞-topos. Nevertheless, we extend these stability results and show that internal injectivity is stable by base change in a certain class of ∞-toposes. (Specifically, in slices of ∞-toposes in which *sets cover*—see Theorem 4.3.25). This class suffices for many of our purposes. From this stability result, we deduce that HoTT-injectivity coincides with internal injectivity for models in this class. In turn, from this we deduce that our Ext groups recover *sheaf Ext* in these models—we return to this point in a moment.

The technique we use to show the stability results just mentioned can be applied more generally to any "set-theoretic" notion in an ∞-topos (i.e., any notion pertaining to 0-truncated objects), not just to internal injectivity of modules. In this generality, the technique lets one extend base change stability results from 1-topos theory to base change stability in the the aforementioned class of ∞-toposes. We leave it to future work to refine this technique into a precise theorem.

Our results give a fair understanding of the semantics of injectivity of modules in HoTT, but this does not change the fact that we do not have enough injectives (nor projectives) in

HoTT itself. Thus we cannot rely on resolutions in our approach to homological algebra, and instead we need to follow and develop "resolution-free" techniques. Such techniques have been previously studied, though they are perhaps more finicky and certainly less mainstream. An important example is Yoneda's resolution-free approach to Ext groups [Yon60] in an arbitrary (pre-)abelian category, nowadays called the *Yoneda Ext groups*. Given two modules B and A over a ring R (or more generally, two objects of a pre-abelian category), Yoneda defines the n-th Ext group $\mathrm{Ext}^n_R(B, A)$ as the path components of the category of length n exact sequences

$$A \to E_1 \to \cdots \to E_n \to B.$$

Such length n extensions may be added via the *Baer sum*, making this set into a group [Bae34]. As defined, these groups are *large* (i.e., proper classes) but can sometimes be shown to be isomorphic to small abelian groups. If the ring R is commutative then $\mathrm{Ext}^n_R(B, A)$ inherits an R-module structure. We stress that $\mathrm{Ext}^n_R(B, A)$ is always an ordinary abelian group; if A and B are modules internal to some topos, then these Ext groups are not objects of the topos in any sense. In contrast, the interpretation of the Ext groups we define below give internal analogs of the above (i.e., module objects) when interpreted into an ∞-topos

In Chapters 3 and 4 we develop Yoneda's approach to Ext groups in our setting. The approach described above works exactly as stated for $n = 1$. We define the type $\mathrm{SES}_R(B, A)$ of short exact sequences $A \to E \to B$, and define $\mathrm{Ext}^1_R(B, A)$ to be its set of path components. This works because the category of short exact sequences is a groupoid, and this groupoid is captured by the type $\mathrm{SES}_R(B, A)$. We then show that the loop space of $\mathrm{SES}_R(B, A)$ is isomorphic to the group of R-module homomorphisms $B \to A$. (To our knowledge, this fact was first observed in [Ret86].) As in the traditional case, our Ext groups are large—however, we construct an equivalence of types

$$\mathrm{SES}_R(B, A) \;\simeq\; (\mathsf{K}(B, 2) \to_* \mathsf{K}(A, 3))$$

from which it follows that $\mathrm{Ext}^1_R(B, A)$ is isomorphic to a small group (since the right-hand side is small). Moreover, for any short exact sequence $B \to E \to B'$ we show that the sequence

$$\mathrm{SES}_R(B', A) \to \mathrm{SES}_R(E, A) \to \mathrm{SES}_R(B, A)$$

where the maps are given by pulling back extensions, is a fibre sequence. The associated long exact sequence of homotopy groups gives the usual six-term exact sequence of Ext groups.

For $n > 1$, the category of length n exact sequences is not a groupoid, as it was for $n = 1$. To get the set of path components of these categories, we quotient out by zig-zags of arbitrary length. This approach is simplified by observing that a general map of length n extensions can be factored into a composite of "tensor moves" (see [Mac63, pp. 83]). This is the approach we take, following [Mac63] to define the higher Ext groups $\mathrm{Ext}^n_R(B, A)$ in HoTT. We also construct the usual long exact sequences associated with these, and the topic of Chapter 3 is the formalization of these results in HoTT. The long exact sequence makes Ext into a δ-functor [Gro57], which we moreover show is universal as such, hence a right derived functor (or *right satellite*) of the hom functor. Whenever R is a principal ideal domain (in the constructive sense of [LQ15]) and B is finitely presented, we show that $\mathrm{Ext}^n_R(B, A)$ vanishes for $n > 1$. However we emphasize that even for $R = \mathbb{Z}$ these higher Ext groups need not vanish in general.

This is because they recover higher Ext sheaves in certain models, which do not necessarily vanish.

In Section 4.3.3 we show that our Ext groups recover sheaf Ext when interpreted into certain ∞-toposes. To show this, we first prove in Section 4.2.5 that our Ext groups can be computed using injective (and projective) resolutions in HoTT. Being a right-derived functor, sheaf Ext can be computed using injective resolutions, which always exist in Grothendieck toposes. Thus, in models where ordinary injectivity implies the interpretation of injectivity from HoTT, we deduce that our Ext groups recover sheaf Ext (Theorem 4.3.29).

The final Chapter 5 is independent of the previous chapters, except for an application of Yoneda Ext groups in Theorem 5.5.13. In it, we introduce the new homotopy-theoretic concepts of *central* types and their associated *banded types* (or *bands*, for short). We also develop some general H-space theory, emphasizing in particular the moduli space of H-space structure on an H-space.

A pointed type A is central when the *evaluation fibration*

$$\mathbf{ev}_{\mathrm{id}} \;:\; (A \to A)_{(\mathrm{id})} \longrightarrow_* A$$

which evaluates a map at the base point of A is an equivalence. The domain of $\mathbf{ev}_{\mathrm{id}}$ is the component of $A \to A$ at the identity map. Intuitively, a central type is one that is equivalent to its own "orientation-preserving symmetries" through $\mathbf{ev}_{\mathrm{id}}$. The simplest central type is the circle $\mathbb{S}^1$. More generally, any Eilenberg–Mac Lane space for an abelian group is central, as can be seen from the following characterization: *a type A is central if and only if it is a connected H-space and $A \to_* A$ is a set* (Proposition 5.3.6).

Centrality of A is a simple condition with remarkable consequences: it implies that A is an infinite loop space (in a unique way) and that any pointed self-map of A can be infinitely delooped (uniquely). We give concrete and curious formulas which describe these deloopings, using the notion of a *banded type*. The latter are types X equipped with a *band* $p : \|A = X\|_0$, and these form the elements of the type $\mathsf{BAut}_1(A)$. Under the centrality assumption on A, we show that $\mathsf{BAut}_1(A)$ is a delooping of A. Moreover, we define a tensor product of banded types

$$X_p \otimes Y_q \;:\equiv\; (X_p^* =_{\mathsf{BAut}_1(A)} Y_q)$$

making $\mathsf{BAut}_1(A)$ into a connected H-space (Theorem 5.4.19), where X_p^* is a certain dual of X_p. By showing that pointed self-maps of A admit unique deloopings, we deduce that $\mathsf{BAut}_1(A)$ is also central, using the characterization mentioned above. Then we can recursively consider bands to get arbitrary deloopings of any central type.

We also prove some results about H-spaces in Section 5.2, all of which are new in our setting, and some of which are new even in traditional H-space theory. For simplicity, we discuss our results in the case of a pointed, *connected* type A. A useful observation underlying our considerations is that H-space structures on A correspond to pointed sections of the evaluation fibration $\mathbf{ev}_{\mathrm{id}}$ above. It is not hard to prove that any such section in fact trivializes this fibration, and from this to deduce an equivalence of types (Theorem 5.2.27)

$$\mathsf{HSpace}(A) \;\simeq\; (A \wedge A \to_* A),$$

where the left-hand side is the type of H-space structures on A. The formula above generalizes a classical formula of [AC63] and [Cop59] which gives a bijection on path components. As an application, we see that the space of H-space structures on any n-sphere which is itself an H-space is equivalent to $\Omega^{2n}\mathbb{S}^n$. In particular, $\mathsf{HSpace}(\mathbb{S}^3) \simeq \Omega^6\mathbb{S}^3$, generalizing a theorem of [Jam57] which says that homotopy classes of H-space structures on the 3-sphere biject with $\pi_6\mathbb{S}^3$ (which is isomorphic to $\mathbb{Z}/12$, classically).

Furthermore, we relate H-space structures to Whitehead products via the notion of an (α, β)-*extension*, due to [Whi46]. Roughly, such an extension is a map $f : A \times B \to C$ which restricts to a given map $\alpha \vee \beta : A \vee B \to C$ along the wedge inclusion. A useful observation is that $(\mathrm{id}_A, \mathrm{id}_A)$-extensions correspond to H-space structures on A, which implies that any H-space admits all (α, β)-extensions. It follows that all Whitehead products vanish on an H-space (Corollary 5.2.17), which in turn implies that no even spheres admit an H-space structure (Proposition 5.2.18) since $[\iota_{2n}, \iota_{2n}] = 2$ by [Bru16]. Lastly, we give an elementary proof that the homotopy groups of H-spaces stabilize one step earlier than usual (Proposition 5.2.19).

Above, we have attempted to give a cohesive account of the content of this thesis. Each of the chapters begins with a more detailed introduction and discussion of relevant background material, which the reader may consult to get a more in-depth overview of the thesis.

Formalization. Many of our results have been formalized using the Coq library for homotopy type theory [CH], and most of these have already been integrated into this library. We list the relevant pull requests (which also include some material not discussed in this thesis):

- projective types and the axiom of choice (#1393);
- theory related to object classifiers and fibre sequences (#1524);
- theory of Yoneda Ext (#1534, #1646, #1663 by J. Ender, #1712, #1718);
- H-space theory (#1697, #1701).

The formalization of the long exact sequence of Yoneda Ext groups currently resides in the repository Yoneda-Ext, and the formalization results related to banded types is at central-types. In total, these results constitute over 11,000 lines of Coq code.

Chapter 2

Univalent categories of modules

Abstract. We show that categories of modules over a ring in Homotopy Type Theory (HoTT) satisfy the internal versions of the AB axioms from homological algebra. The main subtlety lies in proving AB4, which is that coproducts indexed by arbitrary sets are left-exact. To prove this, we replace a set X with the strict category of lists of elements in X. From showing that the latter is filtered, we deduce left-exactness of the coproduct. More generally, we show that exactness of filtered colimits (AB5) implies AB4 for any abelian category in HoTT. Our approach is heavily inspired by Roswitha Harting's construction of the *internal coproduct* of abelian groups in an elementary topos with a natural numbers object [Har82].

To state the AB axioms we define and study filtered (and sifted) precategories in HoTT. A key result needed is that filtered colimits commute with finite limits of sets. This is a familiar classical result, but has not previously been checked in our setting.

Finally, we interpret our most central results into an ∞-topos $\mathcal{X}$. Given a ring R in $\tau_{\leq 0}(\mathcal{X})$—i.e., an ordinary sheaf of rings—we show that the internal category of R-modules in $\mathcal{X}$ represents the presheaf which sends an object $X \in \mathcal{X}$ to the category of $(X{\times}R)$-modules in $\mathcal{X}/X$. In general, our results yield a product-preserving left adjoint to base change of modules over X. When X is 0-truncated, this left adjoint is the internal coproduct. By an internalization procedure, we deduce left-exactness of the internal coproduct as an ordinary functor from its internal left-exactness coming from HoTT.

2.1 Introduction

We study categories of modules over a ring in Homotopy Type Theory (HoTT). Our main result is that these satisfy the (internal) axioms AB3 through AB5 and have a generator, i.e., they are *Grothendieck categories*. By working in HoTT our results hold in any (Grothendieck) ∞-topos [Shu19], and conjecturally in any *elementary* ∞-topos [KL18, Ras22, Shu17]. This work is part of, and motivated by, the development of homological algebra in HoTT and the resulting notions in an ∞-topos, which we discuss in Section 2.4.

In ordinary homological algebra, it is common knowledge that the category of modules over a ring is Grothendieck and satisfies AB4. However, the question is more subtle in a constructive setting such as ours. For example, the category of abelian groups in the type theory of [CS07] is only preabelian (see their Section 4.1 for a discussion). Fortunately for us, R-Mod is abelian in HoTT, and this has already been formalized for $R \equiv \mathbb{Z}$ in the file `CategoryTheory/categories/abgrs.v` of the [UniMath] library.

The main subtlety in verifying that R-Mod is Grothendieck is the existence of coproducts over an arbitrary set X. When assuming the law of the excluded middle, we are accustomed to having a natural monomorphism $\bigoplus_{x:X} A(x) \to \Pi_{x:X} A(x)$ from an arbitrary coproduct of modules to the corresponding product. Indeed, one often defines the coproduct to be the "finitely supported" elements within the product. While the coproduct $\bigoplus_{x:X} A(x)$ still always exists in a constructive setting, it is harder to define, and in contrast to the classical setting there may be no non-trivial maps (let alone monomorphisms) of the form $\bigoplus_{x:X} A(x) \to \Pi_{x:X} A(x)$! This is further discussed in Section 2.3.1.

When Grothendieck first introduced the AB axioms, he remarked that AB4 follows from AB5 [Gro57, p. 129]. This is the second point which is a bit more subtle in our setting, and we prove this in Section 2.3.3. In fact, we prove a bit more: the AB5 axiom implies that the coproduct functor $\bigoplus_X$ is left-exact for arbitrary sets X (Theorem 2.3.19). Our result is analogous to, and inspired by, the *internal coproduct* of a family of abelian groups in an elementary topos (with $\mathbb{N}$) as constructed by Roswitha Harting in [Har82]. Her main result is that the internal coproduct, indexed by an arbitrary object, is left-exact (hence preserves monomorphisms). After Proposition 3.7 in [Ble18], Ingo Blechschmidt remarks that the internal coproduct exists and preserves monomorphisms for families of modules as well. Our work in Section 2.3.3 simultaneously translates and generalizes these results by constructing type-indexed colimits in arbitrary abelian categories in HoTT. We then recover the analogue of Harting's result: when the indexing type is a set, the colimit specializes to the coproduct and is left-exact. In general, however, the colimit fails to be left-exact (Example 2.3.9).

The original construction of the internal coproduct of abelian groups was carried out in the internal language of an elementary topos. This internal language was not well-developed at the time, and the paper [Har82]—which is entirely dedicated to this construction—weighs in at over 60 pages. In contrast, by working in HoTT our generalized construction goes through in just over 2 pages (Section 2.3.3).

The usual proof that AB5 implies AB4 replaces a discrete indexing category X (for a coproduct) by a filtered category (the finite subsets of X) and uses the fact that moving from one to the other does not change the colimit of a diagram. However, in a constructive setting neither the Bishop-finite nor the (ordered) finite subsets of X form filtered categories unless X is decidable. Harting's insight was to work with the category HX of lists of elements in X instead. In Section 2.3.3 we define HX as a precategory associated to a 1-type X in HoTT, then we show that HX is always sifted, and moreover filtered if X is a set. In Section 2.2, we develop the necessary theory of sifted and filtered colimits.

In Section 2.4 we interpret our most central results into a higher topos $\mathcal{X}$ equipped with a ring object R. Specifically, we show that the internal category of R-modules resulting from interpretation represents the presheaf sending an $X \in \mathcal{X}$ to the category of $(X{\times}R)$-modules over X (Theorem 2.4.17). We repackage internal categories as *Rezk* $(1,1)$-*objects* (Definition 2.4.1), which are 2-restricted versions of 0-truncated complete Segal objects. Rezk $(1,1)$-objects are easily seen to represent presheaves of categories, which is their main utility for us.

We also interpret type-indexed colimits of modules, which specialize to coproducts when the indexing type is a set. For an object $X \in \mathcal{X}$, we get an adjunction

$$\mathrm{colim}_X : (X{\times}R)\text{-}\mathsf{Mod} \leftrightarrows R\text{-}\mathsf{Mod} : X \times (-)$$

where the left adjoint preserves products (Theorem 2.4.18). If X is a set, then the left adjoint is left-exact. To deduce (external) left-exactness from internal left-exactness (resulting from interpretation) we use an internalization procedure (Definition 2.4.7) that applies more generally, and may be of independent interest.

Conventions. We use the conventions and notation of [Uni13]. Our terminology for category theory mirrors that of [Uni13, Chapter 9] and [AKS15], in particular we leave the "univalent" implicit when saying category (except in this paper's title). When we consider abelian categories we do assume these are univalent. If $\mathscr{D}$ and $\mathscr{C}$ are precategories, we denote the functor precategory using exponential notation: $\mathscr{C}^{\mathscr{D}}$. For a functor $F : \mathscr{C}^{\mathscr{D}}$ and a morphism $\delta : d \to d'$ in $\mathscr{D}$, we write $F_\delta : F(d) \to F(d')$ for the morphism in $\mathscr{C}$ obtained by applying F. If moreover $\eta : G \Rightarrow G'$ is a natural transformation of functors with domain $\mathscr{C}$, then we will write η_F for the restriction of η along F.

When we say something is a "property of X", we mean it in the formal sense of being a proposition. The standard n-element set is denoted $\mathsf{Fin}(n)$. Section 2.4 has its own section on notation.

2.2 Sifted and filtered precategories

We define sifted and filtered precategories, then prove that sifted (resp. filtered) colimits of sets commute with finite products (resp. finite limits). In fact, we prove the stronger fact that filtered colimits commute with *finitely generated* limits (Definition 2.2.14). This generalization lets us, for example, compute the fixed points of a filtered colimit of G-sets as the filtered colimit of the fixed points, for a finitely generated group G (Corollary 2.2.19).

These are classical results in category theory, and the usual proofs go through in our context with some added care, which is what we supply. The work builds on Chapters 9 and 10 of the HoTT Book [Uni13].

Before we begin, we would like to emphasize that developing 1-category theory in HoTT is unproblematic, as opposed to ∞-category theory. We do not know how, or whether it is even possible, to represent current approaches to the latter in HoTT. Nevertheless, we may speak about ∞-groupoids and functors between them, namely: an ∞-groupoid is simply a type, and a functor is simply a function. In particular, if X is a type and $\mathscr{C}$ is a category, then a function $X \to \mathscr{C}$ is a functor from this point of view, and there is an obvious category $\mathscr{C}^X$.

2.2.1 Limits and colimits of sets

We start by defining limits and colimits indexed by precategories. When the codomain is a category, we show that the (co)limit of a functor is invariant under replacing the domain with its Rezk completion (Lemma 2.2.3). For limits and colimits of sets, we show that the classical descriptions remain valid in our setting (Proposition 2.2.4). Lastly, when the indexing category is a groupoid (i.e., a 1-type; see [Uni13, Example 9.1.16]), we show that the limit and colimit are given respectively by the Π- and Σ-type of the underlying family (Proposition 2.2.6).

Definition 2.2.1. Let $D : \mathscr{D} \to \mathscr{C}$ be a functor between precategories. A **limit of** D is an object $\lim_{\mathscr{D}} D$ of $\mathscr{C}$ representing the functor $\mathscr{C}^{\mathscr{D}}(\mathrm{const}_{\mathscr{D}}(-), D) : \mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$. Dually, a **colimit of** D is an object $\mathrm{colim}_{\mathscr{D}} D$ of $\mathscr{C}$ representing the functor $\mathscr{C}^{\mathscr{D}}(D, \mathrm{const}_{\mathscr{D}}(-))$.

When $\mathscr{C}$ is a category, Theorem 9.5.9 in [Uni13] implies that the type of (co)limits of a functor D is a mere proposition. Thus if a (co)limit exists, it is unique.

Remark 2.2.2. Consider a functor $D : \mathscr{D} \to \mathscr{C}$. The data of a limit of D consists of an object $\lim_{\mathscr{D}} D : \mathscr{C}$ along with a natural isomorphism $\delta : \mathscr{C}^{\mathscr{D}}(\mathrm{const}_{\mathscr{D}}(-), D) \simeq \mathscr{C}(-, \lim_{\mathscr{D}} D)$ witnessing representability. When we say that an object $c : \mathscr{C}$ "is the limit of D", we mean that such a representability witness is specified. Of course, by the Yoneda lemma, such a witness consists exactly of an element in $\mathscr{C}^{\mathscr{D}}(\mathrm{const}_{\mathscr{D}}(c), D)$ defining a universal cocone on D. The dual story applies to colimits.

Given a functor $D : \mathscr{D} \to \mathscr{C}$ from a precategory to a category, we may factor D uniquely via the Rezk completion $\widehat{\mathscr{D}}$ as follows (see [Uni13, Chapter 9.9] for details):

$$
\begin{tikzcd}
\mathscr{D} \arrow[dr, bend left, "D"] \arrow[dd, "\eta_{\mathscr{D}}"'] & \\
& \mathscr{C} \\
\widehat{\mathscr{D}} \arrow[ur, bend right, "\widehat{D}"'] &
\end{tikzcd}
$$

In particular, we have a natural comparison map $\lim_{\widehat{\mathscr{D}}} \widehat{D} \longrightarrow \lim_{\mathscr{D}} D$ induced from $\eta_{\mathscr{D}}$ by precomposition, and dually for the colimit. The following lemma implies that that these comparison maps are isomorphisms, meaning we can freely move between the (co)limit of D and $\widehat{D}$.

Lemma 2.2.3. *Let $D : \mathscr{D} \to \mathscr{C}$ be a functor from a precategory to a category. The two following restriction maps are natural bijections in $c : \mathscr{C}$,*

$$\eta_{\mathscr{D}}^* \;:\; \mathscr{C}^{\widehat{\mathscr{D}}}(\mathrm{const}_{\widehat{\mathscr{D}}}(c), \widehat{D}) \longrightarrow \mathscr{C}^{\mathscr{D}}(\mathrm{const}_{\mathscr{D}}(c), D),$$

$$\eta_{\mathscr{D}}^* \;:\; \mathscr{C}^{\widehat{\mathscr{D}}}(\widehat{D}, \mathrm{const}_{\widehat{\mathscr{D}}}(c)) \longrightarrow \mathscr{C}^{\mathscr{D}}(D, \mathrm{const}_{\mathscr{D}}(c)).$$

Consequently, the (co)limits of D and $\widehat{D}$ coincide, if either exists.

Proof. The functor $\eta_{\mathscr{D}} : \mathscr{D} \to \widehat{\mathscr{D}}$ is a weak equivalence [Uni13, Theorem 9.9.5], thus mapping into $\mathscr{C}$ induces an isomorphism $\eta_{\mathscr{D}}^* : \mathscr{C}^{\widehat{\mathscr{D}}} \to \mathscr{C}^{\mathscr{D}}$ by [Uni13, Theorem 9.9.4]. Clearly, for every $c : \mathscr{C}$, we have that $\mathrm{const}_{\widehat{\mathscr{D}}}(c) \circ \eta_{\mathscr{D}} = \mathrm{const}_{\mathscr{D}}(c)$ and $\widehat{D} \circ \eta_{\mathscr{D}} = D$ by definition. The maps in question are actions of $\eta_{\mathscr{D}}^* : \mathscr{C}^{\widehat{\mathscr{D}}} \to \mathscr{C}^{\mathscr{D}}$ on specific hom-sets, which are (natural) bijections by full faithfullness. $\square$

The usual descriptions of limits and colimits of sets are valid in HoTT.

Proposition 2.2.4. *Let $\mathscr{D}$ be a small category, and $D : \mathscr{D} \to \mathrm{Set}$ a functor.*

1. *The limit of D exists, and is given by the set*

$$\lim_{\mathscr{D}} D = \{x : \Pi_{\mathscr{D}} D \mid \Pi_{d,d':\mathscr{D}} \Pi_{\delta:d\to d'} D_\delta(x_d) = x_{d'}\}$$

equipped with the natural projections $(\lim_{\mathscr{D}} D \to D(d))_{d:\mathscr{D}}$ *forming a universal cocone.*

2. *The colimit of D also exists, and is given by the set-quotient of $\Sigma_{\mathscr{D}} D$ by the relation*

$$(d, x) \sim (d', x') :\equiv \|\Sigma_{\delta:d\to d'} D_\delta(x) = x'\|_{-1}$$

equipped with the natural quotient maps $(D(d) \to \Sigma_{\mathscr{D}} D/{\sim})_{d:\mathscr{D}}$ *forming a universal cocone.*

Proof. The description of the limit (1) results from computing $\lim_{\mathscr{D}} D$ via products and equalizers:

$$\lim_{\mathscr{D}} D \dashrightarrow \Pi_{d,d':\mathscr{D}} \Pi_{\delta:d\to d'} D(d) \rightrightarrows \Pi_{d:\mathscr{D}} D(d)$$

From the explicit descriptions of products and equalizers in Set, we conclude. Dually, the description of colimits (2) is obtained by writing $\mathrm{colim}_D D$ via coproducts and coequalizers and using their respective descriptions as Σ-types and quotients in Set. □

For indexing categories which are groupoids, both limits and colimits have a simpler description, given in Proposition 2.2.6. To show this we use the following lemma, which tells us that for functors from a groupoid into a category, we can choose to simply work with the underlying map of types.

Lemma 2.2.5. *Let $\mathscr{G}$ be a groupoid, and $\mathscr{C}$ a category. The forgetful map $U : \mathscr{C}^{\mathscr{G}} \to (\mathscr{G} \to \mathscr{C})$ which forgets functoriality is an equivalence. The inverse V sends a map $f : \mathscr{G} \to \mathscr{C}$ to the functor $V(f)$ acting as f on objects, and which sends a path $\gamma : g =_{\mathscr{G}} g'$ to* $\mathrm{idtoiso}_{\mathscr{G}}(\mathrm{ap}_f(\gamma))$.

Proof. First of all the reader should convince themselves that the proposed definition of the inverse V indeed constructs a functor $V(f)$ from a general map of types $f : \mathscr{G} \to \mathscr{C}$. It is then clear that V is a section of the forgetful map U, so it remains to show that any functor $F : \mathscr{G} \to \mathscr{C}$ is equal to the functor induced from its map on the underlying types.

Clearly forgetting functoriality of F and then inducing functoriality produces the same map on the underlying types, by definition. Consider a general map $\mathrm{idtoiso}_{\mathscr{G}}(\gamma) : g \to g'$ in $\mathscr{G}$, where $\gamma : g =_{\mathscr{G}} g'$. This is general since $\mathscr{G}$ is a groupoid. We need to show that $F_{\mathrm{idtoiso}_{\mathscr{G}}(\gamma)} = \mathrm{idtoiso}_{\mathscr{G}}(\mathrm{ap}_F(\gamma))$ as morphisms $F(g) \to F(g')$ in $\mathscr{C}$. But this follows by path induction on γ. □

Similar in spirit to Lemma 2.2.3, the following proposition says that a (co)limit of sets is invariant under the change of perspective afforded by the previous lemma.

Proposition 2.2.6. *Suppose $\mathscr{G}$ is a groupoid, and let $D : \mathscr{G} \to$ Set be a functor. The natural maps $\lim_{\mathscr{G}} D \to \Pi_{\mathscr{G}} D$ and $\|\Sigma_{\mathscr{G}} D\|_0 \to \mathrm{colim}_{\mathscr{D}} D$ are bijections.*

Proof. First we consider the limit. A family $d : \Pi_{\mathscr{G}} D$ lies in the limit if and only if the proposition

$$\Pi_{g,g':\mathcal{G}} \Pi_{f:\mathscr{G}(g,g')} D_f(d_g) = d_{g'}$$

holds. Since $\mathscr{G}$ is a groupoid, we can identify $\mathscr{G}(g, g')$ with $g =_{\mathscr{G}} g'$. The above then immediately follows by path induction, meaning the predicate defining the limit is a tautology.

Similarly, we will show that the equivalence relation defining the colimit is trivial so that the set-quotient on $\Sigma_{\mathscr{G}} D$ is simply given by set-truncation. Suppose $(g_0, d_0) \sim (g_1, d_1)$ for the colimit relation defined in Proposition 2.2.4. By definition there merely exists some $f : g_0 \to g_1$ such that $D_f(d_0) = d_1$. We wish to deduce that $(g_0, d_0) = (g_1, d_1)$. Since this is a proposition, we may assume f actually exists. As before, we identify f with a path $g_0 =_{\mathscr{G}} g_1$, using that $\mathscr{G}$ is a groupoid. Then the existence of the path $D_f(d_0) = d_1$ implies exactly that $(g_0, d_0) = (g_1, d_1)$, by characterization of paths in Σ-types. In conclusion, the colimit relation $\sim$ is just equality, hence the set-quotient $\mathrm{colim}_{\mathscr{G}} D$ is simply $\|\Sigma_{\mathscr{G}} D\|_0$. □

2.2.2 Sifted colimits

We define sifted precategories and prove that sifted colimits commute with finite products in Set. In [ARV10], sifted categories are *defined* to have this property. We will instead take the equivalent condition of Theorem 2.15 from loc. cit. as our definition, which is a simpler condition to check. To us, the main interest is that sifted colimits of algebraic structures (such as groups, abelian groups, and modules) may be computed on the underlying sets—see Corollary 2.2.12.

Definition 2.2.7. Let $\mathscr{C}$ be precategory.

1. Let c and c' be objects of $\mathscr{C}$ and let $n : \mathbb{N}$. A **zig-zag from c to c' of length n** is inductively defined as a path $c =_{\mathscr{C}} c'$ when $n \equiv 0$, and for $n \geq 1$ a sequence

$$\begin{tikzcd} c \arrow[dr] & & c_2 \arrow[dl] \arrow[dr] & & c' , \arrow[dl] \\ & c_1 & & \cdots & \end{tikzcd}$$

   where "$\cdots$" signifies a length $n-1$ zig-zag from c_2 to c' (hence just a path if $n \equiv 1$);

2. The precategory $\mathscr{C}$ is **connected** if it is merely inhabited (i.e., the proposition $\|\mathscr{C}\|_{-1}$ holds) and for every two objects in $\mathscr{C}$ there merely exists a zig-zag connecting them;

3. Let $\mathscr{C}'$ be a precategory. A functor $F : \mathscr{C}' \to \mathscr{C}$ between precategories is **final** if for every $c : \mathscr{C}$, the slice precategory c/F is connected.

Being connected is a property of a precategory, and consequently being final is a property of a functor. In [ARV10], a functor is said to be final if restriction along it preserves colimits. We have instead taken the equivalent condition (3) of Lemma 2.13 in loc. cit. as our definition, from which we prove this fact:

Proposition 2.2.8. *Let $F : \mathscr{C}' \to \mathscr{C}$ and $G : \mathscr{C} \to \mathscr{D}$ be functors between precategories. If F is final, then restriction along F is a natural bijection between functors $\mathscr{D} \to$ Set as follows:*

$$F^* : \mathscr{D}^{\mathscr{C}}(G, \mathrm{const}_{\mathscr{C}}(d)) \longrightarrow \mathscr{D}^{\mathscr{C}'}(GF, \mathrm{const}_{\mathscr{C}'}(d))$$

naturally in $d : \mathscr{D}$. Consequently, the colimit of G coincides with the colimit of GF, if either exists.

Proof. Let $d : \mathscr{D}$. First of all, it is straightforward to check that F^* defines a natural transformation as stated. To prove that it is a natural isomorphism, we show that each component is a bijection.

Injectivity: Suppose $\eta, \eta' : G \Rightarrow \mathrm{const}_{\mathscr{C}}(d)$ are such that $\eta_F = \eta'_F$. We want to show that $\eta_c = \eta'_c$ for all $c : \mathscr{C}$, which is a proposition. Let $c : \mathscr{C}$, and pick a morphism $f : c \to F(c')$ using that c/F is merely inhabited and the fact that we're proving a proposition. But then, by naturality of η and η', we have

$$\eta_c = \eta_{F(c')} \circ G_f = \eta'_{F(c')} \circ G_f = \eta'_c$$

where the middle equation comes from $\eta_F = \eta'_F$. Hence F^* is injective.

Surjectivity: Consider a natural transformation $\nu : GF \Rightarrow \mathrm{const}_{\mathscr{C}'}(d)$. For $c : \mathscr{C}$, define the function

$$\begin{aligned} \phi : c/F &\longrightarrow (G(c) \to d) \\ f &\longmapsto \nu_{c'} \circ G_f, \end{aligned}$$

where $f : c \to F(c')$. For $f, f' : c/F$, one can easily show (using naturality of ν) that $\phi(f) = \phi(f')$ by induction over the length of a zig-zag from f to f'. Consequently, $\mathsf{im}(\phi)$ is a proposition and we may therefore factor ϕ via its propositional truncation, producing *ptr*$\phi : \|c/F\|_{-1} \to \mathsf{im}(\phi) \to (G(c) \to d)$. (This argument also appears in [KECA17, Theorem 5.4], which may be consulted for details.) Thus we get a map $g : G(c) \to d$ using the fact that c/F is merely inhabited. Doing this for all $c : \mathscr{C}$ gets us a transformation $\eta : \Pi_{c:\mathscr{C}} G(c) \to d$ which, by construction, satisfies $\eta_F = \nu$.

It remains to prove that η is natural. Let $g : c_0 \to c_1$ be a morphism in $\mathscr{C}$. We need to show that $\eta_{c_0} = \eta_{c_1} \circ G_g$, which is a proposition. By finality of F, we may choose $f_0 : c_0 \to F(c'_0)$ and $f_1 : c_1 \to F(c'_1)$ to obtain the following diagram:

$$\begin{tikzcd}
G(c_0) \arrow[r, "G_{f_0}"] \arrow[dd, "G_g"'] & GF(c'_0) \arrow[d, dotted] \arrow[dr, bend left, "\nu_{c'_0}"] & \\
& \vdots \arrow[r, dotted] & d \\
G(c_1) \arrow[r, "G_{f_1}"] & GF(c'_1) \arrow[u, dotted] \arrow[ur, bend right, "\nu_{c'_1}"'] &
\end{tikzcd}$$

where the outer diagram is the one we wish to show commutes. Since c_0/F is connected, the two maps f_0 and $f_1 \circ g$ are connected by a zig-zag which, after applying G, produces the dotted lines above. The left square then commutes by definition of a zig-zag, and the triangles on the right commute by naturality of ν. Inducting over the length of the zig-zag, we conclude that η is natural, as desired. □

Definition 2.2.9. A precategory $\mathscr{S}$ is **sifted** if it is merely inhabited and $\Delta_{\mathscr{S}} : \mathscr{S} \to \mathscr{S} \times \mathscr{S}$ is final.

There are various equivalent classical definitions of siftedness (see, e.g., [AR01, Theorem 1.6]). We chose the one above to make the connection with final functors immediate, and to facilitate the proof of the following:

Lemma 2.2.10. *If a precategory $\mathscr{C}$ is merely inhabited and has binary coproducts, then $\mathscr{C}$ is sifted.*

Proof. Suppose $\mathscr{C}$ is merely inhabited and has binary coproducts. Then for every $(c_0, c_1) : \mathscr{C}^2$, the slice precategory $(c_0, c_1)/\Delta_{\mathscr{C}}$ has an initial object given by the coproduct. Then we are done, since any category with initial object is connected (by zig-zags of length at most 2). □

This is one direction of [ARV10, Theorem 2.15].

Proposition 2.2.11. *Sifted colimits of sets commute with finite products.*

Proof. Let $\mathscr{S}$ be a sifted precategory. The claim that colimits over $\mathscr{S}$ commute with empty products follows from $\mathscr{S}$ being merely inhabited. Consider two functors $G, H : \mathscr{S} \to \mathrm{Set}$, then we have the following natural bijections:

$$\begin{aligned}
\mathrm{colim}_{s:\mathscr{S}}\,(G_s \times H_s) &\simeq \mathrm{colim}_{(s,t):S\times S}\, G_s \times H_t && \text{(Prop. 2.2.8 applied to } \Delta_{\mathscr{S}})\\
&\simeq \mathrm{colim}_{s:\mathscr{S}}\, \mathrm{colim}_{t:\mathscr{S}}\, G_s \times H_t\\
&\simeq \mathrm{colim}_{s:\mathscr{S}}\,(G_s \times \mathrm{colim}_{t:\mathscr{S}}\, H_t) && (G_s \times - \text{ is cocontinuous})\\
&\simeq \mathrm{colim}_{s:\mathscr{S}}\, G_s \times \mathrm{colim}_{t:\mathscr{S}}\, H_t && (-\times \mathrm{colim}_{t:\mathscr{S}}\, H_t \text{ is cocontinuous})
\end{aligned}$$

where the second step can be checked directly. The product bifunctor $\times$ preserves colimits in each variable, being a left adjoint. □

We deduce that sifted colimits of groups and modules can be computed on the underlying sets. This is true more generally for any *algebraic theory* [ARV10, Proposition 2.5], but we only state and prove the case that we require.

Corollary 2.2.12. *Let $\mathscr{A}$ be the category of groups or of R-modules (for a ring R). The forgetful functor $U : \mathscr{A} \to \mathrm{Set}$ creates sifted colimits.*

Proof. We first consider the case when $\mathscr{A}$ is the category of groups. Let $\mathscr{S}$ be a sifted precategory and let $G : \mathscr{S} \to \mathscr{A}$ be a diagram. The previous proposition implies that the functor $\mathrm{colim}_{\mathscr{S}} : \mathrm{Set}^{\mathscr{S}} \to \mathrm{Set}$ preserves finite products. It follows that it preserves group objects. Since a group object in $\mathrm{Set}^{\mathscr{S}}$ is simply an object-wise group object, we get a natural group structure on $\mathrm{colim}_{s:\mathscr{S}}\, U(G_s)$. To show that this group is the colimit of G, we need to construct a universal cocone $(G_s \to \mathrm{colim}_{s:\mathscr{S}}\, U(G_s))_{s:\mathscr{S}}$ in $\mathscr{A}$.

The multiplication map on $\mathrm{colim}_{s:S}\, U(G_s)$ is induced from all the multiplication maps $G_s \times G_s \to G_s$ using functoriality of the colimit and that $\mathrm{colim}_{\mathscr{S}}$ preserves binary products. Thus the natural maps $i_s : G_s \to \mathrm{colim}_{s:\mathscr{S}}\, U(G_s)$ are group homomorphisms (not just maps), giving a cocone of the desired form. For any other cocone $(f_s : G_s \to H)_{s:\mathscr{S}}$ in $\mathscr{A}$, we get an induced map $f : \mathrm{colim}_{s:\mathscr{S}}\, U(G_s) \to UH$ of sets. Since $f_s(-) \cdot_H f_s(-) = f_s(- \cdot_{G_s} -)$ for any

$s : \mathscr{S}$, uniqueness of maps out of colimits gives that $f(-) \cdot f(-) = f(- \cdot -)$, i.e., f is a group homomorphism. This means that $(i_s : G_s \to \mathrm{colim}_{s:\mathscr{S}} U(G_s))_{s:\mathscr{S}}$ is a universal cocone under G in $\mathscr{A}$, as desired.

Now we consider the case when $\mathscr{A}$ is the category of R-modules for a ring R. It is straightforward to check that colimits of R-modules may be computed on the underlying abelian groups, as in classical algebra. Thus we need only consider the case when $R \equiv \mathbb{Z}$. But if G is a diagram of *abelian* groups in the argument above, then $\mathrm{colim}_{s:\mathscr{S}} U(G_s)$ is also abelian, because $\mathrm{colim}_{\mathscr{S}} : \mathrm{Set}^{\mathscr{S}} \to \mathrm{Set}$ also preserves *abelian* group objects. But then we are done since $\mathrm{colim}_{s:\mathscr{S}} U(G_s)$ has the required universal property. □

2.2.3 Filtered colimits

Filtered colimits of sets have particularly nice descriptions, and it is well known that they commute with finite limits, classically. Less known is that fact that filtered colimits actually commute with *finitely generated limits* (Definition 2.2.14). We start with the relevant definitions.

Definition 2.2.13. A precategory $\mathscr{F}$ is **filtered** if the following propositions all hold:

1. $\mathscr{F}$ is merely inhabited;
2. for any two objects $c, c' : \mathscr{F}$ there merely exists an **upper bound** $c \to c'' \leftarrow c'$;
3. for any two arrows $f, g : c \to c'$ there merely exists some $h : c' \to c''$ such that $hf = hg$.

By definition, filteredness is a property of a precategory. It is straightforward to prove, by induction, that any finite family of objects in a filtered category merely admits an upper bound. Similarly, any finite number of parallel arrows merely admit a (not necessarily universal) coequalizing arrow. Here, by "finite" we mean **Bishop-finite**, i.e., a type X for which there merely exists a natural number $n : \mathbb{N}$ and an equivalence $\mathsf{Fin}(n) \simeq X$, where $\mathsf{Fin}(n)$ denotes the standard n-element set.

More generally, filtered categories admit cones under finitely generated diagrams.

Definition 2.2.14. A precategory $\mathscr{D}$ is **finitely generated** if the underlying type of objects is Bishop-finite, and there exists a family of morphisms $\Phi : \Pi_{i:I}\mathscr{D}(s_i, t_i)$ in $\mathscr{D}$ indexed by a Bishop-finite set I, such that every morphism in $\mathscr{D}$ merely factors as follows:

$$\Pi_{m,m':\mathscr{D}}\Pi_{g:m\to m'}\|\Sigma_{n:\mathbb{N}}\Sigma_{j:\mathsf{Fin}(n)\to I}\; g = \Phi_{j(n-1)} \cdots \Phi_{j(0)}\|_{-1}$$

where $\mathsf{Fin}(n)$ denotes the standard n-element set.

Observe that a finitely generated precategory is automatically a strict category, since Bishop-finite types are necessarily sets.

Proposition 2.2.15. *Let $\mathscr{F}$ and $\mathscr{D}$ be filtered and finitely generated categories, respectively. Any functor $D : \mathscr{D} \to \mathscr{F}$ merely admits a cocone.*

Proof. Lemma 2.13.2 of [Bor94] readily generalizes to the case when $\mathscr{D}$ is finitely generated. □

Using this proposition, we can easily show that filteredness implies siftedness:

Lemma 2.2.16. *Filtered precategories are sifted.*

Proof. Suppose $\mathscr{F}$ is a filtered precategory. To see that $\mathscr{F}$ is sifted, we need to show that the slice precategory $(c_0, c_1)/\Delta_{\mathscr{F}}$ is connected for any $c_0, c_1 : \mathscr{F}$. An object of this slice is precisely an upper bound $c_0 \to c \leftarrow c_1$, so the slice is merely inhabited since $\mathscr{F}$ is filtered. To see that the slice is connected, let c and c' be upper bounds of c_0 and c_1. We may form the following diagram:

$$\begin{tikzcd}
& c_0 \arrow[dl] \arrow[dr] & \\
c & & c' \\
& c_1\,. \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

A cocone of this diagram merely exists, by the previous proposition. This cocone exhibits a zig-zag of length one connecting c and c'. □

Theorem 2.2.17. *Let $\mathscr{F}$ and $\mathscr{D}$ be filtered and finitely generated categories, respectively, and consider a functor $D : \mathscr{F} \times \mathscr{D} \to \mathrm{Set}$. The natural map $\mathrm{colim}_{\mathscr{F}} \lim_{\mathscr{D}} D \to \lim_{\mathscr{D}} \mathrm{colim}_{\mathscr{F}} D$ is a bijection.*

Proof. For finite categories $\mathscr{D}$ the proof of [Bor94, Theorem 2.13.4] goes through since every application of the axiom of choice is used for a *finite* indexing set, which is valid in HoTT (see, e.g., `Spaces.Finite.finite_choice`). The generalization to $\mathscr{D}$ being finitely generated only requires straightforward modifications using Proposition 2.2.15 in the last part of Borceux' argument. □

Remark 2.2.18. That filtered colimits commute with finite limits has been formalized in [mathlib] in the file `category_theory/limits/filtered_colimit_commutes_finite.lean`. Their proof follows that of Theorem 2.13.4 in [Bor94], as we did, however they freely employ classical reasoning (as Borceux does) in their formalization. In particular, they begin their proofs by assuming the law of the excluded middle (using the `classical` tactic) and employ the full axiom of choice (using some to choose elements of merely inhabited sets) in the same places as Borceux. (In fact, in more places than Borceux, if you read his proof constructively.) Our contribution is simply the observation that, with our constructive definitions of limits and colimits of sets, the proof does not require the law of the excluded middle and each application of the axiom of choice is in fact an application of *finite choice*, which is valid for us.

We say that a group G is finitely generated if there exist a Bishop-finite generating set. Recall that a G-set X is simply a map $X : BG \to \mathrm{Set}$ (see, e.g., [Bez+23, Section 4.7]), and the fixed points of X are given by $\Pi_{BG} X$. As an application of our development thus far, we have the following:

Corollary 2.2.19. *Let G be a finitely generated group, and let $X : \mathscr{F} \to (BG \to \mathrm{Set})$ be a filtered diagram of G-sets. The fixed points of the colimit is the colimit of the fixed points:*

$$\Pi_{BG}\, \mathrm{colim}_{\mathscr{F}}\, X \simeq \mathrm{colim}_{x:\mathscr{F}}\, \Pi_{BG} X(x).$$

Proof. The category BG is the Rezk completion of the strict category $B'G$ which has a single object with G as its endomorphisms. If G is a finitely generated group, then $B'G$ is a finitely generated category in the sense of Definition 2.2.14. By Proposition 2.2.6 we have that $\Pi_{BG}(-) = \lim_{BG}(-)$, and by Lemma 2.2.3 we can change the limits to be over $B'G$. We conclude by the previous theorem, since $B'G$ is finitely generated and $\mathscr{F}$ is filtered. □

2.3 The internal AB axioms

The goal of this section is to show that for a ring R in HoTT, the category of R-modules satisfies the axioms AB3 through AB5 and has a generator—meaning it is a Grothendieck category (Definition 2.3.3). Formally, a Grothendieck category is only assumed to satisfy AB3 and AB5, but we show that AB4 follows from AB5 (Theorem 2.3.19). It is straightforward to check that R-Mod is an abelian category in HoTT, and indeed this has already been formalized for $R \equiv \mathbb{Z}$ in the file `CategoryTheory/categories/abgrs.v` of the [UniMath] library. (Some work towards the general case can be found in the file `modules.v` in the same directory.) Moreover, R being a generator is simply a restatement of function extensionality. What remains is to show that R-Mod satisfies the axioms AB3 through AB5.

We wish to treat families $A : X \to \mathscr{A}$ in an abelian category $\mathscr{A}$ indexed by an arbitrary type X. As pointed out at the beginning of Section 2.2, we think of these as functors from an ∞-groupoid into a category. Since $\mathscr{A}$ is a category, its underlying type is 1-truncated, and so we may factor any such family A through the 1-truncation of X. One checks that the 1-truncation map $|-|_1 : X \to \|X\|_1$ induces an equivalence of categories by precomposition:

$$|-|_1^* : \mathscr{A}^{\|X\|_1} \to \mathscr{A}^X.$$

It follows, by an argument similar to the one in Lemma 2.2.3, that the limit (resp. colimit) of a functor $A : X \to \mathscr{A}$ coincides with the limit (resp. colimit) of the 1-truncation $|A|_1 : \|X\|_1 \to \mathscr{A}$. (Limits and colimits of functors from an ∞-groupoid into a category are defined in the obvious way.) Thus when we discuss limits and colimits of such a family A, we may assume that X is a 1-type without loss of generality.

2.3.1 Grothendieck categories

We define Grothendieck abelian categories in homotopy type theory, assuming the reader is familiar with additive and abelian precategories. The traditional definitions of the latter can be directly translated into our setting, as has already been done in [UniMath] under the namespace `CategoryTheory/Abelian`. Be aware that by abelian *category* we do mean that it is a (univalent) category. While much of our discussion likely works for abelian precategories as well, we are particularly interested in discussing families of objects and their (co)limits, which is most naturally done for categories.

Definition 2.3.1. Let $\mathscr{A}$ be an additive category, and X a set. For a family $A : X \to \mathscr{A}$, the **coproduct of A** (if it exists) is the colimit of A, denoted $\bigoplus_{x:X} A(x)$. Dually, the **product of A** (if it exists) is the limit of A, denoted $\Pi_{x:X}A(x)$. If no confusion will arise, we often leave the variable $x : X$ implicit.

Suppose $\mathscr{A}$ is an additive category. Then, by definition, finite products and coproducts in $\mathscr{A}$ coincide, and we call these **biproducts**. The word *finite* here means "finitely iterated," i.e. pairwise biproducts carried out a finite number of times. If X is a decidable set, and $A : X \to \mathscr{A}$ is a family, then there is always a comparison map $m : \bigoplus_X A \to \Pi_X A$ which is a monomorphism. This is straightforward to prove in HoTT (and was shown for families of modules in an elementary topos with $\mathbb{N}$ in [Tav85]). Of course, if X is the standard n-element set $\mathsf{Fin}(n)$ for some $n : \mathbb{N}$, then the map m is an isomorphism. We deduce the following, since m being an isomorphism is a proposition:

Lemma 2.3.2. *Let $\mathscr{A}$ be an additive category, and X a Bishop-finite type. For any family $A : X \to \mathscr{A}$, the natural map $m : \bigoplus_X A \to \Pi_X A$ is an isomorphism.*

It may come as a surprise that no such monomorphism m need exist in general. In fact, Harting demonstrates in [Har82, Remark 2.1] that there might be no non-trivial map like m. Her example is in the Sierpiński 1-topos, which is the 0-truncated fragment of the Sierpiński ∞-topos (see Proposition 2.4.13). The latter is a model of HoTT, and Harting's example therefore shows that it is impossible for us to construct a non-zero map $\bigoplus_X A \to \Pi_X A$ for a general set X. The example also demonstrates that the construction of arbitrary coproducts is tricky; for example, one cannot carve out $\bigoplus_X A$ from $\Pi_X A$ as those families with "finite support." We will have more to say about models of HoTT in Section 2.4 below. For more details about the Sierpiński ∞-topos specifically, we recommend [Shu15, Section 8].

Definition 2.3.3. For an abelian category $\mathscr{A}$ we consider the following axioms.

(AB3) for any small set X and family $A : X \to \mathscr{A}$, the coproduct $\bigoplus_X A$ exists in $\mathscr{A}$;

Assuming $\mathscr{A}$ satisfies AB3, we get a coproduct functor $\bigoplus_X : \mathscr{A}^X \to \mathscr{A}$. It also follows that $\mathscr{A}$ is cocomplete (since it always has coequalizers). Thus, assuming AB3, we may additionally ask for:

(AB4) for any small set X, the functor $\bigoplus_X : \mathscr{A}^X \to \mathscr{A}$ preserves monomorphisms;

(AB5) for any small filtered precategory $\mathscr{F}$, the functor $\mathrm{colim}_{\mathscr{F}} : \mathscr{A}^{\mathscr{F}} \to \mathscr{A}$ preserves finite limits.

A **generator** of $\mathscr{A}$ is an object $G : \mathscr{A}$ such that for any two morphisms $f, f' : A \to B$, we have

$$(\Pi_{g:G\to A} fg = f'g) \to (f = f').$$

When $\mathscr{A}$ has a specified generator and satisfies AB3 and AB5, then $\mathscr{A}$ is a **Grothendieck category**.

We note that monomorphisms in $\mathscr{A}^X$ are object-wise monomorphisms (i.e., families of such). The axiom AB5 implies that the colimit functor $\mathrm{colim}_{\mathscr{F}}$ is exact for filtered precategories $\mathscr{F}$. In the next section we show that R-Mod is Grothendieck for any ring R.

2.3.2 Colimits of R-modules

We consider a 1-type X as a category and construct an adjunction:

$$\mathrm{colim}_X : R\text{-}\mathsf{Mod}^X \leftrightarrows R\text{-}\mathsf{Mod} : \mathrm{const}_X.$$

When X is a set, the colimit is the coproduct of an X-indexed family of modules. In contrast, when $R \equiv \mathbb{Z}$ and X is pointed and connected, $R\text{-}\mathsf{Mod}^X$ is the category of $\pi_1(X)$-modules (Definition 2.3.7). We will see that the functor colim_X computes the *coinvariants* of a $\pi_1(X)$-module. Dually, the functor $\lim_X$ computes the *invariants*.

As in classical algebra, the forgetful functor $U : R\text{-}\mathsf{Mod} \to \mathsf{Ab}$ creates limits and colimits. Thus by constructing colim_X for families of abelian groups, we extend it to families of modules via U. Moreover, the category of abelian groups is equivalent to the category of pointed, 1-connected 2-types [BvDR18, Theorem 5.1] through the functor $\mathsf{K}(-, 2)$ (whose inverse is Ω^2). The latter category is cocomplete, so we can transfer colimits across this equivalence. We now give an explicit account of this procedure.

Proposition 2.3.4. *Let X be a 1-type. We have an adjunction* $\mathrm{colim}_X : \mathsf{Ab}^X \leftrightarrows \mathsf{Ab} : \mathrm{const}_X$.

Proof. We start by constructing the functor colim_X. Let $A : X \to \mathsf{Ab}$. Via [BvDR18, Theorem 5.1], we may instead consider the corresponding family $x \mapsto \mathsf{K}(A(x), 2)$ of pointed, 1-connected 2-types. The colimit of this family among types is then $\Sigma_X \mathsf{K}(A, 2)$ by Lemma 2.2.5, whereas the colimit among pointed types is the pushout

$$\begin{tikzcd} X \arrow[r, "\mathtt{pt}"] \arrow[d] & \Sigma_X \mathsf{K}(A,2) \arrow[d, dashed] \\ 1 \arrow[r, dashed] & \bigvee_X \mathsf{K}(A,2) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd},$$

called the *indexed wedge*. Thus the colimit of $\mathsf{K}(A, 2)$ among pointed 2-types is $\|\bigvee_X \mathsf{K}(A, 2)\|_2$ by [Uni13, Section 7.4]. Moreover, by Theorem 7.3.9 in [Uni13] we have that

$$\|\Sigma_X \mathsf{K}(A, 2)\|_1 \quad\simeq\quad \|\Sigma_{x:X} \|\mathsf{K}(A(x), 2)\|_1\|_1 \quad\simeq\quad X$$

using that $\mathsf{K}(A(x), 2)$ is 1-connected for all $x : X$, and that X is a 1-type. From this we deduce that 1-truncating the pushout square above produces

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d] & X \arrow[d, dashed] \\ 1 \arrow[r, dashed] & 1 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

since pushouts commute with truncation. In particular, $\bigvee_X \mathsf{K}(A, 2)$ is 1-connected. Finally, since $\|\bigvee_X \mathsf{K}(A, 2)\|_2$ has the desired universal property among pointed 2-types, it certainly has it among pointed, 1-connected 2-types, being one itself. Now we apply Ω^2, the inverse of $\mathsf{K}(-, 2)$, and move the truncation to the outermost level, to define our functor on objects:

$$\mathrm{colim}_X(A) :\equiv \pi_2\left(\bigvee_X \mathsf{K}(A, 2)\right).$$

As defined, colim_X is a composite of the functors, hence is itself a functor. $\square$

Corollary 2.3.5. *Let R be a ring. The category R-Mod is complete and cocomplete.*

Proof. We reduce to $R \equiv \mathbb{Z}$ since the forgetful functor creates both limits and colimits. For limits, note that Ab has small products given simply by the Π-type associated to a family $X \to \mathsf{Ab}$ indexed by a set. Since Ab has equalizers, it is complete. Dually, Proposition 2.3.4 produces small coproducts by letting X be a set. Since Ab has coequalizers, it is cocomplete. □

Theorem 2.3.6. *The category R-Mod is Grothendieck.*

Proof. That R is a generator is an immediate consequence of function extensionality. By the previous corollary, R-Mod is cocomplete and therefore satisfies AB3. The axiom AB5 follows from Theorem 2.2.17, since products of modules are computed on the underlying sets, and the forgetful functor R-$\mathsf{Mod} \to \mathrm{Set}$ creates sifted (hence filtered) colimits by Corollary 2.2.12. □

At this point, it is not obvious that R-Mod satisfies AB4. This will be a consequence of Theorem 2.3.19 in the next section. For the remainder of this section, we discuss colim_X when X is the classifying space of a group.

Definition 2.3.7. Let G be a group. A family $A : BG \to \mathsf{Ab}$ is a ***G*-module**. The **invariants** of A comprise the abelian group $A_G :\equiv \lim_{BG} A$, and the **coinvariants** comprise the abelian group $A^G :\equiv \mathrm{colim}_{BG} A$.

Using the fact that limits of abelian groups may be computed on the underlying sets, along with the concrete description of limits in Proposition 2.2.4, we see that $A_G = \{a : A \mid \Pi_{g:G}\, ga = a\}$, which is the usual definition of the invariants. Writing $\mathrm{colim}_{BG} A$ as a coequalizer produces

$$\bigoplus_{g:G} A \underset{a \mapsto a}{\overset{a \mapsto ga}{\rightrightarrows}} A \dashrightarrow A^G$$

from which we see that A^G is the quotient of A by the subgroup $\langle a - ga \mid g : G\rangle$, which is the usual definition of the coinvariants.

We now relate some of Harting's considerations about coproducts to our colimits, which are their higher analogs. To that end, we consider the group $G :\equiv \mathbb{Z}/2$ in the remark and example below. A G-module is then an abelian group equipped with an automorphism which squares to the identity.

Remark 2.3.8. After Definition 2.3.1, we discussed Harting's counterexample to the existence of a monomorphism $\bigoplus_X A \to \Pi_X A$ when X is a set. Of course, this also means that there is in general no monomorphism $\mathrm{colim}_X A \to \lim_X A$ when X is not necessarily a set, but it is much easier to produce a counterexample to this. For example, if we consider the G-module $\mathbb{Z}$ given by the negation automorphism $n \mapsto -n$, then the coinvariants are $\mathbb{Z}^G = \mathbb{Z}/2$ but the invariants are $\mathbb{Z}_G = 0$. Of course, there are no monomorphisms $\mathbb{Z}/2 \to 0$.

In [Har82], Harting carried out her specific construction of the internal coproduct of abelian groups so as to prove that the resulting coproduct functor was left-exact (in particular, it preserves monomorphisms). For us, the internal coproduct is colim_X for a set X. Here we demonstrate that colim_X generally fails to be left-exact when X is not a set.

Example 2.3.9. Consider the G-module $\mathbb{Z}$ equipped with the negation automorphism $n \mapsto -n$, and the G-module $\mathbb{Z}\times\mathbb{Z}$ equipped with the "swap" automorphism $(a, b) \mapsto (b, a)$. We have a G-equivariant monomorphism $1 \mapsto (-1, 1) : \mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$ which fails to induce a monomorphism on the coinvariants. Explicitly, the respective coinvariants are $\mathbb{Z}^G = \mathbb{Z}/2$ and $(\mathbb{Z} \times \mathbb{Z})^G = \mathbb{Z}$. There are of course no non-trivial homomorphisms $\mathbb{Z}/2 \to \mathbb{Z}$, and certainly no monomorphisms. Consequently the functor $\mathrm{colim}_{B(\mathbb{Z}/2)}$, which computes the coinvariants, is not left-exact.

2.3.3 AB5 implies AB4

We prove that AB4 follows from AB5 for any abelian category, as is familiar in ordinary homological algebra. The classical proof proceeds by replacing a discrete indexing category X (for a coproduct) by a filtered category (the finite subsets of X) sharing the same colimit, then applying AB5. However, in a constructive setting neither the category of Bishop-finite subsets of X, nor the category of ordered finite subsets of X, form filtered categories unless X is decidable. For this reason we will work with lists of elements in X, i.e. general maps of the form $\mathsf{Fin}(n) \to X$ as opposed to only the injections.

We wish to point out that this is how Harting constructs the *internal coproduct* of abelian groups in an elementary topos (with $\mathbb{N}$) in [Har82], though she does not phrase things in terms of the AB axioms. While the goal of this construction is to realize a coproduct as a filtered colimit, we find it interesting to observe that Harting's "set-theoretic" description readily generalizes to untruncated indexing types, as well as abelian categories $\mathscr{A}$. Specifically, given a family $A : X \to \mathscr{A}$ indexed by an arbitrary type X, we replace A by a sifted diagram $GA : HX \to \mathscr{A}$ sharing the same colimit (if it exists). If X is a set, so the colimit is the coproduct, then $\bigoplus_{x:X} A(x)$ will be a filtered colimit, as desired.

Our first objective is to define the precategory HX of lists of elements in any 1-type X. In general HX will be sifted, and even filtered when X is a set. The latter situation is essentially the one studied in [JW78, pp. 177–178]. Throughout this section, let X be a 1-type (unless otherwise stated), and let $\mathscr{A}$ be an abelian category. We implicitly identify X^n and $\mathsf{Fin}(n) \to X$ where $\mathsf{Fin}(n)$ is the standard n-element set.

Definition 2.3.10. The 1-type $HX :\equiv \Sigma_{n:\mathbb{N}} X^n$ becomes a precategory as follows. Given two elements $(n, x), (m, y) : \Sigma_{n:\mathbb{N}} X^n$, a morphism is a commuting triangle (with specified witness of commutativity):

$$(n, x) \to (m, y) \;:\equiv\; \Sigma_{f:\mathsf{Fin}(n)\to\mathsf{Fin}(m)} \, (x =_{X^n} y \circ f).$$

Since X is a 1-type, so is X^n, and this hom-type is therefore a set, as required for being a precategory. Identity morphisms are given by identity maps and reflexivity paths. Composing morphisms is done by pasting triangles. Specifically, given morphisms $(f, p) : (n, x) \to (m, y)$ and $(g, q) : (m, y) \to (l, z)$ in HX, then their composite is

$$(g, q) \circ (f, p) \;:\equiv\; (g \circ f,\; p \cdot (q \circ f)),$$

where we view q as a homotopy $\Pi_{i:\mathsf{Fin}(m)} y_i = z_{g(i)}$ via function extensionality. Associativity of composition follows from associativity of function composition and path composition. Clearly the identity maps form left- and right-units for the composition operation, meaning we have defined a precategory structure on HX.

We observe the following lemma (using Lemma 2.2.10), which in general fails for the precategories of Bishop-finite or finite ordered subsets of X.

Lemma 2.3.11. *The precategory HX has binary coproducts. In particular, it is sifted.*

The next proposition is [JW78, Lemma 4.4] translated to our setting. (The precise relation being that a presheaf is flat if and only if its category of elements or "total category" is filtered; see also [JW78, Proposition 1.3].) Note that $\Sigma_{n:\mathbb{N}} X^n$ is a set if X is, and HX is then a strict category. In this situation, when discussing morphisms in HX we may omit references to the commutativity witnesses.

Proposition 2.3.12. *If X is a set, then HX is filtered.*

Proof. Clearly HX is merely inhabited, and coproducts yield upper bounds. It remains to verify that for any two parallel arrows $f, g : (n, x) \to (m, y)$ in HX, there merely exists a an arrow $h : (m, y) \to (l, z)$ making the following diagram commute:

$$\begin{tikzcd} \mathsf{Fin}(n) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[dr, bend right, "x"'] & \mathsf{Fin}(m) \arrow[r, dashed, "h"] \arrow[d, "y"] & \mathsf{Fin}(l) \arrow[dl, dashed, bend left, "z"] \\ & X & \end{tikzcd}$$

In fact, such a morphism h exists (not just merely). We may take $\mathsf{Fin}(l)$ to be the coequalizer of f and g, which is a finite set because the relation induced by f and g on $\mathsf{Fin}(m)$ is decidable and the quotient of a finite set by a decidable relation is itself finite. The induced map $z : \mathsf{Fin}(l) \to X$ follows from the universal property of this coequalizer among sets. $\square$

The following example demonstrates that HX may fail to be filtered when X is not a set.

Example 2.3.13. Let $X :\equiv \mathsf{K}(\mathbb{Z}/2, 1)$ with base point $x_0 : X$. Consider the two parallel arrows in HX:

$$\begin{tikzcd} \mathsf{Fin}(1) \arrow[rr, "\mathrm{id}"] \arrow[dr, bend right, "x_0"'] & \arrow[d, phantom, "\mathtt{refl}_{x_0}"] & \mathsf{Fin}(1) \arrow[dl, bend left, "x_0"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Fin}(1) \arrow[rr, "\mathrm{id}"] \arrow[dr, bend right, "x_0"'] & \arrow[d, phantom, "\sigma"] & \mathsf{Fin}(1) \arrow[dl, bend left, "x_0"] \\ & X & \end{tikzcd}$$

where $\sigma : \Omega X$ is the non-trivial element under the identification $\Omega X \simeq \mathbb{Z}/2$. If we had an upper bound $h : (1, x_0) \to (l, z)$ for these two arrows, then we would obtain an element $p : \Omega X$ such that $\mathtt{refl}_{x_0} \cdot p = \sigma \cdot p$. This would imply that $\mathtt{refl}_{x_0} = \sigma$, which is absurd.

Now we show how to replace diagrams $X \to \mathscr{A}$ by diagrams $HX \to \mathscr{A}$.

Construction 2.3.14. We build a functor $G : \mathscr{A}^X \to \mathscr{A}^{HX}$. For a family $A : X \to \mathscr{A}$, define

$$GA(n, x) :\equiv \bigoplus_{i:\mathsf{Fin}(n)} A(x_i).$$

For a morphism $(f, p) : (n, x) \to (m, y)$ in HX, we have the path $p : \Pi_{i:\mathsf{Fin}(n)} x_i = y_{f(i)}$ which induces a morphism $A_p : \bigoplus_{i:\mathsf{Fin}(n)} A(x_i) \to \bigoplus_{i:\mathsf{Fin}(n)} A(y_{f(i)})$ by transport and functoriality of biproducts. Define $GA_{(f,p)} : \bigoplus_{i:\mathsf{Fin}(n)} A(x_i) \to \bigoplus_{j:\mathsf{Fin}(m)} A(y_j)$ as the composite:

$$\bigoplus_{i:\mathsf{Fin}(n)} A(x_i) \xrightarrow{A_p} \bigoplus_{i:\mathsf{Fin}(n)} A(y_{f(i)}) \xrightarrow{\sim} \bigoplus_{j:\mathsf{Fin}(m)} \bigoplus_{i:\mathsf{fib}_f(j)} A(y_{f(i)}) \longrightarrow \bigoplus_{j:\mathsf{Fin}(m)} A(y_j),$$

where the middle map simply arranges the biproduct using that $\mathsf{Fin}(n)$ is the sum of the fibres of f, and the last map sums over the fibres of f. This sum is well-defined since it is finite: any function between finite types has decidable fibres, and a decidable subset of a finite type is finite, hence $\mathsf{fib}_f(j)$ is finite for all $j : \mathsf{Fin}(m)$. We will write ∇ for the composite of the two last maps above. Checking that GA defines a functor is straightforward.

Lastly, the obvious functor $h_X : X \to HX$ defined by $h_X(x) :\equiv (1, x)$ makes the following diagram commute:

$$\begin{tikzcd} X \arrow[dr, bend left, "A"] \arrow[dd, "h_X"'] & \\ & \mathscr{A} . \\ HX \arrow[ur, bend right, "GA"'] & \end{tikzcd}$$

The following is the analog of [Har82, Proposition 2.5] in our setting.

Lemma 2.3.15. *The functor $G : \mathscr{A}^X \to \mathscr{A}^{HX}$ preserves limits.*

Proof. Let $A : \mathscr{D} \to \mathscr{A}^X$ be a diagram whose limit exists. For all $(n, x) : HX$, we have

$$G(\lim_{d:\mathscr{D}} A_d)(n, x) \equiv \bigoplus_{j:\mathsf{Fin}(n)} \lim_{d:\mathscr{D}} A_d(x_j) = \lim_{d:\mathscr{D}} \bigoplus_{j:\mathsf{Fin}(n)} A_d(x_j) \equiv \lim_{d:\mathscr{D}} GA_d(n, x)$$

using that limits in functor categories are computed object-wise, and that $\bigoplus_{\mathsf{Fin}(n)}$ preserves limits. □

Before the next proposition, we require a lemma:

Lemma 2.3.16. *Let n be a natural number, and let $A : X \to \mathscr{A}$. Consider an object $M : \mathscr{A}$ along with a family $\eta : \Pi_{x:X} A(x) \to M$. For any path $p : x = x'$ in X^n, the following diagram commutes:*

$$\begin{tikzcd} \bigoplus_{i:\mathsf{Fin}(n)} A(x_i) \arrow[dr, bend left, "\bigoplus_i \eta_{x_i}"] \arrow[dd, "A_p"'] & \\ & M \\ \bigoplus_{i:\mathsf{Fin}(n)} A(x'_i) \arrow[ur, bend right, "\bigoplus_i \eta_{x'_i}"'] & \end{tikzcd}$$

Proof. By path induction on p. □

Now we prove that passing between A and GA leaves the colimit unchanged (if it exists).

Proposition 2.3.17. *Let $A : X \to \mathscr{A}$. Restriction along the functor $h_X : X \to HX$ is an isomorphism*

$$h_X^* : \mathscr{A}^{HX}(GA, \mathrm{const}_{HX}(M)) \to \mathscr{A}^X(A, \mathrm{const}_X(M))$$

which is natural in $M : \mathscr{A}$. Consequently, the colimits of A and GA coincide, when they exist.

Proof. We construct an explicit inverse e to h_X^*. Let $M : \mathscr{A}$, and let $\eta : A \Rightarrow \mathrm{const}_X(M)$ be a natural transformation, i.e. a family $\eta : \Pi_{x:X} A(x) \to M$. Given such a family η, we extend it to a natural transformation $e(\eta) : GA \Rightarrow \mathrm{const}_{HX}(M)$ using the biproduct, as follows. For $(n, x) : HX$, let

$$e(\eta)_{(n,x)} := \bigoplus_i \eta_{x_i} : \bigoplus_{i:\mathsf{Fin}(n)} A(x_i) \longrightarrow M.$$

Thus we have defined a transformation $e(\eta)$, and now we check naturality.

Let $(f, p) : (n, x) \to (m, y)$ be a morphism in HX. Our task is to verify that outer triangle in the following diagram commutes:

$$\begin{tikzcd}
\bigoplus_{i:\mathsf{Fin}(n)} A(x_i) \arrow[r, "A_p"] \arrow[dr, bend right, "\bigoplus_i \eta_{x_i}"'] & \bigoplus_{i:\mathsf{Fin}(n)} A(y_{f(i)}) \arrow[r, "\nabla"] \arrow[d, dashed] & \bigoplus_{j:\mathsf{Fin}(m)} A(y_j) \arrow[dl, bend left, "\bigoplus_j \eta_{x_j}"] \\
& M &
\end{tikzcd}$$

where the dashed line is $\bigoplus_{i:\mathsf{Fin}(n)} \eta_{y_{f(i)}}$. The inner-left triangle commutes by Lemma 2.3.16. That the inner-right triangle commutes can be immediately checked on each component $i : \mathsf{Fin}(n)$. Thus we conclude that $e(\eta)$ is a natural transformation.

From the construction it is clear that $h_X^* \circ e = \mathrm{id}$. For the other equality, let $\nu : GA \Rightarrow \mathrm{const}_{HX}(M)$ be a natural transformation. Given some $(n, x) : HX$, then for any $i : \mathsf{Fin}(n)$ we have the morphism in HX on the left, whose filler is the reflexivity path:

$$\begin{tikzcd}
\mathsf{Fin}(1) \arrow[d, "i"] \arrow[dr, bend left, "x_i"] & \\
\mathsf{Fin}(n) \arrow[r, bend right, "x"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
A(x_i) \arrow[d, "GA_i"] \arrow[dr, bend left, "\nu_{(1,x_i)}"] & \\
\bigoplus_{j:\mathsf{Fin}(n)} A(x_j) \arrow[r, bend right, "\nu_{(n,x)}"'] & M
\end{tikzcd}$$

The vertical arrow in the right triangle is the inclusion, which is also given by functoriality of GA. The right triangle commutes by naturality of ν. By the universal property of the n-fold biproduct, we have that $\nu_{(n,x)} = \bigoplus_{i:\mathsf{Fin}(n)} \nu_{(1,x_i)}$. This means that $\nu = e(h_X^*(\nu))$, and consequently $\mathrm{id} = e \circ h_X^*$. □

The proposition tells us that the following diagram commutes, whenever $\mathscr{A}$ is cocomplete:

$$\begin{tikzcd}
& \mathscr{A}^{HX} \arrow[dr, "\mathrm{colim}_{HX}"] & \\
\mathscr{A}^X \arrow[ur, "G"] \arrow[rr, "\mathrm{colim}_X"] & & \mathscr{A} .
\end{tikzcd}$$

From this we deduce the following results.

Corollary 2.3.18. *The functor* $\mathrm{colim}_X : R\text{-}\mathsf{Mod}^X \to R\text{-}\mathsf{Mod}$ *preserves finite products.*

Proof. By Lemma 2.3.15 we know that G preserves limits, so we need only argue that colim_{HX} preserves finite products. Since products of modules are given by products of the underlying sets, and sifted colimits of modules can be computed on the underlying sets by Corollary 2.2.12, this follows from Proposition 2.2.11 since HX is sifted (Lemma 2.3.11). □

Theorem 2.3.19. *Suppose $\mathscr{A}$ is an abelian category satisfying AB3 and AB5. For any set X, the functor $\bigoplus_X : \mathscr{A}^X \to \mathscr{A}$ is left-exact. In particular, $\mathscr{A}$ satisfies AB4.*

Proof. The assumption that $\mathscr{A}$ satisfies AB5 means that the functor colim_{HX} is exact, because HX is filtered when X is a set by Proposition 2.3.12. Since G preserves limits by Lemma 2.3.15, we conclude from the diagram above that colim_X (i.e. $\bigoplus_X$) is left-exact. ◻

2.4 Semantics

We interpret the most central results from the previous sections into an ∞-topos $\mathcal{X}$, as made possible by recent developments on the semantics of homotopy type theory [KL21, LS20, Shu19, dBoe20]. Specifically, we work out the interpretation of categories of modules (Theorem 2.4.17) and colimits of modules indexed by an object (Theorem 2.4.18).

Until now we have studied categories of abelian groups and modules, as well as abstract abelian categories in HoTT. Semantically, these yield structures in our chosen ∞-topos $\mathcal{X}$. For example, we will see that the "internal category" `Ab` obtained by interpretation represents—in the sense of Definition 2.4.3—the presheaf

$$X \longmapsto \mathrm{Ab}_{\mathcal{X}/^{\kappa}X} \;:\; \mathcal{X}^{\mathrm{op}} \longrightarrow \mathrm{Cat}$$

which sends an object $X \in \mathcal{X}$ to the ordinary category of (relatively κ-compact) abelian groups objects in the 1-topos $\tau_{\leq 0}(\mathcal{X}/^{\kappa}X)$. We often refer to structures in this slice over X just as structures over X, e.g., "abelian groups over X."

We will work and interpret our results directly into the ∞-topos $\mathcal{X}$, justified by the following considerations. Any (Grothendieck) ∞-topos $\mathscr{X}$ can be presented by a type-theoretic model topos $\mathscr{M}$ according to [Shu19]. Assuming an inaccessible cardinal κ, the latter admits a univalent universe for relatively κ-presentable fibrations [Shu19, Definition 4.7] supporting the interpretation of HoTT.[1] Constructions in $\mathscr{M}$ present constructions in $\mathcal{X}$, and we are interested in studying our results in the latter. Moreover, Stenzel [Ste23] proves that the univalent universe in $\mathscr{M}$ presents an *object classifier* $\widetilde{\mathcal{U}}^{\kappa} \to \mathcal{U}^{\kappa}$ [Lur09, Section 6.1.6] for relatively κ-compact morphisms[2] in $\mathcal{X}$. This means that the universe $\mathcal{U}^{\kappa}$ represents the presheaf which maps an object $X \in \mathcal{X}$ to the ∞-groupoid $(\mathcal{X}/^{\kappa}X)^{\simeq}$ of relatively κ-compact maps into X. This makes precise the nature of the univalent universe in $\mathcal{X}$ corresponding to the one from HoTT, and we will not need to refer to $\mathscr{M}$ from here on. We mention that [Ver19] gives another example of working out the interpretation of material in HoTT directly into an ∞-topos. However, our case is simpler because we mainly work with truncated objects.

We will require a small fragment of the theory of complete Segal objects [Ras18] in $\mathcal{X}$ (also called *internal ∞-categories* [Mar21] or *Rezk objects* [RV22]). As our model of the (large) ∞-category Cat_∞ of ∞-categories, we choose the ∞-category of complete Segal spaces. Though our arguments will clearly be model-independent, certain constructions require a choice, and this is a convenient one for our purposes.

[1] Modulo certain classes of higher inductive types, which we do not use.

[2] The difference in terminology (κ-presentable vs. κ-compact) is unfortunate. As we work in the ∞-setting, we will employ Lurie's terminology, i.e. "κ-compact" [Lur09, Definition 6.1.6.4], when necessary.

Notation. We write $\mathcal{X}_\kappa$ for the sub-∞-category of κ-compact objects in $\mathcal{X}$, and for an object $X \in \mathcal{X}$ we form the slice $\mathcal{X}/^\kappa X$ of relatively κ-compact morphisms into X. The 1-topos of 0-truncated objects in $\mathcal{X}$ is $\tau_{\leq 0}(\mathcal{X})$. The functor $(-)^\simeq : \mathrm{Cat}_\infty \to \mathscr{S}$ picks out the ∞-groupoid core of an ∞-category, and $\mathscr{S}$ is the ∞-category of spaces (also called ∞-groupoids). For complete Segal spaces, the functor $(-)^\simeq$ simply picks out the zeroth space. The classifier for κ-compact morphisms in $\mathcal{X}$ presented by Shulman's univalent universe will be written $\widetilde{\mathcal{U}} \to \mathcal{U}$, leaving κ implicit. No confusion will arise as no other universes will be around. Internal homs are written using exponential notation Y^X.

Notions in $\mathcal{X}$ resulting from interpretation will be denoted in typewriter font. For example we will be considering the universe `Set` which classifies κ-compact 0-truncated objects (Lemma 2.4.11). In particular, we leave the κ implicit in the notation of the universe of sets (or of abelian groups, or of R-modules).

The 1-category $\Delta_{\leq 2}$ is the full subcategory of the simplex category Δ on the objects [0], [1], and [2]. The ∞-category of (∞-)functors $\Delta_{\leq 2}^{\mathrm{op}} \to \mathcal{X}$ consists of *2-restricted simplicial objects in* $\mathcal{X}$, and is denoted $\mathcal{X}_{\Delta_{\leq 2}}$ for short. We explain how we view 1-categories as ∞-categories just below. The standard n-simplex is the usual simplicial space $\Delta^n : \Delta^{\mathrm{op}} \to \mathscr{S}$ and its 2-restriction is $\Delta^n_{\leq 2} : \Delta^{\mathrm{op}}_{\leq 2} \to \mathscr{S}$. There is also a standard n-simplex $\underline{\Delta}^n$ in $\mathcal{X}$, whose 2-restriction we denote $\underline{\Delta}^n_{\leq 2} : \Delta^{\mathrm{op}}_{\leq 2} \to \mathcal{X}$.

2.4.1 Rezk $(1, 1)$-objects

The first goal of this section is to repackage the internal categories in $\mathcal{X}$ obtained by interpretation into structures which conveniently represent presheaves of 1-categories. We begin by explaining how we associate 1-categories to ∞-categories akin to $\mathcal{X}$. An ordinary category C is incarnated as a simplicial space through its **classifying diagram** $\mathrm{D}(C)$ [Rez01, Section 3.5]:

$$\mathrm{D}(C) := \left(\cdots \;\vdots\!\longrightarrow\; (C^{[2]})^\simeq \;\substack{\longrightarrow\\\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow}\; (C^{[1]})^\simeq \;\substack{\longleftarrow\\\longrightarrow\\\longleftarrow\\\longrightarrow}\; C^\simeq \right) \tag{2.1}$$

where we used $(-)^\simeq$ to denote the Kan complex obtained from the groupoid core of a 1-category, and $[n]$ denotes the usual poset with $n+1$ elements. This classifying diagram is a complete Segal space, and there is a Quillen adjunction $\mathrm{h} : \mathrm{Cat}_\infty \leftrightarrows \mathrm{Cat} : \mathrm{D}$ which exhibits Cat as precisely the 1-truncated complete Segal spaces [CL20, Theorem 5.11]. The left adjoint h is the fundamental category functor. We view ordinary 1-categories as ∞-categories via this construction, and by identifying Cat with its image under the embedding D, we may speak about presheaves of 1-categories on $\mathcal{X}$.

We will argue that precategories and categories in HoTT interpret to the following structures.

Definition 2.4.1. A **Segal (1, 1)-object in $\mathcal{X}$** is a 2-restricted simplicial object $C : \Delta^{\mathrm{op}}_{\leq 2} \to \mathcal{X}$ satisfying the three following conditions:

(truncation) the **structure map** $(\mathrm{dom}, \mathrm{cod}) : C_1 \to C_0 \times C_0$ is 0-truncated in $\mathcal{X}$;

(Segal condition) the natural map $C_2 \to C_1 \times_{C_0} C_1$ is an equivalence;

(associativity) there is a witness that the following two composites agree:

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \times_{C_0} C_1 \arrow[r, shift left, "\mathrm{id}\times\circ"] \arrow[r, shift right, "\circ\times\mathrm{id}"'] & C_1 \times_{C_0} C_1 \arrow[r, "\circ"] & C_1 \end{tikzcd}$$

where $\circ : C_1 \times_{C_0} C_1 \xrightarrow{\sim} C_2 \xrightarrow{\delta_1^2} C_1$.

If moreover the square below below is a pullback, then C is a **Rezk** $(1,1)$**-object**:

$$\begin{tikzcd} C_0 \arrow[r, "(\mathrm{id},\mathrm{id},\mathrm{id})"] \arrow[d, "\Delta"] & C_1 \times_{C_0} C_1 \times_{C_0} C_1 \arrow[d, "(f,g,h)\mapsto(fg,gh)"] \\ C_0 \times C_0 \arrow[r, "\mathrm{id}\times\mathrm{id}"] & C_1 \times C_1 \end{tikzcd} \tag{2.2}$$

The Segal (or Rezk) $(1,1)$-object C is **locally small** if the structure map is relatively κ-compact.

We emphasize that the truncation condition on the structure maps implies that there is at most one witness of associativity (i.e., it is a property).

It is straightforward to interpret Definition 9.1.1 from [Uni13] to see what the the data of a precategory in $\mathcal{X}$ consists of. We allow the underlying type of a precategory to be any object of $\mathcal{X}$, not necessarily classified by $\mathcal{U}$. Our next lemma gives a precise relation between precategories in $\mathcal{X}$ and the structures just defined.

Lemma 2.4.2. *Precategories in $\mathcal{X}$ correspond to locally small Segal $(1,1)$-objects, and categories to locally small Rezk $(1,1)$-objects.*

Here, by "correspond" we mean that from one structure one can construct the other, and vice-versa. In particular, we may apply results about (pre)categories in HoTT to (Segal) Rezk (1,1)-objects in $\mathcal{X}$. However, we are not constructing an equivalence of spaces (or objects) of such structures (though there may well be one).

Proof. Given a precategory C, we define a 2-restricted simplicial object $\mathsf{C}_\bullet$ as follows. Let $\mathsf{C}_0 :\equiv \mathsf{C}$, and write $(\mathrm{dom}, \mathrm{cod}) : \mathsf{C}_1 \to \mathsf{C}_0 \times \mathsf{C}_0$ for the total space of the hom $\mathsf{C}(-,-) : \mathsf{C}\times\mathsf{C} \to \mathsf{Set}$ with its projection. The identity maps $\mathrm{id} : \Pi_{c:\mathsf{C}}\mathsf{C}(c,c)$ give a section $\mathsf{C}_0 \to \mathsf{C}_1$ of both dom and cod. Now let $\mathsf{C}_2 :\equiv \Sigma_{a,b,c:\mathsf{C}_0}\Sigma_{f:\mathsf{C}(a,b)}\Sigma_{g:\mathsf{C}(b,c)}\Sigma_{h:\mathsf{C}(a,c)} gf = h$ be the object of commuting triangles in C. Then $\mathsf{C}_\bullet$ is a 2-restricted simplicial object in $\mathcal{X}$ with face maps given by projections, and degeneracies induced by id. Clearly $\mathsf{C}_\bullet$ satisfies the truncation condition, and is associative (since the precategory C is). In HoTT, it is easy to show that the map $(f,g) \mapsto (f,g,gf,\mathtt{refl}_{gf})$ is inverse to the natural map $\mathsf{C}_2 \to \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1$, thus we conclude that $\mathsf{C}_\bullet$ is a Segal $(1,1)$-object. It is locally small by construction.

It is similarly straightforward to produce a precategory from a locally small Segal $(1,1)$-object. Under this correspondence, univalence of a precategory is equivalent to the square (2.2) being a pullback, so we conclude that categories correspond to Rezk $(1,1)$-objects. ◻

Now we explain in what sense Rezk $(1,1)$-objects represent presheaves of ordinary categories. It is clear that to recover the fundamental category of a classifying diagram it suffices to recover the lower three simplicial levels. This leads us to the following notion of representability.

Definition 2.4.3. Let $C : \mathcal{X}^{\mathrm{op}} \to \mathrm{Cat}$ be a presheaf of 1-categories on $\mathcal{X}$. A Rezk $(1,1)$-object $\mathtt{C} : \Delta^{\mathrm{op}}_{\leq 2} \to \mathcal{X}$ **represents** C if there is a specified natural equivalence $\eta : \mathcal{X}(-, \mathtt{C}_\bullet) \simeq i_2^* C$ of functors $\mathcal{X}^{\mathrm{op}} \to \mathscr{S}_{\Delta_{\leq 2}}$, where i_2^* is the restriction along the inclusion $\Delta_{\leq 2} \to \Delta$.

We will use this notion of representability when working out the semantics of the category of sets and categories of modules in the next sections. The reader who is mainly interested in those representability results (e.g. Theorem 2.4.17) may skip ahead to the next section. The remaining parts of this section are only needed for Theorem 2.4.18.

Any statement about (pre)categories in HoTT yields a statement about locally small (Segal) Rezk $(1,1)$-objects by translating across the correspondence of Lemma 2.4.2. For example, one can check that products of (pre)categories correspond to object-wise products of (Segal) Rezk $(1,1)$-objects. Our next statement is that functor precategories interpret to the internal hom of Segal $(1,1)$-objects.

If $F, G : \mathscr{C} \to \mathscr{D}$ are two functors between (pre)categories in HoTT, then we can represent natural transformations $F \Rightarrow G$ as functors $\eta : \mathscr{C}\times[1] \to \mathscr{D}$ such that $\eta|_{\mathscr{C}\times\{0\}} = F$ and $\eta|_{\mathscr{C}\times\{1\}} = G$. Here $[n]$ denotes the usual precategory (poset) in HoTT with $n+1$ elements. The precategory $[n]$ interprets to the Segal $(1,1)$-object $\underline{\Delta}^n_{\leq 2}$ which is the 2-restriction of the standard n-simplex $\underline{\Delta}^n : \Delta^{\mathrm{op}} \to \mathcal{X}$ in $\mathcal{X}$.

We have the following:

Lemma 2.4.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be locally small Segal $(1,1)$-objects in $\mathcal{X}$.*

1. *the object of functors $\mathtt{Fun}(\mathcal{C}, \mathcal{D})$ obtained by interpretation represents the presheaf*

$$X \longmapsto (\mathcal{X}/X)_{\Delta_{\leq 2}}(X \times \mathcal{C}, X \times \mathcal{D}) \ : \ \mathcal{X}^{\mathrm{op}} \longrightarrow \mathscr{S}$$

   where the base change functor $X \times (-)$ is applied object-wise;

2. *the Segal $(1,1)$-object $\mathtt{Fun}(\mathcal{C}, \mathcal{D})_\bullet$ obtained by interpreting the functor category is equivalent to*

$$\mathtt{Fun}(\mathcal{C} \times \underline{\Delta}^2_{\leq 2}, \mathcal{D}) \; \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} \; \mathtt{Fun}(\mathcal{C} \times \underline{\Delta}^1_{\leq 2}, \mathcal{D}) \; \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow} \; \mathtt{Fun}(\mathcal{C} \times \underline{\Delta}^0_{\leq 2}, \mathcal{D})$$

   where the degeneracy and face maps come from the $\underline{\Delta}^n_{\leq 2}$'s. If $\mathcal{D}$ is Rezk, then so is $\mathtt{Fun}(\mathcal{C}, \mathcal{D})_\bullet$.

Proof. It is straightforward to see that functors between precategories in HoTT interpret to simplicial maps between the corresponding Segal $(1,1)$-objects. Then (1) follows by stability of interpretation across base change.

By representing natural transformations as functors, we see that $\mathtt{Fun}(\mathcal{C} \times \underline{\Delta}^1_{\leq 2}, \mathcal{D})$ is the total space of the map $\mathtt{Fun}(\mathcal{C}, \mathcal{D})^2 \to \mathtt{Set}$ which sends two functors to the set of natural transformations between them. Thus we get the first and second levels of (2). Finally, the third level is naturally equivalent to $\mathtt{Fun}(\underline{\Delta}^2_{\leq 2}, \mathtt{Fun}(\mathcal{C}, \mathcal{D})_\bullet)$ in HoTT, and the latter is clearly equivalent to the space of commuting triangles in $\mathtt{Fun}(\mathcal{C}, \mathcal{D})$. These equivalences assemble to a simplicial map, so we are done. □

We note that by combining part (1) and (2) of the lemma, we get a formula for the presheaf represented by $\mathtt{Fun}(C,\mathcal{D})_\bullet$.

When working with Segal and Rezk $(1,1)$-objects we may use category-theoretical language as long as the relevant interpretation has been worked out, or is apparent from the context. We also note that we can take $\mathcal{X}$ to be the ∞-topos $\mathscr{S}$ of spaces, and in this case we will use the terminology **Segal** and **Rezk (1, 1)-spaces** for emphasis.

Our next proposition asserts that functor categories interpret to the internal hom in $\mathcal{X}_{\Delta_{\leq 2}}$. To prove this, we require a lemma. First recall the terminal geometric morphism $\mathcal{X}(1,-) : \mathcal{X} \to \mathscr{S}$ (with left adjoint ℓ) associated to any ∞-topos. Applying this adjunction object-wise, we get an induced adjunction

$$\ell_* \ : \ \mathscr{S}_{\Delta_{\leq 2}} \leftrightarrows \mathcal{X}_{\Delta_{\leq 2}} \ : \ \mathcal{X}(1,-).$$

The left-exactness of ℓ_* implies that it preserves Segal and Rezk $(1,1)$-objects. Moreover, since ℓ_* preserves finite limits and colimits, it preserves the standard n-simplex so that we have $\ell_*(\Delta^n_{\leq 2}) = \underline{\Delta}^n_{\leq 2}$.

Lemma 2.4.5. *Let $C,\mathcal{D} \in \mathcal{X}_{\Delta_{\leq 2}}$. For $X \in \mathcal{X}$ and $n \in \{0,1,2\}$, we have natural equivalences*

$$\mathcal{X}(X,\mathcal{D}^C)_n \ \simeq \ (\mathcal{X}/X)_{\Delta_{\leq 2}}(X \times C \times \underline{\Delta}^n_{\leq 2}, X \times \mathcal{D}).$$

Proof. By stability of the internal hom across base change, we can assume $X = 1$. We then have:

$$\mathcal{X}_{\Delta_{\leq 2}}(C \times \underline{\Delta}^n_{\leq 2},\mathcal{D}) \ \simeq \ \mathcal{X}_{\Delta_{\leq 2}}(\underline{\Delta}^n_{\leq 2},\mathcal{D}^C) \ \simeq \ \mathscr{S}_{\Delta_{\leq 2}}(\Delta^n_{\leq 2},\mathcal{X}(1,\mathcal{D}^C)) \ \simeq \ \mathcal{X}(1,\mathcal{D}^C)_n$$

where the first equivalence is by cartesian-closedness of $\mathcal{X}_{\Delta_{\leq 2}}$, the second equivalence comes from the adjunction $\ell_* \dashv \mathcal{X}(1,-)$ and that $\ell_*(\Delta^n_{\leq 2}) = \underline{\Delta}^n_{\leq 2}$. The last equivalence is by the Yoneda lemma. □

Proposition 2.4.6. *Let C and $\mathcal{D}$ be locally small Segal $(1,1)$-objects in $\mathcal{X}$. The 2-restricted simplicial objects $\mathcal{D}^C$ and $\mathtt{Fun}(C,\mathcal{D})_\bullet$ in $\mathcal{X}$ are naturally equivalent.*

Proof. By the Yoneda lemma, it suffices to show that the represented functors $\mathcal{X}(-,\mathcal{D}^C)$ and $\mathcal{X}(-,\mathtt{Fun}(C,\mathcal{D}))_\bullet$ of the form $\mathcal{X}^{\mathrm{op}} \to \mathscr{S}_{\Delta_{\leq 2}}$ are naturally equivalent. This follows by combining the two previous lemmas. □

The proposition implies that the internal hom between Segal $(1,1)$-objects is itself a Segal $(1,1)$-object, and even Rezk if the codomain is.

If C is a Rezk $(1,1)$-object in $\mathcal{X}$, then the internal limit of a functor $F : \mathcal{D} \to C$ in $\mathcal{X}_{\Delta_{\leq 2}}$ defines a global point $\mathtt{lim}_{\mathcal{D}} F \in \mathcal{X}(1,C_0)$, if the internal limit exists. Of course, so does the limit of an *external* functor $G : D \to \mathcal{X}(1,C)$ in $\mathscr{S}_{\Delta_{\leq 2}}$. We now explain how such external functors $D \to \mathcal{X}(1,C)$ can be *internalized* to functors in $\mathcal{X}_{\Delta_{\leq 2}}$, and we prove that this procedure does not change the limit or colimit.

Definition 2.4.7. Let C be a Rezk $(1,1)$-object in $\mathcal{X}$, and D a Rezk $(1,1)$-space. The **internalization** of a functor $A : D \to \mathcal{X}(1,C)$ is its transpose $\underline{A} : \ell_*(D) \to C$ across the adjunction $\ell_* \dashv \mathcal{X}(1,-)$.

To show that internalization of a functor does not change its (co)limit, we require a lemma. The reader may find it interesting to compare it with [Joh77, Example 2.39].

Lemma 2.4.8. *Let C be a Rezk $(1,1)$-object in $\mathcal{X}$, and D a Rezk $(1,1)$-space. The Rezk $(1,1)$-spaces $\mathcal{X}(1, C^{\ell_*(D)})$ and $\mathcal{X}(1,C)^D$ are naturally equivalent.*

Proof. Using Lemma 2.4.5 and the adjunction $\ell_* \dashv \mathcal{X}(1,-)$, for $n \in \{0,1,2\}$ we have:

$$\mathcal{X}(1, C^{\ell_*(D)})_n \;\simeq\; \mathcal{X}_{\Delta_{\leq 2}}(\ell_*(D) \times \underline{\Delta}^n_{\leq 2}, C) \;\simeq\; \mathcal{S}_{\Delta_{\leq 2}}(D \times \Delta^n_{\leq 2}, \mathcal{X}(1,C)) \;\simeq\; (\mathcal{X}(1,C)^D)_n$$

where the second equivalence uses that ℓ_* preserves products (being left exact), then transposes across the adjunction. The third equivalence is Lemma 2.4.5 applied to Rezk $(1,1)$-spaces. Using basic properties of adjunctions, one can check that these equivalences assemble to a simplicial map. □

The category of sets in HoTT interprets to a Rezk $(1,1)$-object $\mathtt{Set}_\bullet$ which features in the next proposition, and is the main topic of study in the next section. For the following proof, we only use that $\mathcal{X}(1,\mathtt{Set}_\bullet)$ has a terminal object and therefore a global sections functor $\Gamma : \mathcal{X}(1,\mathtt{Set}_\bullet) \to \tau_{\leq 0}(\mathcal{S})$. Observe that if C is a Rezk $(1,1)$-object in $\mathcal{X}$, then the Rezk $(1,1)$-space $\mathcal{X}(1,C)$ is "enriched" over $\mathcal{X}(1,\mathtt{Set}_\bullet)$. A study of this "enrichment" is beyond the scope of this work, and our convention will be to implicitly apply Γ so that the hom functor $\mathcal{X}(1,C)(-,-)$ lands in $\tau_{\leq 0}(\mathcal{S})$.

Proposition 2.4.9. *Let C be a locally small Rezk $(1,1)$-object in $\mathcal{X}$, and let $A : D \to \mathcal{X}(1,C)$ be a functor between Rezk $(1,1)$-spaces. If the internal limit $\mathtt{lim}_{\ell_*(D)}\underline{A}$ in C exists, so does the limit of A and we have a canonical isomorphism $\mathtt{lim}_{\ell_*(D)}\underline{A} \simeq \lim_D A$ in $\mathcal{X}(1,C)$.*

Proof. Suppose the internal limit of $\underline{A}$ in C exists, meaning we have a natural equivalence of functors $C^{\mathrm{op}} \to \mathtt{Set}_\bullet$

$$C(-, \mathtt{lim}_{\ell_*(D)}\underline{A}) \;\simeq\; C^{\ell_*(D)}(\mathtt{const}_{\ell_*(D)}(-), \underline{A}).$$

Applying $\mathcal{X}(1,-)$, we get an equivalence between certain functors $\mathcal{X}(1,C)^{\mathrm{op}} \to \mathcal{X}(1,\mathtt{Set}_\bullet)$, and by further post-composing with the global sections map $\Gamma : \mathcal{X}(1,\mathtt{Set}_\bullet) \to \tau_{\leq 0}(\mathcal{S})$, we get an equivalence

$$\mathcal{X}(1,C)(-, \mathtt{lim}_{\ell_*(D)}\underline{A}) \;\simeq\; \mathcal{X}(1, C^{\ell_*(D)})(\mathtt{const}_{\ell_*(D)}(-), \underline{A}) \tag{2.3}$$

between functors $\mathcal{X}(1,C)^{\mathrm{op}} \to \tau_{\leq 0}(\mathcal{S})$. We have an equivalence $\mathcal{X}(1, C^{\ell_*(D)}) \simeq \mathcal{X}(1,C)^D$ by the previous lemma, which sends $\mathtt{const}_{\ell_*(D)}$ to const_D and $\underline{A}$ to A. On hom-spaces, this means we have:

$$\mathcal{X}(1, C^{\ell_*(D)})(\mathtt{const}_{\ell_*(D)}(-), \underline{A}) \;\simeq\; \mathcal{X}(1,C)^D(\mathrm{const}_D(-), A). \tag{2.4}$$

Combining the equivalences (2.3) and (2.4), we see that $\mathtt{lim}_{\ell_*(D)}\underline{A}$ is the limit of A, as desired. □

The proposition and its proof dualises to colimits, but we will only need it for limits.

2.4.2 The universe of sets

We show that the Rezk $(1,1)$-object $\mathtt{Set}_\bullet$ produced by interpretation represents the presheaf $\tau_{\leq 0}(\mathcal{X}/^{\kappa}(-)) : \mathcal{X}^{\mathrm{op}} \to \mathrm{Cat}$ in the sense of Definition 2.4.3. First we show a lemma that proves useful for these kinds of representability results.

The universe $\mathcal{U}$ is an object classifier for relatively κ-compact morphisms [Ste23] in the sense of [Lur09, Section 6.1.6] and therefore represents (in the usual sense) the presheaf $(\mathcal{X}/^{\kappa}(-))^{\simeq} : \mathcal{X}^{\mathrm{op}} \to \mathscr{S}$. We will be interested in types which classify certain structures in $\mathcal{X}$. For example, given a ring $R \in \tau_{\leq 0}(\mathcal{X}_\kappa)$, we will see that there is a map $R\text{-}\mathtt{mod}\text{-}\mathtt{str} : \mathtt{Set} \to \mathcal{U}$ which classifies R-modules in $\mathcal{X}_\kappa$, meaning that the mapping space $\mathcal{X}(X, \Sigma_{A:\mathtt{Set}} R\text{-}\mathtt{mod}\text{-}\mathtt{str}(A))$ is the groupoid of R-modules in $(\mathcal{X}/^{\kappa}X)$ (Theorem 2.4.17). The following lemma gives a description of these mapping spaces for general type families.

Lemma 2.4.10. *Let $P : Z \to \mathcal{U}$ be a type family in $\mathcal{X}$, and $X \in \mathcal{X}$. The outer square in the following diagram is a pullback:*

$$\begin{tikzcd}
\mathcal{X}(X, \Sigma_{z:Z} P(z)) \arrow[r] \arrow[d] & \mathcal{X}(X, \widetilde{\mathcal{U}}) \arrow[r, "\sim"] \arrow[d] & (\mathcal{X}/^{\kappa}X)_*^{\simeq} \arrow[r] \arrow[d] & \mathscr{S}_* \arrow[d] \\
\mathcal{X}(X, Z) \arrow[r, "\mathcal{X}(X,P)"] & \mathcal{X}(X, \mathcal{U}) \arrow[r, "\sim"] & (\mathcal{X}/^{\kappa}X)^{\simeq} \arrow[r, "\Gamma^{\simeq}"] & \mathscr{S}
\end{tikzcd}$$

where the functor $\Gamma^{\simeq}$ is the restriction of the global points functor $\Gamma : (\mathcal{X}/^{\kappa}X) \to \mathscr{S}$ to the core, and $(\mathcal{X}/^{\kappa}X)_*$ is the ∞-category of relatively κ-compact maps into X equipped with a section.

Proof. The right square is manifestly a pullback, and so is the left square since $\mathcal{X}(X, -)$ preserves limits. The middle square is a pullback because $\widetilde{\mathcal{U}}$ classifies pointed objects. By pullback pasting we conclude that the outer square is a pullback. □

Recall that $\mathtt{Set}$ is defined as the total space of the map $\mathtt{is}\text{-}0\text{-}\mathtt{type} : \mathcal{U} \to \mathcal{U}$ sending an object A to the proposition that the diagonal map $A \to A \times A$ is an embedding (i.e., (-1)-truncated). By [Lur09, Lemma 5.5.6.15], this proposition holds (i.e., has a global point) if and only if A is 0-truncated (i.e., that $\mathcal{X}(B, A)$ is a 0-truncated space for all $B \in \mathcal{X}$ [Lur09, Definition 5.5.6.1]). We now show that the universe $\mathtt{Set}$ of sets classifies 0-truncated objects:

Lemma 2.4.11. *The object $\mathtt{Set}$ represents the presheaf of spaces $\tau_{\leq 0}(\mathcal{X}/^{\kappa}(-))^{\simeq}$.*

Proof. By applying the previous lemma to the type family $\mathtt{is}\text{-}0\text{-}\mathtt{type} : \mathcal{U} \to \mathcal{U}$, we see that $\mathcal{X}(1, \mathtt{Set})$ is the sub-∞-groupoid of $\mathcal{X}(1, \mathcal{U})$ on those objects $A : \mathcal{U}$ for which $\mathtt{is}\text{-}0\text{-}\mathtt{type}(A)$ holds. Since $\mathtt{is}\text{-}0\text{-}\mathtt{type}(A)$ holds if and only if A is 0-truncated, $\mathcal{X}(1, \mathtt{Set})$ is equivalent to the groupoid of 0-truncated objects in $\mathcal{X}_\kappa$.

For general X, we always have that families $X \to \mathtt{Set}$ correspond to families $X \to X \times \mathtt{Set}$ over X. Base change stability of the universe implies that $X \times \mathtt{Set}$ is a universe of sets in $\mathcal{X}/X$. Thus we reduce to the case $X = 1$ just treated by pulling back over X, using that an arrow $f : Y \to X$ is 0-truncated as a map if and only if it is 0-truncated as an object of $\mathcal{X}/X$ [Lur09, Remark 5.5.6.12]. □

Our next goal is to understand the presheaf represented by $\mathtt{Set}$ when equipped with its categorical structure. Applying [Ras21, Theorem 4.4] to the the universal map $p : \widetilde{\mathcal{U}} \to \mathcal{U}$ yields a complete Segal object $N(p)$ in $\mathcal{X}$ which represents (in the usual sense) the presheaf $(\mathcal{X}/^{\kappa}(-)) : \mathcal{X}^{\mathrm{op}} \to \mathrm{Cat}_\infty$. The 2-restriction of $N(p)$ is equivalent to the following 2-restricted simplicial object $\mathcal{U}_\bullet$ in $\mathcal{X}$:

$$\sum_{X,Y,Z:\mathcal{U}} Y^X \times Z^Y \;\substack{\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow}\; \sum_{X,Y:\mathcal{U}} Y^X \;\substack{\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow}\; \mathcal{U}\,.$$

To be explicit, we know that the function types modelled by the universe interpret to the internal hom in $\mathcal{X}$, so the first simplicial level is simply the type-theoretic notation for Rasekh's description of $N(p)_1$, and the second level is given by the Segal condition. Accordingly, for the Rezk $(1,1)$-object $\mathtt{Set}_\bullet$, the object of morphisms is simply given by internal homs in $\mathcal{X}$:

$$\mathtt{Set}_1 := \Sigma_{X,Y:\mathtt{Set}} Y^X \to \mathtt{Set} \times \mathtt{Set}.$$

The following provides the semantics of the category of sets in HoTT.

Proposition 2.4.12. *The Rezk $(1,1)$-object $\mathtt{Set}_\bullet$ represents the presheaf*

$$\tau_{\leq 0}(\mathcal{X}/^{\kappa}(-)) \;:\; \mathcal{X}^{\mathrm{op}} \longrightarrow \mathrm{Cat}$$

in the sense of Definition 2.4.3.

Proof. Let $X \in \mathcal{X}$. We need to produce a natural equivalence $\eta : \mathcal{X}(X, \mathtt{Set}_\bullet) \simeq i_2^*(\tau_{\leq 0}(\mathcal{X}/^{\kappa}X))$ of 2-restricted simplicial spaces. By Lemma 2.4.11, we get a natural equivalence of the zeroth levels. Since the global points of the internal hom give the external hom, Lemma 2.4.10 tells us that $\mathcal{X}(X, \mathtt{Set}_1)$ is naturally equivalent to the groupoid of arrows in $\tau_{\leq 0}(\mathcal{X}/^{\kappa}X)$. These two equivalences clearly assemble to an equivalence of 1-restricted simplicial objects, whereby we get an induced equivalence of the second simplicial levels via the Segal condition. The latter equivalence automatically respects the face maps δ_0^2 and δ_2^2 as well as the degeneracies. It remains to check that it respects the composition map δ_1^2. But this follows from the fact that function types interpret to the internal hom in $\mathcal{X}$. □

Before making analogous considerations for the universe of modules, we make these results more concrete by relating them to classical 1-topos theory. These considerations are certainly well-known.

By definition, any ∞-topos $\mathcal{X}$ is an accessible, left-exact localization of an ∞-category of ∞-presheaves on some site. The following proposition shows that the associated 1-topos $\tau_{\leq 0}(\mathcal{X})$ of 0-truncated sheaves consists precisely of those sheaves which take values in sets. Thus if C is an ordinary 1-category (with a site structure), then $\tau_{\leq 0}(\mathcal{X})$ is the ordinary 1-topos of sheaves of sets on C. (We also note that the proof works for general truncation levels, but we only use the case $n = 0$.)

Proposition 2.4.13. *Let $L : \mathrm{Psh}_\infty(C) \leftrightarrows \mathcal{X} : i$ be an accessible, left-exact localization for some site C. A sheaf $X \in \mathcal{X}$ is 0-truncated if and only if $X(c)$ is 0-truncated for all $c \in C$.*

Proof. The statement for presheaves is shown on the nLab.[3] Let $X : C^{\mathrm{op}} \to \mathscr{S}$ be a sheaf (i.e., an element of $\mathcal{X}$) and suppose X lands in $\tau_{\leq 0}(\mathscr{S})$. Thus $i(X)$ is a 0-truncated presheaf. By [Lur09, Proposition 5.5.2.28], L preserves truncated objects, and so $X \simeq L(i(X))$ is 0-truncated.

Conversely, suppose X is 0-truncated. Then the space $\mathcal{X}(Y, X)$ is 0-truncated for all $Y \in \mathcal{X}$. For any $c \in C$, the Yoneda lemma and the adjunction $L \dashv i$ gives us equivalences

$$X(c) \;\simeq\; \mathrm{Psh}_\infty(C)(C(-,c), i(X)) \;\simeq\; \mathcal{X}(LC(-,c), X).$$

Thus the leftmost side is 0-truncated, since the rightmost side is. ☐

2.4.3 The universe of R-modules

Let R be a ring object in $\tau_{\leq 0}(\mathcal{X})$, i.e., a sheaf of rings. We show that the Rezk $(1,1)$-object $R\text{-}\mathtt{Mod}_\bullet$ of R-modules in $\mathcal{X}$ represents the presheaf sending an object $X \in \mathcal{X}$ to the ordinary category of modules over the ring $X \times R$ in the 1-topos $\tau_{\leq 0}(\mathcal{X}/^\kappa X)$ (Theorem 2.4.17).

The key ingredient we used to prove that $\mathtt{Set}$ classifies 0-truncated objects was that the predicate $\mathtt{is\text{-}0\text{-}type}(A)$ has a global point if and only if A is a 0-truncated object. Similarly, to say what $R\text{-}\mathtt{Mod}$ classifies we need to understand the global points of $R\text{-}\mathtt{mod\text{-}str}(A)$.

Lemma 2.4.14. *Let R be a ring object in $\tau_{\leq 0}(\mathcal{X})$. For all $A \in \tau_{\leq 0}(\mathcal{X})$, global points of the object $R\text{-}\mathtt{mod\text{-}str}(A)$ biject with R-module structures on the object A in $\mathcal{X}$.*

Proof. It is well-known that the global points of A, A^A, $A^{A\times A}$, and $A^{R\times A}$ biject respectively with the set of points of A, the set of endomorphisms of A, the set of binary operations on A, and set of maps $R \times A \to A$ in $\tau_{\leq 0}(\mathcal{X})$. One can check the global points functor Γ sends the limit diagram carving out the subobject $R\text{-}\mathtt{mod\text{-}str}(A)$ of internal R-module structures on A to the limit diagram carving out the (external) set of R-module structures on A from inside the set

$$\Gamma A \times \Gamma(A^A) \times \Gamma(A^{A\times A}) \times \Gamma(A^{R\times A}).$$

Since Γ preserves limits, we are done. ☐

For a ring $R \in \tau_{\leq 0}(\mathcal{X}_\kappa)$ and an object $X \in \mathcal{X}$, recall that $X{\times}R \in \tau_{\leq 0}(\mathcal{X}/^\kappa X)$ is a ring over X. We now show that $R\text{-}\mathtt{Mod}$ classifies R-modules in $\mathcal{X}_\kappa$.

Proposition 2.4.15. *Let R be a ring in $\tau_{\leq 0}(\mathcal{X}_\kappa)$. The object $R\text{-}\mathtt{Mod}$ represents the space-valued presheaf*

$$X \longmapsto (X{\times}R)\text{-}\mathrm{Mod}^{\simeq} \;:\; \mathcal{X}^{\mathrm{op}} \longrightarrow \mathscr{S}.$$

Proof. First of all, for any $X \in \mathcal{X}$, base change stability of the universe gives a natural equivalence $(X{\times}R)\text{-}\mathtt{Mod} \simeq X{\times}R\text{-}\mathtt{Mod}$ over X. From this we deduce the following natural equivalences:

$$\mathcal{X}(X, R\text{-}\mathtt{Mod}) \;\simeq\; \mathcal{X}/X(\mathrm{id}_X, X \times R\text{-}\mathtt{Mod}) \;\simeq\; \mathcal{X}/X(\mathrm{id}_X, (X{\times}R)\text{-}\mathtt{Mod}).$$

By working with the rightmost space, we reduce to the case $X = 1$.

As defined, $R\text{-}\mathtt{Mod}$ is the total space of $R\text{-}\mathtt{mod\text{-}str}$. Combining Lemmas 2.4.14 and 2.4.10, we see that $\mathcal{X}(1, R\text{-}\mathtt{Mod})$ is naturally equivalent to the groupoid of R-modules in $\mathcal{X}_\kappa$. ☐

[3] See Proposition 4.2 at n-truncated object of an $(\infty,1)$-category (rev. 78) on the nLab.

We recall how internal objects of homomorphisms in $\mathcal{X}$ are constructed. Given an abelian group object A in $\tau_{\le 0}(\mathcal{X})$, we write $+_A : A \times A \to A$ for the addition map.

Definition 2.4.16. 1. Let A and B be abelian group objects in $\tau_{\le 0}(\mathcal{X})$. The **object of group homomorphisms** $\underline{\mathrm{Ab}}(A, B)$ is the following equalizer in $\mathcal{X}$:

$$\underline{\mathrm{Ab}}(A,B) \dashrightarrow B^A \underset{f \longmapsto f\circ(+_A)}{\overset{f\longmapsto f+_B f}{\rightrightarrows}} B^{A\times A}.$$

2. Let R be a ring in $\tau_{\le 0}(\mathcal{X})$, and let A and B be two R-modules. Write $\alpha_X : R \times X \to X$ for the R-action on an R-module X. The **object of R-module morphisms** R-$\underline{\mathrm{Mod}}(A, B)$ is the following equalizer in $\mathcal{X}$:

$$R\text{-}\underline{\mathrm{Mod}}(A,B) \dashrightarrow \underline{\mathrm{Ab}}(A,B) \underset{f\longmapsto \alpha_B(\mathrm{id}_R\times f)}{\overset{f\longmapsto f\circ\alpha_A}{\rightrightarrows}} B^{R\times A}.$$

It is not hard to see, using an argument similar to the proof of Lemma 2.4.14, that the global points of R-$\underline{\mathrm{Mod}}(A, B)$ are actual R-module homomorphisms from A to B. In addition, the object $R\text{-}\mathtt{Mod}(A, B)$ coming from interpretation is equivalent to R-$\underline{\mathrm{Mod}}(A, B)$, since it interprets to the same equalizer.

Theorem 2.4.17. *Let R be a ring in $\tau_{\le 0}(\mathcal{X}_\kappa)$. The Rezk $(1,1)$-object $R\text{-}\mathtt{Mod}_\bullet$ represents the presheaf*

$$X \longmapsto (X{\times}R)\text{-Mod} \ : \ \mathcal{X}^{\mathrm{op}} \longrightarrow \mathrm{Cat}$$

in the sense of Definition 2.4.3.

Proof. Let $X \in \mathcal{X}$. By our definition of representability, we need to produce a natural equivalence $\eta : \mathcal{X}(X, R\text{-}\mathtt{Mod}_\bullet) \simeq i_2^*(X{\times}R)\text{-Mod}$ of 2-restricted simplicial spaces. Proposition 2.4.15 gets us η_0.

For the first level, recall that $R\text{-}\mathtt{Mod}_1$ is the total space of $R\text{-}\mathtt{Mod}(-,-) : R\text{-}\mathtt{Mod}^2 \to \mathtt{Ab}$ in $\mathcal{X}$. By our discussion just above, applying $\mathcal{X}(X,-)$ to this family recovers the internal hom of $(X{\times}R)$-modules restricted to the groupoid core: Since the global points of the internal hom of modules recovers the external hom of modules, Lemma 2.4.10 gives a natural equivalence of spaces

$$\eta_1 \ : \ \mathcal{X}(X, R\text{-}\mathtt{Mod}_1) \ \simeq \ ((X{\times}R)\text{-Mod}^{[1]})^{\simeq}.$$

By construction, this equivalence respects the two projection maps sending a homomorphism to its domain and codomain. We also need to check that η_1 respects the degeneracy map

$$\mathrm{id} \ : \ \mathcal{X}(X, R\text{-}\mathtt{Mod}) \ \longrightarrow \ \mathcal{X}(X, R\text{-}\mathtt{Mod}_1)$$

which picks out the identity. This follows from the corresponding fact for sets, since id here is induced by the degeneracy $\mathcal{X}(X, \mathtt{Set}) \to \mathcal{X}(X, \mathtt{Set}_1)$ and equality of R-module homomorphisms can be checked on the underlying maps. We conclude that η_0 and η_1 assemble to a map of 1-restricted simplicial spaces.

For the second level, we have a candidate equivalence for $\eta_2 : \mathcal{X}(X, R\text{-}\mathtt{Mod}_2) \to (R\text{-Mod}^{[2]})^{\simeq}$ given by $\eta_1 \times_{\eta_0} \eta_1$ using the Segal condition and that $\mathcal{X}(X,-)$ preserves limits. By construction

η_2 respects the two face maps δ^2_0 and δ^2_2, since these are just pullback projections. In addition, η_2 respects the two degeneracy maps since these are induced by id above, and η_1 respects id. Finally, we need to check that η_2 respects composition. But composition of R-module homomorphisms is defined by composing the underlying maps, and since we can check equality of R-module homomorphisms on the underlying maps, this follows from the corresponding statement for sets.

We conclude that η defines a natural equivalence of 2-restricted simplicial objects. □

Finally, we explain the semantics of Theorem 2.3.19 and Corollary 2.3.18 for module categories. To any object X in $\mathcal{X}$ (more generally, any morphism) we have the usual sequence of adjoints

$$\Sigma_X \dashv X \times (-) \dashv \Pi_X.$$

The right adjoints automatically lift to categories of modules, being left-exact. By the internal cocompleteness of categories of modules, we have a corresponding leftmost adjoint colim_X. By Corollary 2.3.18, colim_X preserves internal products, and Theorem 2.3.19 implies that it is internally left-exact whenever X is 0-truncated. On global points, we deduce the following:

Theorem 2.4.18. *Let R be a ring object in $\tau_{\leq 0}(\mathcal{X}_\kappa)$, and let $X \in \mathcal{X}$. We have an adjunction:*

$$\mathrm{colim}_X \ : \ (X{\times}R)\text{-Mod} \leftrightarrows R\text{-Mod} \ : \ X \times (-)$$

where colim_X preserves finite products. If X is 0-truncated, then $\bigoplus_X \equiv \mathrm{colim}_X$ is left-exact.

We emphasize that this is an *external* statement about ordinary categories, and left-exactness refers to preservation of finite limits in the usual (external) sense.

Proof. From Theorem 2.3.19 we get an adjunction between Rezk $(1,1)$-objects in $\mathcal{X}$

$$\mathtt{colim}_X \ : \ R\text{-}\mathtt{Mod}^X \leftrightarrows R\text{-}\mathtt{Mod} \ : \ \mathtt{const}_X,$$

which yields an adjunction on global points. Using the previous theorem and that $\mathtt{const}_X$ corresponds to base change on global points, we obtain the desired adjunction $\mathrm{colim}_X \dashv X{\times}(-)$.

We now argue that colim_X preserves finite products. For any (external) natural number n, the product of an n-element family $A : n \to (X{\times}R)\text{-Mod}$ can be computed as the (internal) limit of the internalization $\underline{A} : \ell_*(n) \to R\text{-}\mathtt{Mod}^X$, by Proposition 2.4.9. The object $\ell_*(n)$ is the standard n-element set in $\mathcal{X}$, which is internally finite (indeed, equivalent to the object $\mathtt{Fin}(n)$). Hence $\mathtt{colim}_X$ preserves the (internal) limit of $\underline{A}$ by Corollary 2.3.18. From this we deduce a sequence of equivalences

$$\lim_{i:n} \mathrm{colim}_X A(i) \ \simeq \ \mathtt{lim}_{i:\ell_*(n)} \mathtt{colim}_X \underline{A}(i) \ \simeq \ \mathtt{colim}_X \mathtt{lim}_{\ell_*(n)} \underline{A} \ \simeq \ \mathrm{colim}_X \lim_n A$$

where the first and third equivalences use Proposition 2.4.9 for limits, and that colim_X is given by $\mathtt{colim}_X$ on global points (as shown at the beginning of this proof). Thus colim_X preserves finite products.

Now suppose that X is a set. To see that $\bigoplus_X$ is left-exact it suffices to show that it equalizers, since we already know that it preserves finite products. Applying ℓ_* to an equalizer diagram in $(X{\times}R)$-Mod produces an internal equalizer diagram in $R\text{-}\mathtt{Mod}^X$. The claim then follows by an argument similar to the one above, using that $\mathtt{colim}_X$ respects internal equalizers when X is a set by Theorem 2.3.19. □

We end by discussing the relation of this theorem to [Har82, Theorem 2.7].

Remark 2.4.19. Harting's construction of the left-exact coproduct applies to any elementary 1-topos with $\mathbb{N}$. The 1-topos $\tau_{\leq 0}(\mathcal{X}_\kappa)$ is, in particular, an elementary topos, hence Theorem 2.7 of loc. cit. implies the 0-truncated and $R \equiv \mathbb{Z}$ case of our theorem above. (The case for general R is covered by [Ble18, Proposition 3.7] for modules.) Conversely, a Grothendieck 1-topos $\mathrm{Sh}_0(C)$ is equivalent to the 0-truncated fragment of the ∞-topos $\mathrm{Sh}_\infty(C)$ of ∞-sheaves on the same site (see Proposition 2.4.13). Consequently, we recover Harting's theorem for $\mathrm{Sh}_0(C)$ by applying our theorem to $\mathrm{Sh}_\infty(C)$.

In [Har82], the construction of the internal coproduct of abelian groups occupies almost 60 pages. The length is largely due to the at the time underdeveloped state of the internal language of an elementary 1-topos. However, once the construction was complete, left-exactness followed by general results in [Joh77]. By contrast, our generalized construction is essentially contained in Section 2.3.3, which weighs in at just over 2 pages. The analogues of the general results of [Joh77] in our setting (or at least the parts we needed) are embodied by Proposition 2.4.9 and the theory about filtered colimits that we developed in Section 2.2.3.

Chapter 3

Formalising Yoneda Ext in univalent foundations

Abstract. Ext groups are fundamental objects from homological algebra which underlie important computations in homotopy theory. We formalise the theory of Yoneda Ext [Yon54] in homotopy type theory (HoTT) using the Coq-HoTT library [CH]. This is an approach to Ext which does not require projective or injective resolutions, though it produces large abelian groups. Using univalence, we show how these Ext groups can be naturally represented in HoTT. We give a novel proof and formalisation of the usual six-term exact sequence via a fibre sequence of 1-types (or groupoids), along with an application. In addition, we discuss our formalisation of the contravariant long exact sequence of Ext, an important computational tool. Along the way we implement and explain the Baer sum of extensions and how Ext is a bifunctor.

3.1 Introduction

The field of homotopy type theory (HoTT) lies at the intersection of type theory and algebraic topology, and serves as a bridge to transfer tools and insights from one domain to the other. In one direction, the formalism of type theory has proven to be a powerful language for reasoning about some of the highly coherent structures occurring in branches of modern algebraic topology. Several of these structures are "natively supported" by HoTT, and we can reason about them much more directly than in classical set-based approaches. This makes HoTT an ideal language in which to formalise results and structures from algebraic topology. Moreover, theorems in HoTT are valid in any ∞-*topos*, not just for ordinary spaces. Details about the interpretation of our constructions into an ∞-topos, and the relation of our Ext groups to *sheaf Ext*, are covered in Section 4.3.

We present a formalisation of Ext groups in HoTT following the approach Yoneda developed in [Yon54, Yon60]. Ext groups are fundamental objects in homological algebra, and they permeate computations in homotopy theory. For example, the universal coefficient theorem relates Ext groups and cohomology, and features in the classical proof that $\pi_5(S^3) \simeq \mathbb{Z}/2$. Much of our formalisation has already been accepted into the Coq-HoTT library under the `Algebra.AbSES` namespace, though we have also contributed to other parts of the library throughout this project. The long exact sequence, along with a few other results we need, are currently in a separate repository named `Yoneda-Ext`. We supply links to formalised statements using a trailing $^\diamond$-sign throughout.

In ordinary mathematics, Ext groups of modules over a ring are usually defined using projective (or injective) resolutions. This is possible because the axiom of choice implies the existence of such projective resolutions, and Ext groups are independent of any particular choice of resolution. (Similarly, categories of sheaves of modules always admit injective resolutions.) In our setting, however, even abelian groups fail to admit projective resolutions. This stems from the fact that some sets fail to be projective, which may be familiar to those working constructively or internally to a topos. Accordingly, to define Ext groups in homotopy type theory we cannot rely on resolutions. Fortunately, Yoneda [Yon54, Yon60] gave such a general approach, whose theory is detailed in [Mac63], our main reference. A drawback of this approach is that it produces *large* abelian groups, as we explain in Section 3.3.1.

We build upon the Coq-HoTT library [CH], which contains sophisticated homotopy-theoretic results, but which is presently lacking in terms of "basic" algebra. For this reason, we have opted to simply develop Ext groups of abelian groups, instead of for modules over a ring or in a more general setup. Nevertheless, it is clear that everything we do could have been done over an arbitrary ring, given a well-developed library of module theory. Moreover, we emphasise that higher Ext groups in HoTT are interesting even for abelian groups. While in classical mathematics such Ext groups of abelian groups are trivial in dimension 2 and up, in HoTT they may be nontrivial in all dimensions! This is because there are models of HoTT in which these Ext groups are nontrivial—this is explained in Section 4.3.

In Section 3.3 we explain how univalence lets us naturally represent Yoneda's approach to Ext in HoTT. We construct the type $\mathtt{AbSES}(B,A)$ of short exact sequences between two abelian groups A and B, and define $\mathtt{Ext}^1(B,A)$ to be the set of path-components of $\mathtt{AbSES}(B,A)$. This definition is justified by characterising the paths in $\mathtt{AbSES}(B,A)$, which crucially uses univalence. We also show that the loop space of $\mathtt{AbSES}(B,A)$ is isomorphic to the group $\mathtt{Hom}(B,A)$ of group homomorphisms, and that $\mathtt{Ext}^1(P,A)$ vanishes whenever P is projective, in a sense we define. These results all play a role in the subsequent sections.

The main content of Section 3.4 is a proof and formalisation of the following:

Theorem 3.4.1. *Let $A \xrightarrow{i} E \xrightarrow{p} B$ be a short exact sequence of abelian groups. For any abelian group G, pullback yields a fibre sequence:* $\mathtt{AbSES}(B,G) \xrightarrow{p^*} \mathtt{AbSES}(E,G) \xrightarrow{i^*} \mathtt{AbSES}(A,G)$.$^\diamond$

We give a novel, direct proof of this result which requires managing considerable amounts of coherence. The formalisation is done for abelian groups, but the proof applies to modules over a general ring. Its formalisation benefited from the `WildCat` library of Coq-HoTT (see Section 3.2.2), which makes it convenient to work with types equipped with an imposed notion of paths. This allows us to work with *path data* in $\mathtt{AbSES}(B,A)$ with better computational properties than actual paths, but which correspond to paths via the aforementioned characterisation. From the fibre sequence of the theorem we deduce the usual six-term exact sequence (Proposition 3.4.7), which we then use to compute Ext groups of cyclic groups:

$$\mathtt{Ext}^1(\mathbb{Z}/n, A) \;\cong\; A/n$$

for any nonzero $n : \mathbb{N}$ and abelian group A (Corollary 3.4.9).$^{\diamond}$ The six-term exact sequence, along with this corollary, have already been applied in [BCFR23] (Theorem 5.5.13). We also discuss how $\mathtt{Ext}^1$ becomes a bifunctor into abelian groups using the Baer sum.

Finally, in Section 3.5 we define $\mathtt{Ext}^n$ for any $n : \mathbb{N}$ and discuss our formalisation of the long exact sequence, in which the connecting maps are given by *splicing*:$^{\diamond}$

Theorem 3.5.5. *Let $A \xrightarrow{i} E \xrightarrow{p} B$ be a short exact sequence of abelian groups. For any abelian group G, there is a long exact sequence by pulling back:*$^{\diamond\ \diamond\ \diamond}$

$$\cdots \xrightarrow{i^*} \mathtt{Ext}^n(A,G) \xrightarrow{-\circledcirc E} \mathtt{Ext}^{n+1}(B,G) \xrightarrow{p^*} \mathtt{Ext}^{n+1}(E,G) \xrightarrow{i^*} \cdots .$$

At present, we have only formalised this long exact sequence of *pointed sets*. It remains to construct the Baer sum making $\mathtt{Ext}^n$ into an abelian group for $n > 1$, however once this is done then we automatically get a long exact sequence of abelian groups. Our proof follows that of Theorem 5.1 in [Mac63], which is originally due to Stephen Schanuel.

Notation and conventions. We use typewriter font for concepts which are defined in the code, such as `AbSES` and `Ext`. In contrast, when we use normal mathematical font, such as $\mathsf{Ext}^n(B,A)$, we mean the classical notion. For mathematical statements we prefer to stay close to mathematical notation by writing for example $\mathtt{Ext}^n(B,A)$ for what means `Ext n B A` in Coq. The symbol $^{\diamond}$ is used to refer to relevant parts of the code.

Our terminology mirrors that of [Uni13]; in particular we say 'path types' for what are also called 'identity types' or 'equality types'. We write $\mathcal{U}_*$ for the universe of pointed types, and `pt` for the base point of a pointed type. The $\equiv$-symbol is for definitional equality.

3.2 Preliminaries

3.2.1 Homotopy type theory

We briefly explain the formal setup of homotopy type theory along with some basic notions that we need. For a thorough introduction to HoTT, the reader may consult [Uni13, Rij23].

Homotopy type theory (HoTT) extends Martin–Löf type theory (MLTT) with the *univalence axiom* and often various higher inductive types (HITs). Of the latter, we simply need propositional truncation and set truncation, which we explain in more detail below.

The univalence axiom characterises the identity types of universes. In ordinary MLTT, there is always a function

$$\mathtt{idtoequiv} : \prod_{X,Y:\mathcal{U}} (X = Y) \to (X \simeq Y)$$

defined by sending the reflexivity path on a type X to the identity self-equivalence on X, using the induction principle of path types. The univalence axiom asserts that `idtoequiv` is an equivalence for all X and Y. In HoTT, the first thing we often do after defining a new type is to characterise its path types. The univalence axiom does this for the universe.

From univalence, a general *structure identity principle* [Uni13, Chapter 9.8] follows which characterises paths between structured types, such as groups and other algebraic structures. In the case of groups, univalence implies that paths between groups correspond to group isomorphisms. Similarly, paths between modules correspond to module isomorphisms.

Propositions, sets, and groupoids. In HoTT there is a hierarchy of ***n*-truncated types** (or ***n*-types**, for short) for any integer $n \geq -2$. In general, a type X is an $(n+1)$-type when all the path types $x_0 =_X x_1$ are n-types. The recursion starts at -2, when the condition is just that the map $X \to 1$ is an equivalence, and in this case X is **contractible**.

We only deal with the bottom four levels of this hierarchy: **contractible types**, **propositions** ((-1)-types), **sets** (0-types) and 1-types. A type X is a proposition when any two points in X are equal (but there may not be any points). A type X is a set when the path types $x_0 =_X x_1$ are all propositions—this amounts to there being "at most" one path between x_0 and x_1. Lastly, a type X is a 1-type when its path types are sets—in particular, for any $x : X$, the **loop space** $\Omega X :\equiv (x =_X x)$ is a set which is a group under path composition. (We leave base points implicit when taking loop spaces.)

There are truncation operations which create a proposition or a set from a given type X. We denote by $\|X\|$ the propositional truncation, and by $\pi_0 X$ the set truncation (or *set of path-components*) of X. In Coq-HoTT, the corresponding notation is `merely X` and `Tr 0 X`. The map `tr` $: X \to \pi_0 X$ sends a point to its connected component. When we say that a type X **merely** holds, then we mean that its propositional truncation $\|X\|$ holds.

3.2.2 The Coq-HoTT library

The Coq-HoTT library [CH] is an open-source repository of formalised mathematics in homotopy type theory using Coq. It is particularly aimed at developing synthetic homotopy theory, and includes theory about spheres, loop spaces, classifying spaces, modalities, "wild ∞-categories," and basic results about abelian groups, to mention a few things. The library is part of the *Coq Platform* and is available through the standard `opam` package repositories.

Below we explain some of the main features of this library, and of Coq itself, which are important for the present work.

Universes and cumulativity. We assume basic familiarity with universes and universe levels in Coq, and in particular that they are *cumulative*: a type X : `Type@{u}` can be resized to live in `Type@{v}` under the constraint `u` $\leq$ `v`. (Here `u` and `v` are **universe levels**.) Resizing is done implicitly by Coq.

In the Coq-HoTT library, we additionally make most of our structures cumulative. This essentially means that resizing commutes with the formation of a data structure—i.e., it does not matter whether you resize the inputs to the data structure or whether you resize the resulting data structure. As an example, consider the data structure `prod` which forms the product of two types in a common (for simplicity) universe level. Suppose we have two universe levels `u` and `v` with the constraint `u < v`. Given `X Y : Type@{u}`, we can form the product at level `u` and then resize, or first resize and then form the product. By making `prod` a cumulative data structure, the two results agree (with implicit resizing):

$$\texttt{prod@\{u\} X Y} \equiv \texttt{prod@\{v\} X Y}.$$

Cumulativity of data structures is an essential Coq feature which facilitates the kind of formalisation we do in this paper. For example, it lets us resize groups and homomorphisms. It also lets us reduce the number of universes in some of our definitions via the following trick: instead of having separate universes for different inputs, we can often use a single universe (which represents the maximum) and leverage cumulativity.

We also make use of universe constraints since our constructions move between various universe levels. The constraints both document and verify the mathematical intent.

The `WildCat` library. The `WildCat` namespace contains the development of "wild ∞-categories," functors between such, and related things. This library was spearheaded by Ali Caglayan, tslil clingman, Floris van Doorn, Morgan Opie, Mike Shulman, and Emily Riehl. The concepts generalise those appearing in [vDoo18, Section 4.3.1], and are not currently present in the literature. We explain the basics of this library which are especially relevant for our formalisation.

Starting from the notion of **graph**◊—a type A with a binary operation (or *correspondence*) `Hom` into $\mathcal{U}$—the notion of a **0-functor**◊ is that of a homomorphism of graphs:

```
Class IsGraph (A : Type) := { Hom : A -> A -> Type }.
Class Is0Functor {A B : Type} `{IsGraph A} `{IsGraph B} (F : A -> B)
  := { fmap : forall {a b : A} (f : Hom a b), Hom (F a) (F b) }.
```

We will often use the notation `Hom` in this text, leaving the graph structure implicit.

From here one could go ahead and define categories by defining a composition operation and using the identity types of the type $\mathtt{Hom}(a,b)$ to express the various laws a category needs to satisfy, such as associativity of composition. A more flexible approach is to instead allow $\mathtt{Hom}(a,b)$ to itself be a graph, making A into a **2-graph**.◊ This is the approach taken by `WildCat`, and this flexibility is important for our formalisation.

```
Class Is2Graph (A : Type) `{IsGraph A}
  := { isgraph_hom : forall (a b : A), IsGraph (Hom a b) }.
```

For a 2-graph A, a category structure can then be defined in a straightforward manner using `isgraph_hom` to express the various laws that need to hold. This structure is bundled into a class called `Is1Cat`.◊ For example, associativity is expressed as follows, using the notation `$==` as a shorthand for the 2-graph structure and `$o` for composition:

```
  cat_assoc : forall (a b c d : A)
    (f : Hom a b) (g : Hom b c) (h : Hom c d),
      (h $o g) $o f $== h $o (g $o f);
```

If all the morphisms in A are invertible, then A is a **groupoid**.◊ Finally, for the notion of a **1-functor** between categories we also express the laws using the 2-graph structure.◊

```
Class Is1Functor {A B : Type} `{Is1Cat A} `{Is1Cat B}
  (F : A -> B) `{!Is0Functor F} := {
```

```
    fmap_id : forall a, fmap F (Id a) $== Id (F a);
    fmap_comp : forall a b c (f : Hom a b) (g : Hom b c),
      fmap F (g $o f) $== fmap F g $o fmap F f;
    fmap2 : forall a b (f g : Hom a b),
      (f $== g) -> (fmap F f $== fmap F g) }.
```

The terms `fmap_id` and `fmap_comp` express that the functor `F` respects identities and composition, as usual. If we had used identity types instead of a 2-graph structure, so that `f $== g` simply meant `f = g`, then `F` would automatically respect equality between morphisms, making `fmap2` redundant. However, in the more general 2-graph setup, this needs to be included as a law.

The adjective "wild" is used for the sort of categories just defined to indicate that they do not capture all the coherence needed to represent ∞-categories, only the 1-categorical structure. However, in our usage we will only encounter genuine 1-categories and groupoids. In particular, any type X defines a groupoid via its identity types$^\diamond$, and if X is a 1-type then this groupoid structure captures everything about X. This enables us to impose our own notion of paths, which we call *path data* below, for certain types of interest.

3.3 Yoneda Ext

As mentioned in the introduction, we will follow Yoneda's approach to Ext groups [Yon54, Yon60], which does not require projective (or injective) resolutions, though it produces *large* groups. This approach and related theory is explained in [Mac63], which is our main reference. At present, the Coq-HoTT library—with which this work has been formalised—does not contain much theory related to modules over a general ring (nor the theory of abelian categories, or anything of the sort). We therefore only formalise and state our results for abelian groups. It is clear, however, that everything we say could be done for modules over a general ring.

For the classically-minded reader, let us also emphasise that in homotopy type theory the category of abelian groups does *not* have global dimension 1, so that the higher Ext groups we define in Section 3.5 do not necessarily vanish.

3.3.1 The type of short exact sequences

Given two abelian groups A and B, Yoneda defines a group $\mathsf{Ext}^1(B, A)$ by considering the large set (or class) of all short exact sequences $A \xrightarrow{i} E \xrightarrow{p} B$ and taking a quotient by a certain equivalence relation. The sequence being exact means that i is injective, p is surjective, that $p \circ i = 0$, and that the image of i is equal to the kernel of p. We usually simply write E for the short exact sequence $A \to E \to B$ when no confusion can arise. The equivalence relation which Yoneda quotients out by is defined as "$E \sim F$ if and only if there exists an isomorphism $E \cong F$ which respects the maps from A and to B." Equivalently, but more topologically, one can consider the *groupoid* of short exact sequences $A \to E \to B$ and define $\mathsf{Ext}^1(B, A)$ to be the set of path-components of this groupoid—see, e.g., [Mac63, Chapter III] for details about both of these descriptions.

In homotopy type theory, given two abelian groups A and B we form the **type of short exact sequences** from A to B as the Σ-type over all abelian groups E equipped with an injection $\mathtt{inclusion}_E : A \to E$, a surjection $\mathtt{projection}_E : E \to B$, and a witness that these two maps form an exact complex. We represent this data as the following record-type:◊

```
Record AbSES@{u v | u < v} (B A : AbGroup@{u}) : Type@{v} := {
    middle : AbGroup@{u};
    inclusion : Hom A middle;
    projection : Hom middle B;
    isembedding_inclusion : IsEmbedding inclusion;
    issurjection_projection : IsSurjection projection;
    isexact_inclusion_projection
      : IsExact (Tr (-1)) inclusion projection;
  }.
```

Note that $\mathtt{AbSES}(B, A)$ denotes short exact sequences *from A to B*. The abelian group `middle` plays the role of E in the prose above. Here, the condition that $\mathtt{projection}_E \circ \mathtt{inclusion}_E = 0$ is baked into the `IsExact` field, which also expresses exactness.[1] We have included universe annotations which express that E lives in the same universe `u` as the abelian groups A and B. Accordingly, the resulting type $\mathtt{AbSES}(B, A)$ lives in a universe `v` which is strictly greater than `u`, as in Yoneda's construction above. The type $\mathtt{AbSES}(B, A)$ is pointed by the **trivial short exact sequence**◊ $A \to A \oplus B \to B$.

We now define $\mathtt{Ext}^1(B, A)$ as the set-truncation of the type of short exact sequences.◊

```
Definition Ext (B A : AbGroup) := Tr 0 (AbSES B A).
```

In Section 3.3.3 we make the set $\mathtt{Ext}^1(B, A)$ into an abelian group via the *Baer sum*. These abelian groups, and their higher variants defined in Section 3.5, are our main objects of study.

Whenever we define a new type in homotopy type theory, the first thing we often do is to characterise its path types. Theorem 7.3.12 of [Uni13] characterises paths in truncations, yielding

$$(|E|_0 =_{\mathtt{Ext}^1} |F|_0) \;\simeq\; \|E = F\|$$

for any $E, F : \mathtt{AbSES}(B, A)$. As such, it suffices to understand paths in $\mathtt{AbSES}(B, A)$. These are in turn characterised by Theorem 2.7.2 of loc. cit., which characterises paths in general Σ-types, combined with the fact that paths in `AbGroup` are isomorphisms. In our case, the result is that paths between short exact sequences correspond to isomorphisms between the `middle`s making the appropriate triangles commute. We refer to this data as *path data*, and bundle it into a separate type (where * denotes products of types):◊

```
Definition abses_path_data_iso {B A : AbGroup} (E F : AbSES B A)
  := {phi : Iso E F & (phi $o inclusion E == inclusion F)
                    * (projection E == projection F $o phi)}.
```

Here `Iso` forms the type of isomorphisms between two groups. From our discussion above, for any $E, F : \mathtt{AbSES}(B, A)$, we get an equivalence of types◊

$$(E =_{\mathtt{AbSES}(B,A)} F) \;\simeq\; \mathtt{abses\_path\_data\_iso}(E, F).$$

[1] The term `Tr (-1)` can safely be ignored; it expresses that the induced map from A to the kernel of $\mathtt{projection}_E$ is (-1)-*connected*, which here just means it is a surjection.

However, a bit more can be said: the *short five lemma*◇ implies that if we replace `Iso` by `Hom` above, then it still follows that `phi` is an isomorphism. We define `abses_path_data`◇ as `abses_path_data_iso` above, but with `Hom` in place of `Iso`. It is convenient to have both types around: it is easier to construct an element of `abses_path_data`; however we will see situations later on where it is convenient to keep track of a *specific* inverse to the underlying map, which `abses_path_data_iso` lets us do.

Definition 3.3.1. *The correspondence* `abses_path_data_iso` *makes the type* $\mathtt{AbSES}(B, A)$ *into a graph with a corresponding category structure. For the* 2*-graph structure, we assert that two path data are equal just when their underlying maps are homotopic.*◇

This definition is justified by the preceding discussion, which yields:

Lemma 3.3.2. *For any* $E, F : \mathtt{AbSES}(B, A)$*, there are equivalences of types*◇

$$(E = F) \simeq \mathtt{abses\_path\_data\_iso}(E, F) \simeq \mathtt{abses\_path\_data}(E, F).$$

Though elementary, this lemma has an interesting consequence. This statement appears as the $n, i = 1$ case of [Ret86, Theorem 1].

Proposition 3.3.3. *The loop space of* $\mathtt{AbSES}(B, A)$ *is naturally isomorphic to* $\mathtt{Hom}(B, A)$.◇

Proof. It suffices, by the previous lemma, to give an isomorphism between $\mathtt{Hom}(B, A)$ and $\mathtt{abses\_path\_data}(A \oplus B, A \oplus B)$. One can easily check that a map $\phi : A \oplus B \to A \oplus B$ subject to the constraints of path data, is uniquely determined by the composite◇

$$B \to A \oplus B \xrightarrow{\phi} A \oplus B \to A.$$

Moreover, this association defines a group isomorphism—details are in the formalisation.◇ □

To formalise the previous proposition, we first developed basic theory about biproducts of abelian groups which now live in `Algebra.AbGroups.Biproduct`.

In ordinary homological algebra, an abelian group P is *projective* if for any homomorphism $f : P \to B$ and epimorphism $p : A \to B$, there exists a *lift* $l : P \to A$ such that $f = e \circ l$. It is well-known that $\mathtt{Ext}^1(P, A)$ always vanishes when P is projective, and that this property characterises projectivity. In our setting, we define an abelian group P to be **projective** if for any homomorphism f and epimorphism p as above, there *merely* exists a lift l such that $f = l \circ l$. The propositional truncation makes this into a *property* of an abelian group, and not a structure. In Coq, we express this as a type-class:◇

```
Class IsAbProjective (P : AbGroup) : Type :=
  isabprojective : forall (A B : AbGroup),
    forall (f : Hom P B), forall (e : Hom A B),
    IsSurjection e -> merely (exists l : P $-> A, f == e $o l).
```

As in the classical case, projectives are characterised by the vanishing of Ext:

Proposition 3.3.4. *An abelian group* P *is projective if and only if* $\mathtt{Ext}^1(P, A) = 0$ *for all* A.◇

From the induction principle of $\mathbb{Z}$ it follows that $\mathbb{Z}$ is projective$^{\diamond}$ in the sense we defined above. Consequently $\mathtt{Ext}^1(\mathbb{Z}, A) = 0$ for any abelian group A, and we will use this later on.

Remark 3.3.5. There is a subtle point related to projectivity that merits discussion. Our definition of projectivity only requires the lift l to *merely* exist (a property), but one could have asked for actual existence (a structure). There is no concept of "mere existence" in ordinary mathematics, and when translating concepts into HoTT we have to carefully choose to make something a structure or a property. In this case, our definition of projectivity is justified by Proposition 3.3.4. If we had made projectivity a structure, then not even $\mathbb{Z}$ would be projective, which we need it to be.

3.3.2 Ext as a bifunctor

Some of the important structure of Ext^1 is captured by the fact that it defines a *bifunctor* $\mathsf{Ext}^1(-,-) : \mathsf{Ab}^{\mathrm{op}} \times \mathsf{Ab} \to \mathsf{Ab}$. This means that $\mathsf{Ext}^1(-,-)$ is a functor in each variable and that the following "bifunctor law" holds:

$$\mathsf{Ext}^1(f,-) \circ \mathsf{Ext}^1(-,g) = \mathsf{Ext}^1(-,g) \circ \mathsf{Ext}^1(f,-). \tag{3.1}$$

We added a basic implementation of bifunctors to the `WildCat` library for our purposes, asserting the bifunctor law using the 2-graph structure:$^{\diamond}$

```
Class IsBifunctor {A B C : Type} `{IsGraph A, IsGraph B, Is1Cat C}
  (F : A -> B -> C) := {
    bifunctor_isfunctor_10 : forall a, Is0Functor (F a);
    bifunctor_isfunctor_01 : forall b, Is0Functor (fun a => F a b);
    bifunctor_isbifunctor :
      forall a0 a1 (f : Hom a0 a1), forall b0 b1 (g : Hom b0 b1),
        fmap (F _) g $o fmap (flip F _) f
          $== fmap (flip F _) f $o fmap (F _) g }.
```

Here `flip` is the map which reverses the order of arguments of a binary function. We note that in order to state the bifunctor law, we only require `F` to be a 0-functor in each variable. As such we only include those instances in this class.

The bifunctor instance of $\mathtt{Ext}^1$ will come from a bifunctor instance of `AbSES`, so we work with the latter. First of all, `AbSES` : $\mathtt{AbGroup}^{\mathrm{op}} \to \mathtt{AbGroup} \to \mathtt{Type}$ becomes a 0-functor in each variable by pulling back and pushing out, respectively.

Lemma 3.3.6. *Let $g : B' \to B$ be a homomorphism of abelian groups. For any short exact sequence $A \to E \to B$, we have a short exact sequence $A \to g^*(E) \to B'$.$^{\diamond}$ Moreover, if E is trivial, then so is the short exact sequence $g^*(E)$.$^{\diamond}$*

Dually, one can push out a short exact sequence $A \to E \to B$ along a map $f : A \to A'$ to get a short exact sequence $A' \to f_*(A) \to B$.$^{\diamond}$

We supply careful proofs that pushout and pullback respect composition of pointed maps$^{\diamond}$ and homotopies between maps,$^{\diamond}$ and that pushing out along the identity map gives the pointed identity map.$^{\diamond}$ These identities could be shown with shorter proofs, however in Section 3.4 we will have to prove coherences involving the paths constructed here, and these coherences are simpler to solve when phrased in terms of path data. In any case, these proofs make `AbSES` into a 1-functor in each variable.◇◇

For the bifunctor law we make use of the following proposition, which is remarkably useful for showing that a given extension is a pullback of another one.

Proposition 3.3.7. *Suppose given the following diagram with short exact rows:*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha"] & E' \arrow[r] \arrow[d] & B' \arrow[d, "g"] \\ A \arrow[r] & E \arrow[r] & B\,. \end{tikzcd}$$

If $\alpha = \mathrm{id}$ *then the top row is equal to the pullback of the bottom row along* g.◇

Proof. Since the right square commutes, we get a map $E' \to g^*(E)$ by the universal property of the pullback. This map respects the inclusions and projections, and therefore defines a path by Lemma 3.3.2. □

There is a dual statement for pushouts in which the rightmost map must be the identity.◇

Corollary 3.3.8. *Any diagram with short exact rows as follows yields a path* $f_*(E) = g^*(F)$.◇

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & E \arrow[r] \arrow[d] & B' \arrow[d, "g"] \\ A' \arrow[r] & F \arrow[r] & B\,. \end{tikzcd}$$

The corollary lets us swiftly show bifunctoriality:

Proposition 3.3.9. *The binary map* `AbSES` : $\texttt{AbGroup}^{\mathrm{op}} \to \texttt{AbGroup} \to \mathcal{U}$ *is a bifunctor.*◇

Proof. Consider a short exact sequence $A \to E \to B$ and two homomorphisms $f : A \to A'$ and $g : B' \to B$. There is an obvious diagram with short exact rows:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & g^*(E) \arrow[r] \arrow[d] & B' \arrow[d, "g"] \\ A' \arrow[r] & f_*(E) \arrow[r] & B\,. \end{tikzcd}$$

which by the previous corollary yields a path $f_*(g^*(E)) = g^*(f_*(E))$, as required. □

Remark 3.3.10. The results from Section 3.3.3 will show that `AbSES` is an *H-space*.◇ Combining this with Lemma 5.2.7◇, we deduce that `AbSES` is a bifunctor into pointed types. This does not play a role in the rest of this paper, however.

3.3.3 The Baer sum

The *Baer sum* is a binary operation on $\mathsf{Ext}^1(B, A)$ which makes it into an abelian group. Given two extensions $E, F : \mathsf{Ext}^1(B, A)$ their Baer sum is defined as

$$E + F :\equiv \Delta^* \nabla_* (E \oplus F)$$

where $E \oplus F$ is the point-wise direct sum, $\nabla(a, b) :\equiv a_0 + a_1 : A \oplus A \to A$ is the codiagonal map, and $\Delta(b) :\equiv (b, b) : B \to B \oplus B$ is the diagonal map.

Together with Dan Christensen and Jacob Ender, we have implemented the Baer sum in `Algebra.AbSES.BaerSum`. We define this operation on the level of short exact sequences and then descend the operation to the set `Ext`^{1} by truncation-recursion.◊

```
Definition abses_baer_sum `{Univalence} {B A : AbGroup}
  : AbSES B A -> ABSES B A AbSES B A := fun E F =>
      abses_pullback ab_diagonal
        (abses_pushout ab_codiagonal (abses_direct_sum E F)).

Definition baer_sum `{Univalence} {B A : AbGroup}
  : Ext B A -> Ext B A -> Ext B A.
Proof.
  intros E F; strip_truncations.
  exact (tr (abses_baer_sum E F)).
Defined.
```

Above, the `strip_truncations` tactic is a helper for doing truncation-recursion; it lets us assume that both E and F are elements of `AbSES`(B, A) in order to map into the set `Ext`$^1(B, A)$. We then simply form the Baer sum of E and F on the level of short exact sequences before applying `tr` to the result.

The formalisation that the Baer sum makes `Ext`$^1(B, A)$ into an abelian group closely follows the "second proof" of [Mac63, Theorem III.2.1].

Theorem 3.3.11. *The set* `Ext`$^1(B, A)$ *is an abelian group under the Baer sum operation.*◊

The proof can be done entirely by chaining together equations once the bifunctoriality of `Ext`^{1} has been established along with its interaction with direct sums. To illustrate this, we prove that pushouts respect the Baer sum:

Proposition 3.3.12. *Let $\alpha : A \to A'$ be a homomorphism of abelian groups. For any abelian group B, pushout defines a group homomorphism α_* :* `Ext`$^1(B, A) \to$ `Ext`$^1(B, A')$.◊

Proof. Using bifunctoriality of `Ext`^{1} and naturality of $\oplus$, we have:

$$\begin{aligned} \alpha_*(E + F) &= \Delta^*(\alpha_* \nabla_*(E \oplus F)) = \Delta^*(\nabla_*(\alpha_* \oplus \alpha_*)_*(E \oplus F)) \\ &= \Delta^*(\nabla_*(\alpha_* E \oplus \alpha_* F)) \equiv \alpha_* E + \alpha_* F. \end{aligned}$$ □

Similarly, pullback defines a group homomorphism as well.◊ These results make `Ext`^{1} into a bifunctor valued in abelian groups.◊

3.4 The pullback fibre sequence

The main goal of this section is to explain and prove the following mathematical result, and to discuss its formalisation$^\diamond$ along with some applications.

Theorem 3.4.1. *Let $A \xrightarrow{i} E \xrightarrow{p} B$ be a short exact sequence of abelian groups. For any abelian group G, pullback yields a fibre sequence:* $\mathtt{AbSES}(B, G) \xrightarrow{p^*} \mathtt{AbSES}(E, G) \xrightarrow{i^*} \mathtt{AbSES}(A, G)$.$^\diamond$

In Section 4.2.3, we give a different proof of this statement via an equivalence between $\mathtt{AbSES}(B, A)$ and pointed maps between Eilenberg–Mac Lane spaces. However, this different proof seems to only work over $\mathbb{Z}$ whereas our proof here works for a general ring (though it has only been formalised for $\mathbb{Z}$).

A sequence of pointed maps $F \xrightarrow{i} E \xrightarrow{p} B$ is a **fibre sequence** if $p \circ i$ is pointed-homotopic to the constant map, and the induced map $F \to \mathsf{fib}_p$ is an equivalence. Any fibre sequence induces a long exact sequence of homotopy groups [Uni13, Theorem 8.4.6]:

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \cdots \to \pi_0(F) \to \pi_0(E) \to \pi_0(B).$$

In the situation of our theorem, it is immediate from functoriality and exactness of E that $i^* \circ p^*$ is constant. Therefore our goal is to show that the induced map $c : \mathtt{AbSES}(B, G) \to \mathsf{fib}_{i^*}$ is an equivalence.[2] We will do this by first constructing a section of c, and then a contraction of the fibres of c to the values of this section. A key part of the formalisation is to work with path data instead of actual paths, since the former has better computational properties. We will simply use $E = F$ to denote path data, and refer to it as such, in this section.

Lemma 3.4.2. *Let $G \to F \to E$ be a short exact sequence. Given path data $p : i^*(F) = \mathtt{pt}$, we construct a short exact sequence $G \to F/A \to B$.*$^\diamond$

Proof. The path data p means that the sequence $i^*(F)$ splits. Thus we can form the cokernel F/A as in the diagram:

$$\begin{tikzcd}
 & i^*(F) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "i"] \arrow[dl, dashed] \\
G \arrow[ur, bend left] \arrow[r, "j"] \arrow[dr, dotted] & F \arrow[r] \arrow[d, dashed] & E \arrow[d, "p"] \\
 & F/A \arrow[r, dotted] & B\,.
\end{tikzcd}$$

The two maps $G \to F/A \to B$ are given by composition and the universal property of the cokernel, respectively. It is clear that this forms a complex and that the second map is an epimorphism, since it factors one. To see that the map $G \to F/A$ is an injection, suppose $g : G$ is sent to $0 : F/A$. Then $j(g)$ is in the image of some $a : A$ by $A \to F$. But the map $i^*(F) \to F$ is an injection, being the pullback of one, and so using the path data we get an equality $(g, 0) = (0, a)$ in $G \oplus A$. Of course, this implies that $g = 0$, as required.

Exactness of $G \to F/A \to B$ follows from a straightforward diagram chase. ∎

[2] The map c is called `cxfib` in the code.

The diagram above exhibits F as the pullback of F/A along p^*, yielding:

Lemma 3.4.3. *We have path data $q : p^*(F/A) = F$.*$^{\diamond}$

Thus we have given a preimage F/A of F under p^*. To show that the fibre of c is inhabited we will show that $c(F/A) = (F, p)$, which is a path in fib_{i^*}. We express all of this in terms of path data, and such a path in fib_{i^*} then corresponds to path data $q : p^*(F/A) = F$ which makes the following triangle commute:$^{\diamond}$

$$\begin{tikzcd} i^*p^*(F/A) \arrow[rr, "i^*(q)"] \arrow[dr] & & i^*(F) \arrow[dl, "p"] \\ & G \oplus A & \end{tikzcd} \tag{3.2}$$

where the rightmost map comes from i^*p^* being trivial. The key reason we have formulated things in terms of path data is so that the maps in the triangle above simply compute, because they have all been concretely constructed.

In the following, c refers to the map which lands in fib_{i^*} expressed in terms of path data.$^{\diamond}$

Lemma 3.4.4. *We have $q : c(F/A) = (F, p)$ in fib_{i^*}.*$^{\diamond}$

Proof. The previous lemma already yields path data $q : p^*(F/A) = F$, thus it remains to show that the triangle in Eq. (3.2) commutes. The way the maps have been constructed, it's easiest to show this after flipping the triangle so that it starts at $G \oplus A$ and ends at $i^*p^*(F/A)$. (This is fine since all the maps are isomorphisms.) Thus we are comparing two maps out of a biproduct into a pullback. To check whether they are equal, we can check it on each inclusion of the biproduct and after projecting out of the pullback. In each of these cases one obtains diagrams which commute, but checking this is somewhat involved. Fortunately, by our having carefully crafted the path data involved, the maps simply compute and Coq is able to reduce the goal to a simple computation. □

Combining the three previous lemmas, we get a section of $c : \mathtt{AbSES}(B, G) \to \mathsf{fib}_{i^*}$. To conclude that c is an equivalence, we contract each fibre over some (F, p) to $(F/A, q)$.

Lemma 3.4.5. *Suppose $G \to Y \to B$ is a short exact sequence, and let $q' : c(Y) = (F, p)$ in fib_{i^*}. Then $(F/A, q) = (Y, q')$ in the fibre of c over (F, p).*$^{\diamond}$

Proof. Under our assumptions, we have the composite map $\phi : G \oplus A \to i^*p^*(Y) \to p^*(Y)^{\diamond}$ which by a diagram chase can be seen to be the inclusion $G \to p^*(Y)$ on one component, and $(0, p) : A \to p^*(Y)$ on the other.$^{\diamond}$. Consequently, the composite $\mathtt{pr}_1 \circ \phi \circ \mathtt{in}_A : A \to Y$ is trivial. By the universal property of the cokernel, we get an induced map $F/A \to Y$. Once again, by our careful construction of all the maps involved, it is straightforward to simply compute that this map defines path data $F/A = Y$ and moreover that this path lifts to a path in the fibre of c. There is a coherence between three paths in $\mathtt{AbSES}(A, G)$ which is trivially satisfied, since $\mathtt{AbSES}(A, G)$ is a 1-type. □

The final lemma implies that the fibres of c are contractible, which means that c is an equivalence and concludes the proof of Theorem 3.4.1. We now turn our attention to two applications of this theorem. The first application requires a lemma.

Lemma 3.4.6. *Let $g : B' \to B$ be a homomorphism of abelian groups. For any A, the following diagram commutes, where the vertical isomorphisms are all given by Proposition 3.3.3:*◊

$$\begin{tikzcd}
\Omega\,\mathtt{AbSES}(B,A) \arrow[r, "\Omega(g^*)"] \arrow[d, "\sim"] & \Omega\,\mathtt{AbSES}(B',A) \arrow[d, "\sim"] \\
\mathtt{Hom}(B,A) \arrow[r, "\phi\mapsto\phi\circ g"] & \mathtt{Hom}(B',A).
\end{tikzcd} \tag{3.3}$$

Proof. Let $p : A \oplus B = A \oplus B$ be an element of the upper left corner, seen as path data. By path induction, one can easily show that the action of $\Omega(g^*)$ on paths is given by pulling back the path data. (Formally, one first proves this for paths with free endpoints, then you can specialise to loops.) This means that the following diagram commutes

$$\begin{tikzcd}
B' \arrow[r] \arrow[d, equal] & A \oplus B' \arrow[r, "\Omega(g^*)(p)"] \arrow[d, "\mathrm{id}\oplus g"] & A \oplus B' \arrow[r] \arrow[d, "\mathrm{id}\oplus g"] & A \arrow[d, equal] \\
B' \arrow[r, "{(0,g)}"] & A \oplus B \arrow[r, "p"] & A \oplus B \arrow[r] & A
\end{tikzcd}$$

where we have used the functions underlying the path data p and $\Omega(g^*)(p)$, and the unlabeled arrows are the natural ones into or out of a biproduct. The composites of the top and bottom rows above are the results of sending p around the top-right and bottom-left corners of Diagram 3.3, respectively. Since this latter diagram commutes, so does Diagram 3.3. □

Proposition 3.4.7 ([Mac63, Theorem III.3.4])**.** *We have an exact sequence of abelian groups:*◊

$$\begin{tikzcd}
0 \arrow[r] & \mathtt{Hom}(B,G) \arrow[r, "p^*"] & \mathtt{Hom}(E,G) \arrow[r, "i^*"] & \mathtt{Hom}(A,G) \arrow[dll, rounded corners] \\
& \mathtt{Ext}^1(B,G) \arrow[r, "p^*"] & \mathtt{Ext}^1(E,G) \arrow[r, "i^*"] & \mathtt{Ext}^1(A,G).
\end{tikzcd}$$

Proof. This sequence comes from the long exact sequence of homotopy groups [Uni13, Theorem 8.4.6] associated to the fibre sequence of Theorem 3.4.1, using Proposition 3.3.3 and the previous lemma to identify $\Omega\,\mathtt{AbSES}(-,G)$ with $\mathtt{Hom}(-,G)$. □

Remark 3.4.8. The connecting map $\mathtt{Hom}(A,G) \to \mathtt{Ext}^1(B,G)$ in the sequence above is given by $\phi \mapsto \phi_* E$. Showing this from the fibre sequence is somewhat tedious; we have a proof on paper, but not yet a formalisation. Instead, we have formalised a direct proof that the map just stated yields exactness of the sequence.◊◊

We apply the six-term exact sequence to compute Ext groups of cyclic groups:

Corollary 3.4.9 ([Mac63, Proposition III.1.1])**.** *For any $n > 0$ and abelian group A, we have*◊

$$\mathtt{Ext}^1(\mathbb{Z}/n, A) \cong A/n.$$

Proof. The short exact sequence $\mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n$ yields a six-term exact sequence

$$\cdots \to \mathtt{Hom}(\mathbb{Z},A) \xrightarrow{n^*} \mathtt{Hom}(\mathbb{Z},A) \to \mathtt{Ext}^1(\mathbb{Z}/n,A) \to \mathtt{Ext}^1(\mathbb{Z},A) \to \cdots$$

in which the term $\mathtt{Ext}^1(\mathbb{Z},A)$ vanishes since $\mathbb{Z}$ is projective.◊◊ This means that the homomorphism $\mathtt{Hom}(\mathbb{Z},A) \to \mathtt{Ext}^1(\mathbb{Z}/n,A)$ is the cokernel of the preceding one. By identifying $\mathtt{Hom}(\mathbb{Z},A)$ with A, the claim follows. □

3.5 The long exact sequence

We describe our formalisation of the higher Ext groups $\mathtt{Ext}^n(B, A)$ and their contravariant long exact sequence, which largely follows [Mac63, Chapter III.5]. The covariant version can be constructed from the arguments in [Mit65, Chapter VII.5], but we have not formalised this. The Baer sum is not yet formalised for $\mathtt{Ext}^n$ ($n > 1$), so we only have a long exact sequence of *pointed sets*. Nevertheless, exactness for pointed sets and abelian groups coincide, so we automatically get a long exact sequence of the latter once we have the higher Baer sum.

The formalisation of this section is in the separate repository `Yoneda-Ext`, whose `README` file explains how to set up and build the code related to this chapter. There are also comments in the code which explain details beyond what we cover here.

3.5.1 The type of length-n exact sequences

We start by defining a type $\mathtt{ES}^n$ which we will equip with an equivalence relation by which $\mathtt{Ext}^n$ will be the quotient. These constructions will yield functors, which we explain.

The type $\mathtt{ES}^n(B, A)$ of **length-n exact sequences** is recursively defined as:◊

```
Fixpoint ES (n : nat) : AbGroup^op -> AbGroup -> Type
  := match n with
    | 0%nat => fun B A => Hom B A
    | 1%nat => fun B A => AbSES B A
    | S n => fun B A => exists M, (ES n M A) * (AbSES B M)
    end.
```

Thus $\mathtt{ES}^0(B, A)$ is definitionally $\mathtt{Hom}(B, A)$, and $\mathtt{ES}^1(B, A)$ is definitionally $\mathtt{AbSES}(B, A)$. One could also have started the induction at $n \equiv 1$ instead of $n \equiv 2$, but it is convenient to have this definitional equality at level $n \equiv 1$. The functoriality of $\mathtt{ES}^n$ is inherited from `AbSES` and defined in the obvious way by pulling back and pushing out. For $n > 0$, an element of $\mathtt{ES}^{n+1}(B, A)$ is denoted by $(F, E)_M$, with the obvious meaning. The type $\mathtt{ES}^n(B, A)$◊ is pointed by recursion, using the trivial abelian group in the place of M in the inductive step.

Definition 3.5.1. *The **splice** operation is defined as*◊

$$F \circledcirc E :\equiv (F, E)_B : \mathtt{ES}^n(B, A) \to \mathtt{AbSES}(C, B) \to \mathtt{ES}^{n+1}(C, A).$$

By induction one can define a general splicing operation in which the second parameter can have arbitrary length◊, but we only need the restricted version above.

Now we equip $\mathtt{ES}^n(B, A)$ with a relation.

Definition 3.5.2. *We define a relation* `es_zig` $: \mathtt{ES}^n(B, A) \to \mathtt{ES}^n(B, A) \to \mathcal{U}$ *recursively as follows. For $n = 0, 1$,* `es_zig` *is the identity type. For $n \geq 2$, a relation between two elements $(F, E)_M$ and $(Y, X)_N$ consists of a homomorphism $f : \mathtt{Hom}(M, N)$ along with a path $f_*(E) = X$ and a relation* `es_zig`$(F, f^*(Y))$ *(using functoriality of $\mathtt{ES}^n$).*◊

The relation `es_zig` generates an equivalence relation `es_eqrel`◊ (denoted ∼ in the code) whose propositional truncation is `es_meqrel`◊. The functoriality of $\mathtt{ES}^n$ respects all of these relations.◊ ◊ Basic results on equivalence relations are contained in `EquivalenceRelation.v`.

We emphasise that equivalence relation `es_eqrel` is *not* equivalent to the identity type of $\mathtt{ES}^n$. Rather, it is an approximation of the identity type of the classifying space of the category $\mathtt{ES}^n$ (which we do not know if one can construct in HoTT). See, e.g., [Mac63, Chapter III.5] for related discussion.

Definition 3.5.3. *The pointed set* $\mathtt{Ext}^n(B, A)$ *is the quotient of* $\mathtt{ES}^n(B, A)$ *by the equivalence relation* `es_meqrel`.◊

The splice operation descends to this quotient.◊ By pushing out◊ and pulling back◊ extensions, Ext^n becomes a functor in each variable as well. Moreover, we have equalities $f^*(F) \circledcirc E = F \circledcirc f_*(E)$ whenever this expression makes sense, by the definition of `es_zig`.◊

Remark 3.5.4. The definition of $\mathtt{Ext}^{n+1}(B, A)$ is, more conceptually, the $(n + 1)$-fold tensor product of functors $\mathtt{Ext}^{n+1}(B, A) = \mathtt{Ext}^n(-, A) \otimes \mathtt{Ext}^1(B, -)$ (see, e.g., [GV83b, Theorem 9.20] or [Yon60, Eq. 4.3.4]). In our setup, this is a tensor product of Set-valued functors, which can be made into an abelian group by a construction similar to the Baer sum of Section 3.3.3 (though we have not yet formalised this). Alternatively, one could define $\mathtt{Ext}^{n+1}(B, A)$ as the $(n + 1)$-fold tensor product of functors *into abelian groups*. [GV83a, Lemma 2.1] implies that these two definitions coincide. We have chosen the present approach because we do not know of a direct construction of the long exact sequence for the latter approach.

3.5.2 The long exact sequence

We now begin working towards the long exact sequence, following the proof of [Mac63, Theorem XII.5.1]. As explained at the beginning of this section, we have only formalised the long exact sequence of *pointed sets*—however, exactness for pointed sets is the same as for abelian groups. Let us first recall the statement:

Theorem 3.5.5. *Let* $A \xrightarrow{i} E \xrightarrow{p} B$ *be a short exact sequence of abelian groups. For any abelian group* G*, there is a long exact sequence by pulling back:*◊ ◊ ◊

$$\cdots \xrightarrow{i^*} \mathtt{Ext}^n(A, G) \xrightarrow{-\circledcirc E} \mathtt{Ext}^{n+1}(B, G) \xrightarrow{p^*} \mathtt{Ext}^{n+1}(E, G) \xrightarrow{i^*} \cdots.$$

The proof in [Mac63] first discusses the six-term exact sequence, which we proved as Proposition 3.4.7. It then reduces the question to exactness at the domain of the connecting map (Lemma XII.5.2, loc. cit.), and proves exactness at that spot using Lemmas XII.5.3, XII.5.4, and XII.5.5. We will show the three latter lemmas, then directly prove exactness at the other spots, essentially "in-lining" Lemma XII.5.2.

The various constructions we need to do are simpler to carry out on the level of $\mathtt{ES}^n$ as opposed to $\mathtt{Ext}^n$. For this reason we work and formulate things in terms of the former, and then deduce the desired statement for the latter.

Before attacking Lemma XII.5.3, we show the following:

Lemma 3.5.6. *Consider two pairs of short exact sequences which can be spliced:*

$$(A \xrightarrow{l} Y \xrightarrow{s} B',\ B' \xrightarrow{k} X \xrightarrow{r} C), \qquad (A \xrightarrow{j} F \xrightarrow{q} B,\ B \xrightarrow{i} E \xrightarrow{p} C).$$

For any element of $\mathtt{es\_zig}(Y \circledcirc X, F \circledcirc E)$*, we have induced maps* $\mathsf{fib}_{s_*}(X) \to \mathsf{fib}_{q_*}(E)$◊ *and* $\mathsf{fib}_{i^*}(F) \to \mathsf{fib}_{k^*}(Y)$◊.

Proof. We only describe the first map since the second is analogous. The zig from $Y \circledcirc X$ to $F \circledcirc E$ gives a homomorphism $f : B' \to B$ along with two paths $f^*(F) = Y$ and $f_*(X) = E$. Let $G : \mathsf{fib}_{s_*}(X)$; by path induction we may assume $q_*(G) \equiv X$. The path $f^*(F) = Y$ means we have a commuting diagram:

$$\begin{tikzcd} A \arrow[r, "l"] \arrow[d, equal] & Y \arrow[r, "s"] \arrow[d, "\phi"] & B' \arrow[d, "f"] \\ A \arrow[r, "j"] & F \arrow[r, "q"] & B\,. \end{tikzcd}$$

Thus $\phi_*(G)$ defines an element of $\mathsf{fib}_{q_*}(E)$ by $q_*(\phi_*(G)) = f_*(s_*(G)) \equiv f_*(X) = E$. □

Lemma 3.5.7 ([Mac63, Lemma XII.5.3]). *Given two short exact sequences $A \xrightarrow{j} F \xrightarrow{q} B$ and $B \xrightarrow{i} E \xrightarrow{p} C$, the following types are logically equivalent:*◊

1. $\mathsf{fib}_{i^*}(F)$;
2. $\mathsf{fib}_{q_*}(E)$;
3. $\mathtt{es\_eqrel}(\mathtt{pt}, F \circledcirc E)$.

Proof. The logical equivalence of between (1) and (2) is as described in [Mac63].◊ Moreover, the implication (2) to (3) is clear by the definition of `es_zig`. We need to show that (3) implies (1), and we proceed by induction on the length of the zig-zag.

In the base case we have an actual equality $\mathtt{pt} = F \circledcirc E$, in which case (1) clearly holds. For the inductive step, suppose we have two short exact sequences $A \xrightarrow{l} Y \xrightarrow{s} B'$ and $B' \xrightarrow{k} X \xrightarrow{r} C$ such that $Y \circledcirc X$ is related to `pt` by a length n zig-zag, and we have either zig or a zag relating $Y \circledcirc X$ to $F \circledcirc E$. If we have a zig, then we use the induction hypothesis to get an element of $\mathsf{fib}_{s_*}(X)$ to which we apply the map $\mathsf{fib}_{s_*}(X) \to \mathsf{fib}_{q_*}(E)$ from the previous lemma. This suffices since (1) and (2) are logically equivalent.

If we have a zag, then the previous lemma gives a map $\mathsf{fib}_{k^*}(Y) \to \mathsf{fib}_{i^*}(F)$, so we are done by the induction hypothesis. □

We reformulate condition (2) in a manner that generalises to $\mathtt{ES}^n$.◊

```
Definition es_ii_family `{Univalence} {n : nat} {C B A : AbGroup}
  : ES n.+1 B A -> ES 1 C B -> Type
  := fun E F => { alpha : { B' : AbGroup & B' $-> B }
                    & (es_eqrel pt (es_pullback alpha.2 E))
                      * (hfiber (abses_pushout alpha.2) F) }.
```

Lemma 3.5.8 ([Mac63, Lemma XII.5.4]). *In the situation of the previous lemma, the types $\mathsf{fib}_{q_*}(E)$ and $\mathtt{es\_ii\_family}(F, E)$ are logically equivalent.*◊

Mac Lane appeals to the six-term exact sequence to prove this lemma, but we give a direct construction. In order to show Lemma XII.5.3, we prove a higher analogue of Lemma 3.5.6. This analogue is phrased in terms of the "relation fibre" `rfiber`, which takes the fibre of a point with respect to a relation.

Lemma 3.5.9. *Let $n > 0$ and consider $Y : \mathtt{ES}^n(B', A)$, $F : \mathtt{ES}^n(B, A)$, and two short exact sequences $B' \xrightarrow{k} X \to C$ and $B \xrightarrow{i} E \to C$. Given* $\mathtt{es\_zig}(Y \circledcirc X, F \circledcirc E)$*, we have maps* $\mathtt{rfiber}_{i^*}(F) \to \mathtt{rfiber}_{k^*}(Y)^\diamond$ *and* $\mathtt{es\_ii\_family}(Y, X) \to \mathtt{es\_ii\_family}(F, E)^\diamond$.

Lemma 3.5.10 ([Mac63, Lemma XII.5.5])**.** *Let $n > 0$, $F : \mathtt{ES}^n(B, A)$, and $E : \mathtt{ES}^1(C, B)$. The following types are equivalent:*$^\diamond$

1. $\mathsf{fib}_{i^*}(E)$;
2. $\mathtt{es\_ii\_family}(F, E)$;
3. $\mathtt{es\_eqrel}(pt, F \circledcirc E)$.

Proof. We first prove an auxiliary lemma which shows that if the three statements are equivalent for a given n, then (1) and (2) are equivalent for $n + 1$. The base case for this lemma is simply Lemma 3.5.7. For the inductive step, our auxiliary lemma gives us that (1) and (2) are equivalent. It is easy to show that (2) always implies (3), so it remains to show that (3) implies either (1) or (2). For this we induct on the length of a zig-zag, and use the equivalence of (1) and (2) along with the previous lemma, similarly (at least in structure) to the proof of Lemma 3.5.7. □

Afterwards, we reformulate this lemma in terms of $\mathtt{Ext}^n$.$^\diamond$ With this lemma at hand, and using similar methods to the ones presented here, we follow the proof of [Mac63, Lemma 5.2] to deduce exactness of the long sequence of Theorem 3.5.5.

3.6 Conclusion

We have presented a formalisation of the theory of Yoneda Ext in the novel setting of homotopy type theory, starting from the basic definition of a short exact sequence and arriving at the (contravariant) long exact sequence, with various related results along the way. At present, the long exact sequence is one of pointed sets, and we leave it to future work to formalise the Baer sum on $\mathtt{Ext}^n$ for $n > 1$, which would promote this into a long exact sequence of abelian groups. (The notion of exact sequence coincides for abelian groups and pointed sets.)

For pragmatic reasons we have worked with abelian groups, though it is clear that everything we have done could be applied to general modules. Even so, the higher Ext groups of abelian groups do not necessarily vanish in HoTT (see Proposition 4.3.12), so these are already interesting. There are various more general approaches that we would like to consider in the future, such as working with *pure* exact sequences (in which the classes of monomorphisms and epimorphisms are appropriately replaced) in an abelian category.

Many of our results have been contributed to the Coq-HoTT library [CH] under the namespace `Algebra.AbSES`, which currently weighs in at about 2900 lines of code (whitespace and comments included). This excludes the various contributions made to other parts of the library; the precise contributions may be seen through the pull requests `#1534`, `#1646`, `#1663`, `#1712`, `#1718`, and `#1738`. In addition, the code for the long exact sequence currently weighs in at about 1350 lines in the separate `Yoneda-Ext` repository.

The formalisation covers a substantial part of chapters III.1-3, III.5, and XII.5 of [Mac63], but also extends beyond the classical theory. In particular, our proof of Theorem 3.4.1 is new even for classical Yoneda Ext (though the theorem is known). This theorem presented the most challenging part of this formalisation, as it required managing considerable amounts of coherence. The other challenging part was the long exact sequence, whose proof involves an intricate induction and numerous constructions. By formalising these theorems we have not only established their correctness but also contributed evidence of the feasibility of dealing with sophisticated mathematical structures in a proof assistant like Coq.

Chapter 4

Ext in homotopy type theory

Abstract. We develop the theory of Yoneda Ext groups [Yon54] over a ring in homotopy type theory (HoTT) and describe their interpretation into an ∞-topos. This is an abstract approach to Ext groups which does not require projective or injective resolutions. While it produces group objects that are a priori *large*, we show that the Ext^1 groups are equivalent to small groups, leaving open the question of whether the higher Ext groups are essentially small as well. We also show that the Ext^1 groups take on the usual form as a product of cyclic groups whenever the input modules are finitely presented and the ring is a PID (in the constructive sense).

When interpreted into an ∞-topos of sheaves on a 1-category, our Ext groups recover (and give a resolution-free approach to) *sheaf* Ext groups, which arise in algebraic geometry [Gro57]. (These are also called "local" Ext groups.) We may therefore interpret results about Ext from HoTT and apply them to sheaf Ext. To show this, we prove that injectivity of modules in HoTT interprets to internal injectivity in these models. It follows, for example, that sheaf Ext can be computed using resolutions which are projective or injective in the sense of HoTT, when they exist, and we give an example of this in the projective case. We also discuss the relation between internal $\mathbb{Z}G$-modules (for a 0-truncated group object G) and abelian groups in the slice over BG, and study the interpretation of our Ext groups in both settings.

4.1 Introduction

We begin the study of homological algebra in homotopy type theory (HoTT) by developing the theory of Ext groups of modules over a ring R. Ext groups are important algebraic invariants, and also have many applications in homotopy theory. Classically, Ext groups are ingredients in the universal coefficient theorem for cohomology, and we hope to use the results here to obtain a universal coefficient spectral sequence in homotopy type theory. In addition, the results discussed here were used in [BCFR23] to show that certain types must be products of Eilenberg–Mac Lane spaces (Theorem 5.5.13).

One cannot assume that there are enough injective or projective modules in HoTT, so we define our Ext groups to be *Yoneda* Ext groups. These were first defined in [Yon54, Yon60], and are also described in [Mac63]. In this approach, given modules A and B over a ring R, the n-th Ext group $\mathrm{Ext}^n_R(B, A)$ is defined as the set of path components of the space (or groupoid) of length-n exact sequences $A \to E_1 \to E_2 \to \cdots E_n \to B$ of R-modules. For $n = 1$, this definition can be elegantly carried out in HoTT, by virtue of univalence. Specifically, we define $\mathsf{SES}_R(B, A)$ to be the type of short exact sequences from A to B (Definition 4.2.2). Using univalence, we see that paths in $\mathsf{SES}_R(B, A)$ correspond to isomorphisms between short exact sequences, so our type is capturing the correct notion. We define (Definition 4.2.5)

$$\mathsf{Ext}^1_R(B, A) \;:\equiv\; \|\mathsf{SES}_R(B, A)\|_0.$$

The definition of Ext^n_R for $n > 1$ is more difficult in HoTT, because we do not know how to correctly represent the type of length-n exact sequences. Instead we define $\mathsf{Ext}^n_R(B, A)$ to be the set-quotient of a certain type ES^n_R equipped with a relation (Definition 4.2.20). This approach is described in [Mac63], and has been formalized in HoTT in [Fla23a] (Section 3.5.1). We show that these types are abelian groups through an operation known as the *Baer sum* [Bae34].

One aspect of these resolution-free definitions of Ext groups is that they produce types lying in a larger universe. We show that for $n = 1$, our Ext groups are essentially small:

Theorem 4.2.12. *Let B and A be abelian groups. We have a natural equivalence*

$$\mathsf{SES}_{\mathbb{Z}}(B, A) \;\simeq\; (\mathsf{K}(B, 2) \to_* \mathsf{K}(A, 3)).$$

In particular, $\mathsf{SES}_{\mathbb{Z}}(B, A)$ and $\mathsf{Ext}^1_{\mathbb{Z}}(B, A)$ are equivalent to small types.

Here $\to_*$ denotes the type of pointed maps. This result easily implies that $\mathsf{SES}_R(B, A)$ is essentially small for modules over a general ring R (Corollary 4.2.13). However, we don't know whether the higher Ext groups are essentially small in general. From this theorem, we also deduce the usual six-term exact sequences of Ext groups from a fibre sequence, in the special case $R \equiv \mathbb{Z}$ (Propositions 4.2.16 and 4.2.17).

The usual long exact sequences also exist, and the contravariant one has been formalized for $\mathbb{Z}$ in [Fla23a] (Theorem 3.5.5). Using these, we show in Proposition 4.2.28 that our Ext groups can be computed using projective (and injective) resolutions, whenever one is at hand. It follows that our Ext groups yield right-derived functors of the hom-functor of modules whenever one has enough projectives or injectives. More generally, we show:

Theorem 4.2.24. *The large δ-functor $\{\mathsf{Ext}^n_R(-, A)\}_{n:\mathbb{N}}$ is universal, for any R-module A.*

The definition of a universal δ-functor is recalled in Definition 4.2.21.

We stress that the higher Ext groups $\mathsf{Ext}^n_{\mathbb{Z}}(B, A)$ need not vanish for $n > 1$ in our setting. Indeed, there are models of HoTT in which these are non-trivial, as we discuss below. Nevertheless, we show that these Ext groups do vanish whenever R is a (constructive) PID and the module B is finitely presented (Corollary 4.2.36). Moreover, when A is also finitely presented, then we get the usual description of $\mathsf{Ext}^1_R(B, A)$ as a product of cyclic groups (Proposition 4.2.33).

For a (0-truncated) group G, it is well-known that HoTT lets one work G-equivariantly by working in the context of the classifying type BG. In an ∞-topos this corresponds to working in the slice over the object BG. We study our constructions from this perspective in Section 4.2.7, which later lets us work out concrete examples of the interpretation of our Ext groups in Section 4.3.4. An abelian group "in the context of BG" is simply a map $BG \to \mathsf{Ab}$. Since the type of modules is 1-truncated, it is equivalent to replace BG with a pointed, connected type X

(and G by $\pi_1(X)$). To emphasize that our proofs do not require any truncation assumptions, we choose to work with such an X. We show that the category $X \to \mathsf{Ab}$ is equivalent to the category of $\mathbb{Z}\pi_1(X)$-*modules* (Proposition 4.2.39), where $\mathbb{Z}\pi_1(X)$ is the usual group ring. When working in the context of X, we carry out operations pointwise. Thus given $B, A : X \to \mathsf{Ab}$, we form the "ΩX-equivariant" type of short exact sequences $x \mapsto \mathsf{SES}_{\mathbb{Z}}(B_x, A_x) : X \to \mathcal{U}$. Of course, we can also consider the type $\mathsf{SES}_{\mathbb{Z}\pi_1(X)}(B, A)$ of short exact sequences of $\mathbb{Z}\pi_1(X)$-modules. These are related:

Theorem 4.2.41. *For any $B, A : X \to \mathsf{Ab}$, we have an equivalence*

$$\prod_{x:X} \mathsf{SES}_{\mathbb{Z}}(B_x, A_x) \;\simeq\; \mathsf{SES}_{\mathbb{Z}\pi_1(X)}(B_{\mathtt{pt}}, A_{\mathtt{pt}}).$$

Here $B_{\mathtt{pt}}$ and $A_{\mathtt{pt}}$ are the $\mathbb{Z}\pi_1(X)$-modules corresponding to the families B and A. We deduce the usual formula relating Ext^1 and cohomology with local coefficients:

Corollary 4.2.42. *For any $M : X \to \mathsf{Ab}$, we have a group isomorphism*

$$H^1(X; M) \;\simeq\; \mathsf{Ext}^1_{\mathbb{Z}\pi_1(X)}(\mathbb{Z}, M_{\mathtt{pt}}),$$

where the left-hand side is the cohomology of X with local coefficients in M, and $\mathbb{Z}$ on the right has trivial $\mathbb{Z}\pi_1(X)$-action.

In Section 4.3, we interpret our main results and constructions from HoTT into an ∞-topos $\mathcal{X}$. Given a ring R in an ∞-topos, it was shown in Theorem 2.4.17 that the interpretation of the category of ("small") R-modules from HoTT yields an internal category in $\mathcal{X}$ which represents the presheaf sending an object $X \in \mathcal{X}$ to the category of $(X{\times}R)$-modules in the slice $\mathcal{X}/X$. (Here $(X{\times}R)$ is a ring object in this slice.) Building on this, in Section 4.3.1 we show that the object of short exact sequences $\mathtt{SES}_R(B, A)$ between two modules A and B in $\mathcal{X}$ represents the presheaf

$$X \longmapsto \mathrm{SES}_{(X\times R)}(X \times B, X \times A) \;:\; \mathcal{X}^{\mathrm{op}} \longrightarrow \mathscr{S},$$

where $\mathrm{SES}_{(X\times R)}$ denotes the (1-truncated) space of short exact sequences between ("small") $(X{\times}R)$-modules. From this description we show how to recover the classical Ext groups in Corollary 4.3.4.

An interesting result is that in certain ∞-toposes, the interpretation of our Ext groups recover *sheaf* Ext (Definition 4.3.28), which has been studied in algebraic geometry [Gro57]. The precise theorem is:

Theorem 4.3.29. *Suppose sets cover in $\mathcal{X}$. For any $X \in \mathcal{X}$, ring $R \in \mathcal{X}/X$ and R-module B, the functor $\mathtt{Ext}^n_R(B, -) : R\text{-Mod} \to \mathrm{Ab}_{\mathcal{X}/X}$ is naturally isomorphic to the sheaf Ext functor $\underline{\mathrm{Ext}}^n_R(B, -)$.*

The meaning of "sets cover" is that any object admits an effective epimorphism from a 0-truncated object (Definition 4.3.21). Sets cover in any ∞-topos of ∞-sheaves on a 1-category.

Sheaf Ext is traditionally defined using injective resolutions (which always exist in these models), however our definition does not rely on the existence of enough injectives. To prove this theorem we show that injectivity in HoTT interprets to *internal injectivity* in these ∞-toposes (Corollary 4.3.27), which in turn follows from showing that internal injectivity is stable by base change in these models. Our proof of stability uses (and partly generalizes) results of Roswitha Harting [Har83b, Theorem 1.1] (for abelian groups) and Blechschmidt [Ble18, Proposition 3.7] (for modules) which show that internal injectivity of modules is stable by base change in any elementary 1-topos. In addition, [Ble18, Theorem 3.8] shows that (externally) injective modules are always internally injective, which means that our Ext groups can be computed using the same resolutions used for sheaf Ext.

We also study various notions of projectivity of modules in $\mathcal{X}$, namely the usual *(external) projectivity*, *internal projectivity*, and the notion of projectivity from HoTT. In order to understand the relation between these notions, we provide examples which demonstrate that neither of external and internal projectivity imply the other (Examples 4.3.15 and 4.3.41). The example of an internal projective module that is not externally projective is an adaptation of an argument by Todd Trimble. Moreover, we show that free modules on internally projective objects satisfy the notion of projectivity of modules from HoTT (Proposition 4.3.8). Using this fact, we demonstrate that our higher Ext groups need not vanish even over $\mathbb{Z}$ by computing a nontrivial $\mathsf{Ext}^2_{\mathbb{Z}}$. There are also known computations of sheaf Ext which demonstrate this.

Finally, in Section 4.3.4 we study the theory developed throughout Section 4.3 in some concrete situations. In particular, we relate our Ext groups of abelian groups in a slice $\mathcal{X}/BG$ to Ext groups of abelian groups in the base $\mathcal{X}$ (Proposition 4.3.35), and deduce a vanishing result (Corollary 4.3.37). We also generalize another result of Harting (Proposition 4.3.38), and discuss the connection between our Ext groups in the slice $\mathscr{S}/BG$ and ordinary Ext groups of $\mathbb{Z}G$-modules (Example 4.3.42).

Open questions. We list some outstanding questions.

1. In HoTT, is the abelian group $\mathsf{Ext}^n_R(B, A)$ equivalent to a small type for $n \geq 2$? Is it independent of the universe for $n \geq 2$? (The case $n = 1$ is answered by Theorem 4.2.12.)

2. In HoTT, are injectivity and projectivity (Definitions 4.2.25 and 4.2.27) independent of the universe?

3. In an ∞-topos, do HoTT-injectivity and HoTT-projectivity only depend on the 1-topos of 0-truncated objects? Do they agree with internal injectivity and internal projectivity? These would follow from proving that internal injectivity and internal projectivity are pullback stable

4. Does the interpretation of $\mathsf{Ext}^n_R(B, A)$ into an ∞-topos depend only on the 1-topos of 0-truncated objects? (For ∞-toposes in which sets cover, this is answered by Theorem 4.3.29.)

Notation and conventions. Our setting is Martin-Löf type theory with higher inductive types (HITs) and a hierarchy of univalent universes, as in the HoTT Book [Uni13], whose notation we generally follow. All of our groups, rings and modules are assumed to be sets. We write $\mathcal{U}$ for a fixed universe, and $\mathcal{U}_*$ for the universe of pointed types. Section 4.3 has its own section on notation.

4.2 Ext in HoTT

In this section, we develop the theory of Yoneda Ext groups in HoTT. Many of the results we show have classical analogues, in which case our contribution is the verification that these results hold in our setting as well. Nevertheless, our proofs and definitions make use of univalence and truncations, and we make constructive considerations (particularly in Section 4.2.6), all of which do not feature in the traditional theory.

Let R be a ring throughout this entire section.

4.2.1 The type of short exact sequences

Fix two left R-modules A and B throughout this section. Below we define the type $\mathsf{SES}_R(B, A)$ whose elements are short exact sequences

$$0 \to A \xrightarrow{i} E \xrightarrow{p} B \to 0$$

in R-Mod. The type $\mathsf{SES}_R(B, A)$ is a 1-type, and we define the set $\mathsf{Ext}^1_R(B, A)$ of extensions to be its set-truncation. By characterising paths in $\mathsf{SES}_R(B, A)$, we will show that an extension is trivial if and only if it is *merely split*.

A homomorphism of R-modules is an **epimorphism** (resp. a **monomorphism**) if and only if its underlying function is surjective (resp. an embedding). We write $\mathsf{Epi}_R(E, B)$ and $\mathsf{Mono}_R(A, E)$ for the set of R-module epimorphisms and monomorphisms, respectively.

Definition 4.2.1. Let $A \xrightarrow{i} E \xrightarrow{p} B$ be two composable homomorphisms in R-Mod. Whenever the composite $p \circ i$ is trivial, there is a unique induced map $i' : A \to \mathsf{ker}(p)$. If i' is an epimorphism, then i and p are **exact**:

$$\mathsf{IsExact}(i, p) :\equiv \sum_{h:\prod_{a:A} p(i(a))=0} \mathsf{IsEpi}(i').$$

Definition 4.2.2. The **type of short exact sequences from A to B** is:

$$\mathsf{SES}_R(B, A) :\equiv \sum_{E:R\text{-}\mathsf{Mod}} \; \sum_{i:\mathsf{Mono}_R(A,E)} \; \sum_{p:\mathsf{Epi}_R(E,B)} \mathsf{IsExact}(i, p).$$

Given a short exact sequence $E : \mathsf{SES}_R(B, A)$ we write $i_E : A \hookrightarrow E$ and $p_E : E \twoheadrightarrow B$ for the inclusion and projection maps. The type is pointed by the split short exact sequence $A \xrightarrow{\mathsf{in}_A} A \oplus B \xrightarrow{\mathsf{pr}_B} B$.

As defined, SES_R quantifies over R-Mod and is therefore a *large* type. It is moreover a 1-type, since R-Mod is and the fibre of the outermost sigma is a set. This mirrors the classical fact that the category of module extensions of B by A—whose maps are homomorphisms $E \to E'$ making the relevant triangles commute—is a groupoid. The following proposition strengthens this connection:

Proposition 4.2.3. *For any two short exact sequences E and F from A to B, we have*

$$(E =_{\mathsf{SES}_R(B,A)} F) \simeq \sum_{\phi:R\text{-}\mathsf{Mod}(E,F)} (\phi \circ i = j) \wedge (p = q \circ \phi).$$

Proof. It follows from the characterization of paths in Σ-types and transport in function types that

$$(E =_{\mathsf{SES}_R(B,A)} F) \;\simeq\; \sum_{\phi : E \cong F} (\phi \circ i = j) \wedge (p = q \circ \phi),$$

where $E \cong F$ denotes R-module isomorphisms. The stated equivalence now follows from the short five lemma. □

This lets us compute the loop space of $\mathsf{SES}_R(B, A)$ as in [Ret86].

Corollary 4.2.4. *We have a natural isomorphism* $\widehat{(-)} : \Omega\mathsf{SES}_R(B, A) \simeq R\text{-}\mathsf{Mod}(B, A)$ *of groups.*

Proof. By the previous proposition, a path $A \oplus B =_{\mathsf{SES}_R} A \oplus B$ corresponds to a homomorphism $\phi : A \oplus B \to A \oplus B$ which respects in_A and pr_B. Thus we get an R-module homomorphism $\widehat{\phi} : B \to A$ as the composite

$$\widehat{\phi} : B \xrightarrow{\mathsf{in}_B} A \oplus B \xrightarrow{\phi} A \oplus B \xrightarrow{\mathsf{pr}_A} A.$$

Conversely, for any homomorphism $f : B \to A$, we can define the homomorphism

$$(a, b) \mapsto (a + f(b), b) : A \oplus B \to A \oplus B$$

which respects in_A and pr_B. These associations are easily shown to be mutually inverse, yielding a bijection $\Omega\mathsf{SES}_R(B, A) \simeq R\text{-}\mathsf{Mod}(B, A)$. To see that it's an isomorphism of groups, consider a composite path $\phi \cdot \psi$. The associated R-module homomorphism $A \oplus B \to A \oplus B$ is given by the composite

$$(a, b) \longmapsto (a + \widehat{\phi}(b), b) \longmapsto (a + \widehat{\phi}(b) + \widehat{\psi}(b), b).$$

Hence $\widehat{\phi \cdot \psi} = \widehat{\phi} + \widehat{\psi}$, as required. □

In [Mac63], Mac Lane produces the set (underlying the abelian group) of extensions by applying π_0 to the groupoid of short exact sequences. We now do the corresponding thing:

Definition 4.2.5. The **set of extensions of B by A** is $\mathsf{Ext}^1_R(B, A) :\equiv \|\mathsf{SES}_R(B, A)\|_0$.

The following proposition characterizes trivial extensions.

Proposition 4.2.6. *Let E be a short exact sequence from A to B. Then E is trivial in* $\mathsf{Ext}^1_R(B, A)$ *if and only if p* merely splits*, i.e., the following proposition holds:*

$$\Big\| \sum_{s : R\text{-}\mathsf{Mod}(B,E)} p \circ s = \mathrm{id}_B \Big\|.$$

Proof. First of all, by the characterisation of paths in truncations [Uni13, Theorem 7.3.12] we have

$$(|E|_0 =_{\mathsf{Ext}^1_R(B,A)} 0) \;\simeq\; \|E =_{\mathsf{SES}_R(B,A)} A \oplus B\|.$$

Forgetting about truncations, the right-hand side holds if and only if p splits, by the usual argument. This in turn implies the statement on the truncations. □

We conclude this section by showing that Ext^1_R defines a bifunctor which lands in abelian groups. This is also shown in Section 3.3.2, but we give a different proof.

Definition 4.2.7. Let $A \to E \to B$ be a short exact sequence of R-modules.

(i) For $f : A \to A'$, the **pushout $f_*(E)$ of E along f** is the short exact sequence defined by the dashed maps below:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & E \arrow[r] \arrow[d] & B . \\ A' \arrow[r, dashed] & f_*(E) \arrow[ur, dashed, bend right] \arrow[ul, phantom, "\ulcorner"] & \end{tikzcd}$$

Here the curved arrow is defined to make the triangle commute and to be zero on A'.

(ii) For $g : B' \to B$, the **pullback $g^*(E)$ of E along g** is the short exact sequence defined by the dashed maps below:

$$\begin{tikzcd} & g^*(E) \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B' \arrow[d, "g"] \\ A \arrow[r] \arrow[ur, dashed, bend left] & E \arrow[r] & B . \end{tikzcd}$$

Here the curved arrow is defined to make the triangle commute and to be zero into B'.

For any R-module M, these operations define maps

$$f_* : \mathsf{SES}_R(M, A) \to \mathsf{SES}_R(M, A') \quad \text{and} \quad g^* : \mathsf{SES}_R(B, M) \to \mathsf{SES}_R(B', M).$$

The pushout and pullback operations commute, in the sense that $f_* g^*(E) = g^* f_*(E)$ whenever this expression makes sense (see Proposition 3.3.9). This means $\mathsf{Ext}^1_R(-,-)$ is a bifunctor into Set. Before making this bifunctor land in Ab, we also need to *detect* pushouts (and pullbacks) of short exact sequences, in the following sense.

Lemma 4.2.8. *Suppose given a diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha"] & E \arrow[r] \arrow[d] & B \arrow[d, "\beta"] \\ A' \arrow[r] & F \arrow[r] & B \end{tikzcd}$$

with short exact rows. If $\beta = \mathrm{id}_B$, then there is a path $\alpha_*(E) = F$ of short exact sequences. □

The dual statement for pullbacks requires the leftmost vertical map to be the identity. For a proof, the reader may consult Proposition 3.3.7 or Lemmas III.1.2 and III.1.4 in [Mac63].

That $\mathsf{Ext}^1_R(B, A)$ is an abelian group follows from the following proposition.

Proposition 4.2.9. *The contravariant functor $\mathsf{Ext}^1_R(-, A)$ takes arbitrary coproducts to products, and the covariant functor $\mathsf{Ext}^1_R(B, -)$ preserves finite products.*

Proof. We first show that $\mathsf{Ext}^1_R(-,A)$ takes arbitrary coproducts to products. To that end, let X be a set and consider a family $B : X \to R\text{-}\mathsf{Mod}$. Theorem 2.3.19 produces an exact coproduct functor $\bigoplus_X : R\text{-}\mathsf{Mod}^X \to R\text{-}\mathsf{Mod}$. We construct a natural bijection

$$\phi \;:\; \mathsf{Ext}^1_R(\bigoplus_{x:X} B_x, A) \longrightarrow \prod_{x:X} \mathsf{Ext}^1_R(B_x, A)$$

for any R-module A, as follows. Since we are defining maps between sets, we may pick representatives of extensions. Given a short exact sequence $A \longrightarrow E \longrightarrow \bigoplus_{x:X} B_x$, define E_x to be the result of pulling back E along the natural map $B_x \to \bigoplus_X B$ for $x : X$. A map in the inverse direction is given as follows. A family $(A \to F_x \to B_x)_{x:X}$ of short exact sequences yields a short exact sequence

$$\bigoplus_X A \to \bigoplus_{x:X} F_x \to \bigoplus_{x:X} B_x$$

by exactness of $\bigoplus_X$, which by pushing out along $\nabla : \bigoplus_X A \to A$ yields an element of $\mathsf{Ext}^1_R(\bigoplus_X B, A)$.

Starting from a short exact sequence $A \to E \to \bigoplus_X B$, the following diagram exhibits the bottom row as the pushout of the top row by Lemma 4.2.8, showing that ϕ is a section:

$$\begin{tikzcd}
\bigoplus_X A \arrow[r] \arrow[d, "\nabla"] & \bigoplus_{x:X} E_x \arrow[r] \arrow[d] & \bigoplus_{x:X} B_x \arrow[d, equal] \\
A \arrow[r] & E \arrow[r] & \bigoplus_{x:X} B_x .
\end{tikzcd}$$

Here the middle vertical arrow is induced from the maps $(E_x \to E)_{x:X}$ coming from the definition of E_x as a pullback.

Similarly, for any family $F :\equiv (A \to F_x \to B_x)_{x:X}$, the following diagram exhibits the top row as the pullback of the bottom row along $B_y \to \bigoplus_X B$, for any $y : X$, since the composite of the left vertical maps is the identity on A:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & F_y \arrow[r] \arrow[d] & B_y \arrow[d] \\
\bigoplus_X A \arrow[r] \arrow[d, "\nabla"] & \bigoplus_{x:X} F_x \arrow[r] \arrow[d] & \bigoplus_{x:X} B_x \arrow[d, equal] \\
A \arrow[r] & \nabla_* \bigoplus_{x:X} F_x \arrow[r] & \bigoplus_{x:X} B_x .
\end{tikzcd}$$

Here the maps from the top row to the middle row are given by the inclusion of the y-summand. This shows that ϕ is a retraction, hence a bijection.

To show that $\mathsf{Ext}^1_R(B,-)$ preserves finite products, it suffices to check that it preserves the empty product and binary products. The former is clear, so we proceed to handle binary products. Pushing out along the two projections of $A_0 \oplus A_1$ yields a homomorphism $\mathsf{Ext}^1_R(B, A_0 \oplus A_1) \to \mathsf{Ext}^1_R(B, A_0) \times \mathsf{Ext}^1_R(B, A_1)$. To get a map in the opposite direction, we take the biproduct of the given extensions (using that biproducts are exact) and then pull back along $\Delta : B \to B \oplus B$. Showing that these two maps are inverses is straightforward. □

Corollary 4.2.10. *Let A and B be R-modules. The set $\mathsf{Ext}^1_R(B, A)$ is naturally an abelian group.*

Proof. We have just shown that the functor $\mathsf{Ext}^1_R(B, -) : R\text{-}\mathsf{Mod} \to \mathrm{Set}$ preserves finite products. It follows that it preserves group objects. But any R-module is itself an abelian group object in R-Mod (in a unique way), so we are done. $\square$

The binary operation on $\mathsf{Ext}^1_R(B, A)$ is called the *Baer sum*. A concrete description of this operation, which has been formalized in joint work with Jacob Ender, is discussed in Section 3.3.3. We also mention that if the ring R is commutative, then $\mathsf{Ext}^1_R(B, A)$ is naturally an R-module.

We also record the following lemma for later use.

Lemma 4.2.11. *Let $f : A \to A'$ and $g : B' \to B$ be isomorphisms of R-modules. For any short exact sequence $A \xrightarrow{i} E \xrightarrow{p} B$, we have $g^* f_*(E) = (A' \xrightarrow{i \circ f^{-1}} E \xrightarrow{g^{-1} \circ p} B')$.* $\square$

4.2.2 Classifying extensions and smallness of Ext^1

We remarked after Definition 4.2.2 that SES_R is a large type, and consequently Ext^1_R is a large abelian group. This is not surprising, since our definition mirrors that of the external Yoneda Ext groups in an abelian category, and examples of abelian categories are known where these are proper classes. However, our Ext^1_R groups turn out to be equivalent to small types.

Recall that [BvDR18, Theorem 5.1] produces the following equivalence of categories for $n \geq 2$:

$$\mathsf{K}(-, n) : \mathsf{Ab} \;\simeq\; \{\text{ pointed, } (n{-}1)\text{-connected } n\text{-types }\} : \Omega^n,$$

under which short exact sequences and fibre sequences correspond.

Theorem 4.2.12. *Let B and A be abelian groups. We have a natural equivalence*

$$\mathsf{SES}_{\mathbb{Z}}(B, A) \;\simeq\; (\mathsf{K}(B, 2) \to_* \mathsf{K}(A, 3)).$$

In particular, $\mathsf{SES}_{\mathbb{Z}}(B, A)$ and $\mathsf{Ext}^1_{\mathbb{Z}}(B, A)$ are equivalent to small types.

We mention that the right-hand side of the equivalence above is moreover equivalent to $(\mathsf{K}(B, n) \to_* \mathsf{K}(A, n+1))$ for $n \geq 2$, since Ω is an equivalence in this range. (See [BvDR18, Theorem 6.7], with their $n = 0$ and their k equal to our n. Their F is our Ω.) The right-hand side is also equivalent to $(\mathsf{K}(B, n) \to_* \mathsf{BAut}(\mathsf{K}(A, n)))$, since $\mathsf{K}(A, n+1)$ is the 1-connected cover of $\mathsf{BAut}(\mathsf{K}(A, n))$ ([Shu14] and Proposition 5.5.9).

Proof. We define maps in both directions and show that they are mutual inverses. To go from left to right, we apply $\mathsf{K}(-, 3)$ to a short exact sequence $A \to E \to B$ to get a fibre sequence, then we negate the maps and take the fibre:

$$\begin{tikzcd} \mathsf{K}(B,2) \arrow[r, dashed] & \mathsf{K}(A,3) \arrow[r, "-\mathsf{K}(i,3)"] & \mathsf{K}(E,3) \arrow[r, "-\mathsf{K}(p,3)"] & \mathsf{K}(B,3)\,. \end{tikzcd}$$

The fibre is naturally equivalent to $\mathsf{K}(B, 2)$, as displayed, since the three rightmost terms form a fibre sequence. This process yields a map from left to right.

Conversely, given a map $f : \mathsf{K}(B,2) \to_* \mathsf{K}(A,3)$, we get a fibre sequence

$$\mathsf{K}(A,2) \longrightarrow F \longrightarrow \mathsf{K}(B,2) \xrightarrow{f} \mathsf{K}(A,3),$$

where F is a pointed, 1-connected 2-type. Taking loop spaces twice yields a short exact sequence $A \to \Omega^2 F \to B$ of abelian groups.

We first consider the composite starting and ending at $\mathsf{SES}_{\mathbb{Z}}(B,A)$. Starting with a short exact sequence $A \to E \to B$, we apply $\mathsf{K}(-,3)$, negate the maps and take three fibres, producing the sequence

$$\mathsf{K}(A,2) \xrightarrow{\mathsf{K}(i,2)} \mathsf{K}(E,2) \xrightarrow{\mathsf{K}(p,2)} \mathsf{K}(B,2)\,.$$

Here we have used that taking three fibres negates maps, by [Uni13, Lemma 8.4.4]. We then apply Ω^2, which yields the original short exact sequence, since $\Omega^2 \circ \mathsf{K}(-,2)$ is the identity.

For the composite starting and ending at $(\mathsf{K}(B,2) \to_* \mathsf{K}(A,3))$, we will use that the map

$$\Omega : (\mathsf{K}(B,3) \to_* \mathsf{K}(A,4)) \longrightarrow (\mathsf{K}(B,2) \to_* \mathsf{K}(A,3))$$

is an equivalence. Write B for the inverse. This equivalence implies that any pointed map $\phi : \mathsf{K}(B,2) \to_* \mathsf{K}(A,3)$ fits into the following fibre sequence:

$$\mathsf{K}(B,2) \xrightarrow{\phi} \mathsf{K}(A,3) \longrightarrow \mathsf{fib}_{-B\phi} \longrightarrow \mathsf{K}(B,3) \xrightarrow{-B\phi} \mathsf{K}(A,4)\,.$$

Applying Ω^3 to the middle three terms produces the short exact sequence associated to ϕ, but with the maps negated. Since Ω^3 is an equivalence and commutes with negation of maps, this means that the middle three terms are equal to $\mathsf{K}(-,3)$ applied to the short exact sequence associated to ϕ with the maps negated. It immediately follows that the composite starting and ending at $(\mathsf{K}(B,2) \to_* \mathsf{K}(A,3))$ is equal to the identity as well, so the maps we defined are mutual inverses.

It is straightforward to check that the map from right to left is natural in both B and A. □

Theorem 4.2.12 is about *abelian* extensions of abelian groups. Results similar to Theorem 4.2.12, but for central extensions, appear in [BvDR18, Mye20, NSS14, Sco20], where one takes $n = 1$.

Corollary 4.2.13. *Let B and A be R-modules. Then both $\mathsf{SES}_R(B,A)$ and $\mathsf{Ext}^1_R(B,A)$ are equivalent to small types.*

Proof. Let $U : R\text{-}\mathsf{Mod} \to \mathsf{Ab}$ denote the forgetful functor. The fibre $\mathsf{fib}_u(E)$ of the induced map $u : \mathsf{SES}_R(B,A) \to \mathsf{SES}_{\mathbb{Z}}(UB,UA)$ over an extension E (of abelian groups) is small, since it is contained in the set of R-module structures on E. Thus $\mathsf{SES}_R(B,A) \simeq \sum_{E:\mathsf{SES}_{\mathbb{Z}}(UB,UA)} \mathsf{fib}_u(E)$, where the latter is equivalent to a small type. □

Remark 4.2.14. Classically, one argument that the external Ext group $\mathrm{Ext}^1_R(B,A)$ is small is that the underlying set of any extension E of A by B is isomorphic to the product set $A \times B$. However, this can fail in models of HoTT, such as in the Sierpiński ∞-topos, as we show in Remark 4.3.14. If we allow more general situations, smallness of external Ext can also fail. For example, [Wof16] describes a locally small abelian category in which the external Yoneda

Ext group $\mathrm{Ext}^1_{\mathbb{Z}}(Z, Z)$ can be a proper class for a certain object Z. We believe that this category can arise as the category of abelian group objects in an elementary ∞-topos of G-spaces, where G is the free abelian group on a proper class of generators, and for each object, all but a set of generators are required to act trivially. In this setting, Z is the interpretation of the integers, and so the interpretation of our $\mathsf{Ext}^1_{\mathbb{Z}}(Z, Z)$ is zero, since the integers are projective in the sense of HoTT (see Definition 4.2.25 and Proposition 4.2.26). This illustrates that it is somewhat surprising that the interpretation of $\mathsf{Ext}^1_R(B, A)$ is small in every model of HoTT.

Remark 4.2.15. It follows from the equivalence $\mathsf{SES}_{\mathbb{Z}}(B, A) \simeq (\mathsf{K}(B, 2) \to_* \mathsf{K}(A, 3))$ in Theorem 4.2.12 that $\mathsf{SES}_{\mathbb{Z}}(B, A)$ is independent of the choice of universe containing A and B. Therefore, the same holds for $\mathsf{Ext}^1_{\mathbb{Z}}(B, A)$. The argument in the proof of Corollary 4.2.13 shows that these statements are also true when $\mathbb{Z}$ is replaced by a general ring R, since the set of R-module structures on an abelian group E is independent of the choice of universe.

4.2.3 The six-term exact sequences

For $A \to E \to B$ a short exact sequence of R-module and M another R-module, there are covariant and contravariant six-term exact sequences of abelian groups

$$\begin{tikzcd} 0 \arrow[r] & R\text{-}\mathsf{Mod}(M, A) \arrow[r, "i_*"] & R\text{-}\mathsf{Mod}(M, E) \arrow[r, "p_*"] & R\text{-}\mathsf{Mod}(M, B) \arrow[dll, rounded corners] \\ & \mathsf{Ext}^1_R(M, A) \arrow[r, "i_*"] & \mathsf{Ext}^1_R(M, E) \arrow[r, "p_*"] & \mathsf{Ext}^1_R(M, B)\,, \end{tikzcd}$$

and

$$\begin{tikzcd} \mathsf{Ext}^1_R(A, M) & \mathsf{Ext}^1_R(E, M) \arrow[l, "i^*"'] & \mathsf{Ext}^1_R(B, M) \arrow[l, "p^*"'] \\ R\text{-}\mathsf{Mod}(A, M) \arrow[urr, rounded corners] & R\text{-}\mathsf{Mod}(E, M) \arrow[l, "i^*"'] & R\text{-}\mathsf{Mod}(M, B) \arrow[l, "p^*"'] & 0. \arrow[l] \end{tikzcd}$$

These can be proved following Theorem 3.2 of [Mac63], and the contravariant version has been formalized in Proposition 3.4.7 for $R \equiv \mathbb{Z}$. These can be proved following Theorem 3.2 of [Mac63], and the contravariant version has been formalized in Section 3.4. Here we find it interesting to give different arguments in the special case where $R \equiv \mathbb{Z}$.

Proposition 4.2.16. *Let $A \xrightarrow{i} E \xrightarrow{p} B : \mathsf{Ab}$ be a short exact sequence, and $M : \mathsf{Ab}$. Then*

$$\mathsf{SES}_{\mathbb{Z}}(M, A) \xrightarrow{i_*} \mathsf{SES}_{\mathbb{Z}}(M, E) \xrightarrow{p_*} \mathsf{SES}_{\mathbb{Z}}(M, B)$$

is a fibre sequence, where the maps are given by pushing out.

Proof. Applying $\mathsf{K}(-, 3)$ to the given short exact sequence produces a fibre sequence

$$\mathsf{K}(A, 3) \to \mathsf{K}(E, 3) \to \mathsf{K}(B, 3).$$

Since $(Z \to_* -)$ preserves fibre sequences for any pointed type Z, we can apply $(\mathsf{K}(M,2) \to_* -)$ to obtain a fibre sequence

$$(\mathsf{K}(M,2) \to_* \mathsf{K}(A,3)) \to (\mathsf{K}(M,2) \to_* \mathsf{K}(E,3)) \to (\mathsf{K}(M,2) \to_* \mathsf{K}(B,3)),$$

where the maps are given by post-composition. Theorem 4.2.12 then gives the desired fibre sequence, since naturality means that post-composition corresponds to pushout of short exact sequences. □

Using Corollary 4.2.4, one can show that the long exact sequence of homotopy groups associated to the fibre sequence above recovers the usual covariant six-term exact sequence of Ext groups mentioned at the beginning of this section.

We now give the dual result, which can similarly be shown to produce the contravariant six-term exact sequence of Ext groups. The construction of the following fibre sequence is more difficult, because we need to map out of a fibre sequence (not into, as in the previous proposition).

Proposition 4.2.17. *Let $A \xrightarrow{i} E \xrightarrow{p} B : \mathsf{Ab}$ be a short exact sequence, and $M : \mathsf{Ab}$. Then*

$$\mathsf{SES}_{\mathbb{Z}}(A, M) \xleftarrow{i^*} \mathsf{SES}_{\mathbb{Z}}(E, M) \xleftarrow{p^*} \mathsf{SES}_{\mathbb{Z}}(B, M)$$

is a fibre sequence, where the maps are given by pulling back.

Proof. Applying $\mathsf{K}(-,2)$ to the given short exact sequence produces a fibre sequence

$$\mathsf{K}(A,2) \to \mathsf{K}(E,2) \to \mathsf{K}(B,2).$$

Let C be the cofibre of the left map, which comes with a natural map $C \to \mathsf{K}(B,2)$. Since $(- \to_* Z)$ sends cofibre sequences to fibre sequences for any Z, we can apply $(- \to_* \mathsf{K}(M,3))$ to obtain a fibre sequence

$$(\mathsf{K}(A,2) \to_* \mathsf{K}(M,3)) \leftarrow (\mathsf{K}(E,2) \to_* \mathsf{K}(M,3)) \leftarrow (C \to_* \mathsf{K}(M,3)).$$

We claim that $(C \to_* \mathsf{K}(M,3)) \simeq (\mathsf{K}(B,2) \to_* \mathsf{K}(M,3))$, from which the statement follows, as in the proof of the previous result. Since $\mathsf{K}(M,3)$ is a 3-type, it suffices to prove that $\|C\|_3 \simeq \|\mathsf{K}(B,2)\|_3$, and for this it suffices to show that the map $C \to \mathsf{K}(B,2)$ is 3-connected, using [Uni13, Lemma 7.5.14]. The map $C \to \mathsf{K}(B,2)$ is the cogap map associated to the map $\mathsf{K}(E,2) \to \mathsf{K}(B,2)$ and the base point inclusion $1 \to \mathsf{K}(B,2)$. Since $\mathsf{K}(B,2)$ is connected, it suffices to check the connectivity of the fibre of this map over the base point. By [Rij17, Theorem 2.2], this fibre is the join $\mathsf{K}(A,2) * \Omega\mathsf{K}(B,2)$ of the fibres, which is (1+0+2)-connected, as required. (This fact about connectivities of joins is proved in [CH, `Join.v`]. It also follows from [CS20, Corollary 2.32], since the join is the suspension of the smash product.) □

4.2.4 Higher Ext groups

The definition of higher Ext groups from [Mac63, Chapter XII] or [Yon54, pp. 216] can be translated to HoTT and has already been formalized (for $R \equiv \mathbb{Z}$, but the arguments work for a general ring) in Section 3.5 along with the contravariant long exact sequence. An account of the covariant long exact sequence that can be carried out in our setting may be found in [Mit65, Chapter VII.5]. We first discuss the definition of Ext^n_R, referring the reader to Section 3.5.1 for further details. The long exact sequence of Ext groups (Theorem 3.5.5) makes the collection $\{\mathsf{Ext}^n_R(-,A)\}_{n:\mathbb{N}}$ into a *(large) δ-functor*, for any A. In Theorem 4.2.24, we show that this δ-functor is universal, as expected.

Definition 4.2.18. Let B and A be R-modules. The type $\mathsf{ES}^n_R(B,A)$ is inductively defined to be

$$\mathsf{ES}^n_R(B,A) :\equiv \begin{cases} R\text{-}\mathsf{Mod}(B,A) & \text{if } n \equiv 0, \\ \mathsf{SES}_R(B,A) & \text{if } n \equiv 1, \\ \sum_{C:R\text{-}\mathsf{Mod}} \mathsf{ES}^m_R(C,A) \times \mathsf{SES}_R(B,C) & \text{if } n \equiv m+1, m > 0. \end{cases}$$

There is an evident **splicing** operation $\circledcirc : \mathsf{ES}^m_R(C,A) \times \mathsf{SES}_R(B,C) \to \mathsf{ES}^{m+1}_R(B,A)$ for any $C : R\text{-}\mathsf{Mod}$, which is given by pushing out along a map when $m \equiv 0$. For $m \geq 1$, we use $\circledcirc$ as a type constructor.

Our splicing operation is written in diagrammatic order, as in [Mac63]. An element of ES^n_R consists of n short exact sequences which can be spliced in succession from left to right. It is straightforward to define a more general splicing operation where the right factor can have arbitrary length.

Definition 4.2.19. Let $n : \mathbb{N}$. For $E, F : \mathsf{ES}^n_R(B,A)$ define a relation inductively by $E \rightsquigarrow F :\equiv$

$$\begin{cases} E = F & \text{if } n \equiv 0, 1; \\ \sum_{\beta:R\text{-}\mathsf{Mod}(C,C')} (E_0 \rightsquigarrow \beta^*(F_0)) \times (\beta_*(E_1) = F_1) & \text{if } n > 1,\ E \equiv (C, E_0, E_1),\ F \equiv (C', F_0, F_1). \end{cases}$$

We now define higher Ext groups as the set-quotient of ES^n_R by this relation.

Definition 4.2.20. Let B and A be R-modules. For $n : \mathbb{N}$, define the **set of length-n extensions of B by A** to be

$$\mathsf{Ext}^n_R(B,A) :\equiv \begin{cases} R\text{-}\mathsf{Mod}(B,A) & \text{if } n \equiv 0, \\ \mathsf{Ext}^1_R(B,A) & \text{if } n \equiv 1, \\ \|\mathsf{ES}^n_R(B,A)/\!\rightsquigarrow\|_0 & \text{if } n > 1. \end{cases}$$

The splicing operation respects the relation $\rightsquigarrow$ and thus passes to the quotient Ext^n_R. The same is true for pushouts and pullbacks of length-n exact sequences, which makes Ext^n_R into a profunctor. We define a Baer sum on $\mathsf{Ext}^{n+2}_R(B,A)$ by $E+F :\equiv \nabla(E \oplus F)\Delta$, and this makes Ext^{n+2}_R into an abelian group, for all $n : \mathbb{N}$. Our next goal is to show that the collection $\{\mathsf{Ext}^n_R(-,A)\}_{n:\mathbb{N}}$ is a *(large) universal δ-functor* for any A.

Definition 4.2.21. A **δ-functor structure** on a collection $\{T^n : R\text{-}\mathsf{Mod}^{\mathrm{op}} \to \mathsf{Ab}\}_{n:\mathbb{N}}$ of additive functors associates to any short exact sequence $A \to E \to B$ of R-modules a **connecting homomorphism** $\delta^n_E : T^n(A) \to T^{n+1}(B)$ for each $n : \mathbb{N}$, such that:

(i) The following long complex is exact:

$$0 \to T^0(B) \to T^0(E) \to T^0(A) \xrightarrow{\delta^0_E} T^1(B) \to \cdots \to T^n(E) \to T^n(A) \xrightarrow{\delta^n_E} T^{n+1}(B) \to \cdots .$$

(ii) For any morphism of short exact sequences as on the left below, the square on the right commutes for every $n : \mathbb{N}$:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha"] & E \arrow[r] \arrow[d] & B \arrow[d, "\beta"] \\ A' \arrow[r] & F \arrow[r] & B' \end{tikzcd} \qquad \begin{tikzcd} T^n(A') \arrow[r, "\delta^n_F"] \arrow[d, "T^n_\alpha"] & T^{n+1}(B') \arrow[d, "T^{n+1}_\beta"] \\ T^n(A) \arrow[r, "\delta^n_E"] & T^{n+1}(B) . \end{tikzcd}$$

A **δ-functor**[1] is such a collection equipped with a δ-functor structure. Replacing Ab above with the category Ab' of large abelian groups[2], we obtain the notion of a **large δ-functor**.

If T and S are (large) δ-functors, then a **morphism** $f : T \to S$ of δ-functors consists of a collection of natural transformations $\{f_n : T^n \Rightarrow S^n\}_{n:\mathbb{N}}$ which respect the connecting maps. The (large) δ-functor T is **universal** if the restriction map $(T \to S) \to (T^0 \Rightarrow S^0)$ is a bijection, for any (large) δ-functor S.

The splicing operation $- \circledcirc E$ defines connecting maps for the family $\{\mathsf{Ext}^n_R(-, A)\}_{n:\mathbb{N}}$ of contravariant functors, and the long exact sequence from Theorem 3.5.5 shows that the first axiom holds. It is straightforward to verify the second axiom. Thus we have a large δ-functor structure on $\{\mathsf{Ext}^n_R(-, A)\}_{n:\mathbb{N}}$. Below, we show that it is universal. This fact is implicit in Yoneda's approach to *satellites* in [Yon60, Chapter 4], though he does not give an explicit proof of universality. (Satellite is another word for (large) universal δ-functor.) However, [Buc60] constructs satellites which can be shown to be isomorphic to Yoneda's definition, and Buchsbaum does prove that his construction produces a universal δ-functor (see his Proposition 4.3).

Proposition 4.2.22. *Let T be a large δ-functor and let A and B be R-modules. For each $n : \mathbb{N}$, there is a homomorphism of abelian groups $d_n : \mathsf{Ext}^n_R(B, A) \to \mathsf{Ab}(T^0(A), T^n(B))$ which is natural in A and B.*

Proof. We proceed by induction on n. Since T^0 is an additive contravariant functor, it gives a homomorphism

$$\phi \longmapsto T^0_\phi \ : \ R\text{-}\mathsf{Mod}(B, A) \longrightarrow \mathsf{Ab}(T^0(A), T^0(B))$$

which is natural in A and B. We can therefore define $d_0(\phi) :\equiv T^0_\phi$.

For $n \equiv 1$, consider the map $E \mapsto \delta^0_E : \mathsf{SES}_R(B, A) \to \mathsf{Ab}(T^0(A), T^1(B))$. Since the codomain is a set, we get our map d_1 out of the set-truncation $\mathsf{Ext}^1_R(B, A)$. We check that d_1 is natural in A; naturality in B is similar. Let $f : A \to A'$ be a homomorphism. Our goal is to show that the square on the left commutes:

$$\begin{tikzcd} \mathsf{Ext}^1_R(B, A) \arrow[r, "d_1"] \arrow[d, "f_*"] & \mathsf{Ab}(T^0(A), T^1(B)) \arrow[d, "(T^0_f)^*"] \\ \mathsf{Ext}^1_R(B, A') \arrow[r, "d_1"] & \mathsf{Ab}(T^0(A'), T^1(B)) \end{tikzcd} \qquad \begin{tikzcd} T^0(A') \arrow[r, "\delta^1_{f_* E}"] \arrow[d, "T^0_f"] & T^1(B) \arrow[d, equal] \\ T^0(A) \arrow[r, "\delta^1_E"] & T^1(B) . \end{tikzcd}$$

[1] More precisely, this is the notion of a *contravariant*, *cohomological* δ-functor [Wei94, Chapter 2.1].

[2] For $A, B : \mathsf{Ab}'$, we still write $\mathsf{Ab}(A, B)$ for the large abelian group of group homomorphisms.

Since naturality is a proposition, we may pick an actual short exact sequence $A \to E \to B$ in the top left corner. The question then is whether the equation $d_1(f_*(E)) = d_1(E) \circ T^0_f$ holds. But this equation underlies the commuting square above on the right, which comes from part (ii) of the δ-functor structure of T applied to the natural morphism $E \to f_*E$ of short exact sequences.

Now let $n \geq 1$ and assume that we have the natural homomorphism d_n. We proceed to construct d_{n+1}. First we define a map $d'_{n+1} : \mathsf{ES}^{n+1}_R(B, A) \to \mathsf{Ab}(T^0(A), T^n(B))$ by

$$d'_{n+1}(F \circledcirc E) \;:\equiv\; \delta^n_E \circ d_n([F]) \;:\; T^0(A) \longrightarrow T^{n+1}(B),$$

where $[F] : \mathsf{Ext}^n_R(C, A)$ is the equivalence class of $F : \mathsf{ES}^n_R(C, A)$. To descend d'_{n+1} to a map d_{n+1} on the quotient Ext^{n+1}_R, we need to show that it respects the relation on ES^{n+1}_R.

Suppose we have a relation $E \rightsquigarrow F$ in $\mathsf{ES}^{n+1}_R(B, A)$. Writing $E \equiv (C, E_0, E_1)$ and $F \equiv (C', F_0, F_1)$, the relation gives a map $\beta : C \to C'$, a relation $E_0 \rightsquigarrow \beta^*(F_0)$ in $\mathsf{ES}^n_R(C, A)$, and a path $\beta_*(E_1) = F_1$ in $\mathsf{SES}_R(B', C)$. We need to argue that the outer square below commutes:

$$\begin{tikzcd}
T^0(A) \arrow[r, "d_n([F_0])"] \arrow[d, "d_n([E_0])"'] & T^n(C') \arrow[dl, "T^n_\beta"'] \arrow[d, "\delta^n_{F_1}"] \\
T^n(C) \arrow[r, "\delta^n_{E_1}"] & T^{n+1}(B).
\end{tikzcd}$$

The lower-right triangle commutes by condition (ii) of the δ-structure of T, using the map of short exact sequences $E_1 \to F_1$ associated to the equality $\beta_*(E_1) = F_1$. For the upper-left triangle, first note that we have $d_n([E_0]) = d_n([\beta^*(F_0)])$ since d_n respects $\rightsquigarrow$ by induction. Naturality of d_n gives us further that $d_n(\beta^*[F_0]) = T^n_\beta \circ d_n([F_0])$, from which we conclude that the upper-left triangle commutes. Thus we get the desired map d_{n+1} by passing to the quotient.

It remains to show that d_{n+1} is natural and a homomorphism. By Lemma 4.2.23 below, the latter follows from the former, so we only check naturality. First we check it in the first variable, so let $f : A \to A'$ be a homomorphism. Since we need to show a proposition, we may consider an actual element $F \circledcirc E : \mathsf{ES}^{n+1}_R(B, A)$. Then, since pushouts of longer exact sequences are defined recursively on the left factor, we have

$$d_{n+1}(f_*(F \circledcirc E)) \;\equiv\; d_{n+1}(f_*(F) \circledcirc E) \;\equiv\; \delta^{n+1}_E \circ d_n(f_*F) \;=\; \delta^{n+1}_E \circ d_n(F) \circ T^0_f$$

where the only rightmost equality uses naturality of d_n. The rightmost term is definitionally equal to $d_{n+1}(F \circledcirc E) \circ T^0_f$, as desired.

For naturality of d_{n+1} in the second variable, let $g : B' \to B$ be a homomorphism. Again, we consider a general element $F \circledcirc E$ as above. Since pullback of longer exact sequences are defined directly on the last splice factor, we have

$$d_{n+1}(g^*(F \circledcirc E)) \;\equiv\; d_{n+1}(F \circledcirc g^*E) \;\equiv\; \delta^{n+1}_{g^*E} \circ d_n(F) \;=\; T^{n+1}_g \circ \delta^{n+1}_E \circ d_n(F) \;\equiv\; T^{n+1}_g \circ d_{n+1}(E \circledcirc F),$$

where the only non-definitional equality comes from part (ii) of the δ-functor structure of T applied to the natural morphism $g^*E \to E$ of short exact sequences. □

The following is Proposition 4.1 in [Yon60], whose proof is easy to translate to our setting. Abelian categories are defined as usual.

Lemma 4.2.23. *Let $\mathscr{A}$ be an abelian category. Consider two additive functors $S, T : \mathscr{A} \to \mathsf{Ab}$. Suppose $\eta_A : S(A) \to T(A)$ is a collection of set-maps, natural in $A \in \mathscr{A}$. Then each η_A is a homomorphism.* □

We come to the main result of this section.

Theorem 4.2.24. *The large δ-functor $\{\mathsf{Ext}^n_R(-, A)\}_{n:\mathbb{N}}$ is universal, for any R-module A.*

Proof. Let $\{T^n : R\text{-}\mathsf{Mod}^{\mathrm{op}} \to \mathsf{Ab}'\}_{n:\mathbb{N}}$ be a large δ-functor. Note that $(\mathsf{Ext}^0_R(-, A) \Rightarrow T^0) \simeq T^0(A)$, by the Yoneda lemma. (To be precise, we view R-Mod as an Ab'-enriched category, and use the Yoneda lemma.) We will construct a morphism of (large) δ-functors $\mathsf{Ext}^n_R(-, A) \to T$ for any element $\eta : T^0(A)$, and show that such morphisms are uniquely determined by their restriction to the zeroth level.

Let $n : \mathbb{N}$, let $\eta : T^0(A)$, and let B be an R-module. Using d_n from the previous proposition, define

$$u_n(-) \equiv d_n(-, \eta) \ : \ \mathsf{Ext}^n_R(B, A) \longrightarrow T^n(B).$$

Clearly u_n is a group homomorphism. Also, since $u_0(\mathrm{id}_A) = T_{\mathrm{id}_A}(\eta) = \eta$, u_0 corresponds to η under the Yoneda lemma.

To see that $\{u_n\}_{n:\mathbb{N}}$ is a morphism of δ-functors, we need to show that it respects the connecting maps. To that end, let $B \to E \to B'$ be a short exact sequence. We proceed by induction. For $n \equiv 0$, we need to show that the following diagram commutes:

$$\begin{tikzcd} R\text{-}\mathsf{Mod}(B, A) \arrow[r, "u_0"] \arrow[d, "- \circledcirc E"] & T^0(B) \arrow[d, "\delta^0_E"] \\ \mathsf{Ext}^1_R(B', A) \arrow[r, "u_1"] & T^1(B') . \end{tikzcd} \tag{4.1}$$

Let $f : B \to A$ be an R-module morphism and recall that $f \circledcirc E \equiv f_*(E)$. By definition, we have that $u_1(f_*(E)) = \delta^0_{f_*(E)}(\eta)$. Using functoriality of the δ-structure of T, the natural map of short exact sequences from E to $f_*(E)$ yields a commuting square

$$\begin{tikzcd} T^0(A) \arrow[r, "T_f"] \arrow[d, "\delta^0_{f_*(E)}"] & T^0(B) \arrow[d, "\delta^0_E"] \\ T^1(B') \arrow[r, equal] & T^1(B'). \end{tikzcd}$$

Thus $\delta^0_{f_*(E)}(\eta) = \delta^0_E(T_f(\eta))$. The right-hand side is precisely $\delta^0_E(u_0(f))$, concluding the base case.

For the inductive step, we need to show that the following square commutes, for $n \geq 1$:

$$\begin{tikzcd} \mathsf{Ext}^n_R(B, A) \arrow[r, "u_n"] \arrow[d, "- \circledcirc E"] & T^n(B) \arrow[d, "\delta^n_E"] \\ \mathsf{Ext}^{n+1}_R(B', A) \arrow[r, "u_{n+1}"] & T^{n+1}(B') . \end{tikzcd} \tag{4.2}$$

Whether the square commutes is a proposition, so we may choose a representative F of an element in the top left corner. But then the square clearly commutes by the definition of u (and d).

It remains to show uniqueness of the δ-functor morphism u. Specifically, we need to show that for any δ-functor morphism $\{v_n : \mathsf{Ext}_R^n(-, A) \Rightarrow T^n\}_{n:\mathbb{N}}$ such that $v_0 = u_0$, we have that $v = u$. To show that $v_1(E) = u_1(E)$ for any $E : \mathsf{Ext}_R^1(B, A)$, we may assume E is a short exact sequence. Then we may consider diagram (4.1), but with the bottom horizontal map being v_1. For this E, the top-left corner is $R\text{-}\mathsf{Mod}(A, A)$ and we may chase id_A around the two sides of the square. Since the square commutes, we get $v_1(E) = \delta_E^0(\eta)$, and the right-hand side is $u_1(E)$, by definition. Similarly, for the inductive step we may write a general element of $\mathsf{Ext}_R^{n+1}(B, A)$ as a splice $F \circledcirc E$ and consider diagram (4.2) with the lower horizontal map being v_{n+1}. (By the induction hypothesis the top horizontal map is $u_n = v_n$.) Chasing F around the two sides of the square, we get $v_{n+1}(F \circledcirc E) = \delta_E^n(u_n(E)) = u_{n+1}(E)$, as desired. □

4.2.5 Computing Ext via projective resolutions

In this section, we use the long exact sequence to show that our Ext groups can be computed using projective resolutions. A dual argument shows the same for injective resolutions. We begin by defining and characterizing projectivity and injectivity of modules in our setting.

Definition 4.2.25. We say that an R-module P is **projective** if for all R-modules A and B (in $\mathcal{U}$), every epimorphism $e : R\text{-}\mathsf{Mod}(A, B)$ and every $f : R\text{-}\mathsf{Mod}(P, B)$, there merely exists a lift of f through e:

$$\Big\| \sum_{g:R\text{-}\mathsf{Mod}(P,A)} e \circ g = f \Big\|.$$

In other words, the postcomposition map $e_* : R\text{-}\mathsf{Mod}(P, A) \to R\text{-}\mathsf{Mod}(P, B)$ is an epimorphism. We write $\mathsf{IsProjective}(P)$ for this property.

It is clear that R^n is a projective R-module, for any ring R and natural number n. More generally, if X is a projective *set*, then the free R-module on X is a projective R-module. In addition, binary coproducts of projective modules are easily seen to be projective.

The following reproduces a classical characterization of projective modules.

Proposition 4.2.26. *Let P be an R-module. The following are equivalent:*

(i) P is projective.

(ii) Every epimorphism $p : R\text{-}\mathsf{Mod}(A, P)$ merely splits, i.e., the following holds:

$$\Big\| \sum_{s:R\text{-}\mathsf{Mod}(P,A)} p \circ s = \mathrm{id}_P \Big\|.$$

(iii) $\mathsf{Ext}_R^1(P, A) = 0$ for all R-modules A.

Proof. The equivalence between (*i*) and (*ii*) mirrors the classical argument, and the equivalence of (*ii*) and (*iii*) follows from Proposition 4.2.6. □

Definition 4.2.27. We say that an R-module I is **injective** if for all R-modules A and B (in $\mathcal{U}$), every monomorphism $m : R\text{-}\mathsf{Mod}(A, B)$ and every $f : R\text{-}\mathsf{Mod}(A, I)$, there merely exists an extension of f along m. In other words, the precomposition map $m^* : R\text{-}\mathsf{Mod}(B, I) \to R\text{-}\mathsf{Mod}(A, I)$ is an epimorphism. We write $\mathsf{IsInjective}(I)$ for this property. A dual argument characterizes the injectives using mere splittings or the condition that $\mathsf{Ext}^1_R(B, I) = 0$ for all B.

In Section 4.3.2, we interpret projectivity into a model of HoTT and study its relation to existing notions of projectivity. We do the same for injectivity in Section 4.3.3.

Now we turn to computing Ext^n_R from a projective resolution. The argument is standard homological algebra, and the content is that it holds with the results available to us in homotopy type theory. In the following, assume we have a projective resolution $P_\bullet$ of B. This is equipped with a surjection $p_0 : P_0 \to B$ inducing an isomorphism $P_0/\operatorname{im}(P_1) = B$, and so P_1 surjects onto $B_1 :\equiv \ker(p_0)$. Continuing inductively, we may factor the projective resolution as follows:

$$\begin{tikzcd}
{} \arrow[r, dotted] & P_2 \arrow[r] \arrow[d, two heads, "p_2"] & P_1 \arrow[r] \arrow[d, two heads, "p_1"] & P_0 \arrow[d, two heads, "p_0"] \\
{} \arrow[ur, dotted] & B_2 \arrow[ur, hook, "i_2"'] & B_1 \arrow[ur, hook, "i_1"'] & B_0\,,
\end{tikzcd} \tag{4.3}$$

where $B_0 :\equiv B$ and $B_{i+1} :\equiv \ker(p_i)$. Let $P_{-1} :\equiv 0$ and $i_0 :\equiv 0$ in the following.

Proposition 4.2.28. *The abelian group* $\mathsf{Ext}^n_R(B, A)$ *is the* n^{th}*cohomology of the cochain complex*

$$R\text{-}\mathsf{Mod}(P_\bullet, A) :\equiv (\cdots \to R\text{-}\mathsf{Mod}(P_{n-1}, A) \to R\text{-}\mathsf{Mod}(P_n, A) \to R\text{-}\mathsf{Mod}(P_{n+1}, A) \to \cdots).$$

Proof. Applying $R\text{-}\mathsf{Mod}(-, A)$ to (4.3) gives a diagram

$$\begin{tikzcd}
R\text{-}\mathsf{Mod}(P_{n-1}, A) \arrow[r] \arrow[dr, "i_n^*"'] & R\text{-}\mathsf{Mod}(P_n, A) \arrow[r] \arrow[dr, "i_{n+1}^*"'] & R\text{-}\mathsf{Mod}(P_{n+1}, A) \\
& R\text{-}\mathsf{Mod}(B_n, A) \arrow[u, hook, "p_n^*"] & R\text{-}\mathsf{Mod}(B_{n+1}, A) \arrow[u, hook, "p_{n+1}^*"']
\end{tikzcd}$$

which has the chain complex across the top. Since p_{n+1} is an epimorphism, p_{n+1}^* is a monomorphism, and so we get that $\ker(R\text{-}\mathsf{Mod}(P_n, A) \to R\text{-}\mathsf{Mod}(P_{n+1}, A)) = \ker(i_{n+1}^*)$. Since $(B_{n+1} \xrightarrow{i_{n+1}} P_k \xrightarrow{p_n} B_n)$ is a short exact sequence, the contravariant long exact sequence implies that $\ker(i_{n+1}^*)$ is $R\text{-}\mathsf{Mod}(B_n, A)$, with p_n^* being the kernel inclusion. Consequently,

$$H^n(R\text{-}\mathsf{Mod}(P_\bullet, A)) = \ker(i_{n+1}^*)/\operatorname{im}(R\text{-}\mathsf{Mod}(P_{n-1}, A)) = R\text{-}\mathsf{Mod}(B_n, A)/\operatorname{im}(i_n^*) = \operatorname{cok}(i_n^*).$$

If $n = 0$, then this is $R\text{-}\mathsf{Mod}(B_0, A) \equiv \mathsf{Ext}^0_R(B, A)$, since $i_0 \equiv 0$. To understand $\operatorname{cok}(i_{n+1}^*)$, we use the full long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & R\text{-}\mathsf{Mod}(B_n, A) \arrow[r, "p_n^*"] & R\text{-}\mathsf{Mod}(P_n, A) \arrow[r, "i_{n+1}^*"] & R\text{-}\mathsf{Mod}(B_{n+1}, A) \arrow[dll, rounded corners] \\
& \mathsf{Ext}^1_R(B_n, A) \arrow[r] & 0 \arrow[r] & \mathsf{Ext}^1_R(B_{n+1}, A) \arrow[dll, rounded corners] \\
& \mathsf{Ext}^2_R(B_n, A) \arrow[r] & 0 \arrow[r] & \mathsf{Ext}^2_R(B_{n+1}, A) \arrow[r] & \cdots
\end{tikzcd}$$

where the zeros down the middle column appear due to P_n being projective, by Proposition 4.2.26. It follows from the first connecting map that $\mathrm{cok}(i_{n+1}^*) = \mathsf{Ext}_R^1(B_n, A)$. The subsequent connecting maps imply that $\mathsf{Ext}_R^k(B_{n+1}, A) = \mathsf{Ext}_R^{k+1}(B_n, A)$ for $k \geq 1$. Applying the second equality recursively gives $\mathsf{Ext}_R^1(B_n, A) = \mathsf{Ext}_R^{n+1}(B, A)$, for $n \geq 0$, and so we conclude that $\mathrm{cok}(i_{n+1}^*) = \mathsf{Ext}_R^{n+1}(B, A)$ for $n \geq 0$. □

While not formally dual, a similar argument using the covariant long exact sequence lets us compute $\mathsf{Ext}_R^n(B, A)$ via an injective resolution of A.

4.2.6 Ext of finitely presented modules over (constructive) PIDs

In Section 4.3.2, we will see examples which demonstrate that higher Ext groups of abelian groups do not necessarily vanish. The main result of this section is that finitely presented abelian groups B merely have projective dimension at most 1, and consequently $\mathsf{Ext}_{\mathbb{Z}}^n(B, -)$ vanishes for $n > 1$. This is true more generally for finitely presented modules over *principal ideal domains*, in the constructive sense of [LQ15]. Before turning to the constructive definition of a PID, we briefly discuss finitely presented modules.

Definition 4.2.29. Let R be a ring and let A be an R-module.

(i) A is **finitely generated** if there merely exists an epimorphism $R^n \twoheadrightarrow A$, for some $n : \mathbb{N}$.

(ii) A is **finitely presented** if there merely exists an epimorphism $p : R^n \twoheadrightarrow A$, for some $n : \mathbb{N}$, such that the kernel of p is finitely generated.

If A is finitely presented, then [LQ15, Lemma 1.0, p. 180] implies that any map $R^n \twoheadrightarrow A$ has finitely generated kernel. Moreover, Proposition 4.2(i) of loc. cit. says that a quotient A/I, where I is a finitely generated submodule of A, is also finitely presented. These facts play a role later in this section.

We now recall the constructive definition of a PID, and other relevant notions from [LQ15].

Definition 4.2.30. Let R be a commutative ring, and write $x \mid y :\equiv (\sum_{a:R} ax = y)$ for $x, y : R$.

(i) R is is an **integral domain** if every element $x : R$ is either equal to 0 or **regular**: the (left) multiplication map $y \mapsto xy : R \to R$ is a monomorphism.

(ii) A **greatest common divisor** of $x, y : R$ is an element g such that the following holds:

$$g \mid x \times g \mid y \times \Big(\prod_{z:R} (z \mid x \times z \mid y) \to z \mid g \Big).$$

(iii) R is a **Bézout ring** if for every $x, y : R$ there merely exist $u, v : R$ such that $ux + vy$ is a greatest common divisor of x and y. The data of such a u and v is called a **Bézout relation** for x and y.

(iv) R is a **Bézout domain** if it is both a Bézout ring and an integral domain.

(v) R is a **principal ideal domain (PID)** if it is a Bézout domain, and every ascending chain of finitely generated ideals merely admits two equal consecutive terms.

This definition of PIDs might seem foreign to classically trained mathematicians, so we take a moment to give some context.

Definition 4.2.31. An ideal I of a ring R is **principal** if the proposition $\|\sum_{a:R} Ra = I\|$ holds.

It is not true in our setting that all ideals of $\mathbb{Z}$ are principal. This is for a good reason: in models, there may be "local" ideals which have no "global" generators. However, all *finitely generated* ideals of $\mathbb{Z}$ are principal in our setting, and it is straightforward to verify that $\mathbb{Z}$ is a PID in the sense of Definition 4.2.30. Indeed, in $\mathbb{Z}$ one can actually *compute* greatest common divisors and Bézout relations—they don't just merely exist. The ascending chain condition actually computes as well: using the following lemma it reduces to checking equality of principal ideals, which one can do since $\mathbb{Z}$ has decidable equality.

Lemma 4.2.32. *Suppose R is a Bézout ring. Any finitely generated ideal of R is principal.*

Proof. The existence of Bézout relations means that every ideal of R that is generated by two elements is principal. The claim follows by recursion. □

The reason for the additional "Noetherianity" condition in our definition of PID is that it is needed to compute Smith normal forms—see [LQ15, p. 209] for further discussion. We also get that any finitely presented module over a PID merely splits into a free part and a product of cyclic modules. Using additivity of $\mathsf{Ext}^1_{\mathbb{Z}}(-,A)$ (Proposition 4.2.9), projectivity of $\mathbb{Z}$, and that $\mathsf{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n,A) \simeq A/n$ (Corollary 3.4.9) we deduce:

Proposition 4.2.33. *Let B be a finitely presented abelian group, and write $B \simeq (\bigoplus_{i=1}^k \mathbb{Z}/d_i) \oplus \mathbb{Z}^r$ for the decomposition which merely exists by [LQ15, Prop. 7.3]. For any abelian group A, we merely have an isomorphism $\mathsf{Ext}^1_{\mathbb{Z}}(B,A) \simeq \prod_{i=1}^k A/d_i$.* □

From the existence of Smith normal forms over PIDs, we also deduce the following.

Proposition 4.2.34. *Suppose R is a PID. For any R-linear morphism $\alpha : R^m \to R^n$, there merely exist R-linear automorphisms ϕ and ψ of respectively R^m and R^n such that $\psi\alpha\phi$ sends the i^{th} basis vector in R^m to a multiple of the i^{th} basis vector in R^n for $1 \le i \le \min(m,n)$.*

Proof. Follows from [LQ15, Proposition 7.3(i)], whose proof is straightforward to carry out in HoTT. □

Using the proposition, we can prove the following generalization of Lemma 4.2.32.

Proposition 4.2.35. *Let R be a PID, and $n : \mathbb{N}$. A finitely generated submodule of R^n is merely free.*

Proof. Let K be a finitely generated submodule of R^n for some $n : \mathbb{N}$. We need to show that there merely exists some $k : \mathbb{N}$ and an isomorphism $R^k \simeq K$. By our assumption that K is finitely generated, there merely exists an epimorphism $R^l \twoheadrightarrow K$ for some $l : \mathbb{N}$. Write $\alpha : R^l \to K \to R^n$ for the composite map. Since we are proving a proposition, we may assume that the matrix of α is diagonal, in the sense of Proposition 4.2.34. The elements on the diagonal are either regular or zero, by integrality, and we may consider the standard basis elements $e_i : R^l$ such that $\alpha(e_i)_i$ is regular. Thus we get an inclusion $R^k \subseteq R^l$ induced by including these basis elements e_i. Finally, the composite map $p : R^k \to R^l \to K$ is necessarily an epimorphism, since we only threw away basis elements of R^l which are sent to 0 by α. By construction, the restriction of α to R^k is an embedding, thus p factors an embedding and must be one itself. It follows that p is an isomorphism. □

Recall that $\mathsf{Ext}^1_R(B, A)$ is itself an R-module whenever R is commutative. For a PID R we deduce from the above that $\mathsf{Ext}^1_R(B, A)$ is finitely presented (as an R-module) whenever A and B are, and moreover that $\mathsf{Ext}^n_R(B, -)$ vanishes for $n > 1$.

Corollary 4.2.36. *Let R be a PID. If B is a finitely presented R-module and A is any R-module, then $\mathsf{Ext}^n_R(B, A) = 0$ for $n > 1$. If A is also finitely presented, then so is the R-module $\mathsf{Ext}^1_R(B, A)$.*

Proof. Let B be a finitely presented R-module and let A be any R-module. Since we are proving a proposition, we may assume we have a short exact sequence $K \to R^n \to B$ where the kernel K is finitely generated. The previous proposition lets us moreover assume that K is actually free of finite rank m. Thus the short exact sequence gives a projective resolution of B, and the claim for $n > 1$ immediately follows by computing $\mathsf{Ext}^n_R(B, A)$ using this projective resolution (Proposition 4.2.28). The calculation of $\mathsf{Ext}^1_R(B, A)$ using this projective resolution yields an exact sequence

$$A^m \to A^n \to \mathsf{Ext}^1_R(B, A) \to 0.$$

Hence, if A is finitely presented, then $\mathsf{Ext}^1_R(B, A)$ is a quotient of the finitely presented module A^n by a finitely generated submodule (the image of $A^m \to A^n$), which is finitely presented. □

4.2.7 Ext of $\mathbb{Z}G$-modules

Any construction in homotopy type theory can be carried out "in context," meaning that the terms going into a particular construction may themselves depend on some extraneous variable. In an ∞-topos, the corresponding thing is to work in a *slice* of your base ∞-topos. In this section, we work in the context of a pointed, connected type X whose base point will be denoted by $\mathtt{pt} : X$. We will see that abelian groups in the context of X correspond to modules over the group ring $\mathbb{Z}\pi_1(X)$, and we will discuss our Ext groups in this setting.

By "an abelian group in the context of X" we mean a family $X \to \mathsf{Ab}$. Since Ab is a 1-type, it is equivalent to consider families on the 1-truncation of X. The latter is equivalent to $B\pi_1(X)$ since X is pointed and connected. To emphasize that no truncation assumptions are needed, we prefer to work with X in this section.

We begin by constructing the group ring $\mathbb{Z}G$ for a group G. For this we use the coproduct of abelian groups, which has various constructions—see, e.g., Section 2.3.3 and [LLM23].

Construction 4.2.37. Let G be a group. We construct the **group ring $\mathbb{Z}G$** as follows. The underlying abelian group of $\mathbb{Z}G$ is the coproduct $\bigoplus_G \mathbb{Z}$, which is the free abelian group on the set G. To define a bilinear map $\mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}G \to \mathbb{Z}G$ it suffices, by the tensor-hom adjunction and the universal property of coproducts, to give a function

$$G \to G \to \mathbb{Z}G.$$

For this we supply the map $(g, h) \mapsto 1_{gh}$, where 1_{gh} is the unit in the gh-summand of $\bigoplus_G \mathbb{Z}$. The resulting binary operation on $\mathbb{Z}G$ is bilinear, by construction. We need to check that $1_e : \mathbb{Z}G$ is a two-sided unit (where $e : G$ is the unit), and that multiplication is associative.

Under the equivalences

$$(\mathbb{Z}G \to \mathbb{Z}G) \;\simeq\; \Big(\prod_{g:G}(\mathbb{Z} \to \mathbb{Z}G)\Big) \;\simeq (G \to \mathbb{Z}G)$$

the identity map on $\mathbb{Z}G$ corresponds to $g \mapsto 1_g : G \to \mathbb{Z}G$. Since $eg = g$ for $g : G$, we see that $1_e \cdot (-) : \mathbb{Z}G \to \mathbb{Z}G$ is the identity. Similarly for $(-) \cdot 1_e$.

For associativity, simply observe that the two maps

$$(-) \cdot (\,- \cdot -\,),\ (\,- \cdot -\,) \cdot (-) : \mathbb{Z}G^3 \to \mathbb{Z}G$$

both correspond to the map $(g, h, k) \mapsto 1_{ghk} : G \to G \to G \to \mathbb{Z}G$, since G is associative.

Before our next statement, we specify that by an **invertible** element of a (possible non-commutative) ring R, we mean an element with a specified two-sided inverse.[3] If one exists, a two-sided inverse is unique, so the type of such defines a proposition. We write $R^\times$ for the group of invertible elements of a ring R.

Proposition 4.2.38. *The group ring functor* $\mathbb{Z}(-) : \mathsf{Grp} \to \mathsf{Ring}$ *is left adjoint to* $(-)^\times$.

Proof. We construct a bijection $\mathsf{Ring}(\mathbb{Z}G, R) \simeq \mathsf{Grp}(G, R^\times)$ which is natural in G and R. By the universal property of the coproduct, we already have a bijection $\mathbb{Z}\text{-}\mathsf{Mod}(\mathbb{Z}G, R) \simeq (G \to R)$. If a map on the left-hand side is a ring homomorphism, then the corresponding map $G \to R$ lands in $R^\times$, since ring homomorphisms are required to preserve the unit. Thus what we need to show is that a map $\phi : G \to R^\times$ is a group homomorphism if and only if the induced map $\hat{\phi} : \mathbb{Z}G \to R$ is a ring homomorphism.

A map $\phi : G \to R^\times$ is a group homomorphism if and only if the two maps

$$\phi(-) \cdot \phi(-),\ \phi((-) \cdot (-)) : G^2 \to R^\times$$

coincide. This happens if and only if the two maps $\hat{\phi}(-) \cdot \hat{\phi}(-),\ \hat{\phi}((-) \cdot (-)) : \mathbb{Z}G^2 \to R$ coincide. In other words, ϕ is a group homomorphism if and only if $\hat{\phi}$ is a ring homomorphism. □

Using the previous proposition, we relate the category $X \to \mathsf{Ab}$ of abelian groups in the context of X to $\mathbb{Z}\pi_1(X)$-modules. Recall that for any abelian group M and ring R, an R-module structure on M corresponds to a ring homomorphism $R \to \mathsf{Ab}(M, M)$.

Proposition 4.2.39. *We have an equivalence of 1-categories* $\mathbb{Z}\pi_1(X)\text{-}\mathsf{Mod} \simeq (X \to \mathsf{Ab})$.

Proof. Using the previous proposition and uniqueness of deloopings of maps between groups, we have the following equivalences of types:

$$\begin{aligned}
\mathbb{Z}\pi_1(X)\text{-}\mathsf{Mod} \;&\simeq\; \sum_{M:\mathsf{Ab}} \mathsf{Ring}(\mathbb{Z}\pi_1(X), \mathsf{Ab}(M, M)) \\
&\simeq\; \sum_{M:\mathsf{Ab}} \mathsf{Grp}(\pi_1(X), \mathrm{Aut}_{\mathbb{Z}}(M)) \\
&\simeq\; \sum_{M:\mathsf{Ab}} (B\pi_1(X) \to_* (\mathsf{Ab}, M)) \\
&\simeq\; (B\pi_1(X) \to \mathsf{Ab}) \;\simeq (X \to \mathsf{Ab}),
\end{aligned}$$

[3] It is equivalent to require separate left and right inverses, since one can prove that these must agree (when both exist).

where the last line uses that $B\pi_1(X)$ is the 1-truncation of X. It is straightforward to make this association into a functor which is an equivalence of categories, keeping in mind that the hom-sets in the category $X \to \mathsf{Ab}$ are of the form $\prod_{x:X} \mathsf{Ab}(A_x, B_x)$ for $A, B : X \to \mathsf{Ab}$. □

Given a family $A : X \to \mathsf{Ab}$, the abelian group underlying the corresponding $\mathbb{Z}\pi_1(X)$-module is $A_{\mathtt{pt}}$, the evaluation of A at the base point $\mathtt{pt} : X$. Accordingly, we may view $A_{\mathtt{pt}}$ either as an abelian group or as a $\mathbb{Z}\pi_1(X)$-module, depending on context.

An example of particular interest to us is the following.

Proposition 4.2.40. *For $B, A : X \to \mathsf{Ab}$, the abelian group $\mathsf{Ext}^n_{\mathbb{Z}}(B_{\mathtt{pt}}, A_{\mathtt{pt}})$ is naturally a $\mathbb{Z}\pi_1(X)$-module.*

Proof. Apply Proposition 4.2.39 to the family $x \mapsto \mathsf{Ext}^n_{\mathbb{Z}}(B_x, A_x)$. □

When $n = 1$, we can understand the action via the following lift to $\mathsf{SES}_{\mathbb{Z}}$. For any $x : X$, consider the type of short exact sequences from A_x to B_x:

$$x \longmapsto \mathsf{SES}_{\mathbb{Z}}(B_x, A_x) \ : \ X \longrightarrow \mathcal{U}\,.$$

This family defines a ΩX-action on $\mathsf{SES}_{\mathbb{Z}}(B_*, A_*)$, and one can check that the action of an element $g : \Omega X$ on a short exact sequence E is given by

$$g \cdot E :\equiv (A_{\mathtt{pt}} \xrightarrow{i_E \circ g^{-1}} E_{\mathtt{pt}} \xrightarrow{g \circ p_E} B_{\mathtt{pt}}), \tag{4.4}$$

where we have used the action of g on $A_{\mathtt{pt}}$ and $B_{\mathtt{pt}}$. Lemma 4.2.11 gives an alternative description in terms of pullbacks and pushouts. On components, this gives the action of $\pi_1(X)$ on $\mathsf{Ext}^1_{\mathbb{Z}}(B_*, A_*)$ from Proposition 4.2.40.

For $n > 1$, one gets a $\pi_1(X)$-action on $\mathsf{Ext}^n_{\mathbb{Z}}(B_{\mathtt{pt}}, A_{\mathtt{pt}})$ which is similar to Eq. (4.4), but with a (representative of a) longer extension in place of a short exact sequence.

The following theorem identifies the type of fixed points of the action (4.4).

Theorem 4.2.41. *For any $B, A : X \to \mathsf{Ab}$, we have an equivalence*

$$\prod_{x:X} \mathsf{SES}_{\mathbb{Z}}(B_x, A_x) \ \simeq \ \mathsf{SES}_{\mathbb{Z}\pi_1(X)}(B_{pt}, A_{pt}).$$

Proof. An element of $\prod_{x:X} \mathsf{SES}_{\mathbb{Z}}(B_x, A_x)$ is easily seen to consist of a family $E : X \to \mathsf{Ab}$ along with two sections $i : \prod_{x:X} \mathsf{Ab}(A_x, E_x)$ and $p : \prod_{x:X} \mathsf{Ab}(E_x, B_x)$ such that the proposition $\prod_{x:X} \mathtt{IsExact}(i_x, p_x)$ holds. The sections i and p correspond to $\mathbb{Z}\pi_1(X)$-module maps $A_{\mathtt{pt}} \to E_{\mathtt{pt}}$ and $B_{\mathtt{pt}} \to E_{\mathtt{pt}}$ under Proposition 4.2.39. The proposition $\prod_{x:X} \mathtt{IsExact}(i_x, p_x)$ holds if and only if it holds at the base point of X, since X is connected. In other words, it holds if and only if $A_{\mathtt{pt}} \to E_{\mathtt{pt}} \to B_{\mathtt{pt}}$ is an exact sequence of abelian groups (and hence of $\mathbb{Z}\pi_1(X)$-modules). □

Corollary 4.2.42. *For any $M : X \to \mathsf{Ab}$, we have a group isomorphism*

$$H^1(X; M) \ \simeq \ \mathsf{Ext}^1_{\mathbb{Z}\pi_1(X)}(\mathbb{Z}, M_{pt}),$$

where the left-hand side is the cohomology of X with local coefficients in M, and $\mathbb{Z}$ on the right has trivial $\mathbb{Z}\pi_1(X)$-action.

Proof. Since $\mathbb{Z}$ is a projective abelian group, by Proposition 4.2.26 we have that $\mathsf{SES}_{\mathbb{Z}}(\mathbb{Z}, M_x)$ is connected, for any $x : X$. By Corollary 4.2.4, its loop space is $\mathsf{Ab}(\mathbb{Z}, M_x) \simeq M_x$. It follows that we have an equivalence $\mathsf{SES}_{\mathbb{Z}}(\mathbb{Z}, M_x) \simeq \mathsf{K}(M_x, 1)$ which is natural in M_x. Thus we get a natural equivalence

$$\prod_{x:X} \mathsf{K}(M_x, 1) \simeq \prod_{x:X} \mathsf{SES}_{\mathbb{Z}}(\mathbb{Z}, M_x).$$

The set-truncation of the left-hand side is $H^1(X; M)$, and the set-truncation of the right-hand side is $\mathsf{Ext}^1_{\mathbb{Z}\pi_1(X)}(\mathbb{Z}, M_{\mathtt{pt}})$ by the previous theorem. After truncating the equivalence above, we get a natural bijection which is an isomorphism by Lemma 4.2.23. □

4.3 Ext in an ∞-topos

Statements in HoTT can be interpreted into an ∞-topos [dBB20, dBoe20, KL18, LS20, Shu19]. In this section, we study the interpretation of the constructions and results from Section 4.2. Our precise setup is explained in the section on foundations just below.

Results about rings and modules in HoTT apply to ring or module objects in $\mathcal{X}$, which we stress are 0-truncated. Accordingly, these objects live in the sub-∞-category of 0-truncated objects in $\mathcal{X}$, which is a 1-topos [Lur09, Theorem 6.4.1.5]. In particular, if R is a ring object in $\mathcal{X}$, then the category of R-modules is equivalent to a category of ordinary sheaves of modules. Such categories have been extensively studied, and the reader may for example refer to [KS06, Chapter 18] for background.

In Section 4.3.1, we work out the interpretation $\mathtt{SES}_R(B, A)$ of the type of short exact sequences $\mathsf{SES}_R(B, A)$, given a ring object R and two R-module objects A and B in $\mathcal{X}$. (Our font usage is explained below.) The object $\mathtt{SES}_R(B, A)$ is shown to classify short exact sequences $A \to E \to B$ of R-modules in $\mathcal{X}$ (Proposition 4.3.3). From this we deduce that the set of path components of the space of global points of this object recovers the usual *external* Yoneda Ext groups (Corollary 4.3.4).

Our next objective is to understand the interpretation $\mathtt{Ext}^n_R(B, A)$ of our Ext groups, which are abelian group objects in $\mathcal{X}$. In special cases (Proposition 4.3.10 and Corollary 4.3.31), we show that the global points of $\mathtt{Ext}^n_R(B, A)$ recover the ordinary Ext groups. But this fails in general. Indeed, we give examples showing that either one can vanish without the other one vanishing (Examples 4.3.15, 4.3.41 and 4.3.42). However, we show that in many cases $\mathtt{Ext}^n_R(B, A)$ recovers a known construction. In any 1-topos, one can define *sheaf* Ext groups (Definition 4.3.28) by taking the right derived functors of the internal hom of modules, using the existence of enough (external) injectives. (The name "sheaf" Ext is used because one often works in a category of sheaves; the name "local" Ext is also used.) We can extend this to an ∞-topos $\mathcal{X}$, by considering sheaf Ext in the 1-topos of 0-truncated objects in $\mathcal{X}$. When sets cover in $\mathcal{X}$ (see Definition 4.3.21), we show that $\mathtt{Ext}^1_R(B, A)$ agrees with sheaf Ext (Theorem 4.3.29). We do this by showing that for such $\mathcal{X}$, injectivity of modules in HoTT corresponds to internal injectivity (Corollary 4.3.27). Since external injectives are always internally injective, it follows that our Ext groups can also be computed using externally injective resolutions, and therefore that they agree with sheaf Ext. A consequence of this is that in this setting our Ext groups only depend on the 1-topos of 0-truncated objects in $\mathcal{X}$ (Corollary 4.3.33).

We also study various notions of projectivity in Section 4.3.2, and provide a computation of our Ext groups using a resolution which is projective in the sense of HoTT in Proposition 4.3.12. This computation demonstrates, in particular, that our higher Ext groups need not vanish over the ring object $\mathbb{Z}$. Lastly, Section 4.3.4 contains a detailed study of our Ext groups over a pointed, connected type X and over a group ring $\mathbb{Z}G$. The considerations in this final section are meant to illustrate and exemplify the theory developed throughout Section 4.3, in addition to being of interest in their own right.

Foundations. We explain our setup for interpreting HoTT into the ∞-topos $\mathcal{X}$. We assume an inaccessible cardinal κ for the entirety of Section 4.3. Formally, the interpretation of HoTT lands in a type-theoretic model topos $\mathscr{M}$ presenting $\mathcal{X}$, which always exists [Shu19]. The model topos $\mathscr{M}$ admits a univalent universe which classifies relatively κ-presentable fibrations. This universe allows us to interpret HoTT with a single universe. Constructions in $\mathscr{M}$ present constructions in $\mathcal{X}$, and we are interested in studying the fruits of our labour in the latter. These constructions are all uniquely determined up to equivalence by their universal properties coming from the interpretation of the various type constructors, and this obviates the need to explicitly work with $\mathscr{M}$. Moreover, it is shown in [Ste23] that the univalent universe in $\mathscr{M}$ presents an object classifier $\mathtt{u} : \tilde{\mathtt{U}} \to \mathtt{U}$ for relatively κ-compact morphisms in $\mathcal{X}$ [Lur09, Section 6.1.6]. This means that the mapping space $\mathcal{X}(X, \mathtt{U})$ is naturally equivalent to the space $(\mathcal{X}/^{\kappa}X)^{\simeq}$ of relatively κ-compact maps into X in $\mathcal{X}$, and lets us precisely determine the objects and structures in $\mathcal{X}$ which are classified by the universes (of types, and of modules) that we consider. We write $\mathcal{X}_\kappa$ for the sub-∞-category of κ-compact objects in $\mathcal{X}$.

Our results from the previous section concern truncated objects such as modules, and types of short exact sequences. The truncation level makes the interpretation particularly straightforward, and there is not much higher coherence to manage. For this reason—and for reasons of space and interest—we allow ourselves to state and work with the result of our interpretation directly in $\mathcal{X}$ and not make any further mention of $\mathscr{M}$.

Notation and conventions. We write $\mathcal{X}$ for a fixed ∞-topos throughout this section. By "topos" we mean *Grothendieck* topos unless otherwise specified. Fonts are used to distinguish types in HoTT, the objects obtained by interpretation in $\mathcal{X}$, and the classical counterparts. For example, $\mathsf{Ext}^n_R(B, A)$ will continue to mean the Ext group in HoTT constructed in Section 4.2. Its interpretation $\mathtt{Ext}^n_R(B, A)$ into $\mathcal{X}$ is written in typewriter font. The classical external Ext groups are written in normal font $\mathrm{Ext}^n_R(B, A)$, whereas the classical sheaf Ext groups are denoted with an underline $\underline{\mathrm{Ext}}^n_R(B, A)$. In general, we use underlines to denote traditional constructions internal to $\mathcal{X}$, such as the internal hom $R\text{-}\underline{\mathrm{Mod}}(A, B)$ between two R-module objects A and B. The (external) set of R-module homomorphisms is $R\text{-}\mathrm{Mod}(A, B)$.

The 1-topos of 0-truncated objects in $\mathcal{X}$ is denoted $\tau_{\leq 0}(\mathcal{X})$, and we write $\mathrm{Set}_{\mathcal{X}}$ for $\tau_{\leq 0}(\mathcal{X}_\kappa)$. We write $\mathrm{Ab}_{\mathcal{E}}$ for the (abelian) category of abelian group objects in a (possibly elementary) 1-topos $\mathcal{E}$, and define $\mathrm{Ab}_{\mathcal{X}} :\equiv \mathrm{Ab}_{\tau_{\leq 0}(\mathcal{X}_\kappa)}$. The ∞-topos of spaces is denoted $\mathscr{S}$, and we simply write Ab for $\mathrm{Ab}_{\mathscr{S}}$, the category of ordinary (κ-compact) abelian groups. Our abelian group and module objects in $\mathcal{X}$ are always be assumed to be κ-compact.

Base points are denoted by $\mathtt{pt}$, unless another name is given.

4.3.1 The object of short exact sequences

Let R be a ring object in $\mathcal{X}_\kappa$, i.e., a ring object in the 1-topos $\mathrm{Set}_{\mathcal{X}}$, and write R-Mod for the category of (κ-compact) R-modules. Statements from HoTT about rings can be interpreted into $\mathcal{X}$ using R. In particular, Theorem 2.4.17 shows that the category of modules (in $\mathcal{U}$) over R interprets to an internal category $R\text{-}\mathtt{Mod}$ which represents the presheaf of 1-categories

$$X \longmapsto (X{\times}R)\text{-Mod} \ : \ \mathcal{X}^{\mathrm{op}} \longrightarrow \mathrm{Cat},$$

where $X \times R$ is the ring object in $\mathcal{X}/X$ obtained by pulling back. Thus a family of modules $X \to R\text{-}\mathtt{Mod}$ in $\mathcal{X}$ corresponds precisely to a (relatively κ-compact) $(X{\times}R)$-module in the slice $\mathcal{X}/X$.

For any two R-modules A and B in $\mathcal{X}_\kappa$, we interpret the type $\mathsf{SES}_R(B, A)$ into $\mathcal{X}$ to get an object $\mathtt{SES}_R(B, A)$ of short exact sequences. We start by describing this and the interpretation of our Ext groups, for spaces:

Proposition 4.3.1. *Let R be a ring object in $\mathcal{S}$ (i.e., an ordinary ring), and let B and A be R-modules.*

(i) The interpretation of $\mathsf{SES}_R(B, A)$ into $\mathcal{S}$ is equivalent to the ordinary (1-truncated) space of short exact sequences from A to B;

(ii) The interpretation of $\mathtt{Ext}^n_R(B, A)$ into $\mathcal{S}$ is isomorphic to $\mathrm{Ext}^n_R(B, A)$, i.e., the ordinary Ext group.

In spaces, we will also use a slight generalization of this statement where the category of R-modules is replaced by an arbitrary abelian (univalent) category.

Proof. Using that $R\text{-}\mathtt{Mod}$ classifies R-modules (Proposition 2.4.15), it is straightforward to check that the interpretation of $\mathtt{SES}_R(B, A)$ is equivalent to the usual space $\mathrm{SES}_R(B, A)$ of short exact sequences. It follows that the interpretation of our $\mathsf{Ext}^1_R(B, A)$ recovers the ordinary Yoneda Ext group $\mathrm{Ext}^1_R(B, A)$.

For $n > 1$, the interpretation $\mathtt{Ext}^n_R(B, A)$ into $\mathcal{S}$ recovers Yoneda's definition of $\mathrm{Ext}^n_R(B, A)$ as a quotient of the space of length-n exact sequences, which is well-known to give the usual Ext groups defined using resolutions. □

Our present goal is to relate $\mathtt{SES}_R(B, A)$ to $\mathrm{SES}_R(B, A)$ for ring and module objects in $\mathcal{X}$. To do this, we require a lemma which characterizes the interpretation of the objects of epimorphisms and monomorphisms from HoTT.

Lemma 4.3.2. *Let A and B be R-modules in $\mathcal{X}$. The object $\mathtt{Epi}_R(B, A)$ resulting from interpretation classifies R-module epimorphisms $B \to A$ in $\mathcal{X}$. Likewise, the object $\mathtt{Mono}_R(B, A)$ classifies R-module monomorphisms.*

Proof. The statement that $\mathtt{Epi}_R(B, A)$ classifies epimorphisms means that there is a natural equivalence

$$\mathcal{X}(X, \mathtt{Epi}_R(B, A)) \ \simeq \ \mathrm{Epi}_{(X\times R)}(X \times B, X \times A)$$

of (0-truncated) spaces, for all $X \in \mathcal{X}$. Here the right-hand side is the set of epimorphisms of $(X{\times}R)$-modules. A map $f : X \to \mathtt{Epi}_R(B, A)$ corresponds to an $(X{\times}R)$-module homomorphism

$f' : X \times B \to X \times A$ in $\mathcal{X}/X$ that satisfies the interpretation of being (-1)-connected from HoTT. Since we know that (-1)-connected maps in HoTT correspond to (-1)-connected maps in an ∞-topos, we have that f' is a (-1)-connected map over X. This means that f' is an (effective) epimorphism between sets (hence modules), as desired.

The statement for $\mathtt{Mono}_R(B, A)$ is shown similarly, but using that (-1)-truncated maps in HoTT correspond to (-1)-truncated maps in $\mathcal{X}$ (which are monomorphisms between sets). □

We use this lemma in the proof of the following proposition, which says that the object of short exact sequences from HoTT classifies short exact sequences in $\mathcal{X}$. Recall that base change functors are exact and therefore preserve ring and module objects, as well as exact sequences of the latter. This means that any morphism $f : X \to Y$ in $\mathcal{X}$ induces a map

$$f^* \ : \ \mathrm{SES}_{(Y\times R)}(Y \times B, Y \times A) \longrightarrow \mathrm{SES}_{(X\times R)}(X \times B, X \times A)$$

by base change, for any two R-modules A and B in $\mathcal{X}$.

Proposition 4.3.3. *Let A and B be R-modules in $\mathcal{X}_\kappa$. The object $\mathtt{SES}_R(B, A)$ represents the presheaf*

$$X \longmapsto \mathrm{SES}_{(X\times R)}(X \times B, X \times A) \ : \ \mathcal{X}^{\mathrm{op}} \longrightarrow \mathscr{S}.$$

In particular, the (1-truncated) space $\mathrm{SES}_R(B, A)$ is equivalent to the global points of the object $\mathtt{SES}_R(B, A)$.

Proof. Let $X \in \mathcal{X}$. Our goal is to produce equivalences of spaces

$$\mathcal{X}(X, \mathtt{SES}_R(B, A)) \ \simeq \ \mathrm{SES}_{(X\times R)}(X \times B, X \times A)$$

which are natural in $X \in \mathcal{X}$. Using the adjunction $\Sigma_X \dashv X \times (-)$ and base-change stability of interpretation, we may replace the left-hand side above via the following natural equivalences:

$$\mathcal{X}(X, \mathtt{SES}_R(B, A)) \ \simeq \ \mathcal{X}/X(\,\mathrm{id}_X, X \times \mathtt{SES}_R(B, A)) \ \simeq \ \mathcal{X}/X(\,\mathrm{id}_X, \mathtt{SES}_{(X\times R)}(X \times B, X \times A)).$$

The rightmost space is the global points of the object $\mathtt{SES}_{(X\times R)}(X \times B, X \times A)$. It therefore suffices to consider the case $X = 1$, and to construct an equivalence

$$\mathcal{X}(1, \mathtt{SES}_R(B, A)) \ \simeq \ \mathrm{SES}_R(B, A).$$

The right-hand side above is the domain of a (-1)-truncated map into the (1-truncated) space G consisting of κ-compact R-modules E equipped with a monomorphism $i : A \to E$ and an epimorphism $p : E \to B$. The object $\mathtt{SES}_R(B, A)$ is the domain of a (-1)-truncated map into the corresponding object G' of such things in $\mathcal{X}$. By Proposition 4.3.1(i) applied to the abelian category R-Mod, the previous lemma, and Theorem 2.4.17, the space of global points of G' is naturally equivalent to G. Thus both sides of the equivalence above are fibered over G, and we can therefore obtain the desired equivalence from a fibrewise bi-implication (which yields an equivalence since the maps are (-1)-truncated). We need to check that the internal proposition $\mathtt{IsExact}(i, p)$ holds if and only i and p define an exact complex in the usual sense.

The proposition $\mathtt{IsExact}(i, p)$ consists of a witness that the internally induced homomorphism $A \to \mathtt{ker}(p)$ is (-1)-connected. The module $\mathtt{ker}(p)$ is clearly equivalent to the externally defined kernel $\ker(p)$, both being given by the fibre over the global point $0 : 1 \to B$.

Under this equivalence, the aforementioned witness implies that the induced map $A \to \ker(p)$ is surjective (i.e., (-1)-connected), and vice-versa.

In conclusion, $\mathcal{X}(1, \mathtt{SES}_R(B, A))$ and $\mathrm{SES}_R(B, A)$ are naturally fibrewise equivalent as spaces over G, which yields the desired natural equivalence on total spaces. □

Recall that $\pi_0 \,\mathrm{SES}_R(B, A)$ is the definition of the Yoneda Ext groups (see, e.g., [Mac63]), which recover the Ext groups defined in terms of resolutions. Thus we have the following:

Corollary 4.3.4. *We have a natural isomorphism $\pi_0(\mathcal{X}(1, \mathtt{SES}_R(B, A))) \simeq \mathrm{Ext}^1_R(B, A)$ of ordinary abelian groups, for any R-modules A and B in $\mathcal{X}_\kappa$.* □

Since we do not have a good description of the (untruncated) type of length-n exact sequences, we do not have a corresponding statement for the higher Ext groups.

Note that taking global points and taking components do not commute, and it is important for the above result that we take global points before taking components. If we reverse the order, we get the claim that $\mathcal{X}(1, \mathtt{Ext}^1_R(B, A)) \simeq \mathrm{Ext}^1_R(B, A)$. We show in Examples 4.3.41 and 4.3.42 that this is false in general. However, we will see in Proposition 4.3.10 and Corollary 4.3.31 that there are situations in which the global points of $\mathtt{Ext}^n_R$ agree with Ext^n_R for all n.

4.3.2 Comparing various notions of projectivity

It is well-known that ordinary Ext groups of (say) modules can be computed using projective resolutions, whenever one is at hand. In Section 4.2.5 we showed that the same thing holds for our Ext groups in HoTT. Accordingly, we can compute our internal Ext groups in $\mathcal{X}$ using resolutions which consist of modules that satisfy the interpretation of the predicate $\mathsf{IsProjective}$ from Definition 4.2.25. An example of such a computation is given in Proposition 4.3.12. In addition, we compare internal projectivity to ordinary (external) projectivity. There are no implications either way in general, which we demonstrate through Examples 4.3.15 and 4.3.41.

Definition 4.3.5. Let R be a ring object in $\mathcal{X}$. An R-module P...

(i) ... is **(externally) projective** if for every epimorphism $e : A \twoheadrightarrow B$ in R-Mod, the map $e_* : R\text{-Mod}(P, A) \to R\text{-Mod}(P, B)$ of ordinary sets (or abelian groups) is an epimorphism;

(ii) ... is **internally projective** if for every epimorphism $e : A \twoheadrightarrow B$ in R-Mod, the map $e_* : R\text{-}\underline{\mathrm{Mod}}(P, A) \to R\text{-}\underline{\mathrm{Mod}}(P, B)$ in $\mathrm{Ab}_\mathcal{X}$ is an epimorphism;

(iii) ... is **HoTT-projective** if the interpretation of the proposition $\mathsf{IsProjective}(P)$ from Definition 4.2.25 holds.

The external and internal notions are the usual ones which pertain to modules in a 1-topos, which for us is the 1-topos $\tau_{\le 0}(\mathcal{X})$. However, in an ∞-topos we also have the third notion of HoTT-projectivity resulting from interpretation. We mention, to be concrete, that if $\mathcal{X}$ is the sheaf ∞-topos on some 1-site, then $\tau_{\le 0}(\mathcal{X})$ is the category of ordinary set-valued sheaves on the same site. In this situation, ring and module objects are ordinary sheaves of rings and modules.

In general, when we say that the interpretation of a statement in HoTT "holds" we mean that the resulting object of $\mathcal{X}$ has a global point. If the statement is a proposition, then this means that the object is terminal. Our first objective is to make a useful reformulation of HoTT-projectivity.

Proposition 4.3.6. *An R-module P is HoTT-projective if and only if the $(X{\times}R)$-module $X \times P$ is internally projective in $(X \times R)$-*Mod *for all $X \in \mathcal{X}$.*

Proof. Let P be an R-module in $\mathcal{X}$. According to Definition 4.2.25, we have

$$\mathsf{IsProjective}(P) \;:=\; \prod_{A:R\text{-}\mathsf{Mod}} \prod_{B:R\text{-}\mathsf{Mod}} \prod_{e:\mathsf{Epi}_R(A,B)} \mathsf{IsEpi}(e_* : R\text{-}\mathsf{Mod}(P,A) \to R\text{-}\mathsf{Mod}(P,B)).$$

Interpreting $\mathsf{IsProjective}(P)$, we get an object of $\mathcal{X}$. It has a global point if and only if the projection

$$Q \;:\; \sum_{A,B:R\text{-}\mathsf{Mod}} \; \sum_{e:\mathtt{Epi}_R(A,B)} \mathtt{IsEpi}(e_*) \longrightarrow \sum_{A,B:R\text{-}\mathtt{Mod}} \mathtt{Epi}_R(A,B)$$

admits a section. This map admits a section if and only if for every map

$$f \;:\; X \longrightarrow \sum_{A,B:R\text{-}\mathsf{Mod}} \mathtt{Epi}_R(A,B)$$

there is a section of the pullback $f^*(Q) \in \mathcal{X}/X$, since we can take f to be the identity map. Such a map f is equivalent to the data of two $(X{\times}R)$-modules A and B over X along with an epimorphism $e : A \to B$. Here we have used Theorem 2.4.17 which says that R-$\mathtt{Mod}$ classifies module objects, and Lemma 4.3.2. By definition, we have that $f^*(Q) = \mathtt{IsEpi}(e_*)$, where $e_* : (X{\times}R)\text{-}\underline{\mathrm{Mod}}(X \times P, A) \to (X{\times}R)\text{-}\underline{\mathrm{Mod}}(X \times P, B)$ is the post-composition map. This proposition $f^*(Q)$ holds if and only if e_* is an epimorphism.

In summary, the statement $\mathtt{IsProjective}(P)$ holds if and only if for every $X \in \mathcal{X}$, all $(X{\times}R)$-modules A and B, and every R-module epimorphism $e : A \to B$, the aforementioned post-composition map e_* is an epimorphism. But this is exactly the statement that $X \times P$ is an internally projective $(X{\times}R)$-module for every $X \in \mathcal{X}$. □

Clearly, HoTT-projectivity always implies internal projectivity. The converse holds for ∞-toposes in which internal projectivity of modules is stable by base change. We do not know whether this is always true, but it is true for spaces, as we show in Proposition 4.3.40.

Next we show that certain free modules are HoTT-projective. Our proof uses the following lemma, due to Alex Simpson for internal injectivity of *objects* in a *1*-topos [Sim13], written up on the nLab[4]. The definition of internal projectivity of objects is the same as for modules, but using the internal hom of objects. It is straightforward to check that Simpson's proof goes through for objects of an ∞-topos as well, providing us with:

Lemma 4.3.7. *Let $P \in \mathcal{X}$ be a internally projective object. Then $X{\times}P$ is an internally projective object in $\mathcal{X}/X$ for all $X \in \mathcal{X}$.* □

Given a 0-truncated object S in $\mathcal{X}$, we can form the free R-module $R(S)$ on this object, for any ring object R. This free R-module is the interpretation of the free R-module on a set in HoTT.

[4] See internally projective object (rev. 13) on the nLab.

Proposition 4.3.8. *Let R be a ring object in $\mathcal{X}$, and let P be an internally projective object. The free R-module $R(P)$ on P is HoTT-projective.*

Proof. By the previous lemma, P is internally projective *as an object* in each slice of $\mathcal{X}$. From an argument similar to that of Proposition 4.3.6, we deduce that P satisfies the interpretation of being a projective set in HoTT. The free R-module $R(P)$ on a projective set is projective in HoTT, so we are done. □

We use this proposition to compute an example of our Ext groups in Proposition 4.3.12. Before turning to this example, we observe that internal projectivity (and thus HoTT-projectivity) implies external projectivity in certain situations. This has some interesting consequences.

Proposition 4.3.9. *Let $\mathcal{E}$ be a (possibly elementary) 1-topos, equipped with a ring object R. If the global points functor $\Gamma : \mathcal{E} \to \mathrm{Set}$ preserves epimorphisms, then internal projectivity of R-modules implies external projectivity.*

Proof. The statement easily follows by identifying the external hom of R-modules as the global points of the corresponding internal hom, and then using the assumption on Γ. □

The previous proposition applies, for example, to any topos of presheaves on a category with a terminal object 1. In that case Γ is represented by evaluation at 1, which respects both limits and colimits of presheaves.

Proposition 4.3.10. *Let R be a ring object in $\mathcal{X}$, and consider two R-modules B and A. Suppose that $\Gamma : \mathrm{Set}_{\mathcal{X}} \to \mathrm{Set}$ preserves epimorphisms. If B has a HoTT-projective resolution $P_\bullet$, then we get an isomorphism $\Gamma\mathtt{Ext}^n_R(B, A) \simeq \mathrm{Ext}^n_R(B, A)$.*

One can check that our assumption on Γ holds if and only if the induced functor $\Gamma : \mathrm{Ab}_{\mathcal{X}} \to \mathrm{Ab}$ is exact, and it is this latter condition that we use in the proof.

Proof. By the interpretation of Proposition 4.2.28, we can compute $\mathtt{Ext}^n_R(B, A)$ with the HoTT-projective resolution $P_\bullet$. Specifically, taking internal homs we get a complex

$$\cdots \to R\text{-}\underline{\mathrm{Mod}}(P_{n-1}, A) \to R\text{-}\underline{\mathrm{Mod}}(P_n, A) \to R\text{-}\underline{\mathrm{Mod}}(P_{n+1}, A) \to \cdots \tag{4.5}$$

of abelian groups in $\mathcal{X}$, and we have isomorphisms $H^n(P_\bullet; A) \simeq \mathtt{Ext}^n_R(B, A)$ where the left-hand side is the cohomology of the above complex in $\mathrm{Ab}_{\mathcal{X}}$.

Now, by our assumption on Γ, the previous proposition tells us that $P_\bullet$ is an externally projective resolution of B (since HoTT-projective always implies internally projective). Thus we may also compute $\mathrm{Ext}^n_R(B, A)$ using $P_\bullet$, which amounts to taking the cohomology (in Ab) of the global points of the complex (4.5) above. Since $\Gamma : \mathrm{Ab}_{\mathcal{X}} \to \mathrm{Ab}$ is exact, it commutes with taking cohomology, and we therefore obtain the desired isomorphism. □

In the presence of enough HoTT-projectives, we deduce:

Corollary 4.3.11. *Let R be a ring object in $\mathcal{X}$, and suppose that R-Mod has enough HoTT-projectives. If $\Gamma : \mathrm{Set}_{\mathcal{X}} \to \mathrm{Set}$ preserves epimorphisms, then we have natural isomorphisms*

$$\Gamma\mathtt{Ext}^n_R(B, A) \simeq \mathrm{Ext}^n_R(B, A)$$

for any two R-modules B and A. □

We now turn to our computation of a non-trivial $\mathtt{Ext}^2_{\mathbb{Z}}$ in the Sierpiński ∞-topos $\mathcal{X}$ using a HoTT-projective resolution. Taking the arrow category $0 \to 1$ to be our site, an abelian group in $\mathcal{X}$ consists of a homomorphism $A_0 \leftarrow A_1$ between two ordinary abelian groups. We write y for the Yoneda embedding.

Note that $\mathrm{y}(0)$ is an internally projective object in this ∞-topos, since it represents the functor sending a presheaf $F_0 \leftarrow F_1$ to the presheaf $F_0 \xleftarrow{\mathrm{id}} F_0$, which preserves epimorphisms. Accordingly, the corresponding free abelian group $\mathbb{Z}\,\mathrm{y}(0)$ is HoTT-projective, by Proposition 4.3.8.

Proposition 4.3.12. *Consider the abelian group $B :\equiv (0 \leftarrow \mathbb{Z}/2)$ in the Sierpiński ∞-topos. We have*

$$\mathtt{Ext}^2_{\mathbb{Z}}(B, \mathbb{Z}y(0)) \;\simeq\; B.$$

Proof. We will compute this internal Ext group using a HoTT-projective resolution of B, as justified by the interpretation of Proposition 4.2.28. Drawing objects of $\mathcal{X}$ vertically and morphisms horizontally, the following is such a resolution $P_*(B)$:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "2"] \arrow[d, "{(0,1)}"] & \mathbb{Z} \arrow[r, two heads] \arrow[d, equal] & \mathbb{Z}/2 \arrow[d] \\
\mathbb{Z} \arrow[r, "{(2,-1)}"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "1+2"] & \mathbb{Z} \arrow[r] & 0\,.
\end{tikzcd}$$

Here, $P_0(B) = \mathbb{Z}\,\mathrm{y}(1)$ is the integer object in $\mathcal{X}$, which is always HoTT-projective. Moreover, $P_2(B) = \mathbb{Z}\,\mathrm{y}(0)$ is HoTT-projective by the discussion just above; and $P_1(B) = \mathbb{Z}\,\mathrm{y}(0) \oplus \mathbb{Z}\,\mathrm{y}(1)$ is a direct sum of HoTT-projectives, which is HoTT-projective. We therefore have that $\mathtt{Ext}^2_{\mathbb{Z}}(B, \mathbb{Z}\,\mathrm{y}(0)) \simeq H^2(P_*(B), \mathbb{Z}\,\mathrm{y}(0))$. The latter computes to $B = (0 \leftarrow \mathbb{Z}/2)$, as one can check. □

Remark 4.3.13. The Sierpiński site $(0 \to 1)$ has a terminal object (the object 1), which implies that its global sections functor preserves epimorphisms between abelian group objects. Specifically, the global points of an object $A_0 \leftarrow A_1$ is simply A_1. From Proposition 4.3.10 and the computation in the previous proposition, we deduce that $\mathrm{Ext}^2_{\mathbb{Z}}(B, \mathbb{Z}\,\mathrm{y}(0)) \simeq \mathbb{Z}/2$ for the external Ext group.

Remark 4.3.14. The first part of the resolution above gives rise to a short exact sequence

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "2"] \arrow[d, "2"] & \mathbb{Z} \arrow[r, two heads] \arrow[d, equal] & \mathbb{Z}/2 \arrow[d] \\
\mathbb{Z} \arrow[r, "1"] & \mathbb{Z} \arrow[r] & 0\,.
\end{tikzcd}$$

The object in the center is clearly not the product of the kernel and the cokernel, even ignoring the group structures. In the Sierpiński ∞-topos, an object is merely inhabited if and only if it is inhabited, so it follows that it does not merely hold that the central object is the product of the kernel and the cokernel. So while it is true that the type of length-1 extensions is essentially small, this cannot be proved by assuming that the underlying type of the middle object is the product of the other two types.

We conclude this section by studying the relation between internal and external projectivity in general. Example 4.3.41 gives an internally projective module which is not externally projective. Here we give an example of an externally projective abelian group that fails to be internally projective, which is an additive version of an example due to Todd Trimble.[5]

Consider the poset $\mathscr{C} := \mathbb{N} * \{a, b\}$, where a and b are greater than all $n \in \mathbb{N}$,

$$\begin{tikzcd}[row sep=small] & & & & a \\ 0 \arrow[r] & 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[ur] \arrow[dr] & \\ & & & & b \end{tikzcd}$$

and let $\mathscr{X}$ be the ∞-topos of presheaves on $\mathscr{C}$. As above, we write $\mathrm{y} : \mathscr{C} \to \mathscr{X}$ for the Yoneda embedding. The functor $\mathbb{Z}(-) : \mathrm{Set}_{\mathcal{X}} \to \mathbb{Z}\text{-Mod}$ constructs the free abelian group on a 0-truncated object in $\mathcal{X}$. In particular, we may depict $\mathbb{Z}\,\mathrm{y}(a)$ as follows:

$$\begin{tikzcd}[row sep=small] & & & & \mathbb{Z} \arrow[dl, equal] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \arrow[r, equal] & \cdots & \\ & & & & 0 \arrow[ul, hook'] \end{tikzcd}$$

The integer object $\mathbb{Z}$ over $\mathscr{C}$ is simply the constant presheaf on the ordinary integers.

Example 4.3.15. $\mathbb{Z}\,\mathrm{y}(a)$ is externally projective but not internally projective in $\mathbb{Z}$-Mod. It follows from Proposition 4.2.26 that there exists an A in $\mathbb{Z}$-Mod so that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}\,\mathrm{y}(a), A) = 0$ and $\mathtt{Ext}^1_{\mathbb{Z}}(\mathbb{Z}\,\mathrm{y}(a), A) \neq 0$.

Proof. It is immediate that $\mathbb{Z}\,\mathrm{y}(a)$ is externally projective, since it represents evaluation at a, which preserves epimorphisms of presheaves of modules. To show that $\mathbb{Z}\,\mathrm{y}(a)$ is not internally projective, we construct an epimorphism $\sigma : F \twoheadrightarrow G$ that isn't preserved by $\mathbb{Z}\text{-}\underline{\mathrm{Mod}}(\mathbb{Z}\,\mathrm{y}(a), -)$.

Let $F : \mathbb{Z}$-Mod be defined as follows, with index-wise inclusions:

$$\begin{tikzcd}[row sep=small] & & & & 0 \arrow[dl] \\ \bigoplus_{n\in\mathbb{N}} \mathbb{Z} & \bigoplus_{n\geq 1} \mathbb{Z} \arrow[l, hook'] & \bigoplus_{n\geq 2} \mathbb{Z} \arrow[l, hook'] & \cdots \arrow[l, hook'] & \\ & & & & 0\,. \arrow[ul, hook'] \end{tikzcd}$$

Define $G(n) := \mathbb{Z}$ for $n \in \mathbb{N}$ and $G(a) := 0 =: G(b)$, with all maps between natural numbers inducing identify maps. Then we have an epimorphism $\sigma : F \twoheadrightarrow G$ given by addition at $n \in \mathbb{N}$ and identities (zero maps) at a and b. However $\mathbb{Z}\text{-Mod}(G, F) = 0$ since any such map must factor through $\lim_n F(n) = 0$ on the $\mathbb{N}$-part of $\mathscr{C}$ (and is necessarily 0 on a and b).

Using that $\mathbb{Z}(-) : \mathrm{Set}_{\mathcal{X}} \to \mathbb{Z}$-Mod is strong monoidal, we see that

$$G = \mathbb{Z}(\mathrm{y}(a) \times \mathrm{y}(b)) = \mathbb{Z}\,\mathrm{y}(a) \otimes_{\mathbb{Z}} \mathbb{Z}\,\mathrm{y}(b),$$

[5] See presentation axiom (rev. 46) on the nLab.

where $\otimes_{\mathbb{Z}}$ is the tensor product of presheaves of modules, which is pointwise (see, e.g., [Sta23, Section 6.6]). Using the tensor-hom adjunction (see Section 17.22 of loc. cit.), we have isomorphisms

$$\mathbb{Z}\text{-}\underline{\mathrm{Mod}}(\mathbb{Z}\,\mathrm{y}(a), F)(b) \;\simeq\; \mathbb{Z}\text{-}\mathrm{Mod}(\mathbb{Z}\,\mathrm{y}(a) \otimes_{\mathbb{Z}} \mathbb{Z}\,\mathrm{y}(b), F) \;\simeq\; \mathbb{Z}\text{-}\mathrm{Mod}(G, F) \;\simeq\; 0.$$

On the other hand, $\mathbb{Z}\text{-}\underline{\mathrm{Mod}}(\mathbb{Z}\,\mathrm{y}(a), G)(b) = \mathbb{Z}\text{-}\mathrm{Mod}(\mathbb{Z}\,\mathrm{y}(a)\otimes_{\mathbb{Z}}\mathbb{Z}\,\mathrm{y}(b), G) = \mathbb{Z}\text{-}\mathrm{Mod}(G, G)$ contains at least two elements: 0 and id_G. This means that $\sigma_* : \mathbb{Z}\text{-}\underline{\mathrm{Mod}}(\mathbb{Z}\,\mathrm{y}(a), F) \to \mathbb{Z}\text{-}\underline{\mathrm{Mod}}(\mathbb{Z}\,\mathrm{y}(a), G)$ cannot be an epimorphism, since it isn't one at b. □

4.3.3 Internal injectivity and sheaf Ext

The goal of this section is to show that in certain ∞-toposes, our Ext groups recover *sheaf* Ext groups. (We recall these in Definition 4.3.28.) In order to achieve this, we first must compare three different notions of injectivity in a model.

Definition 4.3.16. Let R be a ring object in $\mathcal{X}_\kappa$. An R-module I...

(i) ... is **(externally) injective** if for every R-module monomorphism $m : A \to B$, the map $m^* : R\text{-}\mathrm{Mod}(B, I) \to R\text{-}\mathrm{Mod}(A, I)$ in Ab is an epimorphism;

(ii) ... is **internally injective** if for every R-module monomorphism $m : A \to B$, the map $m^* : R\text{-}\underline{\mathrm{Mod}}(B, I) \to R\text{-}\underline{\mathrm{Mod}}(A, I)$ in $\mathrm{Ab}_{\mathcal{X}}$ is an epimorphism;

(iii) ... is **HoTT-injective** if it satisfies the interpretation of the proposition $\mathsf{IsInjective}(I)$ from Definition 4.2.27 in $\mathcal{X}$.

As in the projective case, injectivity and internal injectivity are the familiar notions from the 1-topos $\mathrm{Set}_{\mathcal{X}}$. In surprising contrast to the projective case we considered above, external injectivity always implies internal injectivity in a 1-topos. This theorem is due to [Har83b] for abelian groups and [Ble18, Theorem 3.8] for modules. The converse holds in any localic 1-topos (as Blechschmidt shows), however not every internally injective module is externally injective in general. For example, in [Har83a, pp. 259] it is shown that the $\mathbb{Z}/2$-module $\mathbb{Q}/\mathbb{Z}$ with trivial action is not externally injective, though it is internally injective (as an abelian group over $B\mathbb{Z}/2$) by [Har81, Proposition 1.2(i)] (see also Proposition 4.3.38 below).

Remark 4.3.17. Let R be a ring object in $\mathcal{X}_\kappa$. The category R-Mod is equivalent to a category of modules in a 1-topos [Lur09, Theorem 6.4.1.5], and is therefore Grothendieck abelian [KS06, Theorem 18.1.6]. Consequently, it has enough external injectives. Since external injective are internally injective by [Ble18, Theorem 3.8], there are also enough internal injectives in R-Mod.

To relate HoTT-injectivity to internal injectivity we proceed as we did in Section 4.3.2 for projectivity. A proof similar to the one of Proposition 4.3.6 gives us the following:

Proposition 4.3.18. *An R-module I in $\mathcal{X}$ is HoTT-injective if and only if the $(X{\times}R)$-module $X \times I$ is internally injective in $\mathcal{X}/X$ for all $X \in \mathcal{X}$.* □

Clearly, every HoTT-injective module is internally injective. If we could show that internal injectivity is stable by base change in an ∞-topos, then the two notions would coincide. However, we do not know whether internal injectivity is stable by base change in a general ∞-topos. We will show below that it holds in certain situations.

In [Har83b], Harting showed that internal injectivity is stable by base change for abelian groups in any elementary 1-topos. The same holds for modules (see the discussion immediately after [Ble18, Proposition 3.7]). It follows that the same is true in an ∞-topos for base change by a 0-truncated object:

Lemma 4.3.19. *Let $X \in \mathcal{X}$ be* 0*-truncated. Base change $X \times (-) : R\text{-Mod} \to (X{\times}R)\text{-Mod}$ over X preserves internal injectivity.* □

A key fact in the converse direction is that internal injectivity *descends along effective epimorphisms*. This is essentially a corollary of the fact that base change along effective epimorphisms reflects effective epimorphisms (more generally, connected maps) [Lur09, Proposition 6.5.1.16(6)].

Lemma 4.3.20. *Let V be a* (-1)*-connected object of $\mathcal{X}$, and let I be an R-module. If $V \times I$ is internally injective as a* $(V{\times}R)$*-module over V, then I is internally injective.*

The same result holds for internal projectivity as well, with only minor changes to the proof.

Proof. Suppose $V \times I$ is internally injective as a $(V{\times}R)$-module, and let $i : A \hookrightarrow B$ be a monomorphism of R-modules. Using that base change preserves internal homs, consider the following diagram:

$$\begin{tikzcd}
(V{\times}R)\text{-}\underline{\mathrm{Mod}}(V \times B, V \times I) \arrow[r, two heads] \arrow[d, "(V\times i)^*"'] \arrow[dr, phantom, "\lrcorner", very near start] & R\text{-}\underline{\mathrm{Mod}}(B, I) \arrow[d, "i^*"] \\
(V{\times}R)\text{-}\underline{\mathrm{Mod}}(V \times A, V \times I) \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & R\text{-}\underline{\mathrm{Mod}}(A, I) \arrow[d] \\
V \arrow[r, two heads] & 1\,.
\end{tikzcd}$$

Here the two-headed arrows signify effective epimorphisms (which are pullback-stable). By assumption, $V \times I$ is internally injective, so $(V \times i)^*$ is an effective epimorphism (since monomorphisms are stable by base change). Thus i^* factors an effective epimorphism, and must therefore be one itself. We conclude that I is internally injective, as desired. □

One can also prove the previous lemma by using that pullback along an effective epimorphism is conservative [Lur09, Lemma 6.2.3.16]. We apply this to the map $\mathtt{IsEpi}(i^*) \to 1$, which is an equivalence precisely when i^* is an epimorphism.

We now introduce conditions on $\mathcal{X}$ which will imply that internal injectivity is stable by base change.

Definition 4.3.21. Let $n \geq -1$ be a truncation level. An object $X \in \mathcal{X}$ is **covered by an *n*-type** if there exists an n-type V along with an effective epimorphism $V \twoheadrightarrow X$. If all objects of $\mathcal{X}$ are covered by n-types, then ***n*-types cover** in $\mathcal{X}$. When $n = 0$, we say that **sets cover**.

Sets cover in any ∞-topos of ∞-sheaves on a 1-category, since any such sheaf can be covered by a coproduct of representables.

Note that the condition that n-types cover in $\mathcal{X}$ is not the interpretation of the corresponding concept from HoTT. (See [Uni13, Exercise 7.9], [Chr21, Definition 5.2] and the nLab[6].) The HoTT notion only requires that V and the effective epimorphism *merely* exist. On the other hand, it requires this in every slice.

By combining the two previous lemmas with this definition above, we obtain the following result.

Proposition 4.3.22. *Let I be an internally injective R-module in $\mathcal{X}$, and let $X \in \mathcal{X}$. If X is covered by a set, then $X \times I$ is an internally injective $(X{\times}R)$-module.*

Proof. Let I be an internally injective R-module in $\mathcal{X}$, and let $X \in \mathcal{X}$. We wish to show that $X \times I$ is an internally injective $(X{\times}R)$-module in $\mathcal{X}/X$. Since X is covered by a set, we have an effective epimorphism $V \twoheadrightarrow X$ with 0-truncated domain. By Lemma 4.3.19, $V \times I$ is an internally injective $(V{\times}R)$-module. But $V \twoheadrightarrow X$ is a (-1)-connected object over X, thus by Lemma 4.3.20 we can descend internal injectivity from $V \times I$ to $X \times I$. $\square$

Corollary 4.3.23. *If sets cover in $\mathcal{X}$, then internal injectivity of R-modules is stable by base change.* $\square$

Our next goal is to extend this result to any slice of an ∞-topos in which sets cover. This generalization will let us understand the interpretation of internal injectivity when working in a non-empty context in HoTT, which corresponds to working in a slice of the chosen ∞-topos model. The key lemma is the following:

Proposition 4.3.24. *If n-types cover in $\mathcal{X}$, then n-types cover in $\mathcal{X}/X$ for any $(n+1)$-type $X \in \mathcal{X}$.*

Proof. Consider an object $Y \to X$ in $\mathcal{X}/X$. Since n-types cover in $\mathcal{X}$, there is an effective epimorphism $e : V \twoheadrightarrow Y$ with V an n-type in $\mathcal{X}$. The map e defines an effective epimorphism over X with domain the composite $V \to Y \to X$. The latter is a map from an n-type to an $(n+1)$-type, which is necessarily n-truncated (since its fibres are all n-types, as is easily shown in HoTT). Hence the domain of e is n-truncated as an object of $\mathcal{X}/X$, so Y is covered by an n-type. $\square$

Theorem 4.3.25. *Suppose sets cover in $\mathcal{X}$, and let $X \in \mathcal{X}$. For any ring $R \in \mathcal{X}/^{\kappa}X$, internal injectivity of R-modules is stable by base change in $\mathcal{X}/X$.*

Proof. Let $I \in \mathcal{X}/X$ be an internally injective R-module. The truncation map $X \to \|X\|_1$ is 1-connected, and therefore induces an equivalence

$$\tau_{\le 0}(\mathcal{X}/\|X\|_1) \xrightarrow{\sim} \tau_{\le 0}(\mathcal{X}/X)$$

by base change [Lur09, Lemma 7.2.1.13]. Moving back along this equivalence, I becomes an internally injective module over $\|X\|_1$. Since $\|X\|_1$ is a 1-type, the previous proposition implies that sets cover in $\mathcal{X}/\|X\|_1$. By Corollary 4.3.23, this means I is internally injective when pulled back to any slice of $\mathcal{X}/\|X\|_1$. But any slice of $\mathcal{X}/X$ is a slice of $\mathcal{X}/\|X\|_1$, so we are done. $\square$

[6]See n-types cover (rev. 6) on the nLab.

Remark 4.3.26. These methods apply in much greater generality than just internal injectivity. Consider any "internal notion" P in a 1-topos which is stable by base change and descends along (effective) epimorphisms. Then, since the 0-truncated objects in $\mathcal{X}$ form a 1-topos, we can ask whether $P(Y)$ holds for some given 0-truncated object Y in any slice $\mathcal{X}/X$. The arguments above show that if sets cover in $\mathcal{X}$, then P is stable by base change in $\mathcal{X}/X$. (Making precise the meaning of "internal notion" is beyond our current scope.)

Corollary 4.3.27. *Suppose sets cover in $\mathcal{X}$. Consider an object X, a ring $R \in \mathcal{X}/^{\kappa}X$ and an R-module I. Then I is HoTT-injective if and only if it is internally injective.* □

Using Corollary 4.3.27, we explain how our Ext groups recover sheaf Ext.

Definition 4.3.28. Let $\mathcal{E}$ be a 1-topos, let R be a ring object in $\mathcal{E}$, and let B be an R-module. We define the functor $\underline{\mathrm{Ext}}^n_R(B,-) : R\text{-Mod} \to \mathrm{Ab}_{\mathcal{X}}$ to be the n^{th} right derived functor of $R\text{-}\underline{\mathrm{Mod}}(B,-)$, where we use an external injective resolution to define the derived functor. We refer to $\underline{\mathrm{Ext}}^i_R(B,A)$ as **sheaf Ext**. We extend this definition to an ∞-topos $\mathcal{X}$ by applying it to the 1-topos $\tau_{\leq 0}(\mathcal{X})$.

The sheaf Ext groups arise in algebraic geometry ([Gro57, Chapitre IV], [Har77, Section III.6]) and are also considered in [KS06, Section 18.4] and [Ble17, Section 13.4].

For any R-module B in $\mathcal{X}_\kappa$, we obtain an internal functor $\mathtt{Ext}^n_R(B,-)$ in $\mathcal{X}$ by interpretation, which yields an ordinary functor $\mathtt{Ext}^n_R(B,-) : R\text{-Mod} \to \mathrm{Ab}'_{\mathcal{X}}$. Here the $'$ indicates that the codomain consists of abelian group objects in $\mathcal{X}$, not just $\mathcal{X}_\kappa$. The dual of Proposition 4.2.28 lets us compute $\mathtt{Ext}^n_R(B,A)$ via a HoTT-injective resolution of A. Combining the results of this section, we obtain the following:

Theorem 4.3.29. *Suppose sets cover in $\mathcal{X}$. For any $X \in \mathcal{X}$, ring $R \in \mathcal{X}/^{\kappa}X$ and R-module B, the functor $\mathtt{Ext}^n_R(B,-) : R\text{-Mod} \to \mathrm{Ab}'_{\mathcal{X}/X}$ is naturally isomorphic to the sheaf Ext functor $\underline{\mathrm{Ext}}^n_R(B,-)$. In particular, we may take $\mathtt{Ext}^n_R(B,-)$ to land in $\mathrm{Ab}_{\mathcal{X}/X}$.*

Proof. Since (external) injectives in R-Mod are always internally injective by [Ble18, Theorem 3.8], and moreover internal and HoTT-injectives coincide in $\mathcal{X}/X$ by Corollary 4.3.27, we can use an (externally) injective resolution to compute $\mathtt{Ext}^n_R(B,-)$ by the dual of Proposition 4.2.28. But this means that $\mathtt{Ext}^n_R(B,-)$ is the n^{th} right derived functor of the internal hom of R-modules, meaning it is naturally isomorphic to $\underline{\mathrm{Ext}}^n_R(B,-)$. □

It follows that the computation in Proposition 4.3.12 can be regarded as a computation of sheaf Ext in the Sierpiński ∞-topos, using a HoTT-projective resolution.

Remark 4.3.30. In his thesis, Blechschmidt gives a definition of sheaf Ext groups in the internal language of a localic 1-topos using the existence of enough injectives [Ble17, Section 13.4]. In contrast, our internal Ext is the interpretation of Definition 4.2.5, which does not rely on injectives.

Since there are always enough internally injective R-modules, we deduce the following:

Corollary 4.3.31. *Let $X \in \mathcal{X}$. Suppose that sets cover in $\mathcal{X}$ and that the (set-restricted) global points functor $\Gamma_X : \mathrm{Set}_{\mathcal{X}/X} \to \mathrm{Set}$ preserves epimorphisms. For any ring $R \in \mathcal{X}/^{\kappa}X$, R-modules B and A, and $n \geq 0$, we have a natural isomorphism $\Gamma_X \mathtt{Ext}^n_R(B,A) \simeq \mathrm{Ext}^n_R(B,A)$. In particular, the ordinary Ext groups are obtained as the global points of sheaf Ext.*

Proof. An argument similar to Proposition 4.3.9 shows that internal injectivity implies external injectivity of R-modules under our assumption on Γ_X. Thus internal and external injectivity coincide, and are equivalent to HoTT-injectivity by Corollary 4.3.27. The statement follows by the same proof as in Proposition 4.3.10, but using an (internally) injective resolution of A. □

Remark 4.3.32. The previous corollary is well-known for sheaf Ext, and is an easy consequence of the ("local-to-global") Grothendieck spectral sequence which relates sheaf Ext and ordinary Ext, specifically:

$$(\mathrm{R}^p\,\Gamma)\underline{\mathrm{Ext}}^q_R(B,A) \implies \mathrm{Ext}^{p+q}_R(B,A).$$

(Here R^p denotes the p^{th} right derived functor.) Our assumption on Γ_X implies that $\mathrm{R}^p\,\Gamma_X$ vanishes for $p > 0$, which means this spectral sequence collapses at the E_2-page. It immediately follows that we have an isomorphism $\Gamma_X\underline{\mathrm{Ext}}^n_R(B,A) \simeq \mathrm{Ext}^n_R(B,A)$, for all $n \in \mathbb{N}$.

We also record the following corollary of Theorem 4.3.29.

Corollary 4.3.33. *Suppose sets cover in $\mathcal{X}$, and let $X \in \mathcal{X}$. The interpretation of* $\mathsf{Ext}^n_R(B,A)$ *into $\mathcal{X}/X$ depends solely on $\tau_{\leq 0}(\mathcal{X}/X)$.* □

There are many ∞-toposes which share 1-toposes of 0-truncated objects. For example, if $X \in \mathscr{S}$ is a pointed, connected space, then 0-truncated objects in the slice ∞-topos $\mathscr{S}/X$ are $\pi_1(X)$-sets. Thus if X is simply connected, these are just sets. Since sets cover in $\mathscr{S}$, the corollary tells us that interpreting $\mathtt{Ext}^n_R$ into any slice $\mathscr{S}/X$ with X simply connected yields the same result (up to equivalence). This means in particular that we can move between $\mathscr{S}$ and $\mathscr{S}/X$ when computing $\mathtt{Ext}^n_R$—a potentially useful trick.

4.3.4 Ext over BG

Let X be a pointed, connected object in $\mathcal{X}$, with base point $\mathtt{pt} : X$. In this final section, we study Ext groups of abelian group objects in the slice $\mathcal{X}/X$, and relate them to Ext groups in the base $\mathcal{X}$. As we will see, these considerations are intimately related with those of Section 4.2.7, and they illustrate the theory developed thus far in Section 4.3.

We refer to abelian group objects in $\mathcal{X}/X$ as $(X{\times}\mathbb{Z})$-modules, as this makes the base clear. We mention that the 1-truncation map $X \to B\pi_1(X)$ is 1-connected and therefore induces an equivalence between the 1-topos of sets over X and the 1-topos of sets over $B\pi_1(X)$ by pulling back [Lur09, Lemma 7.2.1.13]. Accordingly, we get an equivalence $X \times \mathbb{Z}\text{-Mod} \simeq B\pi_1(X) \times \mathbb{Z}\text{-Mod}$ of categories of modules. To emphasize that no truncation assumptions are needed, we work with X rather than $B\pi_1(X)$.

The category of $(X{\times}\mathbb{Z})$-modules has another description. However, as we will see in a moment, this other description is not equivalent when working internally, since it changes the ambient topos. For any (0-truncated) group object in $\mathcal{X}$, we can form the internal group ring $\mathbb{Z}G$ as the object $\bigoplus_G \mathbb{Z}$ with its natural ring structure. This is the result of interpreting the group ring of Construction 4.2.37 into $\mathcal{X}$. Being a G-set, $\mathbb{Z}G$ defines an object of $\mathcal{X}/BG$ which may be seen to be the free abelian group on the base point (or universal cover) $1 \to BG$, though we will not make use of this description.

Proposition 4.3.34. *Restriction to the base point of X gives an equivalence of 1-categories*

$$X \times \mathbb{Z}\text{-Mod} \;\simeq\; \mathbb{Z}\pi_1(X)\text{-Mod},$$

where the left-hand side is the category of abelian groups in $\mathcal{X}/^{\kappa}X$ and the right-hand side is the category of $\mathbb{Z}\pi_1(X)$-modules in $\mathcal{X}_\kappa$. In particular, we obtain an isomorphism

$$\mathrm{Ext}^n_{X\times\mathbb{Z}}(B,A) \;\simeq\; \mathrm{Ext}^n_{\mathbb{Z}\pi_1(X)}(B_{\mathtt{pt}},A_{\mathtt{pt}})$$

of external Ext groups for all $(X{\times}\mathbb{Z})$-modules A and B.

We point out that this statement does *not* imply that $\mathtt{Ext}_{X\times\mathbb{Z}}$ coincides with $\mathtt{Ext}_{\mathbb{Z}\pi_1(X)}$, as the former is an abelian group in $\mathcal{X}/X$, whereas the latter is an abelian group in $\mathcal{X}$. However, the relation between these objects is interesting and is further discussed below.

Proof. By Theorem 2.4.17, the category $X \times \mathbb{Z}$-Mod is equivalent to the category $\mathcal{X}(X, \mathbb{Z}\text{-}\mathtt{Mod})$ (whose categorical structure comes from $\mathbb{Z}$-$\mathtt{Mod}$). The interpretation of Proposition 4.2.39 yields an equivalence of categories $\mathbb{Z}\text{-}\mathtt{Mod}^X \simeq \mathbb{Z}\pi_1(X)\text{-}\mathtt{Mod}$ in $\mathcal{X}$, which on global points yields the desired equivalence of categories $\mathcal{X}(X,\mathbb{Z}\text{-}\mathtt{Mod}) \simeq \mathbb{Z}\pi_1(X)$-Mod. It follows that the stated (external) Ext groups are isomorphic. □

Given a $(X{\times}\mathbb{Z})$-module A, we call its restriction $A_{\mathtt{pt}}$ along $1 \to X$ the **underlying abelian group object** of A. This has a natural $\pi_1(X)$-action, so it can also be regarded as a $\mathbb{Z}\pi_1(X)$-module in $\mathcal{X}$. Note that the equivalence $X \times \mathbb{Z}$-Mod $\simeq \mathbb{Z}\pi_1(X)$-Mod sends the ring $X \times \mathbb{Z}$ to $\mathbb{Z}$ with the trivial $\pi_1(X)$-action, and not to the ring $\mathbb{Z}\pi_1(X)$. We also warn the reader that care must be taken when moving across this equivalence. For example, the category $X \times \mathbb{Z}$-Mod is enriched over itself via the internal hom of abelian groups in $\mathcal{X}/^{\kappa}X$, while $\mathbb{Z}\pi_1(X)$-Mod is naturally enriched over $\mathrm{Ab}_{\mathcal{X}}$. The hom-objects of this second enrichment are given by taking the fixed points of the former enrichment, using the equivalence. Because of the difference between the internal homs, many internal properties are not preserved by the equivalence. An example of this is given in Example 4.3.41, as explained in the discussion immediately after it.

We record the following fact, which immediately follows from base-change stability of interpretation.

Proposition 4.3.35. *Let B and A be $(X{\times}\mathbb{Z})$-modules. The underlying abelian group object of the $(X{\times}\mathbb{Z})$-module $\mathtt{Ext}^n_{X\times\mathbb{Z}}(B,A)$ is $\mathtt{Ext}^n_{\mathbb{Z}}(B_{\mathtt{pt}},A_{\mathtt{pt}})$.* □

A concrete description of the $\pi_1(X)$-action on $\mathtt{Ext}^n_{\mathbb{Z}}(B_{\mathtt{pt}},A_{\mathtt{pt}})$ can be worked out from Proposition 4.2.40 and the discussion surrounding it.

Remark 4.3.36. It might be tempting to believe that the abelian group object $\mathtt{Ext}^n_{\mathbb{Z}\pi_1(X)}(B_{\mathtt{pt}},A_{\mathtt{pt}})$ is isomorphic to the fixed points of the $\mathbb{Z}\pi_1(X)$-module $\mathtt{Ext}^n_{X\times\mathbb{Z}}(B,A)_{\mathtt{pt}}$ described by the previous theorem. In general, this is not the case, as we will see in Example 4.3.42.

We deduce a vanishing result for $\mathtt{Ext}_{X\times\mathbb{Z}}$.

Corollary 4.3.37. *Let n be a natural number. Suppose that $\mathtt{Ext}^n_{\mathbb{Z}}(B,A)$ vanishes for all abelian groups B and A in $\mathcal{X}_\kappa$. Then $\mathtt{Ext}^n_{X\times\mathbb{Z}}(N,M)$ also vanishes for all $(X{\times}\mathbb{Z})$-modules N and M.* □

Our next result characterizes internal injectivity of abelian groups in the slice $\mathcal{X}/X$, and generalizes [Har81, Proposition 1.2(i)] for ordinary sheaves on a space.

Proposition 4.3.38. *An $(X{\times}\mathbb{Z})$-module is internally injective if and only if its underlying abelian group object is internally injective. The same holds for internal projectivity.*

Proof. We prove the injective case, as the projective case is shown similarly.

($\to$) Let I be an internally injective $(X{\times}\mathbb{Z})$-module. A monomorphism $i : A \hookrightarrow B$ of abelian groups in $\mathcal{X}$ pulls back to a monomorphism $i : X \times A \hookrightarrow X \times B$ between $(X{\times}\mathbb{Z})$-modules (with trivial action). Thus we get an epimorphism $i^* : X \times \mathbb{Z}\text{-}\underline{\mathrm{Mod}}(X{\times}B, I) \twoheadrightarrow X \times \mathbb{Z}\text{-}\underline{\mathrm{Mod}}(X{\times}A, I)$ of $(X{\times}\mathbb{Z})$-modules. This map is given by $i^* : \mathbb{Z}\text{-}\underline{\mathrm{Mod}}(B, I_{\mathtt{pt}}) \to \mathbb{Z}\text{-}\underline{\mathrm{Mod}}(A, I_{\mathtt{pt}})$ on the underlying abelian group objects, since base change (here along $1 \twoheadrightarrow X$) respects internal homs. The latter map is therefore an epimorphism, as desired.

($\leftarrow$) By Lemma 4.3.20 we can descend internal injectivity along the effective epimorphism $1 \twoheadrightarrow X$, meaning I is an internally injective $(X{\times}\mathbb{Z})$-module whenever I_* is an internally injective abelian group object. ◻

We note that the proof of ($\to$) only used that X was pointed.

Remark 4.3.39. The previous proposition sheds new light on Proposition 4.3.35. Namely, if one computes $\mathtt{Ext}^n_{X\times\mathbb{Z}}(B, A)$ using an internally injective resolution of $(X{\times}\mathbb{Z})$-modules (which always exists), then the underlying abelian resolution can be used to compute $\mathtt{Ext}^n_{\mathbb{Z}}(B_{\mathtt{pt}}, A_{\mathtt{pt}})$. Thus we see that the latter is the underlying abelian group object of the former.

Before our next examples, we show that internal projectivity and HoTT-projectivity coincide in spaces.

Proposition 4.3.40. *In the ∞-topos of spaces, HoTT-projectivity and internal projectivity of modules coincide.*

Proof. Firstly, note that external and internal projectivity coincide in spaces. Now, suppose that P is an (internally) projective R-module. By Proposition 4.3.6, we need to show that $X \times P$ is an internally projective $(X{\times}R)$-module in $\mathcal{X}/X$, for any $X \in \mathcal{X}$. Since sets cover in spaces, and internal projectivity descends along effective epimorphisms by a proof analogous to Lemma 4.3.20, we can assume that X is a set. Then an $(X{\times}R)$-module is simply an X-indexed collection of R-modules, and the internal hom of such is the index-wise hom. The axiom of choice implies that an X-indexed collection of epimorphisms defines an epimorphism between the collections, so $X \times P$ is internally projective. ◻

We now give examples of modules which are internally projective, but not externally projective.

Example 4.3.41. Let G be a non-trivial (0-truncated) group in $\mathcal{S}$. The $(BG{\times}\mathbb{Z})$-module $BG{\times}\mathbb{Z}$ is internally projective, but not externally projective. It follows that there exists a $(BG{\times}\mathbb{Z})$-module A so that $\mathtt{Ext}^1_{BG\times\mathbb{Z}}(BG \times \mathbb{Z}, A) = 0$ and $\mathrm{Ext}^1_{BG\times\mathbb{Z}}(BG \times \mathbb{Z}, A) \neq 0$.

Proof. Any ring is HoTT-projective as a module over itself, and is therefore internally projective. In particular, $BG \times \mathbb{Z}$ is internally projective. (This can also be seen from Proposition 4.3.38.) External projectivity of $(BG{\times}\mathbb{Z})$-modules corresponds to ordinary projectivity of

$\mathbb{Z}G$-modules by Proposition 4.3.34. As a $\mathbb{Z}G$-module, $BG\times\mathbb{Z}$ corresponds to the abelian group $\mathbb{Z}$ with trivial G-action. Since G is non-trivial, the augmentation map $\mathbb{Z}G \to \mathbb{Z}$ cannot split, thus $\mathbb{Z}$ is not projective. The last claim follows from Proposition 4.2.26. □

As mentioned above, for a module in $\mathscr{S}$, internal and external projectivity agree. So the example shows that while $BG \times \mathbb{Z}$ is internally projective as a $(BG{\times}\mathbb{Z})$-module, the corresponding $\mathbb{Z}G$-module $\mathbb{Z}$ is not internally projective. This demonstrates the sense in which the equivalence of Proposition 4.3.34 does not respect internal properties, since the ambient topos changes. This is also demonstrated by the following example, which ties together many of our results and remarks into a concrete example in spaces.

Example 4.3.42. Take $\mathcal{X}$ to be $\mathscr{S}$ and G to be a (0-truncated) group. Since $\mathrm{Ab}_{\mathscr{S}} = \mathrm{Ab}$ has global dimension 1 and $\mathtt{Ext}^n_{\mathbb{Z}}$ interprets to ordinary $\mathrm{Ext}^n_{\mathbb{Z}}$ by Proposition 4.3.1, we deduce that $\mathtt{Ext}^n_{\mathbb{Z}}$ vanishes for $n > 1$. Corollary 4.3.37 then says that $\mathtt{Ext}^n_{BG\times\mathbb{Z}}$ vanishes for $n > 1$. Even more, $\mathtt{Ext}^n_{BG\times\mathbb{Z}}(BG \times \mathbb{Z}, M)$ vanishes for all $n \geq 1$ and every M, since $BG \times \mathbb{Z}$ is internally projective. On the other hand, by Proposition 4.3.34, the ordinary Ext groups $\mathrm{Ext}^n_{BG\times\mathbb{Z}}(BG \times \mathbb{Z}, A)$ are the same as the ordinary Ext groups $\mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, A_{\mathtt{pt}})$, which need not vanish. For example, it is well known that $\mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M) \cong H^n(BG; M)$, which may be nonzero for all $n \in \mathbb{N}$. Indeed, as we saw in Example 4.3.41, $\mathbb{Z}$ with trivial action is not a projective $\mathbb{Z}G$-module. Note also that $\mathtt{Ext}^n_{\mathbb{Z}G}$ agrees with $\mathrm{Ext}^n_{\mathbb{Z}G}$, again using Proposition 4.3.1. In particular, as mentioned in Remark 4.3.36, it is not the case in general that $\mathtt{Ext}^n_{\mathbb{Z}G}$ can be described as the fixed points of the $\mathbb{Z}G$-module corresponding to $\mathtt{Ext}^n_{BG\times\mathbb{Z}}$, even for $n = 1$.

We explain this phenomenon in a bit more detail. A short exact sequence of $(BG{\times}\mathbb{Z})$-modules

$$0 \to A \to E \to B \to 0$$

may be seen both as an element of the abelian group $\mathrm{Ext}^1_{BG\times\mathbb{Z}}(B, A) \cong \mathrm{Ext}^1_{\mathbb{Z}G}(B_{\mathtt{pt}}, A_{\mathtt{pt}})$, and as an element (indeed, G-fixed point) of the $\mathbb{Z}G$-module $\mathtt{Ext}^1_{BG\times\mathbb{Z}}(B, A)_{\mathtt{pt}}$. This extension E is trivial as an element of the former if and only if the map $E \to B$ admits a $(BG{\times}\mathbb{Z})$-module section (i.e., a G-equivariant section). In contrast, E is trivial as an element of the latter if and only if the underlying map $E_{\mathtt{pt}} \to B_{\mathtt{pt}}$ admits an *abelian* section, by Proposition 4.3.35.

Chapter 5

Central H-spaces and banded types

Abstract. We introduce and study *central* types, which are generalizations of Eilenberg–Mac Lane spaces. A type is central when it is equivalent to the component of the identity among its own self-equivalences. From centrality alone we construct an infinite delooping in terms of a tensor product of *banded types*, which are the appropriate notion of torsor for a central type. Our constructions are carried out in homotopy type theory, and therefore hold in any ∞-topos. Even when interpreted into the ∞-topos of spaces, our main results and constructions are new.

Along the way, we further develop the theory of H-spaces in homotopy type theory, including their relation to *evaluation fibrations* and Whitehead products. These considerations let us, for example, rule out the existence of H-space structures on the $2n$-sphere for $n > 0$. We also give a novel description of the moduli space of H-space structures on an H-space. Using this description, we generalize a formula of Arkowitz–Curjel and Copeland for counting the number of path components of this moduli space. As an application, we deduce that the moduli space of H-space structures on the 3-sphere is $\Omega^6\mathbb{S}^3$.

5.1 Introduction

In this paper we study H-spaces and their deloopings. By working in homotopy type theory, our results apply to any ∞-topos [dBB20, dBoe20, KL18, LS20, Shu19]. While most of the theory we develop is new to homotopy type theory, much of the theory related to H-spaces in Section 5.2 is inspired by classical topology. However, our main results and constructions in Sections 5.3 and 5.4 are new even for the ∞-topos of spaces.

A key concept is that of a central type. A pointed type A is **central** if the map

$$f \longmapsto f(\mathtt{pt}) \;:\; (A \to A)_{(\mathrm{id})} \longrightarrow A$$

which evaluates a function at the base point of A is an equivalence. Here $(A \to A)_{(\mathrm{id})}$ denotes the identity component of the type of all self-maps of A, and $\mathtt{pt}$ denotes the base point of A. Since the domain of this map is a connected H-space (see Definition 5.2.1(2)), so is any central type. We give a list of equivalent conditions for a connected H-space to be central in Proposition 5.3.6, and one of them is that the type $A \to_* A$ of pointed self-maps is a set. It follows, for example, that every Eilenberg–Mac Lane space $\mathsf{K}(G, n)$, with G abelian and

$n \geq 1$, is central. We show in Section 5.5.3 that some, but not all, products of Eilenberg–Mac Lane spaces are central. In Section 5.5.4, we show that every truncated, central type with at most two non-zero homotopy groups, both of which are finitely presented, is a product of Eilenberg–Mac Lane spaces. In general, we don't know whether every central type is a product of Eilenberg–Mac Lane spaces, and we leave this as an open question.

Our first result is:

Theorem 5.4.6. *Let A be a central type. Then A has a unique delooping.*

The key ingredient of this result—and much of the paper—is that we have a concrete description of the delooping of A. It is given by the type $\mathsf{BAut}_1(A) :\equiv \sum_{X:\mathcal{U}} \|A = X\|_0$ of types *banded* by A, which is the 1-connected cover of $\mathsf{BAut}(A)$. As an example, since $\mathsf{K}(G,n)$ is central for G abelian and $n \geq 1$, this gives an alternative way to define $\mathsf{K}(G, n+1)$ in terms of $\mathsf{K}(G,n)$, as previously indicated by the first author [Buc19]. Banded types are denoted X_p, where $p : \|A = X\|_0$ is the band.

We also show:

Theorem 5.4.10. *Let A be a central type. Every pointed map $f : A \to_* A$ has a unique delooping*

$$X_p \longmapsto (X \to A)_{(f^* \circ \tilde{p}^{-1})} \; : \; \mathsf{BAut}_1(A) \longrightarrow_* \mathsf{BAut}_1(A).$$

The formula for the delooping is explained in Definition 5.4.7. It follows that the type of pointed self-maps of $\mathsf{BAut}_1(A)$ is a set, since it is equivalent to $A \to_* A$.

One of the motivations for studying $\mathsf{BAut}_1(A)$ is that one can define a *tensoring* operation. Given two banded types X and Y in $\mathsf{BAut}_1(A)$, the type $X^* = Y$ has a natural banding, where X^* denotes a certain dual of X. We write $X \otimes Y$ for this banded type, and show in Theorem 5.4.19 that it makes $\mathsf{BAut}_1(A)$ into an abelian H-space. Combined with Theorem 5.4.6, Theorem 5.4.10, and the characterization of central types mentioned earlier, we therefore deduce:

Corollary 5.4.20. *For a central type A, the type $\mathsf{BAut}_1(A)$ is again central. Therefore, A is an infinite loop space, in a unique way. Moreover, every pointed map $A \to_* A$ is infinitely deloopable, in a unique way.*

Our tensoring operation gives a new description of the H-space structure on $\mathsf{K}(G,n)$, which will be helpful for calculations of Euler classes in work in progress and is what originally motivated this work.

We also give an alternate description of the delooping of a central type A as a certain type of A-torsors (Section 5.4.3), and give an analogous description of $\mathsf{K}(G,1)$ for any group G (Section 5.5.1).

To prove the above results, we first need to further develop the theory of H-spaces. One difference between our work and classical work in topology is that we emphasize the moduli space $\mathsf{HSpace}(A)$ of H-space structures on a pointed type A, rather than just the set of components. For example, we prove:

Theorem 5.2.27. *Let A be an H-space such that for all $a : A$, the map $a \cdot -$ is an equivalence. Then the type $\mathsf{HSpace}(A)$ of H-space structures on A is equivalent to the type $A \wedge A \to_* A$ of pointed maps.*

This generalizes a classical formula of Arkowitz–Curjel and Copeland, which plays a key role in classical results on the number of H-space structures on various spaces. The classical formula only determines the path components of the type of H-space structures, while our formula gives an equivalence of types. From our formula it immediately follows, for example, that the type of H-space structures on the 3-sphere is $\Omega^6\mathbb{S}^3$. The proof of Theorem 5.2.27 uses *evaluation fibrations*, which generalize the map appearing in the definition of "central." In fact, these evaluation fibrations play an important role in much of the paper, and underlie our main results about central types. We also relate the existence of sections of an evaluation fibration to the vanishing of Whitehead products. Related considerations let us show that no even spheres besides $\mathbb{S}^0$ admit H-space structures.

In Proposition 5.3.3 we show that every central type has a unique H-space structure, in the strong sense that the type $\mathsf{HSpace}(A)$ is contractible. We prove several results about types with unique H-space structures. For example, we show that such H-space structures are associative and coherently abelian, and that every pointed self-map is an H-space map, a weaker version of the delooping above. We also give an example, pointed out to us David Wärn, which shows that not every type with a unique H-space structure is central (Example 5.5.12).

We note that these results rely on us defining "H-space" to include a coherence between the two unit laws (see Definition 5.2.1).

Formalization. Many of the main results of this paper have been formalized in the Coq proof assistant using the Coq-HoTT library [CH]. This includes much of the basic theory that we develop related to H-spaces, and these results have already been accepted into the Coq-HoTT library under the `Homotopy.HSpace` namespace. Notably, Theorem 5.2.27 has been formalized as `equiv_cohhspace_ppmap`, modulo the smash-hom adjunction which has been formalized by Floris van Doorn in Lean 2 [vDoo18]. In the separate repository central-types, we have formalized Theorems 5.4.6 and 5.4.19, and that $\mathsf{BAut}_1(A)$ is central whenever A is (i.e., one part of Corollary 5.4.20), plus various other results.

Outline. In Section 5.2, we give results about H-spaces which do not depend on centrality, including a description of the moduli space of H-space structures, results about Whitehead products and H-space structures on spheres, and results about unique H-space structures. In Section 5.3, we define central types, show that central types have a unique H-space structure, give a characterization of which H-spaces are central, and prove other results needed in the next section. Section 5.4 is the heart of the paper. It defines the type $\mathsf{BAut}_1(A)$ of bands for a central type A, shows that it is a unique delooping of A, proves that it is an H-space under a tensoring operation, and shows that central types and their self-maps are uniquely infinitely deloopable. We also give the alternate description of the delooping in terms of A-torsors. Finally, Section 5.5 gives examples and non-examples of central types, mostly related to Eilenberg–Mac Lane spaces and their products. It also contains a construction of $\mathsf{K}(G,1)$ as a type of torsors, and defines an H-space structure on this type when G is abelian, paralleling the H-space structure on $\mathsf{BAut}_1(A)$. It ends by showing that every truncated, central type with at most two non-zero homotopy groups, both of which are finitely presented, is a product of Eilenberg–Mac Lane spaces.

Notation and conventions. In general, we follow the notation used in [Uni13]. For example, we write path composition in diagrammatic order: given paths $p : x = y$ and $q : y = z$, their composite is $p \cdot q$. The reflexivity path is written $\mathtt{refl}$.

We write $\mathcal{U}$ for a fixed univalent universe of types, and frequently make use of univalence. We also use function extensionality without always explicitly mentioning it.

Given a type A and an element $a : A$, we write (A, a) for the type A pointed at a. If A is already a pointed type with unspecified base point, then we write $\mathtt{pt}$ for the base point. If A and B are pointed types, and $f, g : A \to_* B$ are pointed maps, then $f =_* g$ is the type of pointed homotopies between f and g.

If A is an H-space, then we write $x \cdot y$ for the product of two elements $x, y : A$ (unless another notation for the multiplication is given). For a pointed type A, we write $\mathsf{HSpace}(A)$ for the type of H-space structures on A with the base point as the identity element (Definition 5.2.1).

We write $\mathbb{S}^n$ for the n-sphere.

5.2 H-spaces and evaluation fibrations

In Section 5.2.1, we begin by recalling the notion of a (coherent) *H-space structure* on a pointed type A, give several equivalent descriptions of the type of H-space structures, and prove basic results that will be useful in the rest of the paper.

In Section 5.2.2, we discuss the type of pointed extensions of a map $B \vee C \to_* A$ to $B \times C$, and show that the type of H-space structures on A is equivalent to the type of pointed extensions of the fold map. We relate the existence of extensions to the vanishing of Whitehead products, and use this to show that there are no H-space structures on even spheres except $\mathbb{S}^0$. In addition, we show that for any n-connected H-space A, the Freudenthal map $\pi_{2n+1}(A) \to \pi_{2n+2}(\Sigma A)$ is an isomorphism, not just a surjection.

In Section 5.2.3, we study *evaluation fibrations*. We show that for a left-invertible H-space A, various evaluation fibrations are trivial, and use this to show that the type of H-space structures is equivalent to $A \wedge A \to_* A$, generalizing a classical formula of Arkowitz–Curjel and Copeland. It immediately follows, for example, that the type of H-space structures on the 3-sphere is $\Omega^6 \mathbb{S}^3$.

Section 5.2.4 is a short section which studies the case when the type of H-space structures is contractible. We stress that this is not the same as $\mathsf{HSpace}(A)$ having a single component, which is what is classically meant by "A has a unique H-space structure." This situation is interesting in its own right. We show that such H-space structures are associative and coherently abelian, and we prove that all pointed self-maps of A are automatically H-space maps.

5.2.1 H-space structures

We begin by giving the notion of H-space structure that we will consider in this paper.

Definition 5.2.1. Let A be a pointed type.

1. A **non-coherent H-space structure** on A consists of a binary operation $\mu : A \to A \to A$, a **left unit law** $\mu_l : \mu(\mathtt{pt}, -) = \mathrm{id}_A$ and a **right unit law** $\mu_r : \mu(-, \mathtt{pt}) = \mathrm{id}_A$.

2. A **(coherent) H-space structure** on A consists of a non-coherent H-space structure μ on A along with a **coherence** $\mu_{lr} : \mu_l(\mathtt{pt}) =_{\mu(\mathtt{pt},\mathtt{pt})=\mathtt{pt}} \mu_r(\mathtt{pt})$.

3. We write $\mathsf{HSpace}(A)$ for the type of (coherent) H-space structures on A.

When the H-space structure is clear from the context we may write $x \cdot y :\equiv \mu(x, y)$. Any H-space structure yields a non-coherent H-space structure by forgetting the coherence. Suppose A has a (non)coherent H-space structure μ.

4. If $\mu(a, -) : A \to A$ is an equivalence for all $a : A$, then μ is **left-invertible**, and we write $x\backslash y :\equiv \mu(x, -)^{-1}(y)$. **Right-invertible** is defined dually, and we write $x/y :\equiv \mu(-, y)^{-1}(x)$.

5. The **twist** μ^T of μ is the natural (non)coherent H-space structure with operation

$$\mu^T(a_0, a_1) :\equiv \mu(a_1, a_0).$$

When we say "H-space" we mean the coherent notion—we will only say "coherent" for emphasis. The notion of H-space structure considered in [Uni13, Def. 8.5.4] corresponds to our *non-coherent* H-space structures. While many constructions can be carried out for non-coherent H-spaces (such as the Hopf construction), the coherent case is more natural for our purposes.

The type of H-space structures on a pointed type has several equivalent descriptions.

Proposition 5.2.2. *Let A be a pointed type. The following types are equivalent:*

1. *The type $\mathsf{HSpace}(A)$ of H-space structures on A.*

2. *The type of pointed sections of the pointed map* $\mathbf{ev} : (A \to A, \mathrm{id}_A) \to_* A$ *which sends an unpointed map f to $f(\mathtt{pt})$.*

3. *The type of families $\mu : \prod_{a:A}(A, \mathtt{pt}) \to_* (A, a)$ of pointed maps equipped with a pointed homotopy $\mu(\mathtt{pt}) =_* \mathrm{id}_A$.*

Moreover, the type of non-coherent H-space structures on A is equivalent to the type of families $\mu : \prod_{a:A}(A, \mathtt{pt}) \to_* (A, a)$ of pointed maps equipped with an unpointed homotopy $\mu(\mathtt{pt}) = \mathrm{id}_A$.

Proof. All of these claims are simply reshuffling of data combined with function extensionality. For example, given a pointed section s of $\mathbf{ev}$, the underlying map of s gives the binary operation μ, the pointedness of s gives the left unit law μ_l, the homotopy $\mathbf{ev} \circ s = \mathrm{id}_A$ gives the right unit law, and the pointedness of that homotopy gives the coherence. The data in (3) is almost identical. In particular, it is the pointedness of the homotopy $\mu(\mathtt{pt}) =_* \mathrm{id}_A$ that corresponds to coherence, and omitting this gives the description of non-coherent H-space structures. □

Remark 5.2.3. Note that A is left-invertible if and only if the maps in (3) are equivalences. We say that A is a **homogeneous type** if it is equipped with a family $\mu : \prod_{a:A}(A, \mathtt{pt}) \simeq_* (A, a)$, and so we see that every left-invertible H-space is homogeneous.

We have the following converse.

Lemma 5.2.4. *Let A be a pointed type equipped with a family $\mu : \prod_{a:A}(A, \mathtt{pt}) \to_* (A, a)$ such that $\mu(\mathtt{pt})$ is an equivalence. Then A can be given the structure of an H-space.*

Proof. The new family defined by $\mu'(a) = \mu(a) \circ \mu(\mathtt{pt})^{-1}$ has the property that $\mu'(\mathtt{pt}) =_* \mathrm{id}_A$, and therefore gives an H-space structure on A. □

Note that this lemma shows that every homogeneous type can be given the structure of a (left-invertible) H-space, and that every non-coherent H-space can be given the structure of an H-space. In the latter case, since the original $\mu(\mathtt{pt})$ is equal to the identity map (as unpointed maps), the new H-space retains the same binary operation μ and left unit law μ_l, but has a different right unit law μ_r. While the types of non-coherent and coherent H-space structures on a pointed type are logically equivalent, they are not generally equivalent as types (see Remark 5.3.4).

We'll be interested in abelian and associative H-spaces later on.

Definition 5.2.5. Let A be an H-space with multiplication μ.

1. If there is a homotopy $h : \prod_{a,b} \mu(a, b) = \mu(b, a)$, then μ is **abelian**.
2. If $\mu = \mu^T$ in $\mathsf{HSpace}(A)$, then μ is **coherently abelian**.
3. If there is a homotopy $\alpha : \prod_{a,b,c:A} \mu(\mu(a, b), c) = \mu(a, \mu(b, c))$, then μ is **associative**.

The following lemma gives a convenient way of constructing abelian H-space structures, and will be used in Theorem 5.4.19.

Lemma 5.2.6. *Let A be a pointed type with a binary operation μ, a symmetry $\sigma_{a,b} : \mu(a, b) = \mu(b, a)$ for every $a, b : A$ such that $\sigma_{\mathtt{pt},\mathtt{pt}} = \mathtt{refl}$, and a left unit law $\mu_l : \mu(\mathtt{pt}, -) = \mathrm{id}_A$. Then A becomes an abelian H-space with the right unit law induced by symmetry.*

Proof. For $b : A$, the right unit law is given by the path $\sigma_{b,\mathtt{pt}} \cdot \mu_l(b)$ of type $\mu(b, \mathtt{pt}) = b$. For coherence we need to show that the following triangle commutes:

$$\begin{tikzcd}
\mu(\mathtt{pt},\mathtt{pt}) \arrow[rr, "\sigma_{\mathtt{pt},\mathtt{pt}}"] \arrow[dr, "\mu_l"'] & & \mu(\mathtt{pt},\mathtt{pt}) \arrow[dl, "\mu_l"] \\
& \mathtt{pt}. &
\end{tikzcd}$$

By our assumption that $\sigma_{\mathtt{pt},\mathtt{pt}} = \mathtt{refl}$, the triangle is filled $\mathtt{refl}_{\mu_l}$. □

We collect a few basic facts about H-spaces. The following lemma generalizes a result of Evan Cavallo, who formalized the fact that unpointed homotopies between pointed maps into a *homogeneous type A* can be upgraded to pointed homotopies. Being a homogeneous type is logically equivalent to being a left-invertible H-space [Cav21]. Here we do not need to assume left-invertibility, and we factor this observation through a further generalization.

Lemma 5.2.7. *Let A be a pointed type, and consider the following three conditions:*

1. A is an H-space.

2. *The evaluation map* $(\mathrm{id}_A = \mathrm{id}_A) \to (\mathtt{pt} = \mathtt{pt})$ *sending a homotopy* h *to* $h_{\mathtt{pt}}$ *has a section.*

3. *For every pointed type* B *and pointed maps* $f, g : B \to_* A$*, there is a map* $(f = g) \to (f =_* g)$ *which upgrades unpointed homotopies to pointed homotopies.*

Then (1) implies (2) and (2) implies (3).

Proof. To show that (1) implies (2), suppose that A is an H-space. By Proposition 5.2.2 we have a pointed section s of $\mathbf{ev} : (A \to A, \mathrm{id}_A) \to_* A$. The evaluation map in (2) is $\Omega\,\mathbf{ev}$, and has a (pointed) section Ωs.

We next show that (2) implies (3). Let $f, g : B \to_* A$ be pointed maps and let $H : f = g$ be an unpointed homotopy. By path induction on H, we can assume we have a single function $f : B \to A$ with two pointings, $f_{\mathtt{pt}}$ and $f'_{\mathtt{pt}} : f(\mathtt{pt}) = \mathtt{pt}$. Our goal is to define a homotopy $K : f = f$ such that $K_{\mathtt{pt}} = r$, where $r :\equiv f_{\mathtt{pt}} \cdot \overline{f'_{\mathtt{pt}}} : f(\mathtt{pt}) = f(\mathtt{pt})$. By path induction on $f_{\mathtt{pt}}$, we can assume that the base point of A is $f(\mathtt{pt})$. By (2), we have $s : (f(\mathtt{pt}) = f(\mathtt{pt})) \to (\mathrm{id}_A = \mathrm{id}_A)$ such that $s(p, f(\mathtt{pt})) = p$ for all $p : f(\mathtt{pt}) = f(\mathtt{pt})$. For $b : B$, define K_b to be $s(r, f(b))$. Then $K_{\mathtt{pt}} = r$, as required. □

The following result is straightforward and has been formalized, so we do not include a proof.

Proposition 5.2.8. *Suppose* A *is a (left-invertible) H-space. For any pointed type* B*, the mapping type* $B \to_* A$ *based at the constant map is naturally a (left-invertible) H-space under pointwise multiplication. Similarly, for any type* B*, the mapping type* $B \to A$ *based at the constant map is a (left-invertible) H-space under pointwise multiplication.* □

In particular, if A is left-invertible then for any $f : B \to_* A$ there is a self-equivalence of $B \to_* A$ which sends the constant map to f—namely, the pointwise multiplication by f on the left.

Another way to produce H-space structures is via the following result:

Proposition 5.2.9. *If* A *is an H-space and* A' *is a pointed retract of* A*, then* A' *is an H-space.*

Proof. Assume we have $s : A' \to_* A$, $r : A \to_* A'$ and $h : r \circ s =_* \mathrm{id}$. We define a multiplication on A' by sending $(a, b) : A' \times A'$ to $r(s(a) \cdot s(b))$. The left unit law is the composite path

$$r(s(\mathtt{pt}) \cdot s(b)) \xlongequal{\mathrm{ap}_r\, \mathrm{ap}_{\mu(-,s(b))}\, s_{\mathtt{pt}}} r(\mathtt{pt} \cdot s(b)) \xlongequal{\mathrm{ap}_r\, \mu_l(s(b))} r(s(b)) \xlongequal{h_b} b\,,$$

and the right unit law is $\mathrm{ap}_r\, \mathrm{ap}_{\mu(s(a),-)}\, s_{\mathtt{pt}} \cdot \mathrm{ap}_r\, \mu_r(s(a)) \cdot h_a$. To show coherence, we must show that these are equal when $a \equiv b \equiv \mathtt{pt}$. By cancelling the common $h_{\mathtt{pt}}$ and removing the ap_r, we see that it suffices to prove that

$$\mathrm{ap}_{\mu(-,s(\mathtt{pt}))}\, s_{\mathtt{pt}} \cdot \mu_l(s(\mathtt{pt})) = \mathrm{ap}_{\mu(s(\mathtt{pt}),-)}\, s_{\mathtt{pt}} \cdot \mu_r(s(\mathtt{pt})).$$

To prove this, we compose both sides with $s_{\mathtt{pt}}$ on the right, use naturality of μ_l and μ_r, coherence, and the naturality of ap_μ in its two variables. (We can also appeal to Lemma 5.2.4 to avoid this part of the argument.) □

5.2.2 (α,β)-extensions and Whitehead products

We begin by defining (α,β)-*extensions*, and then use them to give a different description of the type of H-space structures on a pointed type A. Then we relate them to Whitehead products, and use Brunerie's computation of Whitehead products to rule out H-space structures on even spheres. To do this, we prove some results about Whitehead products from [Whi46] which relate to H-spaces. Finally, we also show that for an n-connected H-space, the Freudenthal map $\pi_{2n+1}(A) \to \pi_{2n+2}(\Sigma A)$ is an isomorphism, not just a surjection. None of the results in this section are used in the rest of the paper.

Definition 5.2.10. Let $\alpha : B \to_* A$ and $\beta : C \to_* A$ be pointed maps. An $(\boldsymbol{\alpha},\boldsymbol{\beta})$**-extension** is a pointed map $f : B \times C \to_* A$ equipped with a pointed homotopy filling the following diagram:

$$\begin{tikzcd} B \vee C \arrow[rr, "\alpha\vee\beta"] \arrow[dr] & & A \\ & B \times C . \arrow[ur, "f"'] & \end{tikzcd}$$

Remark 5.2.11. It is equivalent to consider the type of *unpointed* (α,β)-extensions consisting of unpointed maps $f : B \times C \to A$ and unpointed fillers. The additional data in a pointed extension is a path $f_{\mathtt{pt}} : f(\mathtt{pt},\mathtt{pt}) = \mathtt{pt}$ and a 2-path that determines $f_{\mathtt{pt}}$ in terms of the other data. These form a contractible pair.

When α and β are maps between spheres, Whitehead instead says that f is "of type (α,β)" but we prefer to stress that we work with a structure and not a property.

We now relate (α,β)-extensions to the following generalization of the map $\mathbf{ev}$. Given a pointed map $f : B \to_* A$, we again write $\mathbf{ev}$ for the map $(B \to A, f) \to_* A$ which evaluates at $\mathtt{pt} : B$. This map is pointed since f is. Recall that the case where $f \equiv \mathrm{id}_A$ played a key role in Proposition 5.2.2.

Definition 5.2.12. Let $e : X \to_* A$ and $g : Y \to_* A$ be pointed maps. A **pointed lift of $\boldsymbol{g}$ through $\boldsymbol{e}$** consists of a pointed map $s : Y \to_* X$ along with a pointed homotopy $e \circ s =_* g$. When $g \equiv \mathrm{id}$, then we recover the notion of a pointed section of e.

Proposition 5.2.13. *Let $\alpha : B \to_* A$ and $\beta : C \to_* A$ be pointed maps. The type of (α,β)-extensions is equivalent to the type of pointed lifts of β through* $\mathbf{ev} : (B \to A, \alpha) \to_* A$. ◻

The proof of the statement is a straightforward reshuffling of data along with cancellation of a contractible pair. Diagrammatically, it gives a correspondence between the dashed arrows below, with pointed homotopies filling the triangles:

$$\begin{tikzcd} B \vee C \arrow[r, "f\vee g"] \arrow[d] & A \\ B \times C \arrow[ur, dashed] & \end{tikzcd} \qquad\qquad \begin{tikzcd} & (B \to A, f) \arrow[d, "\mathbf{ev}"] \\ C \arrow[ur, dashed] \arrow[r, "g"] & A \end{tikzcd}$$

Proposition 5.2.14. *H-space structures on a pointed type A correspond to* $(\mathrm{id}_A, \mathrm{id}_A)$*-extensions.*

Proof. This follows from Propositions 5.2.2 and 5.2.13. ◻

Lemma 5.2.15. *If A is an H-space, then there is an (α,β)-extension for every pair $\alpha : B \to_* A$ and $\beta : C \to_* A$ of pointed maps.*

Proof. Using naturality of the left and right unit laws and coherence, one can show that the map $(b,c) \mapsto \alpha(b) \cdot \beta(c) : B \times C \to A$ is an (α,β)-extension. Alternatively, observe that the (α,β)-extension problem factors through the $(\mathrm{id}_A, \mathrm{id}_A)$-extension problem via the map $\alpha \times \beta : B \times C \to A \times A$. □

These results explain the relation between H-space structures and (α,β)-extensions, which are in turn related to Whitehead products via the next two results. (See [Bru16, Section 3.3] for the definition of Whitehead products.)

Proposition 5.2.16 ([Whi46, Corollary 3.5])**.** *Let $m, n > 0$ be natural numbers and consider two pointed maps $\alpha : \mathbb{S}^m \to_* A$ and $\beta : \mathbb{S}^n \to_* A$. The type of (α,β)-extensions is equivalent to the type of witnesses that the map $[\alpha,\beta] : \mathbb{S}^{m+n-1} \to_* A$ is constant (as a pointed map).*

Proof. Consider the diagram of pointed maps below, where the composite of the top two maps is $[\alpha,\beta]$ and the left diamond is a pushout of pointed types by [Bru16, Proposition 3.2.2]:

$$\begin{tikzcd}
 & \mathbb{S}^m \vee \mathbb{S}^n \arrow[dr] \arrow[drr, bend left, "\alpha\vee\beta"] & & \\
\mathbb{S}^{m+n-1} \arrow[ur] \arrow[dr] & & \mathbb{S}^m \times \mathbb{S}^n \arrow[r, dashed, "f"] & A\,. \\
 & 1 \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}$$

An (α,β)-extension is the same as a pointed map f along with a pointed homotopy filling the top-right triangle. Since the bottom-right triangle is filled by a unique pointed homotopy, an (α,β)-extension thus corresponds exactly to the data of a filler in the outer diagram, i.e., a homotopy witnessing that $[\alpha,\beta]$ is constant as a pointed map. □

With the notation of the previous proposition, we have the following:

Corollary 5.2.17 ([Whi46, Corollary 3.6])**.** *Suppose A is an H-space. Then $[\alpha,\beta]$ is constant.*

Proof. This follows from Lemma 5.2.15 and Proposition 5.2.16. □

Using the above results, we can rule out H-space structures on even spheres in positive dimensions.

Proposition 5.2.18. *The n-sphere merely admits an H-space structure if and only if $[\iota_n, \iota_n] = 0$. In particular, there are no H-space structures on the n-sphere when $n > 0$ is even.*

Proof. The implication $(\to)$ is immediate by Corollary 5.2.17. Conversely, Proposition 5.2.16 implies that $[\iota_n, \iota_n] = 0$ if and only if an $(\mathrm{id}_{\mathbb{S}^n}, \mathrm{id}_{\mathbb{S}^n})$-extension merely exists, which by Proposition 5.2.14 happens if and only if $\mathbb{S}^n$ merely admits an H-space structure.

Finally, Brunerie showed that $[\iota_n, \iota_n] = 2$ in $\pi_{2n-1}(\mathbb{S}^n)$ for even $n > 0$ [Bru16, Proposition 5.4.4], which by the above implies that $\mathbb{S}^n$ cannot admit an H-space structure. □

We also record the following result and a corollary.

Proposition 5.2.19. *Let A be a left-invertible H-space. The unit $\eta : A \to_* \Omega\Sigma A$ has a pointed retraction, given by the connecting map $\delta : \Omega\Sigma A \to_* A$ associated to the Hopf fibration of A.*

Proof. Let $\delta : \Omega\Sigma A \to_* A$ be the connecting map associated to the Hopf fibration of A. Recall that for a loop $p : N = N$, we have $\delta(p) :\equiv p_*(\mathtt{pt})$ where $p_* : A \to A$ denotes transport and A is the fibre above N. By definition of the Hopf fibration, a path $\mathrm{merid}(a) : N =_{\Sigma A} S$ sends an element x of the fibre A to $a \cdot x$. Now define a homotopy $\delta \circ \eta = \mathrm{id}$ by

$$\delta(\eta(a)) \equiv \delta(\mathrm{merid}(a) \cdot \mathrm{merid}(\mathtt{pt})^{-1}) = \mathrm{merid}(\mathtt{pt})_*^{-1}(\mathrm{merid}(a)_*(\mathtt{pt})) \equiv \mathtt{pt}\backslash(a \cdot \mathtt{pt}) = a.$$

Finally, we promote this to a pointed homotopy using Lemma 5.2.7. □

It follows that for any n-connected H-space A, the Freudenthal map $\pi_{2n+1}(A) \to \pi_{2n+2}(\Sigma A)$ is an isomorphism, not just a surjection. In particular, we have:

Corollary 5.2.20. *The natural map $\pi_5(\mathbb{S}^3) \to \pi_6(\mathbb{S}^4)$ is an isomorphism.* □

The fact that the unit $\eta : A \to_* \Omega\Sigma A$ has a retraction when A is a left-invertible H-space also follows from James' reduced product construction, as shown in [Jam55]. Using [Bru16], one can see that this goes through in homotopy type theory. However, the above argument is much more elementary. We don't know if this argument had been observed before.

5.2.3 Evaluation fibrations

We now begin our study of *evaluation fibrations* and their relation to H-space structures.

Definition 5.2.21. Let A be a type and $a : \|A\|_0$. The **path component of a in A** is

$$A_{(a)} :\equiv \sum_{a':A} (|a'|_0 = a).$$

If $a : A$ then we abuse notation and write $A_{(a)}$ for $A_{(|a|_0)}$, and in this case $A_{(a)}$ is pointed at $(a, \mathtt{refl})$.

Definition 5.2.22. For any pointed map $\alpha : B \to_* A$, the **evaluation fibration (at α)** is the pointed map $\mathbf{ev}_\alpha : (B \to A)_{(\alpha)} \to_* A$ induced by evaluating at the base point of B.

Note that the component $(B \to A)_{(\alpha)}$ consists of maps that are merely equal to α as unpointed maps. Also observe that the component $(A \to A)_{(\mathrm{id})}$ is equivalent to $(A \simeq A)_{(\mathrm{id})}$, since being an equivalence is a property of a map. We permit ourselves to pass freely between the two.

Since pointed maps out of connected types land in the component of the base point of the codomain, we have the following consequence of Proposition 5.2.2.

Corollary 5.2.23. *Let A be a pointed, connected type. The type of H-space structures on A is equivalent to the type of pointed sections of $\mathbf{ev}_{\mathrm{id}} : (A \simeq A)_{(\mathrm{id})} \to_* A$.* □

For left-invertible H-spaces, various evaluation fibrations become trivial:

Proposition 5.2.24. *Suppose A is a left-invertible H-space. We have a pointed equivalence over A*

$$\begin{tikzcd} (A \to A) \arrow[rr, "\sim"] \arrow[dr, "\mathbf{ev}"'] & & (A \to_* A) \times A \arrow[dl, "\mathrm{pr}_2"] \\ & A, & \end{tikzcd}$$

where the mapping spaces are both pointed at their identity maps. This pointed equivalence restricts to a pointed equivalence $(A \simeq A) \simeq_* (A \simeq_* A) \times A$ over A, and a pointed equivalence $(A \to A)_{(\mathrm{id})} \simeq_* (A \to_* A)_{(\mathrm{id})} \times A_{(\mathtt{pt})}$ over $A_{(\mathtt{pt})}$.

Proof. Define $e : (A \to A) \to (A \to_* A) \times A$ by $e(f) :\equiv (a \mapsto f(\mathtt{pt})\backslash f(a), f(\mathtt{pt}))$ where the first component is a pointed map in the obvious way. Clearly e is a map over A, and moreover e is pointed. It is straightforward to check that the triangle above is filled by a pointed homotopy. (One could also apply Lemma 5.2.7, but a direct inspection suffices in this case.)

Finally, it's straightforward to check that e has an inverse given by

$$(g, a) \mapsto (x \mapsto a \cdot g(x)).$$

Hence e is an equivalence, as desired. The restrictions to equivalences and path components follow by functoriality. □

The hypotheses of the proposition are satisfied, for example, by connected H-spaces.

Example 5.2.25. We obtain three pointed equivalences for any abelian group A and $n \geq 1$:

$$\begin{aligned} (\mathsf{K}(A,n) \to \mathsf{K}(A,n)) &\simeq_* \mathsf{Ab}(A,A) \times \mathsf{K}(A,n), \\ (\mathsf{K}(A,n) \simeq \mathsf{K}(A,n)) &\simeq_* \mathsf{Aut}_{()}\mathsf{Ab}(A) \times \mathsf{K}(A,n), \text{ and} \\ (\mathsf{K}(A,n) \to \mathsf{K}(A,n))_{(\mathrm{id})} &\simeq_* \mathsf{K}(A,n). \end{aligned}$$

Example 5.2.26. Taking $A :\equiv \mathbb{S}^3$ in the previous proposition, by virtue of the H-space structure on the 3-sphere constructed in [BR18], we get three pointed equivalences:

$$(\mathbb{S}^3 \to \mathbb{S}^3) \simeq_* \Omega^3\mathbb{S}^3 \times \mathbb{S}^3, \quad (\mathbb{S}^3 \simeq \mathbb{S}^3) \simeq_* \Omega^3_{\pm 1}\mathbb{S}^3 \times \mathbb{S}^3, \quad \text{and} \quad (\mathbb{S}^3 \simeq \mathbb{S}^3)_{(\mathrm{id})} \simeq_* (\mathbb{S}^3 \simeq_* \mathbb{S}^3)_{(\mathrm{id})} \times \mathbb{S}^3,$$

where $\Omega^3_{\pm 1}\mathbb{S}^3 :\equiv (\Omega^3\mathbb{S}^3)_{(1)} \sqcup (\Omega^3\mathbb{S}^3)_{(-1)}$ and 1 and -1 refer to the corresponding elements of $\pi_3(\mathbb{S}^3) = \mathbb{Z}$.

By combining our results thus far, we obtain the following equivalence which generalizes a classical formula of [Cop59, Theorem 5.5A], independently shown by [AC63], for counting *homotopy classes* of H-space structures on certain spaces.

Theorem 5.2.27. *Let A be a left-invertible H-space. The type $\mathsf{HSpace}(A)$ of H-space structures on A is equivalent to $A \wedge A \to_* A$.*

Proof. By Proposition 5.2.2, the type of H-space structures on A is equivalent to the type of pointed sections of $\mathbf{ev} : (A \to A) \to A$. By Proposition 5.2.24, this type is equivalent to the type of pointed sections of $\mathrm{pr}_2 : (A \to_* A) \times A \to A$, which are simply pointed maps $A \to_* (A \to_* A, \mathrm{id})$, where the codomain is pointed at the identity. The latter type is equivalent to $A \to_* (A \to_* A)$, where the codomain is pointed at the constant map, by Proposition 5.2.8. Finally, this type is equivalent to $A \wedge A \to_* A$, by the smash–hom adjunction for pointed types [vDoo18, Theorem 4.3.28]. □

Example 5.2.28. It follows, for example, that $\mathsf{HSpace}(\mathbb{S}^1) \simeq \mathbf{1}$ and $\mathsf{HSpace}(\mathbb{S}^3) \simeq \Omega^6\mathbb{S}^3$.

5.2.4 Unique H-space structures

We collect results about H-space structures which are unique, in the sense that the type of H-space structures is contractible. In particular, we give elementary proofs that such H-space structures are coherently abelian and associative. Moreover, pointed self-maps of such types are always H-space maps.

There are many examples of types with a unique H-space structure. We will see in Proposition 5.3.3 that the "central types" we study in the next section have unique H-space structures. The following proposition is another source of examples, which will be used in Example 5.5.12 and Section 5.5.4.

Proposition 5.2.29. *Let $k \geq 0$ and let A be an $(k-1)$-connected, pointed type. If A is $(2k-1)$-truncated, then A has at most one H-space structure. If A is $(2k-2)$-truncated, then A has a unique H-space structure.*

This also appears in [Wär23], with a different argument.

Proof. Assume A is $(k-1)$-connected and $(2k-1)$-truncated. By [CS20, Corollary 2.32], $A \wedge A$ is $(2k-1)$-connected, and therefore $A \wedge A \to_* A$ is contractible. Therefore, if A has an H-space structure, then $\mathsf{HSpace}(A)$ is contractible, by Theorem 5.2.27.

If we make the stronger assumption that A is $(2k-2)$-truncated, then by [BvDR18, Theorem 6.7], A is an infinite loop space, so it is in particular an H-space. □

Now we show that unique H-space structures are particularly well-behaved.

Lemma 5.2.30. *Let A be a pointed type and suppose $\mathsf{HSpace}(A)$ is contractible. Then the unique H-space structure μ on A is coherently abelian.*

Proof. Since $\mathsf{HSpace}(A)$ is contractible, there is an identification $\mu = \mu^T$ of H-space structures. (Here, μ^T is the twist, defined in Definition 5.2.1.) □

Remark 5.2.31. In fact, a unique H-space structure is coherently abelian in a stronger sense, which we now explain. The type of H-space structures on A can be given a $\mathbb{Z}/2$-action, following the general procedure of equipping a type X with a G-action by constructing a type family $Y : BG \to \mathcal{U}$ equipped with an equivalence $Y(\mathtt{pt}) \simeq X$. To construct the $\mathbb{Z}/2$-action on $\mathsf{HSpace}(A)$ we first need to construct a $\mathbb{Z}/2$-action on the identity type. Recall that $B\mathbb{Z}/2$ can be described as $\sum_{X:\mathcal{U}} \|X = \mathbf{2}\|$, where $\mathbf{2}$ is the two-element type. We define the **symmetric identity type**

$$\widetilde{\mathsf{Id}}_A : \prod_{X:B\mathbb{Z}/2} A^X \to \mathcal{U}$$

by $\widetilde{\mathsf{Id}}_A(X, f) :\equiv \sum_{a:A} \prod_{x:X} f(x) = a$. One can easily check that $\widetilde{\mathsf{Id}}_A(\mathbf{2}, f) \simeq (f(0) = f(1))$, so that the symmetric identity type indeed defines a $\mathbb{Z}/2$-action on the ordinary identity type.

Now we define for any 2-element type $X : B\mathbb{Z}/2$ the type

$$\mathsf{HSpace}^X(A) :\equiv \sum_{\mu:A^X \to A} \sum_{H:\mathrm{unital}(\mu)} \widetilde{\mathsf{Id}}(X, x \mapsto H(\mathrm{const}_{\mathtt{pt}}, x, \mathtt{refl})),$$

where $\mathrm{unital}(\mu) :\equiv \prod_{f:X\to A}\prod_{x:X}\prod_{p:f(x)=\mathtt{pt}} \mu(f) = f(\sigma(x))$ asserts that μ satisfies unit laws in both variables and $\sigma : X \to X$ is the transposition. It is straightforward to check that $\mathsf{HSpace}^{\mathbf{2}}(A) \simeq \mathsf{HSpace}(A)$. The type of **symmetric H-space structures** on A is defined to be

$$\prod_{X:B\mathbb{Z}/2} \mathsf{HSpace}^X(A),$$

i.e., the type of fixed points of the $\mathbb{Z}/2$-action on $\mathsf{HSpace}(A)$. If $\mathsf{HSpace}(A)$ is contractible, then each $\mathsf{HSpace}^X(A)$ is contractible, and so the type of symmetric H-space structures on A is also contractible. Since $\sigma : \mathbf{2} \to \mathbf{2}$ transports an H-space structure μ to μ^T, it follows that a symmetric H-space structure is coherently abelian, as in Lemma 5.2.30. Furthermore, by applying the first projection, we obtain an operation

$$\prod_{X:B\mathbb{Z}/2} A^X \to A.$$

Put another way, we obtain an operation

$$\Big(\sum_{X:B\mathbb{Z}/2} A^X\Big) \to A$$

defined on the type of **unordered pairs**. In other words, unique H-spaces are abelian in a fully coherent way.

For the next result, we use the definition of the smash product from [vDoo18, Definition 4.3.6] (see also [CS20, Definition 2.29]) which avoids higher paths. For pointed types (X, x_0) and (Y, y_0), the **smash product** $X \wedge Y$ is the higher inductive type with point constructors $\mathrm{sm} : X \times Y \to X \wedge Y$ and $\mathrm{auxl}, \mathrm{auxr} : X \wedge Y$, and path constructors $\mathrm{gluel} : \prod_{y:Y} \mathrm{sm}(x_0, y) = \mathrm{auxl}$ and $\mathrm{gluer} : \prod_{x:X} \mathrm{sm}(x, y_0) = \mathrm{auxr}$. It is pointed by auxl. The smash product was shown to be associative in [vDoo18, Definition 4.3.33].

Proposition 5.2.32. *Suppose A is a pointed type with a unique H-space structure which is left-invertible. Any pointed map $f : A \to_* A$ is an H-space map, i.e., we have $f(a \cdot b) = f(a) \cdot f(b)$ for all $a, b : A$.*

Proof. Let $f : A \to_* A$ be a pointed map. We will define an associated map $\nu : A \wedge A \to_* A$, which records how f deviates from being an H-space map. We define $\nu(\mathrm{sm}(a, b)) :\equiv (f(a) \cdot f(b))\backslash f(a \cdot b)$, $\nu(\mathrm{auxl}) :\equiv \mathtt{pt}$, and $\nu(\mathrm{auxr}) :\equiv \mathtt{pt}$. For $b : A$, we have a path $\nu(\mathrm{sm}(\mathtt{pt}, b)) \equiv (f(\mathtt{pt}) \cdot f(b))\backslash f(\mathtt{pt} \cdot b) = (\mathtt{pt} \cdot f(b))\backslash f(b) = f(b)\backslash f(b) = \mathtt{pt}$, and similarly for the other path constructor. Since A admits a unique H-space structure, the type $A \wedge A \to_* A$ is contractible by Theorem 5.2.27. Consequently, ν is constant, whence for all $a, b : A$ we have $(f(a) \cdot f(b))\backslash f(a \cdot b) = \mathtt{pt}$, and therefore

$$f(a \cdot b) = f(a) \cdot f(b). \qquad \square$$

Remark 5.2.33. Note that when A and B are two pointed types, each with unique H-space structures, it is not necessarily the case that every pointed map $f : A \to_* B$ is an H-space map. For example, the squaring operation gives a natural transformation $H^2(X;\mathbb{Z}) \to H^4(X;\mathbb{Z})$ which is represented by a map $\mathsf{K}(\mathbb{Z}, 2) \to_* \mathsf{K}(\mathbb{Z}, 4)$. But since squaring isn't a homomorphism, this map isn't an H-space map.

Proposition 5.2.34. *Suppose A is a pointed type with a unique H-space structure which is left-invertible. Then the H-space structure is necessarily associative.*

Proof. Let $a : A$. Define a map $\nu : A \wedge A \to_* A$ as follows. Let $\nu(\mathrm{sm}(b, c)) :\equiv ((a \cdot b) \cdot c) \backslash (a \cdot (b \cdot c))$, $\nu(\mathrm{auxl}) :\equiv \mathtt{pt}$, and $\nu(\mathrm{auxr}) :\equiv \mathtt{pt}$. For $c : A$, we have a path

$$\nu(\mathrm{sm}(\mathtt{pt}, c)) \equiv ((a \cdot \mathtt{pt}) \cdot c) \backslash (a \cdot (\mathtt{pt} \cdot c)) = (a \cdot c) \backslash (a \cdot c) = \mathtt{pt},$$

and similarly for the other path constructor. Since A admits a unique H-space structure, the type $A \wedge A \to_* A$ is contractible by Theorem 5.2.27. Consequently, for each a, the map ν is constant. It follows that for all $a, b, c : A$ we have $((a \cdot b) \cdot c) \backslash (a \cdot (b \cdot c)) = \mathtt{pt}$, and therefore

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c. \qquad \square$$

Note that if $A \wedge A \to_* A$ is contractible, then it follows from the smash-hom adjunction that $A^{\wedge n} \to_* A$ is contractible for each $n \geq 2$, where $A^{\wedge n}$ denotes the smash power.

5.3 Central types

In this and the next section we focus on pointed types which we call *central*. Centrality is an elementary property with remarkable consequences. For example, in the next section we will see that every central type is an infinite loop space (Corollary 5.4.20). To show this, we require a certain amount of theory about central types. We first show that every central type has a unique H-space structure. When A is already known to be an H-space, we give several conditions which are equivalent to A being central. From this, it follows that every Eilenberg–Mac Lane space $\mathsf{K}(G, n)$, with G abelian and $n \geq 1$, is central. We also prove several other results which we will need in the next section.

Definition 5.3.1. Let A be a pointed type. The **center of A** is the type $ZA :\equiv (A \to A)_{(\mathrm{id})}$, which comes with a natural map $\mathbf{ev}_{\mathrm{id}} : ZA \to_* A$ (see Definition 5.2.22). If the map $\mathbf{ev}_{\mathrm{id}}$ is an equivalence, then A is **central**.

Remark 5.3.2. The terminology "central" comes from higher group theory. Suppose $A :\equiv BG$ is the delooping of an ∞-group G. The center of G is the ∞-group $ZG :\equiv \prod_{x:G}(x = x)$, with delooping $BZG :\equiv (BG \simeq BG)_{(\mathrm{id})}$, which is our ZA.

Central types and H-spaces are connected through evaluation fibrations:

Proposition 5.3.3. *Suppose that A is central. Then A admits a unique H-space structure. In addition, A is connected, so this H-space structure is both left- and right-invertible.*

Proof. Since $\mathbf{ev}_{\mathrm{id}}$ is an equivalence, it has a unique section. By Corollary 5.2.23, we deduce that A has a unique H-space structure μ. It follows from Lemma 5.2.30 that it is coherently abelian. Finally, the equivalence $\mathbf{ev}_{\mathrm{id}} : (A \to A)_{(\mathrm{id})} \simeq A$ implies that A is connected. Then, since $\mu(\mathtt{pt}, -)$ and $\mu(-, \mathtt{pt})$ are both equal to the identity, it follows that μ is left- and right-invertible. $\square$

It follows from Proposition 5.2.34 and Lemma 5.2.30 that the unique H-space structure on a central type is associative and coherently abelian.

Remark 5.3.4. In contrast, the type of non-coherent H-space structures on a central type A is rarely contractible. We'll show here that it is equivalent to the loop space ΩA. First consider the type of binary operations $\mu : A \to (A \to A)$ which *merely* satisfy the left unit law. This is equivalent to the type of maps $A \to (A \to A)_{(\mathrm{id})}$, since A is connected. Such a map μ satisfies the right unit law if and only if the composite $\mathbf{ev}_{\mathrm{id}} \circ \mu : A \to A$ is the identity map. In other words, μ must be a section of the equivalence $\mathbf{ev}_{\mathrm{id}}$, so there is a contractible type of such μ.

The left unit law says that μ sends $\mathtt{pt}$ to id. After post-composing with $\mathbf{ev}_{\mathrm{id}}$, it therefore says that it sends $\mathtt{pt}$ to $\mathrm{id}(\mathtt{pt})$, which equals $\mathtt{pt}$. So the type of left unit laws is $\mathtt{pt} = \mathtt{pt}$, i.e., the loop space ΩA. Note that we imposed the left unit law both merely and purely, but that doesn't change the type. So it follows that the type of all non-coherent H-space structures on a central type A is ΩA.

We give conditions for an H-space to be central, in which case the H-space structure is the unique one coming from centrality. For the next two results, write

$$F :\equiv \sum_{f : A \to_* A} \|f = \mathrm{id}\|$$

for the fibre of $\mathbf{ev}_{\mathrm{id}} : (A \to A)_{(\mathrm{id})} \to_* A$ over $\mathtt{pt} : A$. Note that the equality $f = \mathrm{id}$ is in the type of *unpointed* maps $A \to A$.

Lemma 5.3.5. *Suppose that A is an H-space. Then $F \simeq (A \to_* A)_{(\mathrm{id})}$.*

Proof. This follows immediately from Lemma 5.2.7. □

Proposition 5.3.6. *Let A be a pointed type. Then the following are logically equivalent:*

1. *A is central;*
2. *A is a connected H-space and $A \to_* A$ is a set;*
3. *A is a connected H-space and $A \simeq_* A$ is a set;*
4. *A is a connected H-space and $A \to_* \Omega A$ is contractible;*
5. *A is a connected H-space and $\Sigma A \to_* A$ is contractible.*

Proof. (1) $\implies$ (2): Assume that A is central. Then Proposition 5.3.3 implies that A is a connected H-space. Since A is a left-invertible H-space, so is $A \to_* A$, by Proposition 5.2.8. Therefore all components of $A \to_* A$ are equivalent to $(A \to_* A)_{(\mathrm{id})}$, and thus to F by Lemma 5.3.5. Now, F is contractible since $\mathbf{ev}_{\mathrm{id}}$ is an equivalence, and consequently $A \to_* A$ is a set since all of its components are contractible.

(2) $\implies$ (3): This follows from the fact that $A \simeq_* A$ embeds into $A \to_* A$.

(3) $\implies$ (1): If $(A \simeq_* A)$ is a set, then its component $(A \to_* A)_{(\mathrm{id})}$ is contractible. Therefore F is contractible, by Lemma 5.3.5. It follows that $\mathbf{ev}_{\mathrm{id}}$ is an equivalence, since A is connected. Hence A is central.

(3) $\iff$ (4): Since A is a left-invertible H-space, so is $A \to_* A$. The latter is therefore a set if and only if the component of the constant map is contractible, which is true if and only if the loop space $\Omega(A \to_* A)$ is contractible. Finally, the equivalence $\Omega(A \to_* A) \simeq (A \to_* \Omega A)$ shows that this is true if and only if $A \to_* \Omega A$ is contractible.

(4) $\iff$ (5): This follows from the equivalence $(A \to_* \Omega A) \simeq (\Sigma A \to_* A)$. □

Corollary 5.3.7. *If A is central and A' is a pointed retract of A, then A' is central.*

Proof. By Proposition 5.2.9, A' is an H-space. Also note that A' is connected, as a retract of the connected type A.

By Proposition 5.3.6, it suffices to show that $A' \to_* \Omega A'$ is contractible. We do this by showing that it is a retract of the type $A \to_* \Omega A$, which is contractible by Proposition 5.3.6. We define a map $(A' \to_* \Omega A') \to (A \to_* \Omega A)$ by sending f to $(\Omega s) f r$ and a map in the other direction by sending g to $(\Omega r) g s$. The composite sends f to $(\Omega r)(\Omega s) f r s$, which is equal to f because $r \circ s =_* \mathrm{id}$. □

Example 5.3.8. Consider the Eilenberg–Mac Lane space $\mathsf{K}(G, n)$ for $n \geq 1$ and G an abelian group. It is a pointed, connected type. Since $\mathsf{K}(G, n) \simeq \Omega\mathsf{K}(G, n+1)$, it is an H-space. By [BvDR18, Theorem 5.1], $\mathsf{K}(G, n) \simeq_* \mathsf{K}(G, n)$ is equivalent to the set of automorphisms of G. It therefore follows from Proposition 5.3.6 that $\mathsf{K}(G, n)$ is central. We will see in Proposition 5.5.9 a more self-contained proof of this result.

Example 5.3.9. Brunerie showed that $\pi_4(\mathbb{S}^3) \simeq \mathbb{Z}/2$ [Bru16]. Therefore, $\mathbb{S}^4 \to_* \mathbb{S}^3$ is not contractible, and so $\mathbb{S}^3$ is not central, by Proposition 5.3.6(5). Since this is in the stable range, it follows that $\mathbb{S}^n$ is not central for $n \geq 3$. By Corollary 5.3.7, any product $\mathbb{S}^n \times X$ with X pointed is not central for $n \geq 3$.

Remark 5.3.10. For a pointed type A, we have seen that A being central is logically equivalent to A being a connected H-space such that $A \simeq_* A$ is a set. It is natural to ask whether the reverse implication holds without the assumption that A is an H-space. However, this is not the case. Consider, for example, the pointed, connected type $\mathsf{K}(G, 1)$ for a non-abelian group G. Then $\mathsf{K}(G, 1) \simeq_* \mathsf{K}(G, 1)$ is equivalent to the set of group automorphisms of G. If $\mathsf{K}(G, 1)$ were central, then G would be twice deloopable, which would contradict G being non-abelian.

Remark 5.3.11. In Remark 5.2.31 we saw that every unique H-space is a symmetric H-space. In particular, every central H-space is a symmetric H-space, and therefore the binary operation of a central H-space extends to an operation on unordered pairs

$$\begin{tikzcd} A^2 \arrow[d] \arrow[dr, "\mu"] & \\ \sum_{X:B\mathbb{Z}/2} A^X \arrow[r, "\tilde{\mu}"'] & A\,. \end{tikzcd}$$

We claim that the binary operation furthermore extends to the type of *genuine* unordered pairs [Buc23]. The type $\mathrm{GUP}(A)$ of **genuine unordered pairs** of elements of A is defined to be the pushout

$$\begin{tikzcd} B\mathbb{Z}/2 \times A \arrow[r, "\pi_2"] \arrow[d, "\delta"'] & A \arrow[d] \\ \sum_{X:B\mathbb{Z}/2} A^X \arrow[r] & \mathrm{GUP}(A)\,, \end{tikzcd}$$

where $\delta(X, a) := (X, \mathrm{const}_a)$. To see that $\tilde{\mu}$ extends to the genuine unordered pairs, we have to show that

$$\tilde{\mu}_X(\mathrm{const}_a) = \mu(a, a)$$

for every $X : B\mathbb{Z}/2$ and every $a : A$. To see this, note that both $a \mapsto \tilde{\mu}_X(\mathrm{const}_a)$ and $a \mapsto \mu(a, a)$ are pointed maps $A \to_* A$. Since A is assumed to be a central H-space, the type $A \to_* A$ is a set, so the type of pointed homotopies

$$(a \mapsto \tilde{\mu}_X(\mathrm{const}_A)) =_* (a \mapsto \mu(a, a))$$

is a proposition. Therefore it suffices to construct a pointed homotopy

$$(a \mapsto \tilde{\mu}_{\mathbf{2}}(\mathrm{const}_A)) =_* (a \mapsto \mu(a, a))$$

We clearly have such a pointed homotopy, since $\tilde{\mu}$ is an extension of μ. Note that in this argument we made essential use of the assumption that A is central. We do not currently know whether the binary operation of a unique H-space can be extended to the genuine unordered pairs.

By the previous proposition, the type $A \to_* A$ is a set whenever A is central. Presently we observe that it is in fact a ring.

Corollary 5.3.12. *For any central type A, the set $A \to_* A$ is a ring under pointwise multiplication and function composition.*

Proof. It follows from A being a commutative and associative H-space that the set $A \to_* A$ is an abelian group. The only nontrivial thing we need to show is that function composition is linear. Let $f, g, \phi : A \to_* A$, and consider $a : A$. By Proposition 5.2.32, ϕ is an H-space map. Consequently,

$$(\phi \circ (f \cdot g))(a) \equiv \phi(f(a) \cdot g(a)) = \phi(f(a)) \cdot \phi(g(a)) \equiv ((\phi \circ f) \cdot (\phi \circ g))(a). \qquad \square$$

The following remark gives some insight into the nature of the ring $A \to_* A$.

Remark 5.3.13. If BG is an ∞-group and X is a pointed type, recall that a bundle over X is *G-principal* if it is classified by a map $X \to_* BG$ (see e.g. [Sco20, Def. 2.23] for a formal definition which easily generalizes to arbitrary ∞-groups). In particular, it is not hard to see that the Hopf fibration of G (as the loop space of BG) is a G-principal bundle, i.e., classified by a map $\Sigma G \to_* BG$.

In Proposition 5.4.4 we will see that any central type A has a delooping $\mathsf{BAut}_1(A)$. This means we have equivalences

$$(A \to_* A) \;\simeq\; (A \to_* (A \simeq A)_{(\mathrm{id})}) \;\simeq\; (\Sigma A \to_* \mathsf{BAut}_1(A)).$$

Thus we see that $A \to_* A$ is *the ring of principal A-bundles over* ΣA. The equivalence above maps the identity $\mathrm{id} : A \to_* A$ to the Hopf fibration of A (as a principal A-bundle), meaning the Hopf fibration is the multiplicative unit from this perspective.

In the remainder of this section we collect various results which are needed later on. The first result is that "all" of the evaluation fibrations of a central type A are equivalences:

Proposition 5.3.14. *Let A be a central type and let $f : A \to_* A$. The evaluation fibration* $\mathbf{ev}_f : (A \to A)_{(f)} \to_* A$ *is an equivalence, with inverse given by $a \mapsto a \cdot f(-)$.*

Proof. The type $A \to A$ is a left-invertible H-space via pointwise multiplication, by Proposition 5.2.8. So there is an equivalence $(A \to A)_{(\mathrm{id})} \to (A \to A)_{(f)}$ sending g to $f \cdot g$. Since f is pointed, we have

$$\mathbf{ev}_f(f \cdot g) \equiv (f \cdot g)(\mathtt{pt}) \equiv f(\mathtt{pt}) \cdot g(\mathtt{pt}) = \mathtt{pt} \cdot g(\mathtt{pt}) = g(\mathtt{pt}) = \mathbf{ev}_{\mathrm{id}}(g).$$

In other words, $\mathbf{ev}_f \circ (f \cdot -) = \mathbf{ev}_{\mathrm{id}}$, which shows that $\mathbf{ev}_f$ is an equivalence. Since f is pointed, the stated map is a section of $\mathbf{ev}_f$, hence is an inverse. □

Corollary 5.3.15. *Let A be a central type, let $f : A \to_* A$, and let $g : (A \to A)_{(f)}$. Then for all $a : A$, we have $g(a) = g(\mathtt{pt}) \cdot f(a)$.* □

Any central type has an inversion map, which plays a key role in the next section.

Definition 5.3.16. Suppose that A is central. The **inversion map** $\mathrm{id}^* : A \to A$ sends a to $a^* :\equiv a\backslash\mathtt{pt}$.

The defining property of a^* is that $a \cdot a^* = \mathtt{pt}$. Since A is abelian, we also have $a^* \cdot a = \mathtt{pt}$, so it would have been equivalent to define the inversion to be $a \mapsto \mathtt{pt}/a$. Because $\mathtt{pt} \cdot \mathtt{pt} = \mathtt{pt}$, it follows that $\mathtt{pt}^* = \mathtt{pt}$, and from commutativity of a central H-space it follows that and $a^{**} = a$ for all a. Thus the inversion map id^* is a pointed self-equivalence of A and an involution.

A curious property is that on the component of id^*, inversion of equivalences is homotopic to the identity. This comes up in the next section.

Proposition 5.3.17. *The map $\phi \mapsto \phi^{-1} : (A \simeq A)_{(\mathrm{id}^*)} \to (A \simeq A)_{(\mathrm{id}^*)}$ is homotopic to the identity.*

Proof. Let $\phi : (A \simeq A)_{(\mathrm{id}^*)}$. We need to show that $\phi = \phi^{-1}$, or equivalently that $\phi(\phi(\mathtt{pt})) = \mathtt{pt}$, since $\mathbf{ev}_{\mathrm{id}}$ is an equivalence. (Note that $\phi \circ \phi : (A \simeq A)_{(\mathrm{id})}$.) Using Corollary 5.3.15, we have that

$$\phi(\phi(\mathtt{pt})) = \phi(\mathtt{pt}) \cdot \phi(\mathtt{pt})^* = \mathtt{pt}.$$ □

5.4 Bands and torsors

We begin in Section 5.4.1 by defining and studying types *banded* by a central type A, also called *A-bands*. We show that the type $\mathsf{BAut}_1(A)$ of banded types is a delooping of A, that A has a unique delooping, and that every pointed self-map $A \to_* A$ has a unique delooping.

In Section 5.4.2, we show that $\mathsf{BAut}_1(A)$ is itself an H-space under a tensoring operation, from which it follows that it is again a central type. Thus we may iteratively consider banded types to obtain an infinite loop space structure on A, which is unique. As a special case, taking A to be $\mathsf{K}(G, n)$ for some abelian group G produces a novel description of the infinite loop space structure on $\mathsf{K}(G, n)$, as described in Section 5.5.2.

In Section 5.4.3, we define the type of A-torsors, which we show is equivalent to the type of A-bands when A is central, thus providing an alternate description of the delooping of A. The type of A-torsors has been independently studied by David Wärn [Wär23], who has shown that it is a delooping of A under the weaker assumption that A has a unique H-space structure.

5.4.1 Types banded by a central type

We now study types *banded* by a central type A. On this type we will construct an H-space structure, which will be seen to be central.

Definition 5.4.1. For a type A, let $\mathsf{BAut}_1(A) :\equiv \sum_{X:\mathcal{U}} \|A = X\|_0$. The elements of $\mathsf{BAut}_1(A)$ are types which are **banded** by A or **A-bands**, for short. We denote A-bands by X_p, where $p : \|A = X\|_0$ is the **band**. The type $\mathsf{BAut}_1(A)$ is pointed by $A_{|\mathtt{refl}|_0}$.

Given a band $p : \|A = X\|_0$, we will write $\tilde{p} : \|X \simeq A\|_0$ for the associated equivalence.

Remark 5.4.2. It's not hard to see that $\mathsf{BAut}_1(A)$ is a connected, locally small type—hence essentially small, by the join construction [Rij17].

The characterization of paths in Σ-types tells us what paths between banded types are.

Lemma 5.4.3. *Consider two A-bands X_p and Y_q. A path $X_p = Y_q$ of A-bands corresponds to a path $e : X = Y$ between the underlying types making the following triangle of truncated paths commute:*

$$\begin{tikzcd} & A \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, "|e|_0"] & & Y . \end{tikzcd}$$

In other words, there is an equivalence $(X_p = Y_q) \simeq (X = Y)_{(\tilde{p} \cdot q)}$. □

For the remainder of this section, let A be a central type. We begin by showing that the type of A-bands is a delooping of A.

Proposition 5.4.4. *We have that $\Omega\, \mathsf{BAut}_1(A) \simeq A$.*

Proof. We have $\Omega\, \mathsf{BAut}_1(A) \simeq (A = A)_{(\mathtt{refl})} \simeq (A \simeq A)_{(\mathrm{id})} \simeq A$, where the first equivalence makes use of Lemma 5.4.3 and the last equivalence is by centrality. □

Corollary 5.4.5. *The unique H-space structure on A is deloopable.* □

Note that this gives an independent proof that it is associative (cf. Proposition 5.2.34).

Theorem 5.4.6. *The type A has a unique delooping.*

Proof. We must show that the type $\sum_B (\Omega B \simeq_* A)$ of deloopings of A is contractible, where B ranges over the universe of pointed, connected types. We will use $\mathsf{BAut}_1(A)$, with the equivalence ψ from Proposition 5.4.4, as the center of contraction. (More precisely, we use a small type BA which is equivalent to $\mathsf{BAut}_1(A)$, along with the naturally associated equivalence $\Omega(BA) \simeq_* A$, as the center. See Remark 5.4.2. We will suppress this from the rest of the proof.)

Let B be a delooping of A, i.e., a pointed, connected type with a pointed equivalence $\phi : \Omega B \simeq_* A$. Given $x : B$, consider $\mathtt{pt} =_B x$. Since A is connected, B is simply connected. Therefore, to give a banding on $\mathtt{pt} =_B x$, it suffices to do so when x is $\mathtt{pt}$, in which case we use ϕ. This defines a map $f : B \to \mathsf{BAut}_1(A)$. It is easy to show that the equivalence $\phi : (\mathtt{pt} =_B \mathtt{pt}) \simeq A$ is an equivalence of bands, making f into a pointed map.

We claim that the following triangle commutes:

$$\begin{tikzcd} \Omega B \arrow[rr, "\Omega f"] \arrow[dr, "\sim", "\phi"'] & & \Omega\,\mathsf{BAut}_1(A) \arrow[dl, "\sim"', "\psi"] \\ & A\,. & \end{tikzcd}$$

Let $q : \mathtt{pt} =_B \mathtt{pt}$. Then $(\Omega f)(q)$ is the path associated to the equivalence

$$A \;\simeq\; (\mathtt{pt} =_B \mathtt{pt}) \;\simeq\; (\mathtt{pt} =_B \mathtt{pt}) \;\simeq\; A.$$

The first equivalence is ϕ^{-1} and the last is ϕ, as these give the pointedness of f. The middle equivalence is the map sending p to $p \cdot q$. The map ψ comes from the evaluation fibration, so to compute $\psi((\Omega f)(q))$ we compute what happens to the base point of A. It gets sent to $\mathtt{refl}$, then q, and then $\phi(q)$. This shows that the triangle commutes.

It follows that Ωf is an equivalence. Since B and $\mathsf{BAut}_1(A)$ are connected, f is an equivalence as well. So f and the commutativity of the triangle provide a path from (B, ϕ) to $(\mathsf{BAut}_1(A), \psi)$ in the type of deloopings. □

We conclude this section by showing how to deloop maps $A \to_* A$.

Definition 5.4.7. Given $f : A \to_* A$, define $Bf : \mathsf{BAut}_1(A) \to_* \mathsf{BAut}_1(A)$ by

$$Bf(X_p) :\equiv (X \to A)_{(f^* \circ \tilde{p}^{-1})},$$

where $f^* :\equiv f \circ \mathrm{id}^*$, and we have used that $f^* \circ \tilde{p}^{-1}$ is well-defined as an element of the set-truncation. To give a banding of $(X \to A)_{(f^* \circ \tilde{p}^{-1})}$ we may induct on p and use Proposition 5.3.14. The same argument shows that Bf is a pointed map.

Note that $f(a^*) = f(a)^*$ for any $a : A$, since f is an H-space map by Proposition 5.2.32, so there's no choice involved in this definition.

Let $g : \mathsf{BAut}_1(A) \to_* \mathsf{BAut}_1(A)$. Given a loop $q : \mathtt{pt} = \mathtt{pt}$, the loop $(\Omega g)(q)$ is the composite

$$\mathtt{pt} \;=\; g(\mathtt{pt}) \;=\; g(\mathtt{pt}) \;=\; \mathtt{pt},$$

which uses pointedness of g and $\mathrm{ap}_g(q)$. We will identify $(\mathtt{pt} = \mathtt{pt})$ with A and then write

$$\Omega' g \;:\; A \;\simeq_*\; (\mathtt{pt} = \mathtt{pt}) \;\xrightarrow{\Omega g}_*\; (\mathtt{pt} = \mathtt{pt}) \;\simeq_*\; A.$$

Proposition 5.4.8. *We have that $\Omega' Bf = f$ for any $f : A \to_* A$.*

Proof. The following diagram describes how Bf acts on a loop $p : \mathtt{pt} =_{\mathsf{BAut}_1(A)} \mathtt{pt}$:

$$\begin{tikzcd} A_{\mathtt{refl}} \arrow[d, equal, "p"] & (A \to A)_{(f^*)} \arrow[d, equal, "g \mapsto g \circ \tilde{p}^{-1}"] & A \arrow[l, "\sim"'] \\ A_{\mathtt{refl}} & (A \to A)_{(f^*)} \arrow[r, "\sim"] & A \end{tikzcd}$$

Since $\tilde{p}$ is in the component of the identity, Corollary 5.3.15 tells us that $\tilde{p}(a) = x \cdot a$ for all $a : A$, where $x = \tilde{p}(\mathtt{pt})$. So $\tilde{p}^{-1}(a) = x \backslash a$. Then the composite $A \simeq A$ on the right is seen to be

$$a \mapsto \mathbf{ev}_{f^*}\Big((a \cdot f^*(-)) \circ \tilde{p}^{-1}\Big) = \mathbf{ev}_{f^*}\Big(a \cdot f^*(x\backslash(-))\Big) = a \cdot f(x^{**}) = a \cdot f(x).$$

The domain $A_{\mathtt{refl}} = A_{\mathtt{refl}}$ is identified with A by sending a path p to $\tilde{p}(\mathtt{pt})$, which in this case is the x above. The codomain $(A \simeq A)_{(\mathrm{id})}$ is identified with A using $\mathbf{ev}_{\mathrm{id}}$, which sends the displayed function to $\mathtt{pt} \cdot f(x)$, which equals $f(x)$. So we have that $\Omega B f = f$. By Lemma 5.2.7, they are equal as pointed maps. □

Proposition 5.4.9. *We have that $B\Omega' g = g$ for any $g : \mathsf{BAut}_1(A) \to_* \mathsf{BAut}_1(A)$.*

Proof. Given an A-band X_p, we need to show that $g(X_p) = (X \to A)_{((\Omega' g)^* \circ \tilde{p}^{-1})}$. First we construct a map of the underlying types from left to right. For $y : g(X_p)$, define the map

$$G_y : X \xrightarrow{\sim} (\mathtt{pt} = X_p) \simeq (X_p = \mathtt{pt}) \xrightarrow{\mathrm{ap}_g} (g(X_p) = g(\mathtt{pt})) \simeq (\mathtt{pt} = \mathtt{pt}) \to A,$$

where the second map is path inversion, and the fourth map uses the trivialization of $g(X_p)$ associated to y and pointedness of g. The identification $\mathtt{pt} = g(\mathtt{pt})$ corresponds to a unique point $y_0 : g(\mathtt{pt})$. To check that G_y lies in the right component, we may induct on p and assume $y \equiv y_0$, since $g(\mathtt{pt})$ is connected. We then get the map

$$G_{y_0} : A \xrightarrow{\mathrm{id}^*} A \simeq (\mathtt{pt} = \mathtt{pt}) \xrightarrow{\Omega g} (\mathtt{pt} = \mathtt{pt}) \to A,$$

since path inversion on $(\mathtt{pt} = \mathtt{pt})$ corresponds to inversion on A, and y_0 corresponds to the pointing of g. This map is precisely the definition of $(\Omega' g)^*$, so G lands in the desired component.

To check that G defines an equivalence of bands we may again induct on p. We write $\widetilde{y_0} : \mathtt{pt} \simeq g(\mathtt{pt})$ for the equivalence associated to the point $y_0 : g(\mathtt{pt})$, which is a lift of the (equivalence associated to the) banding of $g(\mathtt{pt})$. It then suffices to check that the diagram

$$\begin{tikzcd} g(\mathtt{pt}) \arrow[rr, "G"] \arrow[dr, "\widetilde{y_0}^{-1}"'] & & (A \to A)_{((\Omega' g)^*)} \arrow[dl, "\mathbf{ev}_{(\Omega' g)^*}"] \\ & \mathtt{pt} & \end{tikzcd}$$

commutes. Let $y : g(\mathtt{pt})$, which we identify with a trivialization $y' : \mathtt{pt} = g(\mathtt{pt})$. Chasing through the definition of G and using that $\mathrm{ap}_g(\mathtt{refl}) = \mathtt{refl}$, we see that

$$G_y(\mathtt{pt}) = \mathbf{ev}(y' \cdot \overline{y_0}) = \widetilde{y_0}^{-1}(y'(\mathtt{pt})) \equiv \widetilde{y_0}^{-1}(y),$$

where $\mathbf{ev} : (\mathtt{pt} = \mathtt{pt}) \to A$ is the last map in the definition of G_y, which transports $\mathtt{pt}$ along a path. Thus we see that the triangle above commutes, whence G is an equivalence of bands, as required. □

Theorem 5.4.10. *We have inverse equivalences*

$$\Omega' : (\mathsf{BAut}_1(A) \to_* \mathsf{BAut}_1(A)) \simeq (A \to_* A) : B.$$

In particular, the type $\mathsf{BAut}_1(A) \to_* \mathsf{BAut}_1(A)$ is a set.

Proof. Combine Propositions 5.4.8 and 5.4.9. □

5.4.2 Tensoring bands

In this section, we will construct an H-space structure on $\mathsf{BAut}_1(A)$, where we continue to assume that A is a central type. This H-space structure is interesting in its own right, and also implies that $\mathsf{BAut}_1(A)$ is itself central. It that follows that A is an infinite loop space. The tensor product of banded types is analogous to the classical tensor product of torsors over an abelian group. An account of the latter is given in Section 5.5.1.

This elementary lemma will come up frequently.

Lemma 5.4.11. *Let $P : \mathsf{BAut}_1(A) \to \mathcal{U}$ be a set-valued type family. Then $\prod_{X_p} P(X_p)$ is equivalent to $P(\mathtt{pt})$.*

Proof. Since each $P(X_p)$ is a set, $\prod_{X_p} P(X_p)$ is equivalent to $\prod_{X:\mathcal{U}} \prod_{p:A=X} P(X_{|p|_0})$. By path induction, this is equivalent to $P(A_{|\mathtt{refl}|_0})$, i.e., $P(\mathtt{pt})$. □

A consequence of the following result is that any pointed A-band is trivial.

Proposition 5.4.12. *Let X_p be an A-band. Then there is an equivalence $(\mathtt{pt} =_{\mathsf{BAut}_1(A)} X_p) \to X$.*

Proof. By Lemma 5.4.3, there is an equivalence $(\mathtt{pt} =_{\mathsf{BAut}_1(A)} X_p) \simeq (A \simeq X)_{(\tilde{p})}$. We will show that $\mathbf{ev}_p : (A \simeq X)_{(\tilde{p})} \to X$ is an equivalence. By Lemma 5.4.11, it's enough to prove this when $X_p \equiv \mathtt{pt}$, and this holds because A is central. □

We now show that path types between A-bands are themselves banded. This underlies the main results of this section.

Proposition 5.4.13. *Let X_p and Y_q be A-bands. The type $X_p =_{\mathsf{BAut}_1(A)} Y_q$ is banded by A.*

Proof. We need to construct a band $\|A = (X_p = Y_q)\|_0$. Since the goal is a set, we may induct on p and q, thus reducing the goal to $\|A = (\mathtt{pt} =_{\mathsf{BAut}_1(A)} \mathtt{pt})\|_0$. Using that $(\mathtt{pt} =_{\mathsf{BAut}_1(A)} \mathtt{pt}) \simeq (A \simeq A)_{(\mathrm{id})}$ and that A is central, we may give the set truncation of the inverse of the evaluation fibration at id_A. □

The following is an immediate corollary of Proposition 5.4.12.

Corollary 5.4.14. *For any A-band X_p, the A-band $(X_p = X_p)$ is trivial.* □

We next define a tensor product of banded types, using the notion of duals of bands. Write $\mathrm{refl}^* : A = A$ for the self-identification of A associated to the inversion map id^* (Definition 5.3.16) via univalence.

Definition 5.4.15. Let X_p be an A-band. The band $p^* :\equiv |\mathtt{refl}^*| \cdot p$ is the **dual of p**, and the A-band $X_p^* :\equiv X_{p^*}$ is the **dual of X_p**.

Since id^* is an involution, it follows that taking duals defines an involution on $\mathsf{BAut}_1(A)$, meaning that $X_p^{**} = X_p$.

Lemma 5.4.16. *We have $\mathtt{pt} = \mathtt{pt}^*$ in $\mathsf{BAut}_1(A)$.*

Proof. The underlying type of $\mathtt{pt}^*$ is A, which has a base point, so this follows from Proposition 5.4.12. □

We now show how to tensor types banded by A.

Definition 5.4.17. For X_p, $Y_q : \mathsf{BAut}_1(A)$, define $X_p \otimes Y_q :\equiv (X_p^* = Y_q)$, with the banding from Proposition 5.4.13.

It follows from Lemma 5.4.3 that the type $X_p \otimes Y_q$ is equivalent to $(X = Y)_{(\overline{p^*} \cdot q)}$. Since taking duals is an involution, we also have identifications

$$X_p \otimes Y_q \;\equiv\; (X_p^* = Y_q) \;\simeq\; (X_p = Y_q^*) \;\simeq\; (X = Y)_{(\overline{p} \cdot q^*)}.$$

Moreover, from Corollary 5.4.14, we see that $X_p^* \otimes X_p = \mathtt{pt}$.

Tensoring defines a binary operation on $\mathsf{BAut}_1(A)$, and we now show that this operation is symmetric.

Proposition 5.4.18. *For any $X_p, Y_q : \mathsf{BAut}_1(A)$, there is a path $\sigma_{(X_p, Y_q)} : X_p \otimes Y_q =_{\mathsf{BAut}_1(A)} Y_q \otimes X_p$ such that $\sigma_{\mathtt{pt},\mathtt{pt}} = 1$.*

Proof. By univalence and the characterization of paths between bands, we begin by giving an equivalence between the underlying types. The equivalence will be path-inversion, as a map

$$(X = Y)_{(\overline{p} \cdot q^*)} \longrightarrow (Y = X)_{(\overline{q} \cdot p^*)}.$$

To see that this is valid it suffices to show that the inversion of $\overline{p} \cdot q^*$ is $\overline{q} \cdot p^*$. We have:

$$\overline{\overline{p} \cdot q^*} \;\equiv\; \overline{\overline{p} \cdot \mathtt{refl}^* \cdot q} \;=\; \overline{\mathtt{refl}^* \cdot q} \cdot p \;=\; \overline{q} \cdot \overline{\mathtt{refl}^*} \cdot p \;=\; \overline{q} \cdot \mathtt{refl}^* \cdot p \;\equiv\; \overline{q} \cdot p^*,$$

where we have used associativity of path composition, and that $\overline{\mathrm{refl}^*} = \mathrm{refl}^*$ by Proposition 5.3.17.

To prove the transport condition, we may path induct on both p and q which then yields the following triangle:

$$\begin{tikzcd}
(A = A)_{(\mathrm{refl}^*)} \arrow[rr, "p \mapsto \overline{p}"] \arrow[dr, "\mathbf{ev}_{\mathrm{refl}^*}"'] & & (A = A)_{(\mathrm{refl}^*)} \arrow[dl, "\mathbf{ev}_{\mathrm{refl}^*}"] \\
& A . &
\end{tikzcd}$$

Here we are writing $\mathbf{ev}_{\mathrm{refl}^*}$ for the composite $(A = A)_{(\mathrm{refl}^*)} \simeq (A \simeq A)_{(\mathrm{id}^*)} \xrightarrow{\mathbf{ev}_{\mathrm{id}^*}} A$. The horizontal map is given by path-inversion, which is homotopic to the identity by Proposition 5.3.17, hence the triangle commutes.

Paths between paths between banded types correspond to homotopies between the underlying equivalences. Thus $\sigma_{\mathtt{pt},\mathtt{pt}} = 1$ since path-inversion on $(A = A)_{(\mathrm{refl}^*)}$ is homotopic to the identity. □

We now use Lemma 5.2.6 to make $\mathsf{BAut}_1(A)$ into an H-space.

Theorem 5.4.19. *The binary operation $\otimes$ makes $\mathsf{BAut}_1(A)$ into an abelian H-space.*

Proof. We start by showing the left unit law. Since $\mathtt{pt}^* = \mathtt{pt}$, we instead consider the goal $(\mathtt{pt} = X_p) = X_p$. An equivalence between the underlying types is given by Proposition 5.4.12, which after inducting on p clearly respects the bands. Using Proposition 5.4.18 and Lemma 5.2.6, we obtain the desired H-space structure. □

Corollary 5.4.20. *For a central type A, the type $\mathsf{BAut}_1(A)$ is again central. Therefore, A is an infinite loop space, in a unique way. Moreover, every pointed map $A \to_* A$ is infinitely deloopable, in a unique way.*

Proof. That $\mathsf{BAut}_1(A)$ is central follows from condition (2) of Proposition 5.3.6, using Theorems 5.4.10 and 5.4.19 as inputs. That A is a infinite loop space then follows from Proposition 5.4.4: writing $\mathsf{BAut}_1^0(A) :\equiv A$ and $\mathsf{BAut}_1^{n+1}(A) :\equiv \mathsf{BAut}_1(\mathsf{BAut}_1^n(A))$, we see that $\mathsf{BAut}_1^n(A)$ is an n-fold delooping of A. The uniqueness follows from Theorem 5.4.6. That every pointed self-map is infinitely deloopable in a unique way follows by iterating Theorem 5.4.10. □

Note that $\mathsf{BAut}_1(A)$ is essentially small (Remark 5.4.2), so these deloopings can be taken to be in the same universe as A.

From Theorem 5.4.19 we deduce another characterization of central types:

Proposition 5.4.21. *A pointed, connected type A is central if and only if $\sum_{X:\mathsf{BAut}_1(A)} X$ is contractible.*

Proof. If A is central, then by the left unit law of Theorem 5.4.19, we have

$$\sum_{X:\mathsf{BAut}_1(A)} X \simeq \sum_{X:\mathsf{BAut}_1(A)} (\mathtt{pt}^* =_{\mathsf{BAut}_1(A)} X) \simeq 1.$$

Conversely, if $\sum_{X:\mathsf{BAut}_1(A)} X$ is contractible, then so is its loop space. But the loop space is equivalent to $\sum_{f:A\to_* A} \|f = \mathrm{id}\|$, i.e., the fibre of $\mathbf{ev}_{\mathrm{id}}$ over the base point. Thus $\mathbf{ev}_{\mathrm{id}}$ is an equivalence, since A is connected. □

5.4.3 Bands and torsors

Let A be a central type. We define a notion of A-torsor which turns out to be equivalent to the notion of A-band from the previous section. Under our centrality assumption, it follows that the resulting type of A-torsors is a delooping of A. An equivalent type of A-torsors has been independently studied by David Wärn [Wär23], who has also shown that it gives a delooping of A under the weaker hypothesis that A has a unique H-space structure.

Definition 5.4.22. An **action** of A on a type X is a map $\alpha : A \times X \to X$ such that $\alpha(\mathtt{pt}, x) = x$ for all $x : X$. If X has an A-action, we say that it is an ***A*-torsor** if it is merely inhabited and $\alpha(-, x)$ is an equivalence for every $x : X$. The type of ***A*-torsor structures** on a type X is

$$T_A(X) :\equiv \sum_{\alpha:A\times X\to X} (\alpha(\mathtt{pt}, -) = \mathrm{id}_X) \times \|X\|_{-1} \times \prod_{x:X} \mathrm{IsEquiv}\, \alpha(-, x),$$

and the type of A-torsors is $\sum_{X:\mathcal{U}} T_A(X)$.

Since A is connected, an A-action on X is the same as a pointed map $A \to_* (X \simeq X)_{(\mathrm{id})}$. Normally one would require at a minimum that this map sends multiplication in A to composition. We explain in Remark 5.4.28 why our definition suffices.

The condition that $\alpha(-, x)$ is an equivalence for all x is equivalent to requiring that for every x_0, $x_1 : X$, there exists a unique $a : A$ with $\alpha(a, x_0) = x_1$. It is also equivalent to saying that $(\alpha, \mathrm{pr}_2) : A \times X \to X \times X$ is an equivalence.

For any type X, write $\mathbf{ev}_\simeq : (A \simeq X) \to X$ for the evaluation fibration which sends an equivalence e to $e(\mathtt{pt})$. For a map f, write $\mathrm{Sect}(f)$ for the type of (unpointed) sections of f.

Lemma 5.4.23. *For any X, we have an equivalence*

$$T_A(X) \simeq \|X\|_{-1} \times \mathrm{Sect}(\mathbf{ev}_{\simeq}).$$

Proof. This is simply a reshuffling of the data. The map from left to right sends a torsor structure with action $\alpha : A \times X \to X$ to the map $X \to (A \to X)$ sending x to $\alpha(-, x)$. By assumption, this lands in the type of equivalences, and the condition $\alpha(\mathtt{pt}, -) = \mathrm{id}_X$ says that it is a section. We leave the remaining details to the reader. □

Lemma 5.4.24. *Let X be an A-torsor. Then X is connected.*

Proof. Since X is merely inhabited and our goal is a proposition, we may assume that we have $x_0 : X$. Then we have an equivalence $\alpha(-, x_0) : A \to X$. A is connected by Proposition 5.3.3, so it follows that X is. □

Proposition 5.4.25. *Let X be an A-torsor. Then X is banded by A.*

Proof. Associated to the torsor structure on X is a section $X \to (A \simeq X)$ of $\mathbf{ev}_{\simeq}$. Since X is 0-connected, it lands in a component of $A \simeq X$. By univalence, this determines a banding of X. □

Theorem 5.4.26. *Let X be a type. There is an equivalence $T_A(X) \simeq \|A = X\|_0$. Therefore, there is an equivalence between the type of A-torsors and $\mathsf{BAut}_1(A)$.*

Proof. Proposition 5.4.25 gives a map f. We check that the fibres are contractible. Let $p : \|A = X\|_0$ be a banding of X. An A-torsor structure t on X with $f(t) = p$ consists of a section s of $\mathbf{ev}_{\simeq}$ that lands in the component $(A \simeq X)_{(\tilde{p})}$, where $\tilde{p}$ denotes the equivalence associated to p. But by Proposition 5.4.12, the evaluation fibration $(A \simeq X)_{(\tilde{p})} \to X$ is an equivalence, so it has a unique section. □

Remark 5.4.27. It follows that $T_A(X)$ is a set. One can also show this using Corollary 5.4.14 and Proposition 5.3.6.

Remark 5.4.28. Let X be an A-torsor, or equivalently, an A-band. By Corollary 5.4.14, we have an equivalence $e : A \simeq (X \simeq X)_{(\mathrm{id})}$. Since A has a unique H-space structure, this equivalence is an equivalence of H-spaces, where the codomain has the H-space structure coming from composition. Since A is connected, the A-action on X gives a map $\alpha' : A \to_* (X \simeq X)_{(\mathrm{id})}$. (In fact, $\alpha' = e$, but we won't use this fact.) Using the equivalence e, it follows from Theorem 5.4.10 that any map with the same type as α' is deloopable in a unique way. That is, it has the structure of a group homomorphism in the sense of higher groups (see [BvDR18]). This explains why our naive definition of an A-action is correct in this situation.

5.5 Examples and non-examples

We show that the Eilenberg–Mac Lane spaces $\mathsf{K}(G, n)$ are central whenever G is abelian and $n > 0$, and we use our results to give a self-contained, independent construction of Eilenberg–Mac Lane spaces. The base case $\mathsf{K}(G, 1)$ is discussed in Section 5.5.1 and the other cases in Section 5.5.2. In Section 5.5.3, we produce examples of products of Eilenberg–Mac Lane spaces which are central and examples which are not central. At present, we do not know whether there exist central types which are not products of Eilenberg–Mac Lane spaces. In Section 5.5.4, we show that any truncated, central type with just two non-zero homotopy groups, both of which are finitely presented, is a product of Eilenberg–Mac Lane spaces.

5.5.1 The H-space of G-torsors

Given a group G, we construct the type TG of G-torsors and show that it is a $\mathsf{K}(G,1)$. Specifically, a pointed type X **is a** $\mathbf{K}(G,1)$ if it is connected and comes equipped with a pointed equivalence $\Omega X \simeq_* G$ which sends composition of loops to multiplication in G. (We always point ΩX at `refl`.) Another account of this fact may be found in [Bez+23, Section 4.9].

When G is abelian, we can tensor G-torsors to obtain an H-space structure on TG which is analogous to the tensor product of bands of Theorem 5.4.19. These constructions are all classical and we therefore omit some details.

Definition 5.5.1. Let G be a group. A ***G*-set** is a set X with a group homomorphism $\alpha : G \to \mathsf{Aut}(X)$. If the set X is merely inhabited and the map $\alpha(-,x) : G \to X$ is an equivalence for every $x : X$, then (X,α) is a ***G*-torsor**. We write TG for the type of G-torsors. Given two G-sets X and Y, we write $X \to_G Y$ for the set of G-equivariant maps from X to Y, defined in the usual way.

We may write $g \cdot x$ instead of $\alpha(g,x)$ when no confusion can arise. The following is straightforward to check:

Lemma 5.5.2. *Let X and Y be G-torsors. We have an equivalence $(X =_{TG} Y) \simeq (X \to_G Y)$, natural in X and Y. In particular, a G-equivariant map between G-torsors is automatically an equivalence.* □

Any group G acts on itself by left translation, making G into a G-torsor which constitutes the base point `pt` of both TG and the type of G-sets. Since a G-equivariant map $\mathtt{pt} \to_G X$ is determined by where it sends $1 : G$, the map $(\mathtt{pt} \to_G X) \to X$ that evaluates at 1 is an equivalence. It is clear that the type TG is a 1-type, which implies that its loop space is a group.

Proposition 5.5.3. *We have a group isomorphism $\Omega TG \simeq G$.*

We only sketch a proof since this is a classical result.

Proof. Since paths between G-torsors correspond to G-equivariant maps, we have equivalences of sets

$$(\mathtt{pt} =_{TG} \mathtt{pt}) \simeq (\mathtt{pt} \to_G \mathtt{pt}) \simeq G,$$

where the second equivalence is given by evaluation at 1. The first equivalence sends path composition to composition of maps, which reverses the order—i.e., it's an anti-isomorphism. The second equivalence evaluates a map at $1 : G$. Thus, for $\phi, \psi : \mathtt{pt} \to_G \mathtt{pt}$ we have

$$\phi(\psi(1)) = \phi(\psi(1) \cdot 1) = \psi(1) \cdot \phi(1),$$

where $\cdot$ denotes the multiplication in G. In other words, evaluation at 1 is an anti-isomorphism, meaning the composite $(\mathtt{pt} =_{TG} \mathtt{pt}) \simeq G$ is an isomorphism of groups. □

The following proposition says that the G-torsors are precisely those G-sets which lie in the component of the base point.

Proposition 5.5.4. *A G-set (X, α) is a G-torsor if and only if there merely exists a G-equivariant equivalence from $\mathtt{pt}$ to X.*

Proof. Suppose X is a G-torsor. To produce a mere G-equivariant equivalence $\mathtt{pt} \simeq_G X$ we may assume we have some $x : X$, since X is merely inhabited. Then $(-) \cdot x : G \to X$ yields an equivalence which is clearly G-equivariant, as required.

Conversely, assume that there merely exists a G-equivariant equivalence from $\mathtt{pt}$ to X. Since being a G-torsor is a proposition, we may assume we have an actual G-equivariant equivalence. But then we are done since $\mathtt{pt}$ is a G-torsor. □

It follows that TG is connected. Thus by Proposition 5.5.3 we deduce:

Corollary 5.5.5. *The type TG is a $\mathsf{K}(G, 1)$.* □

For the remainder of this section, let G be an abelian group.

Proposition 5.5.6. *For any two G-torsors S and T, the path type $S =_{TG} T$ is again a G-torsor.*

Proof. First we make $S =_{TG} T$ into a G-set. This path type is equivalent to the type $S \to_G T$. Using that G is abelian, it's easy to check that the map

$$(g, \phi) \longmapsto (s \mapsto g \cdot \phi(s)) \ : \ G \times (S \to_G T) \longrightarrow (S \to_G T)$$

is well-defined and makes $S \to_G T$ into a G-set.

To check that the above yields a G-torsor, we may assume that $S \equiv \mathtt{pt} \equiv T$, by the previous lemma. One can check that Proposition 5.5.3 gives an equivalence of G-sets, where $\mathtt{pt} \to_G \mathtt{pt}$ is equipped with the G-action just described. Thus $\mathtt{pt} \to_G \mathtt{pt}$ is a G-torsor, as required. □

In order to describe the tensor product of G-torsors, we first need to define duals.

Definition 5.5.7. Let (X, α) be a G-torsor. The **dual** $\boldsymbol{X^*}$ of X is the G-torsor X with action

$$\alpha^*(g, x) :\equiv \alpha(g^{-1}, x).$$

The tensor product of G-torsors is now defined as $X \otimes Y :\equiv (X^* =_{TG} Y)$.

Proposition 5.5.8. *The tensor product of G-torsors makes TG into an H-space.*

Proof. We verify the hypotheses of Lemma 5.2.6. Thus our first goal is to construct a symmetry

$$\sigma_{X,Y} : (X^* =_{TG} Y) \ =_{TG} \ (Y^* =_{TG} X).$$

After identifying paths of G-torsors with G-equivariant equivalences, we may consider the map which inverts such an equivalence. A short calculation shows that if $\phi : X^* \to_G Y$ is G-equivariant, then $\phi^{-1} : Y^* \to_G X$ is again G-equivariant. We need to check that the map sending ϕ to ϕ^{-1} is itself G-equivariant, so let $\phi : X^* \to_G Y$ and let $g : G$. Since the inverse of $g \cdot (-)$ is $g^{-1} \cdot (-)$, we have:

$$(g \cdot \phi)^{-1} = \phi^{-1}(g^{-1} \cdot (-)) = g \cdot \phi^{-1}(-),$$

using that $\phi^{-1} : Y^* \to_G X$ is G-equivariant. Thus inversion is G-equivariant, yielding the required symmetry σ.

Now we argue that $\sigma_{\mathtt{pt},\mathtt{pt}} = \mathtt{refl}$, or, equivalently, that maps $\mathtt{pt}^* \to_G \mathtt{pt}$ are their own inverses. Such a map is uniquely determined by where it sends $1 : G$, so it suffices to show that $\phi(\phi(1)) = 1$ for every $\phi : \mathtt{pt}^* \to_G \mathtt{pt}$. Fortunately, we have

$$\phi(\phi(1)) = \phi(\phi(1) \cdot 1) = \phi(1)^{-1} \cdot \phi(1) = 1.$$

Lastly, it is straightforward to check that the map $(\mathtt{pt}^* \to_G X) \to X$ which evaluates at $1 : G$ is G-equivariant, for any G-torsor X. This yields the left unit law for the tensor product $\otimes$. As such we have fulfilled the hypotheses of Lemma 5.2.6, giving us the desired H-space structure. □

Using Proposition 5.3.6, one can check that TG is a central H-space.

5.5.2 Eilenberg–Mac Lane spaces

We now use our results to give a new construction of Eilenberg–Mac Lane spaces. For an abelian group G, recall that a pointed type X **is a $\mathsf{K}(G,1)$** if it is connected and there is a pointed equivalence $\Omega X \simeq_* G$ which sends composition of paths to multiplication in G. For $n > 1$, a pointed type X **is a $\mathsf{K}(G, n+1)$** if it is connected and ΩX is a $\mathsf{K}(G,n)$. It follows that such an X is an n-connected $(n+1)$-type with $\Omega^{n+1} X \simeq_* G$ as groups.

In the previous section we saw that the type TG of G-torsors is a $\mathsf{K}(G,1)$ and is central whenever G is abelian. The following proposition may be seen as a higher analog of this fact.

Proposition 5.5.9. *Let G be an abelian group and let $n > 0$. If a type A is a $\mathsf{K}(G,n)$ and an H-space, then A is central and $\mathsf{BAut}_1(A)$ is a $\mathsf{K}(G,n+1)$ and an H-space.*

The fact that $\mathsf{BAut}_1(A)$ is a $\mathsf{K}(G,n+1)$ also follows from [Shu14], using the fact that $\mathsf{BAut}_1(A)$ is the 1-connected cover of $\mathsf{BAut}(A)$.

Proof. Suppose that A is a $\mathsf{K}(G,n)$ and an H-space. Then $A \to_* \Omega A$ is contractible, since it is equivalent to $\|A\|_{n-1} \to_* \Omega A$, and $\|A\|_{n-1}$ is contractible. So Proposition 5.3.6 implies that A is central. By Proposition 5.4.4, $\Omega\,\mathsf{BAut}_1(A) \simeq A$, so $\mathsf{BAut}_1(A)$ is a $\mathsf{K}(G,n+1)$. By Theorem 5.4.19, $\mathsf{BAut}_1(A)$ is also an H-space. □

We can use the previous proposition to *define* $\mathsf{K}(G,n)$ for all $n > 0$ by induction. For the base case $n \equiv 1$ we let $\mathsf{K}(G,1) :\equiv TG$, the type of G-torsors from the previous section. When G is abelian, we saw that TG is an H-space, which lets us apply the previous proposition. By induction, we obtain a $\mathsf{K}(G,n)$ for all n. Note that this construction produces a $\mathsf{K}(G,n)$ which lives $n-1$ universes above the given $\mathsf{K}(G,1)$, but that it is essentially small by the join construction [Rij17].

5.5.3 Products of Eilenberg–Mac Lane spaces

Here is our first example of a central type that is not an Eilenberg–Mac Lane space.

Example 5.5.10. Let $K = \mathsf{K}(\mathbb{Z}/2, 1) = \mathbb{R}P^\infty$ and $L = \mathsf{K}(\mathbb{Z}, 2) = \mathbb{C}P^\infty$, and consider $A = K \times L$. This is a connected H-space, and

$$\begin{aligned}
(K \times L \to_* \Omega(K \times L)) &\simeq (K \to_* \Omega(K \times L)) && \text{since } K = \|K \times L\|_1 \\
&\simeq (K \to_* \Omega L) && \text{since } K \text{ is connected} \\
&\simeq (\mathbb{Z}/2 \to_{\mathsf{Ab}} \mathbb{Z}) && \text{by [BvDR18, Theorem 5.1]} \\
&\simeq 1.
\end{aligned}$$

So it follows from Proposition 5.3.6(4) that A is central.

On the other hand, not every product of Eilenberg–Mac Lane spaces is central.

Example 5.5.11. Let $K = \mathsf{K}(\mathbb{Z}/2, 1) = \mathbb{R}P^\infty$ and $L' = \mathsf{K}(\mathbb{Z}/2, 2)$. A calculation like the above shows that $K \times L' \to_* \Omega(K \times L')$ is not contractible, so $K \times L'$ is not central.

As another example, [Cur68, Proposition Ia] shows that $\mathsf{K}(\mathbb{Z}, 1) \times \mathsf{K}(\mathbb{Z}, 2))$ (i.e., $\mathbb{S}^1 \times \mathbb{C}P^\infty$) has infinitely many distinct H-space structures classically. So it is not central, by Proposition 5.3.3.

Clearly both of these examples can be generalized to other groups and shifted to higher dimensions. Moreover, by Corollary 5.3.7, the product of any non-central type with any pointed type is again not central.

By Proposition 5.3.3, centrality of a type implies that it has a unique H-space structure. The converse fails, as we now demonstrate. We are grateful to David Wärn for bringing our attention to this example.

Example 5.5.12. The type $A :\equiv \mathsf{K}(\mathbb{Z}, 2) \times \mathsf{K}(\mathbb{Z}, 3)$ is not central, by a computation similar to the one in the previous example. However, we note that it admits a unique H-space structure. Since A is a loop space it admits an H-space structure. Then, by the first claim in Proposition 5.2.29, with $k = 2$, we see that type of H-space structures on A is contractible.

5.5.4 Truncated types with two non-zero homotopy groups

All the examples we have of central types so far are generalized Eilenberg–Mac Lane spaces (GEMs), i.e., products of Eilenberg–Mac Lane spaces. We do not know whether all central types are GEMs. In this section we rule out a class of potential counterexamples. Specifically, we show that any truncated central type with only two non-zero homotopy groups, both of which are finitely presented, is a product of Eilenberg–Mac Lane spaces.

We first show that one can reduce to the stable range. Let X be a $(k+1)$-truncated central type, which is in particular 0-connected. Since X is an infinite loop space, we may consider an $(n-1)$-fold delooping $B^{n-1}X$, for any $n > k+1$. This is a central, $(n-1)$-connected, $(n+k)$-truncated, pointed type, and thus represents a stable k-type, i.e., a stable $(k+1)$-group [BvDR18]. If $B^{n-1}X$ is a GEM, then so is X, so for our goal of ruling out non-GEM, truncated central types, it suffices to consider stable $(k+1)$-groups for $k \geq 1$.

Here is the main result of the section.

Theorem 5.5.13. *Let X be a truncated central type and let $n, k \geq 1$. Suppose that $\pi_n(X)$ and $\pi_{n+k}(X)$ are non-trivial groups and that all of the other homotopy groups vanish. Assume that $\pi_n(X)$ is finitely presented and if $k > 1$ that $\pi_{n+k}(X)$ is as well. Then X is merely equivalent to $\mathsf{K}(\pi_n(X), n) \times \mathsf{K}(\pi_{n+k}(X), n + k)$.*

Proof. Write $A :\equiv \pi_n(X)$ and $B :\equiv \pi_{n+k}(X)$. By the argument above, we can assume that $n > k + 1$. Since X is truncated and has no other non-trivial homotopy groups, the fibre of the truncation map $X \to \|X\|_n \simeq \mathsf{K}(A, n)$ is a $\mathsf{K}(B, n + k)$. Since we are in the stable range, we can deloop the next map in the fibre sequence, so we see that X is the homotopy fibre of a pointed map $c : \mathsf{K}(A, n) \to_* \mathsf{K}(B, n + k + 1)$. We will show that c is merely homotopic to the constant map, which implies that X splits as claimed.

Since X is central, $X \to_* \Omega X$ is contractible, so $X \to_* \Omega^i X$ is connected for all $i \geq 1$. In particular, taking $i = k$, we get that $\mathsf{Hom}(A, B) \simeq \|\mathsf{K}(A, n) \to_* \mathsf{K}(B, n)\|_0 \simeq 0$. Since A is finitely presented and $\mathbb{Z}$ is a PID (in the constructive sense), A is merely equivalent to a finite direct sum of cyclic groups. (See [MRR88, Theorem V.2.3] or [LQ15, Proposition 7.3].) Our goal is a proposition, so we can assume that A is explicitly given as such a direct sum. Since $\mathsf{Hom}(A, B)$ is trivial and B is non-trivial, we must have that A is finite, with torsion coprime to the torsion of B. Let r be the cardinality of A. Since X is deloopable, so is c, and in particular, the square

$$\begin{array}{ccc} \mathsf{K}(A, n) & \xrightarrow{c} & \mathsf{K}(B, n + k + 1) \\ {\scriptstyle 0}\big\downarrow & & \big\downarrow{\scriptstyle r} \\ \mathsf{K}(A, n) & \xrightarrow{c} & \mathsf{K}(B, n + k + 1) \end{array} \tag{5.1}$$

commutes, where we write r for the map induced by multiplication by r on B.

Now we split into cases. First assume that $k > 1$, which means we also know that B is finitely presented. Since $X \to_* \Omega^{k-1} X$ is connected, we deduce that $H^{n+1}(\mathsf{K}(A, n); B) \simeq \|\mathsf{K}(A, n) \to_* \mathsf{K}(B, n + 1)\|_0 \simeq 0$. By Theorem 4.2.12, the cohomology group is isomorphic to $\mathsf{Ext}^1_{\mathbb{Z}}(A, B)$, so the latter must also vanish. It follows that B is finite as well, as $\mathsf{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/s, \mathbb{Z}) \simeq \mathbb{Z}/s$ (see Corollary 3.4.9) and $\mathsf{Ext}^1_{\mathbb{Z}}$ respects direct sums. Since the torsion of A is coprime to the torsion of B, multiplication by r on B is an isomorphism. Therefore, the right-hand map in (5.1) is an equivalence. It follows that c is trivial.

Now we consider the case when $k = 1$. In the square (5.1), we no longer know that the map on the right is an equivalence. However, it does follow that c factors through the fiber of r, which we analyze next.

We claim that multiplication by r on B is injective. It suffices to show that the kernel is trivial. So let $b : B$ be such that $rb = 0$. Let p be a prime factor of r. Then $p((r/p)b) = 0$, so there is a homomorphism $A \to B$ which hits $(r/p)b$. So $(r/p)b = 0$. Continuing with the remaining prime factors of r, one eventually gets that $b = 0$.

Thus we have a short exact sequence

$$0 \longrightarrow B \xrightarrow{r} B \xrightarrow{q} B/r \longrightarrow 0. \tag{5.2}$$

This gives rise to a fibre sequence

$$\mathsf{K}(B, n + 1) \xrightarrow{q} \mathsf{K}(B/r, n + 1) \xrightarrow{f} \mathsf{K}(B, n + 2) \xrightarrow{r} \mathsf{K}(B, n + 2).$$

The map c is a composite

$$\mathsf{K}(A, n) \xrightarrow{c'} \mathsf{K}(B/r, n+1) \xrightarrow{f} \mathsf{K}(B, n+2).$$

The short exact sequence (5.2) also gives rise to a six-term exact sequence ending in

$$\cdots \longrightarrow \mathsf{Ext}^1_{\mathbb{Z}}(A, B) \longrightarrow \mathsf{Ext}^1_{\mathbb{Z}}(A, B) \longrightarrow \mathsf{Ext}^1_{\mathbb{Z}}(A, B/r) \longrightarrow 0.$$

Constructively, for general A, the sequence would continue with Ext^2, but since A is finitely presented, $\mathsf{Ext}^2_{\mathbb{Z}}(A, C)$ vanishes for all C by Corollary 4.2.36. So the map $\mathsf{Ext}^1_{\mathbb{Z}}(A, B) \to \mathsf{Ext}^1_{\mathbb{Z}}(A, B/r)$ is surjective. Using that maps of degree one are the same as $\mathsf{Ext}^1_{\mathbb{Z}}$, we can identify that map with the map

$$\|\mathsf{K}(A, n) \to_* \mathsf{K}(B, n+1)\|_0 \longrightarrow \|\mathsf{K}(A, n) \to_* \mathsf{K}(B/r, n+1)\|_0$$

induced by q. This means that c' merely factors through q, and therefore that the composite $c = f \circ c'$ is merely zero. $\square$

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Curriculum Vitae

Name:	Jarl G. Taxerås Flaten
Education:	Ph.D., University of Western Ontario (UWO), 2019–2023. M.Sc., Norges Teknisk-Naturvitenskapelige Universitet (NTNU), 2017–2019. B.Sc., École Polytechnique Fédérale de Lausanne (EPFL), 2013–2017.
Publications:	2. *Formalising Yoneda Ext in univalent foundations* (arXiv:2302.12678), accepted to *ITP 2023*. 1. *Univalent categories of modules* (arXiv:2207.03261), in *Mathematical Structures in Computer Science*.
Preprints:	2. *Ext groups in homotopy type theory* (arXiv:2305.09639), with Dan Christensen. Submitted. 1. *Central H-spaces and banded types* (arXiv:2301.02636), with Ulrik Buchholtz, Dan Christensen, and Egbert Rijke. Submitted.
Selected talks:	*Central H-spaces and their bands* (invited), Rochester Topology Seminar, University of Rochester, Februrary 2023. *Central H-spaces and banded types*, Homotopy Type Theory Electronic Seminar Talks, November 2022. *Internal Yoneda Ext groups*, Category Theory Octoberfest 2022, October 2022. *The moduli space of H-space structures*, AMS Fall 2022 Eastern Sectional Meeting, October 2022. *Internal injectivity of modules in higher toposes*, ASL 2022 North American Annual Meeting, Cornell University, April 2022. *Internally injective modules in higher toposes*, Logic and higher structures, CIRM, Marseille, February 2022.
Awards:	*Teaching Assistant award* for the academic year 2021/22.
Teaching:	*Calculus 1000A instructor*, Fall 2022. Teaching Assistant for the *HoTTEST Summer School 2022*.