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Laurie A. Florio
DEVCOM Armaments Center U.S. Army Armament Graduate S

George L. Fischer
DEVCOM Armaments Center U.S. Army Armament Graduate S

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Undetermined Coefficients with Hyperbolic Sines and Cosines

Laurie A. Florio and George L. Fischer
DEVCOM-Armaments Center United States Army Armament Graduate School

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Abstract: The method of undetermined coefficients is commonly applied to solve linear, constant coefficient, non-homogeneous ordinary differential equations when the forcing function is from a selected class of functions. Often the hyperbolic sine and cosine functions are not explicitly included in this list of functions. Through a set of guided examples, this work argues that the hyperbolic sine and cosine ought to be included in the select class of functions. Careful explanation is provided for the necessary treatment of the cases where the argument of the hyperbolic sine and/or cosine functions matches one or both of the roots of the characteristic equation of the differential equation. Finally, a generalized approach where the hyperbolic and trigonometric sine and cosine functions are written in their exponential form illustrates the connections between the exponential, trigonometric, and hyperbolic sine and cosine functions. This exploration leads to a deeper understanding of the method of undetermined coefficients and can be adapted into coursework on the undetermined coefficients topic.

1 Introduction

Hyperbolic sines and cosines are not typically included in the lists of allowable functions that are consulted when attempting to solve nonhomogeneous differential equations using the method of undetermined coefficients [4, 3, 1].The function space of the derivatives of the hyperbolic cosine and sine functions encompasses the hyperbolic cosine and sine functions and so involves a finite set of functions as required for the method of undetermined coefficients. The need for brevity in tables coupled with the fact that these functions can be re-written in exponential form as $\cosh ax = 1/2\ (e^{ax} + e^{-ax})$ and $\sinh ax = 1/2\ (e^{ax} - e^{-ax})$ might be the reason why these functions are not included in the standard curriculum. However, our understanding of the as-taught method of undetermined coefficients is bolstered by exploring what needs to be altered when working with $\cosh ax$ and $\sinh ax$ as nonhomogeneous terms. Recognizing the potential pitfalls during the adoption of the method of undetermined coefficients here could also help in the case of determining gradients with equivalent expressions that might appear different [2]. To demonstrate the applicability of the undetermined coefficients method to the hyperbolic sine and cosine, three general cases must be evaluated. Three major cases that demonstrate the treatment of the hyperbolic sine and cosine non-homogeneous term in the differential equation include: the case where the argument of the hyperbolic function does not match the absolute value of any of the roots of the characteristic equation, where the argument matches the absolute value of one root of the characteristic equation, including a characteristic equation root of multiplicity two, and where the argument matches the absolute value of two independent roots of the characteristic equation. After working through an example of each of these cases, the solution is generalized in a separate section. The discussion can enhance student learning of the method and might be considered for adoption in the method of undetermined coefficient curriculum.

2 Case I Roots of characteristic equation not equal absolute value of the coefficient in the hyperbolic function argument

In this case, we would solve $y'' + Cy' - Dy = E \sinh ax$, with neither $\cosh ax$ nor $\sinh ax$ in the homogeneous solution as neither the positive or negative value of the coefficient of the hyperbolic function argument appears in the roots of the characteristic equation of the differential equation. Differential equation will be denoted by DE henceforth. We put forth as the trial solution:

$$y_P(x) = A \sinh ax + B \cosh ax.$$

An example where Case I would be utilized is $y'' + y' - 6y = \sinh x$. First, solving for the homogeneous form of the DE yields:

$$y_H(x) = c_1 e^{-3x} + c_2 e^{2x}.$$

The positive and negative of the coefficient in the argument of the $\sinh x$ appears in the exponential form of the function. However, neither of these exponentials appear in the coefficient in the argument of the exponential functions in the homogeneous form of the DE, so we can try a particular solution of the form:

$$y_P(x) = A \sinh x + B \cosh x.$$

Substituting this particular solution into the DE, we obtain:

$$B = -\frac{1}{24}; A = -\frac{5}{24}.$$

So the solution to the DE can be expressed as:

$$y(x) = c_1 e^{-3x} + c_2 e^{2x} - \frac{5}{24} \sinh x - \frac{1}{24} \cosh x.$$

After seeing this example, we are somewhat persuaded that we can add hyperbolic sines and cosines to our short but important set of functions when checking to see if we can use the method of undetermined coefficients when the nonhomogeneous term is not part of the homogeneous solution.

3 Case II The Absolute value of one of the roots of the characteristic equation matches the coefficient in the argument of the hyperbolic function, including a root of multiplicity of two

Now, consider the case where one of either e^{ax} or e^{-ax} is found in the homogeneous solution of a DE of the form $y'' + Cy' - Dy = E \sinh ax$. Then, simply selecting a particular solution as the linear combination of the $\sinh ax$ and the $\cosh ax$ will not yield linearly independent particular solution as the exponentials appear in the hyperbolic functions, for example,

$$\sinh ax = \frac{1}{2}\left(e^{ax} - e^{-ax}\right).$$

3.1 Coefficient of hyperbolic function is a single root of characteristic equation

First, the case where the solutions a or $-a$ is a single root of the characteristic equation of the DE is considered. An example of this case is the following:

$$y'' - y' - 6y = \sinh 2x.$$

Factoring the characteristic equation, we arrive at

$$y_H(x) = c_1 e^{3x} + c_2 e^{-2x}.$$

At first glance, these solutions seem different than $A \sinh 2x$ and $B \cosh 2x$, but when we re-write the nonhomogeneous term we see that the $c_2 e^{-2x}$ is duplicated. An attempt at using a particular solution of the form:

$$y_P(x) = A \sinh 2x + B \cosh 2x.$$

will yield an inconsistent set of equations when solving for the A and B coefficients. Similarly, a particular solution of the form below where the $\sinh ax$ is multiplied by x in an attempt to achieve a linearly independent set of functions also fails to yield a consistent set of equations for the coefficients A and B.

$$y_P(x) = Ax \sinh 2x + Bx \cosh 2x.$$

Then, recognizing the issue is the proper handling of the isolated repeated function e^{-2x}, the exponential terms can be written as:

$$e^{2x} = \sinh 2x + \cosh 2x$$

and

$$e^{-2x} = \cosh 2x - \sinh 2x.$$

The particular solution to the DE can then be written as below, where in this problem, the negative exponential term is the function repeated in the homogeneous solution:

$$y_P(x) = Ax\left[\cosh 2x - \sinh 2x\right] + B\left[\cosh 2x + \sinh 2x\right].$$

Substituting into the DE, the coefficients are:

$$B = -\frac{1}{8}; A = \frac{1}{10}.$$

The solution is then:

$$y(x) = c_1 e^{3x} + c_2 e^{-2x} - \frac{1}{8}\left[\cosh 2x + \sinh 2x\right] + \frac{1}{10}x\left[\cosh 2x - \sinh 2x\right].$$

So with this modified form of the particular solution to account for repetition in the homogeneous solution, the hyperbolic sine and cosine functions have been adapted for use with the method of undetermined coefficients.

3.2 Coefficient of hyperbolic function is a root of multiplicity of two of the characteristic equation

Within the scope of either a positive or a negative of the coefficient of the argument of the hyperbolic function appearing in the roots of the characteristic equation of the DE, a second case of a root of multiplicity of two of the characteristic equation is considered.

$$y'' + 4y' + 4y = \sinh 2x.$$

The roots of the characteristic equation of homogeneous equation are a double root of -2. The homogeneous form of the DE is then:

$$y_H(x) = c_1 x e^{-2x} + c_2 e^{-2x}.$$

Using the same technique, the form of the particular solution is now:

$$y_P(x) = Ax^2\left[\cosh 2x - \sinh 2x\right] + B\left[\cosh 2x + \sinh 2x\right].$$

Substituting into the DE, the coefficients are:

$$B = \frac{1}{32}; A = -\frac{1}{4}.$$

The solution is then:

$$y(x) = c_1 x e^{-2x} + c_2 e^{-2x} + \frac{1}{32}\left[\cosh 2x + \sinh 2x\right] - \frac{1}{4}x^2\left[\cosh 2x - \sinh 2x\right].$$

With this, the special case where the absolute value of the double root of the characteristic equation of the DE matches the coefficient in the hyperbolic function has been handled using the method of undetermined coefficients.

4 Case III The Absolute value of two independent roots of the characteristic equation matches the coefficient in the argument of the hyperbolic function

The third possibility is that the absolute value of two independent roots of the characteristic equation of the DE match the coefficient in the argument of the hyperbolic function. This case will arise for differential equations with the homogeneous form $y'' - D^2y = 0$ where D is real. An example of this case is:

$$y'' - 4y = \sinh 2x.$$

Now the homogeneous solution is $y(x) = c_1e^{2x} + c_2e^{-2x}$. Both of these exponentials are included in both $A \sinh 2x$ and $B \cosh 2x$. Because the homogeneous solution space is just a linear combination of the $\cosh 2x$ and the $\sinh 2x$ functions, we can simply multiply our trial solution by x to achieve the linearly independent function set for our undetermined coefficient solution set.

$$y_P(x) = Ax \sinh 2x + Bx \cosh 2x.$$

The solution for the coefficients is then:

$$B = 1/4; A = 0.$$

The solution to $y'' - 4y = \sinh 2x$ is

$$y(x) = c_1e^{2x} + c_2e^{-2x} + \frac{x}{4} \cosh 2x.$$

The solution to the differential equation where the absolute value of both independent roots of the characteristic equation match with the coefficient in the argument of the differential equation has been achieved using the method of undetermined coefficients.

5 Unified approach

An alternative solution method where the non-homogeneous term of the DE comprises combined trigonometric and hyperbolic functions is presented here with the aim to unify the approach to solving undetermined coefficient problems whether an exponential, sine, cosine, hyperbolic sine, or hyperbolic cosine term appears as the non-homogenous term in the linear, ordinary constant coefficient differential equation. When non-homogeneous cosine and sine, hyperbolic cosine, hyperbolic sine based non-homogeneous terms are found, these terms can be treated as combined exponential term pair sets as shown in this section. For the DE below where F, G, H, a, and b are real constants:

$$y'' + Fy' + Gy = H \cosh ax \cos bx.$$

The non-homogeneous term can be re-expressed as:

$$y'' + Fy' + Gy = \frac{H}{4} \left[\left(e^{ax} + e^{-ax}\right) \left(e^{ibx} + e^{-ibx}\right) \right],$$

or re-grouping:

$$y'' + Fy' + Gy = \frac{H}{4}\left[e^{(-a+ib)x} + e^{(-a-ib)x} + e^{(a+ib)x} + e^{(a-ib)x}\right].$$

Now, the method of undetermined coefficients based particular solution for this equation can be written as below following the typical procedure for exponential terms, assuming there are no repetitions of the homogeneous solution in the particular solution form. Note the coefficients in the arguments of the exponential are complex accounting for both the cosine/sine and the hyperbolic cosine/sine components of the form of the undetermined coefficient particular solution.

$$y_p = A_1 e^{(-a+ib)x} + A_2 e^{(-a-ib)x} + A_3 e^{(a+ib)x} + A_4 e^{(a-ib)x}.$$

The coefficients are:

$$\begin{aligned}
A_1 &= \frac{H}{4}\left[\frac{\left(a^2 - b^2 - Fa + G\right) + i\,(-Fb + 2ab)}{\left(a^2 - b^2 - Fa + G\right)^2 + (Fb - 2ab)^2}\right]\\
A_2 &= \frac{H}{4}\left[\frac{\left(a^2 - b^2 - Fa + G\right) - i\,(-Fb + 2ab)}{\left(a^2 - b^2 - Fa + G\right)^2 + (-Fb + 2ab)^2}\right]\\
A_3 &= \frac{H}{4}\left[\frac{\left(a^2 - b^2 + Fa + G\right) - i\,(Fb + 2ab)}{\left(a^2 - b^2 + Fa + G\right)^2 + (Fb + 2ab)^2}\right]\\
A_4 &= \frac{H}{4}\left[\frac{\left(a^2 - b^2 + Fa + G\right) + i\,(Fb + 2ab)}{\left(a^2 - b^2 + Fa + G\right)^2 + (-Fb - 2ab)^2}\right].
\end{aligned}$$

Note that the coefficient groups A1 and A2 and A3 and A4 are complex conjugate pairs as are the arguments of the exponentials each of these coefficients are paired with, giving rise to the cosine/sine and cosh/sinh function combinations when Euler's formula is applied to the exponentials as shown below. Letting:

$$\begin{aligned}
A^* &= Re\,(A_1)\,; B^* = Im\,(A_1)\\
C^* &= Re\,(A_3)\,; D^* = Im\,(A_3)\,.
\end{aligned}$$

Then, the particular solution can be written in the form:

$$y_p = 2e^{-ax}\left[A^* \cos bx - B^* \sin bx\right] + 2e^{ax}\left[C^* \cos bx - D^* \sin bx\right].$$

Examining the terms of the solution, the terms can be regrouped into a familiar form. The particular solution then falls into the expected form of:

$$\begin{aligned}
y_p =& B_1 \cosh ax \cos bx + B_2 \cosh ax \sin bx\\
&+ B_3 \sinh ax \cos bx + B_4 \sinh ax \sin bx.
\end{aligned}$$

This result illustrates how the exponential, cosine, sine, hyperbolic cosine, and hyperbolic sine based non-homogeneous terms in a linear, constant coefficient ordinary differential equation can all be treated in a similar manner when applying the method of undetermined coefficients.

The Appendix provides an additional example with a more complicated non-homogeneous term in the differential equation to further illustrate the treatment of the hyperbolic cosine and sine function with the method of undetermined coefficients.

6 Summary

Hyperbolic sines and cosines may be introduced into the undetermined coefficient curriculum as part of the acceptable functions for nonhomogeneous terms in the differential equation as illustrated through the examples covering the cases where the roots of the characteristic equation are completely independent of the coefficient in the argument of the hyperbolic function, when the absolute value of a root matches this coefficient and when this may be a root of multiplicity two, and finally when the absolute value of two independent roots of the characteristic equation both match with the coefficient in the argument of the hyperbolic function. Care must be taken, especially with the second case where the absolute value of one root, including a root of multiplicity two, matches with the coefficient of the hyperbolic function, to ensure linearly independent functions are provided as the trial form of the particular solution of the differential equation. For this second case, the exponential function with the matching coefficient can be re-expressed in terms of the hyperbolic functions. A unified solution for general nonhomogeneous terms including products was presented. There, simple algebraic solutions for the unknown coefficients were derived. Knowing how to rightly handle each case as well as the unified solution results in a firmer understanding of the method of undetermined coefficients for students learning and applying methods of solutions of ordinary differential equations.

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Appendix

This appendix further demonstrates how the hyperbolic cosine and sine and trigonometric cosine and sine functions appearing in the non-homogeneous term of the ordinary differential equation are applicable to the method of undetermined coefficients. As an example of the unified technique let us try:

$$y'' - y' - 6y = 13\left(sin\ x \sinh 2x - sin\ x \cosh 2x + cos\ x \sinh 2x - cos\ x \cosh 2x\right).$$

The homogeneous form of the solution is:

$$y_H(x) = c_1 e^{3x} + c_2 e^{-2x}.$$

Knowing how to apply undetermined coefficients, with hyperbolic functions, we can proceed. Since it is a product, the sine has the hyperbolic sine covered and similarly for the cosine and hyperbolic cosine, so we can solve without using exponential form.

$$y_P(x) = \sin x\,(A \sinh 2x + B \cosh 2x) + \cos x\,(C \sinh 2x + D \cosh 2x).$$

Our particular solution is, therefore,

$$y_P(x) = 3 \sin x\,(-\sinh 2x + \cosh 2x) + 2 \cos x\,(\sinh 2x - \cosh 2x).$$