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Special Case of Partial Fraction Expansion with Laplace Transform Application

Laurie A. Florio
DEVCOM Armaments Center United States Army Armament Graduate School

Ryan D. Hanc
DEVCOM Armaments Center United States Army Armament Graduate School

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Laurie A. Florio and Ryan D. Hanc
DEVCOM Armaments Center United States Army Armament Graduate School

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Abstract: Partial fraction expansion is often used with the Laplace transforms to formulate algebraic expressions for which the inverse Laplace transform can be easily found. This paper demonstrates a special case for which a linear, constant coefficient, second order ordinary differential equation with no damping term and a harmonic function non-homogeneous term leads to a simplified partial fraction expansion due to the decoupling of the partial fraction expansion coefficients of s and the constant coefficients. Recognizing this special form can allow for quicker calculations and automation of the solution to the differential equation form which is commonly used to model physical systems.

1 Introduction

Laplace transforms allow for the analysis of systems with discontinuous non-homogeneous terms including segmented distributed loads and concentrated loads. The transformation of the differential equations that govern such systems into an algebraic expression through the Laplace transform then requires that the algebraic expression for the Laplace transform of the dependent variable be found. The algebraic expression must be reformulated so that familiar inverse Laplace transforms can be found directly from tables, using special rules like shifts or s multiplication or division, through convolution, or through direct application of the definition of the inverse transform. Often the calculation of the inverse transform is algebra intensive. Second order differential equations with constant coefficients are a prevalent means of modeling physical system operation and are the core of mathematical curriculum related to the solution of ordinary differential equations. Identifying patterns in the partial fraction expansion of such differential equations can result in a more direct and efficient process for the determination of the partial fraction expansion of a given function and the subsequent inverse transform to return to a meaningful solution [1, 2, 3].This work presents a special case for partial fraction expansion of the Laplace transform resulting from a linear, second order ordinary differential equation with constant coefficients with no damping term and with a harmonic function non-homogeneous term. The pattern described was identified while preparing for the Advanced Mathematics I course at the United States Army Armament Graduate School.

2 Solution of non-homogeneous second order, linear ordinary differential equation with constant coefficients through Laplace transform

A second order linear equation with constant coefficients can be used to represent a variety of physical systems. A few examples of basic systems that can be modeled using a constant coefficient, linear, second order ordinary differential equation include a spring-mass-damper, heat conduction with constant properties and a constant cross-section, and an electrical circuit [1, 2, 3]. Because these equations are so commonly used, methods that can speed their analytical solution through recognizing simplifying patterns have merit. The Laplace transform can readily be applied to solving such linear equations with constant coefficients. The solution to the differential equation requires inverting a Laplace transform of the solution which contains terms related to the initial conditions, the transfer function, and the Laplace transform of the non-homogeneous term. Inverting the Laplace transform of the dependent variable becomes more complex when the non-homogeneous term is present in the differential equation. Thus, in order to introduce the special case and solution method and to understand its benefits, the solution of the general form linear, constant coefficient, homogeneous second order ordinary differential equation is described first, followed by the solution with a general non-homogeneous form. Then, the special case of the non-homogeneous form with no damping term in the homogeneous form and with harmonic non-homogeneous terms is presented and the variations of the form of the solution are described. This discussion provides insight into the differential equations, the Laplace transform, and the ability to recognize patterns in the solution of differential equations.

2.1 Homogeneous form of the differential equation

The homogeneous form of the differential equation will be addressed first. The general, constant coefficient, linear second order homogeneous ordinary differential equation (ODE) is given in (2.1) with the initial conditions of $y(0) = \Phi$ and $y'(0) = \Psi$.

$$ay'' + by' + cy = 0 \tag{2.1}$$

Dividing through by a, the equation can be written as,

$$y'' + \beta y' + \xi y = 0.$$

Taking the Laplace transform of the differential equation yields

$$\left(s^2 Y(s) - s y(0) - y'(0)\right) + \beta\left(sY(s) - y(0)\right) + \xi Y(s) = 0$$

or,

$$\left(s^2 Y(s) - \Phi s - \Psi\right) + \beta\left(sY(s) - \Phi\right) + \xi Y(s) = 0.$$

Grouping like terms leads to,

$$Y(s)\left(s^2 + \beta s + \xi\right) = \Phi s + \Psi + \Phi\beta.$$

The type of solutions that results depends upon the roots of the reciprocal of the transfer function, with the transfer function denoted by H(s),

$$Y(s) = \frac{\Phi s + \Psi + \Phi\beta}{s^2 + \beta s + \xi}; \;\; H(s) = \frac{1}{s^2 + \beta s + \xi}. \tag{2.2}$$

From (2.2), the transform of Y(s) will involve one term with a constant coefficient in the numerator and another term with s multiplying a constant coefficient in the numerator. First, the inverse transform of the constant coefficient term is discussed. Four different cases of root types are discussed: real, unique roots, real, non-unique roots(multiplicity of 2), a complex conjugate pair, and an imaginary pair. If the roots of the denominator of the transfer function called r_1 and r_2 are real and unique, then the partial fraction expansion of the transfer function becomes,

$$H(s) = \frac{1}{s^2 + \beta s + \xi} = \frac{1}{(s - r_1)(s - r_2)} = \frac{A}{(s - r_1)} + \frac{B}{(s - r_2)}. \tag{2.3}$$

The inverse transform of the transfer function becomes,

$$h(t) = Ae^{r_1 t} + Be^{r_2 t}.$$

If, instead, the roots of the denominator of the transfer function are real and non-unique, r_1, then the partial fraction expansion of the transfer function is just the transfer function re-written as,

$$H(s) = \frac{1}{s^2 + \beta s + \xi} = \frac{1}{(s - r_1)^2}. \tag{2.4}$$

The inverse transform for this case with repeated roots can then be found implementing the s shift property of the Laplace transform, yielding an inverse transform of the form,

$$h(t) = te^{r_1 t}.$$

If the roots are a complex conjugate pair, since the roots remain unique, then the partial fraction expansion follows that of (2.3) with the complex conjugate pair roots $r_1 = p + iq$ and $r_2 = p - iq$. This gives,

$$A = -B; \;\; -Ar_2 - Br_1 = 1.$$

Yielding,

$$A = -\frac{i}{2q}; \;\; B = \frac{i}{2q}.$$

So that the inverse transform is,

$$h(t) = \frac{e^{pt}}{q} sin(qt).$$

For imaginary roots, this inverse transform reduces to the form below, assuming constants in the numerator.

$$h\left(t\right)=\frac{1}{q}sin\left(qt\right)$$

Note the inverse transform for the imaginary pair root case involves only the sine function whereas the case with the general complex conjugate pair involves both the exponential and the sine function. With this, the cases associated with a term with a constant of 1 in the numerator of the Y(s) function have been described and can be modified for any constant value.

Next, the second type of transform term found with the solution to the homogeneous form of the differential equation, the term with s multiplied by a constant in the numerator as seen in (2.2) is addressed. In attempting to find a partial fraction expansion of this term, if the function H(s) in (2.3) is multiplied by s on the left side of the equation, multiplying the partial fraction expansion found on right side of the equation by s would generate polynomials of the same order of s in the numerator and denominator in this expansion. Such a condition is not valid for a partial fraction expansion as the numerator must be a lower order than the denominator for a partial fraction expansion. The partial fraction expansion of sH(s) for unique roots, real or complex, can be represented through the expression below,

$$sH\left(s\right)=\frac{s}{s^2+\beta s+\xi}=\frac{s}{(s-r_1)(s-r_2)}=\frac{A}{s-r_1}+\frac{B}{s-r_2}.$$

For the case of s in the numerator, this expression yields,

$$A=-\frac{r_1}{r_2-r_1};B=\frac{r_2}{r_2-r_1}.$$

and an inverse transform, $k(t)$, of the form,

$$k\left(t\right)=Ae^{r_1t}+Be^{r_2t}.$$

For the case of an imaginary root pair, this becomes,

$$A=\frac{1}{2};B=\frac{1}{2}.$$

For this A and B for the imaginary root pair, the inverse transform of $sH(s)$, $k(t)$, upon application of Euler's formula is then,

$$k\left(t\right)=\mathscr{L}^{-1}\left(sH(s)\right)=cos\left(qt\right),$$

which is consistent with the well-known Laplace transform of $s/(s^2+q^2)$. For the case of the repeated real root,

$$sH\left(s\right)=\frac{s}{s^2+\beta s+\xi}=\frac{s}{\left(s-r_1\right)^2},$$

the derivative rule can be applied to find the inverse transform below,

$$sF\left(s\right)-f\left(0\right)=\mathscr{L}\left(\frac{df}{dt}\right).$$

From (2.4) with $F(s) = \frac{1}{(s-r_1)^2}$, $f(t) = te^{r_1 t}$ and so $f(0) = 0$, yielding, $\frac{df}{dt} = (1 + r_1 t)\, e^{r_1 t}$. Then, defining $k(t)$ as the inverse transform of sH(s):

$$\text{k(t)} = \mathscr{L}^{-1}(sH(s)) = (1 + r_1 t)\, e^{r_1 t}.$$

This inverse Laplace transform takes the general form of the repeated root for a constant coefficient second order ODE using standard analytical solution methods.

For this homogeneous differential equation, when solved using Laplace transforms, the above discussion has shown partial fractions are useful for determining the inverse transforms given an initial condition set for $y(t)$, resulting in transform terms that involve a constant in the numerator and a constant multiplied by s. Care must be taken to ensure the partial fraction expansion functions chosen are valid.

For the case of the transfer function multiplied by a constant, $QH(s)$, the transfer function multiplied by s multiplied by a constant, $(PsH(s))$, or the transfer function multiplied by the sum of a constant and a constant multiplied by s, $(Ps{+}Q)H(S)$, the partial fraction expansion takes the same form as those described with the specific coefficient values then multiplying the solutions already found.

2.2 Non-homogeneous second order equations

If a non-homogeneous term, $m(t)$, is introduced into the second order differential equation with constant coefficients, an additional term is present in the Laplace transform of the differential equation and therefore in the Laplace transform of $y(t)$, introducing some additional complexity in finding the inverse of the Laplace transform. For the non-homogeneous form of the differential equation below,

$$ay'' + by' + cy = m(t),$$

dividing through by a, the equation can be written as:

$$y'' + \beta y' + \xi y = g(t).$$

Often the non-homogeneous terms applied to the systems are oscillatory, sinusoidal, or harmonic in nature with the amplitude possibly varying with time or position. Some examples of an oscillatory type of non-homogeneous terms are provided below:

$$g(t) = sin(\omega t)\ \ OR\ cos(\omega t)\ \ OR\ e^{-\lambda t} sin(\omega t)\ \ OR\ e^{-\lambda t} cos(\omega t)\,.$$

The oscillatory non-homogeneous functions can involve the summation of these functions as well. The Laplace transforms of these functions are,

$$G(s) = \frac{1}{\omega}\left(\frac{1}{s^2+\omega^2}\right)\ OR\ \left(\frac{s}{s^2+\omega^2}\right)\ OR\ \frac{1}{\omega}\left(\frac{1}{(s+\lambda)^2+\omega^2}\right)\ OR\ \left(\frac{(s+\lambda)}{(s+\lambda)^2+\omega^2}\right). \quad (2.5)$$

For the non-homogeneous differential equation, the Laplace transform of the differential equation in (2.2) is,

$$Y(s)\,(s^2 + \beta s + \xi) = \Phi s + \Psi + \Phi\beta + G(s).$$

The transfer function, $H(s)$, is then,

$$H(s) = \frac{1}{s^2 + \beta s + \xi},$$

and the Laplace transform of $y(t)$ can be written as,

$$Y(s) = \frac{\Phi s + \Psi + \Phi\beta + G(s)}{s^2 + \beta s + \xi} = H(s)\left(\Phi s + \Psi + \Phi\beta + G(s)\right). \tag{2.6}$$

Examining the $G(s)$ functions in (2.5), when the $G(s)$ is multiplied by the $H(s)$ function in (2.6), the denominator of the Laplace transform of the dependent variable $y(t)$ will involve the product of two second order polynomials in s, one from the H(s) and one from the $G(s)$. The most general form of the Laplace transform of y will involve a term that looks like the expression in (2.7) below

$$\frac{G(s)}{s^2 + \beta s + \xi} = \frac{Ps + Q}{(s^2 + \beta s + \xi)((s + \lambda)^2 + \omega^2)}. \tag{2.7}$$

Partial fractions may be used to generate an algebraic expression equivalent to this expression that is or approaches a familiar Laplace transform or one to which some Laplace properties can be applied. In this way, a closed-form expression for the inverse Laplace transform of $y(t)$ can be found. Re-expressing the second polynomial in the denominator, the general form of the partial fraction expansion of this term is given below where $\eta = 2\lambda$ and $\rho = \omega^2 + \lambda^2$. Once the A, B, C, and D coefficients are found, completing the squares is commonly implemented to invert the transform.

$$\frac{Ps + Q}{(s^2 + \beta s + \xi)(s^2 + \eta s + \rho)} = \frac{As + B}{s^2 + \beta s + \xi} + \frac{Cs + D}{s^2 + \eta s + \rho}.$$

The standard procedure to find the coefficients for such a partial fraction expansion involves multiplying the equation by the terms in the denominator of the original function on the left, yielding the following expressions,

$$Ps + Q = (As + B)\left(s^2 + \eta s + \rho\right) + (Cs + D)\left(s^2 + \beta s + \xi\right) \tag{2.8a}$$

$$Ps + Q = \left(As^3 + A\eta s^2 + A\rho s + Bs^2 + B\eta s + B\rho\right) + \left(Cs^3 + C\beta s^2 + C\xi s + Ds^2 + D\beta s + D\xi\right). \tag{2.8b}$$

The coefficients of the terms of the same order of s can then be equated, yielding, in this case, four equations to solve for the four unknown coefficients. As can be seen in the formulations for the coefficients, the coefficients of the s (A and C) and the constant terms (B and D) in the partial fraction expansion are dependent upon one another. This interdependence arises since the coefficients of the s^2 and s order terms in (2.8) involve both expansion coefficients of s and the constant

$$\begin{aligned}
s^3 &: \quad 0 = A + C \\
s^2 &: \quad 0 = A\eta + B + C\beta + D \\
s^1 &: \quad P = A\rho + B\eta + C\xi + D\beta \\
s^0 &: \quad Q = B\rho + D\xi.
\end{aligned}$$

2.2.1 Special case of two imaginary set of roots

Leading to the special case that is the purpose of this paper, a more restrictive case is considered where the roots of the denominator associated with the transfer function and those associated with the non-homogeneous term in the differential equation are imaginary roots. First, the case where ξ cannot be equal to ρ is handled. Let $\xi = \tau^2$ and $\rho = \upsilon^2$. The algebraic expression for the transform involving the non-homogeneous term in the differential equation can then be written as the term on the left, with the partial fraction expansion on the right,

$$\frac{Ps + Q}{(s^2 + \tau^2)(s^2 + \upsilon^2)} = \frac{As + B}{s^2 + \tau^2} + \frac{Cs + D}{s^2 + \upsilon^2}.$$

The partial fraction numerators are of a lower order than those in the denominator and so this expansion is valid. Multiplying through by the denominator of the non-homogeneous term,

$$Ps + Q = (As + B)\,(s^2 + \upsilon^2) + (Cs + D)\,(s^2 + \tau^2).$$

Grouping together the like terms and equating the coefficients yields:

$$\begin{aligned} s^3 :& \quad 0 = A + C \\ s^2 :& \quad 0 = B + D \\ s^1 :& \quad P = A\upsilon^2 + C\tau^2 \\ s^0 :& \quad Q = B\upsilon^2 + D\tau^2. \end{aligned}$$

Upon examination of this set of equations, some interesting conditions are found.

1. The terms that involve the coefficient of s, or the A and C terms in the partial fraction expansion, are completely independent of the terms that involve the constant coefficient, or the B and D terms.

2. The A and C coefficients are of opposite sign as are the B and D coefficients.

3. The form of the solutions for A and C matches the form of the solution for B and D, but that the constant for A and C is P and the constant for B and D is Q.

Hence, for the conditions for which the denominator root sets are both imaginary numbers, instead of using the general partial fraction expansion described, the partial fraction expansion form may be divided into two independent parts. First, an independent treatment of the terms that involve s in the numerator may be made,

$$\frac{Ps}{(s^2 + \tau^2)(s^2 + \upsilon^2)} = \frac{As}{s^2 + \tau^2} + \frac{Cs}{s^2 + \upsilon^2}. \tag{2.9}$$

In (2.9), the s in each term cancels. The partial fraction expansion of this form without the s is valid, unlike the case with only one second order term in the denominator described in the homogeneous differential equation solution where multiplying through by the s

created an invalid expansion since the s expressions in the numerator were not an order lower than for those in the denominator. Instead, here, the s expression in the numerator is one order lower than the s^2 polynomial in the denominator in each expansion term,

$$\frac{P}{(s^2+\tau^2)(s^2+\upsilon^2)} = \frac{A}{s^2+\tau^2} + \frac{C}{s^2+\upsilon^2}. \tag{2.10}$$

Next, the term involving the constant in the numerator in (2.9) is handled

$$\frac{Q}{(s^2+\tau^2)(s^2+\upsilon^2)} = \frac{B}{s^2+\tau^2} + \frac{D}{s^2+\upsilon^2}. \tag{2.11}$$

Examining (2.10) and (2.11), the equivalent nature of the solutions for A and C and B and D can be clearly seen. The general solutions for the A, B, C, and D coefficients for this special case are,

$$A = \frac{P}{\upsilon^2-\tau^2};\ \ B = \frac{Q}{\upsilon^2-\tau^2};\ \ C = -\frac{P}{\upsilon^2-\tau^2};\ \ D = -\frac{Q}{\upsilon^2-\tau^2}.$$

When a fraction of the form in (2.11) is observed, instead of setting the general form of the partial fraction expansion for this function, the partial fraction expansion can be compared to the form and the coefficient found by inspection as,

$$\frac{Ps+Q}{(s^2+\tau^2)(s^2+\upsilon^2)} = \frac{1}{(\upsilon^2-\tau^2)}\frac{Ps+Q}{(s^2+\tau^2)} - \frac{1}{(\upsilon^2-\tau^2)}\frac{Ps+Q}{(s^2+\upsilon^2)}. \tag{2.12}$$

The inverse transform resulting in two sine and two cosine terms can be easily found. Recognizing this unique case can assist with the quicker evaluation of the partial fraction expansion of functions commonly found when modeling physical systems operating in accordance with a constant coefficient, linear, second order ordinary differential equation with no damping term.

2.2.2 Application of the method to a problem

Consider a one degree of freedom spring-mass problem as described in Figure 1. The spring constant k is some arbitrary positive value and for simplicity, the mass, M, is equal to 1 kg. The figure shows the system as well as a sinusoidal forcing function or non-homogeneous term.

A force balance of the system yields the following second-order differential equation of motion,

$$Mx'' + kx = g(t),$$

or, after substitution,

$$x'' + kx = sin\,(\omega t) + cos(\omega t).$$

The Laplace transform of the above yields the following,

$$s^2X\,(s) - sx\,(0) - x'\,(0) + kX\,(s) = \frac{\omega}{s^2+\omega^2} + \frac{s}{s^2+\omega^2}.$$

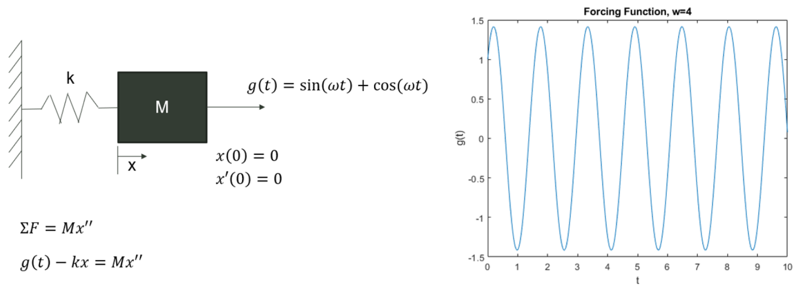

Figure 1: Spring-Mass with Sinusoidal Forcing Function: ω =4 and $k = M$ =1

With application of the initial conditions

$$s^2X(s) + kX(s) = X(s)\left(s^2 + k\right) = \frac{\omega}{s^2 + \omega^2} + \frac{s}{s^2 + \omega^2}.$$

Solving for X(s),

$$X(s) = \frac{\omega}{(s^2 + k)(s^2 + \omega^2)} + \frac{s}{(s^2 + k)(s^2 + \omega^2)}.$$

Noting that the numerator of this function fits the form of $Ps + Q$, where $P = 1$ and $Q = \omega$, that k is a positive constant, and that the denominator yields two sets of imaginary roots, the simplification described in Section 2.2.1 can be applied,

$$A = \frac{1}{\omega^2 - k}; \quad B = \frac{\omega}{\omega^2 - k}; \quad C = -\frac{1}{\omega^2 - k}; \quad D = -\frac{\omega}{\omega^2 - k}.$$

Substituting these coefficients into the partial fraction expansion form of the transformed function,

$$X(s) = \frac{1}{(\omega^2 - k)}\frac{s + \omega}{(s^2 + k)} - \frac{1}{(\omega^2 - k)}\frac{s + \omega}{(s^2 + \omega^2)}.$$

Performing the inverse transform yields the solution to the original ODE,

$$x(t) = \frac{1}{(\omega^2 - k)}\left(cos\left(\sqrt{k}t\right) + \frac{\omega}{\sqrt{k}}sin\left(\sqrt{k}t\right)\right) - \frac{1}{(\omega^2 - k)}\left(cos(\omega t) + sin(\omega t)\right). \quad (2.13)$$

The first two terms stem from the sine non-homogeneous term and the third and fourth terms stem from the cosine non-homogeneous term so that solutions with arbitrary constants in the numerator can easily be determined. For $\sqrt{k/M} \neq \omega$, $M = 1$, the resulting $x(t)$ function involves the oscillations due to the natural frequency of the system and the oscillations due to the applied oscillatory forcing function. If this special case of the ODE is recognized and the Laplace transform method applied, then the simplified form of the partial fraction expansion can be found, and the solution can be readily determined. A plot of the solution with $k = 1$, $M = 1$, and $\omega = 4$ is provided in Figure 2. This figure shows the results for the case with unique imaginary roots as handled in Section 2.2.1

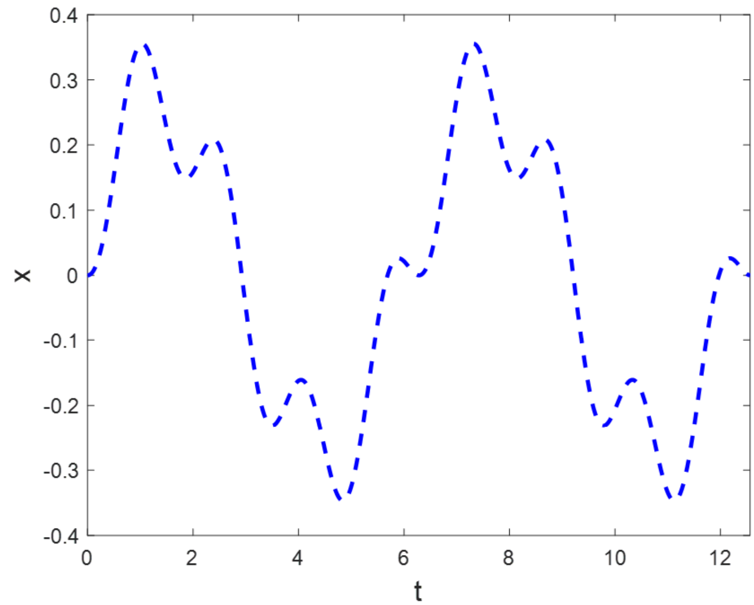

Figure 2: Solution of x(t) from (2.13)

2.2.3 Two imaginary roots of the denominator match

Under the unique conditions that the roots of the two second order polynomials in the denominator of the transform in (2.7) are both imaginary pairs and the roots of both polynomials match, solution form in (2.12) cannot be used. Instead, with $\xi = \tau^2$ the transform of the solution and its partial fraction form are,

$$\frac{Ps+Q}{(s^2+\tau^2)^2} = \frac{Ps}{(s^2+\tau^2)^2} + \frac{Q}{(s^2+\tau^2)^2}. \tag{2.14}$$

A differential equation producing such a set of roots occurs when the applied oscillation frequency matches the natural frequency of the system, yielding a resonance condition. The inverse transform of the first term in (2.14) can be found by applying the derivative property $\mathscr{L}(tf(t)) = -F'(s)$ where $f(t) = \frac{P}{2\tau} sin\,(\tau t)$ to the transform $\frac{Ps}{(s^2+\tau^2)^2}$ giving,

$$\mathscr{L}^{-1}\left(\frac{Ps}{(s^2+\tau^2)^2}\right) = \frac{P}{2\tau} t sin\,(\tau t)\,.$$

The inverse transform of the second term in (2.14) can be found by applying the divide by s/integral property, $\mathscr{L}\left(\int_0^t f(t)\right) = \frac{1}{s}\mathscr{L}(f(t))$ with $f(t) = \frac{Q}{2\tau} t sin\,(\tau t)$, giving,

$$\mathscr{L}^{-1}\left(\frac{Q}{(s^2+\tau^2)^2}\right) = \frac{Q}{2\tau}\left[\frac{1}{\tau^2} sin\,(\tau t) - \frac{t}{\tau} cos\,(\tau t)\right].$$

For the same example mass-spring system as described in Section 2.2.2, the applied non-homogeneous term is modified so that the natural frequency and applied frequency are equal so $\sqrt{k/M} = \omega$, $M = 1$, $\tau = \omega$. The solution with $P = Q = 1$ is,

$$x(t) = \frac{1}{2\omega} t sin(\omega t) + \frac{1}{2\omega}\left[\frac{1}{\omega^2} sin(\omega t) - \frac{t}{\omega} cos(\omega t)\right]. \tag{2.15}$$

Figure 3. depicts the solution for the case where $\omega = 4$ and demonstrates the expected resonance displacement when non-unique imaginary roots of the denominator of (2.7) exist.

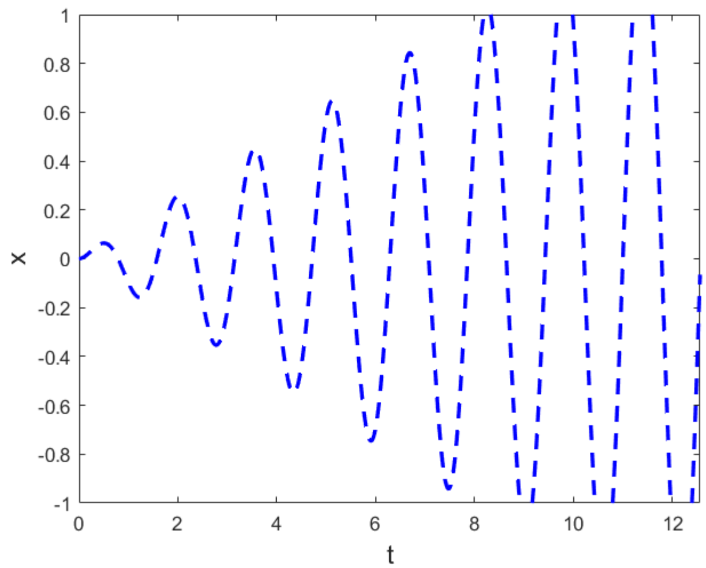

Figure 3: Solution of x(t) from (2.15)

2.2.4 Application to a shifted transform

For a damped system, the partial fraction based method described might be used under limited circumstances. For a differential equation of the form,

$$y'' + \beta y' + \xi y = Pe^{-\lambda t} cos(\omega t) + Qe^{-\lambda t} sin(\omega t).$$

The Laplace transform of the differential equation will involve terms of the form below:

$$\frac{G(s)}{s^2 + \beta s + \xi} = \frac{Ps + Q}{(s^2 + \beta s + \xi)((s + \lambda)^2 + \omega^2)} \tag{2.16a}$$

$$\frac{G(s)}{s^2 + \beta s + \xi} = \frac{Ps + Q}{(s^2 + 2(1/2\beta)s + \xi - 1/4\beta^2)((s + \lambda)^2 + \omega^2)} \tag{2.16b}$$

$$\frac{G(s)}{s^2 + \beta s + \xi} = \frac{Ps + Q}{((s + \sigma)^2 + \tau^2)((s + \lambda)^2 + \omega^2)}. \tag{2.16c}$$

When the parameter $\tau = \sqrt{\xi - 1/4\beta^2}$ in the homogeneous portion of the differential equation and the applied frequency, ω, are equal and the parameter, $\beta/2$ in the homogeneous portion and the damping or decay coefficient for the non-homogeneous terms, λ, are equal, then the two shift levels in the two terms in the denominator of (2.16), $\sigma = \beta/2$ and λ, are equal, and the "frequency", ω, and the $\tau = \sqrt{\xi - \beta^2}$ parameters are equal. Accordingly, with a change of variables to $s + \lambda$, the repeated imaginary root solution method described in Section 2.2.3 can be applied

$$\frac{G(s)}{s^2 + \beta s + \xi} = \frac{Ps + Q}{\left((s+\lambda)^2 + \omega^2\right)^2}.$$

3 Conclusions

The decoupled partial fraction expansion coefficients that come about for the special form of the linear, second order ODE with no $y'(t)$ damping term and the harmonic non-homogeneous term allows for a simple solution using Laplace transforms. The solution of the homogeneous form of the differential equation is comprised of cosine and sine functions. With the cosine and sine function based non-homogeneous terms in the differential equation, the solution of the non-homogeneous form of the differential equation also involves cosine and sine functions or sine and cosine functions multiplied by t if the natural and applied function frequencies are the same. Hence, for any non-homogeneous function that meets the requirements described, the solution of the differential equation can be easily found through Laplace transforms.

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