Study of Singularly Perturbed Models and its Applications in Ecology and Epidemiology

Milaine Sergine Seuneu Tchamga

Submitted in fulfilment of the academic requirement for the degree of Doctor of Philosophy in the School of Mathematics, Statistics and Computer Sciences

UNIVERSITY OF
KWAZULU-NATAL

2017

Abstract

In recent years the demand for a more accurate description of real life processes and advances in experimental techniques have resulted in construction of very complex mathematical models, consisting of tens, hundreds, if not thousands, of highly coupled differential equations. The sheer size and complexity of such models often preclude any robust, theoretical or numerical, analysis of them. Fortunately, often such models describe phenomena occurring on vastly different time or size scales. We focused on complex processes with two time/size scales described by systems of ordinary differential equations. In such a case, there is a small parameter that multiplies one or more derivatives. Using the Tikhonov Theorem, we have been able to understand the asymptotic behaviour of the solution to some complex epidemiological models. Furthermore, we present analysis based on the Butuzov theorem, which, for the purpose of the discussed models, was generalized to two dimensional non-autonomous problems. We applied the developed theory on an ecological model with interactions given by the mass action law.

Preface and Declaration

The work described in this thesis was carried out in the School of Mathematics, Statistics and Computer Sciences, Westville campus at the university of Kwazulu-Natal from February 2014 to July 2017, under the supervision of Professor Jacek Banasiak.

These studies represent original work by the author and have not otherwise been submitted in any form for any degree or diploma to any tertiary institution. Where use has been made of the work of others it is duly acknowledged in the text.

Milaine Sergine Seuneu Tchamga,

25 July 2017

[redacted]

As the candidate's supervisor I have approved the thesis for submission

Signed:

Date:

Declaration 1- Plagiarism

I Milaine Sergine Seuneu Tchamga declare that

1. The research in this thesis, except where otherwise indicated, is my original research.
2. This thesis has not been submitted for any degree or examination at any other university.
3. This thesis does not contain other person's data, pictures, graphs or other information, unless specifically acknowledged as being sourced from persons.
4. This thesis does not contain other person's writing, unless specifically acknowledged as being sourced from other researches. Where other written sources have been quoted, then
   - Their words have been rewritten but the general information attributed to them has been referenced
   - where their exact words have been used, then their writing has been placed in italics and inside quotation marks, and referenced.
5. This thesis does not contain text, graphics or tables copied and pasted from internet unless specifically acknowledged, and the source being detailed in the dissertation and in the Reference section.

Signed in Durban this 25 July 2017

[redacted]

Declaration 2- Publications

DETAILS OF CONTRIBUTION TO PUBLICATIONS that form part and/or include research presented in this dissertation (include publication, submitted, in *press* and published and give details of the contributions of each author to the experimental work and writing of each publication)

Publication 1: Delayed stability switches in singularly perturbed predator-prey models, Non linear analysis: real world applications, Volume 35, Pages 312-335, http://www.arxiv.org/abs/1605.07519v1, 2017.

Publication 2: I co-authored a chapter in the book "Applying systems analysis to complex global problems" that is currently under review.

Signed: [redacted]

Contents

1 Introduction and Background

1.1 Introduction

In recent years the increasing demand for a more accurate description of real life processes and advances in experimental techniques have resulted in construction of very complex mathematical models, consisting of tens, hundreds, if not thousands of highly coupled differential equations. The sheer size and complexity of such models often precludes any robust, theoretical or numerical analysis of them. Fortunately, often such models describe phenomena occurring on vastly different time or size scales. Examples include ecological processes with vital dynamics of individuals coupled with much faster (or much slower) migratory processes, or epidemiological processes in which a quick disease may influence much slower demographic or evolutionary processes in the population. Such a structure of the model offers hope that it can be simplified by focusing at one chosen scale and averaging over the others without affecting too much the salient features of its dynamics. It is often the case that the existence of, say, different time scales in a model is reflected by a scale parameter given by the ratio of the typical scales of different process driving the model. Then the model operates close to the slow or the fast regime if the scale parameter is, respectively, small or large. Sometimes it is possible to obtain a good approximation of the model in the slow regime by simply putting the scale parameter equal to zero (and in the fast regime by letting its reciprocal to be infinity). Equations describing such models commonly are referred to as *regularly perturbed*. Often, however, simply replacing in the model the relevant parameter by its critical value results in a dramatic change of the properties of the model, rendering this approach useless. In such a case to understand the behaviour of the model close to the boundaries between different regimes one has to carefully analyse the limits of the solutions as the scale parameter tends to the critical value and to construct an approximation of a different type which would describe this limit. In other words, in most cases complex models cannot be viewed as simple compositions of basic models such that, if we are interested in one of them, it can be obtained by switching off the unwanted ones. Rather, a complex model is a network of interrelated models describing different regimes and the passage between these regimes can only be achieved by a careful application of appropriate limit relations. In such a case, the limit model still is linked to the other regimes, though rather through appropriate aggregated quantities constructed in the limit process and not through explicit couplings.

1.2 Problem Statement

In this thesis we mainly focus on complex processes described by systems of ordinary differential equations. Particularly, we look at the phenomena occurring at two time scales and modelled by equations in the so-called Tikhonov's form

$$\begin{cases} \frac{dx}{dt} &= f(x, z, \epsilon), \\ \epsilon\frac{dz}{dt} &= g(x, z, \epsilon), \end{cases} \tag{1.2.1}$$

with initial condition $(x(0), z(0)) = (x_0, z_0)$, $t \in [0, T]$, $T \in (0, \infty)$, and where f and g are sufficiently regular functions from a subset $\bar{\mathcal{X}} \times \mathcal{Z} \times (0, \epsilon_0) \subset \mathbb{R}^n \times \mathbb{R}^m \times [0, \infty)$, ϵ_0 is a small parameter, to, respectively, $\mathbb{R}^n$ and $\mathbb{R}^m$, for some $n, m \in \mathbb{N}$. The presence of the small parameter multiplying the derivative makes the problem singular in the sense that setting $\epsilon = 0$ makes the model much simpler but, at the same time, completely changes the system structure and therefore does not always preserve the salient features of the phenomenon that is being described by the original model. A good question will be: under which assumptions can the solution of the original system be approximated by the solution of the system obtained using $\epsilon = 0$? To answer this question, Tikhonov's theory provides conditions under which one can safely approximate the solution to (1.2.1) by the solution to the model

$$\begin{cases} \dot{x} = & f(x, z, 0), & x(t_0) = x_0, \\ 0 = & g(x, z, 0), & z(t_0) = z_0, \end{cases} \tag{1.2.2}$$

also called the *degenerate* or *reduced* system. In this thesis, we will refer to the equation $g(x, z, 0) = 0$ as *the degenerate or the reduced* equation.

It is of interest to determine the behaviour of the solutions to (1.2.1) as ϵ tends to zero and, in particular, to show that they converge to the solutions of the degenerate system. There are several reasons for this. First, taking such a limit in some sense "incorporates" fast processes into the slow dynamics. Hence, it links models acting at different time scales and often leads to new descriptions of natural processes, see e.g. [7]. Second, letting formally $\epsilon = 0$ in (1.2.1) yields a lower order system, whose solutions in many cases offer an approximation to the solution of (1.2.1) that retains the main dynamical features of the latter but can be obtained with less computational effort. In other words, often the qualitative properties of the solutions of the degenerate system can be "lifted" to $\epsilon > 0$ to provide a good description of dynamics of (1.2.1). There have been many approaches developed in order to understand the behaviour of solutions of (1.2.1) as compared to that of the degenerate system. There are, for instance, the non-standard methods [12, 21] and the standard approaches [4, 5, 31, 38, 42, 71, 28]. In the Chapter 2, we will give an overview of some of them.

One of the first asymptotic theories is the *Tikhonov Theory.* Apart from some technical conditions, one of the most important assumptions of the Tikhonov theorem is that the degenerate equation has an isolated solution, $z = \phi(x)$, on a closed domain, $\bar{\mathcal{X}}$. Furthermore, it is required of the equilibrium solution $\hat{z} = \phi(x)$ of the auxiliary equation (2.3.1) to be uniformly asymptotically stable on $\bar{\mathcal{X}}$. In such a case, the solution of (1.2.1), $(x(t,\epsilon), z(t,\epsilon))$, converges to the solution $(x(t), z(t))$ of the reduced system (1.2.2) as ϵ tends to zero for $t \in (0, T]$. In many cases, these assumptions are not satisfied. For example, consider a SIS model with vital dynamics, given by the following set of equations:

$$\begin{cases} \frac{dS}{dt} = \beta N - \mu_1 S - \frac{1}{\epsilon}(\lambda IS - \gamma I), \\ \frac{dI}{dt} = -\mu_2 I + \frac{1}{\epsilon}(\lambda IS - \gamma I), \end{cases} \tag{1.2.3}$$

where S is the number of susceptibles, I is the number of infectives, β, μ_1, μ_2 are, respectively, the birth rate and the death rates for susceptible and infective populations, and λ, γ are, respectively, the transmission rate of the disease and the recovery rate from the disease. The parameter ϵ is the ratio of the time scale of the disease and that of the demographic processes (the birth and the death) of the population, and can be considered small for "quick" diseases such as flu or cold. In this case, considering $\mu_1 = \mu_2$ and assuming that $\beta - \mu > 0$ and $\frac{\gamma}{\lambda} > N_0$, we get two quasi steady states, $I_1 = 0$ and $I_2 = N - \frac{\gamma}{\lambda}$ which intersect each other and switch stability at the point of intersection, $N = \frac{\gamma}{\lambda}$. That is, I_1 is attractive for $N < \frac{\gamma}{\lambda}$ and becomes repelling for $N > \frac{\gamma}{\lambda}$, while I_2 is repelling for $N < \frac{\gamma}{\lambda}$ and attractive for $N > \frac{\gamma}{\lambda}$. A simple expectation could be that, as ϵ tends to zero, the solution tends to the attractive branch of the closest quasi steady state and immediately switches to the other attractive branch after passing close to the intersection point. Indeed, such a behaviour is often observed, [19, 48, 49, 57]. However, in many cases the system behaves in a different way; that is, having passed by the intersection of the quasi steady states, the solution follows the now repelling branch of the first quasi steady state and only after a finite time, say t^*, which is independent on ϵ, it suddenly jumps to the attractive branch of the other quasi steady state.

Furthermore, we notice that numerical simulations of the solution to such a problem can give a wrong approximation of the dynamics. For instance, we plotted the dynamics of

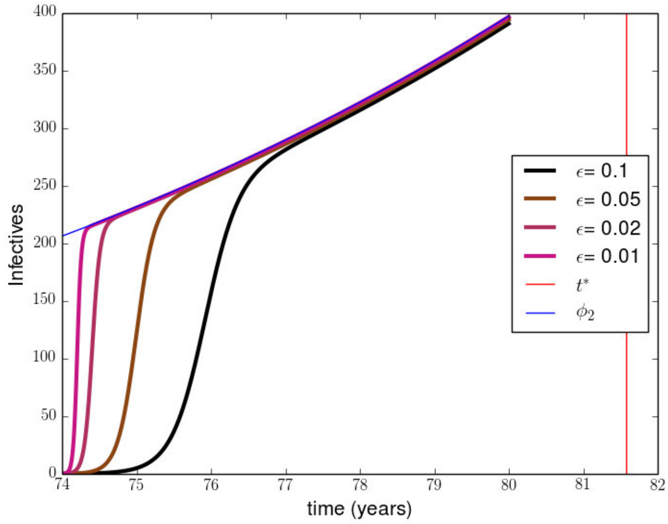

Figure 1.1: This figure shows the graph of the solution to the one dimensional SIS model obtained from (1.2.3) by setting $\mu_1 = \mu_2$. The solution plotted is that of the infective population for different value of ϵ. We use the standard numerical method in Python. It can be observed that the solution tends to the second quasi steady state, ϕ_2, before $t^* = 81.7$ years.

the infective population of (1.2.3) using standard packages in Python. Figure 1.1 and Figure 1.2 show the results for the one dimensional problem in the case of a growing population N, obtained by setting $\mu_1 = \mu_2$. On Figure 1.1 we have the plot obtained by using the standard numerical approach and on Figure 1.2, we have the result obtained by using the analytical solution studied in Chapter 5. The difference between both figures is that the standard approach suggests that the jump occurs much earlier than in reality, that is, at t^*. Figure 1.2 presents the correct behaviour of the solution as predicted by the theory. Thus it is important to provide a more accurate description of the behaviour of the solution as the small parameter tends to its critical value.

1.3 Outline of the Thesis

Presenting conditions under which a delay occurs in two dimensional non-autonomous problems is the main focus of this thesis. However, we will start by studying in detail the application of the theory for some particular epidemiological problems before launching into the generalisation of the method for two dimensional non-autonomous case. So, in Chapter 2 we give a comprehensive literature review of what has been done so far in the field of perturbation theory; in Chapter 3 we provide some mathematical background

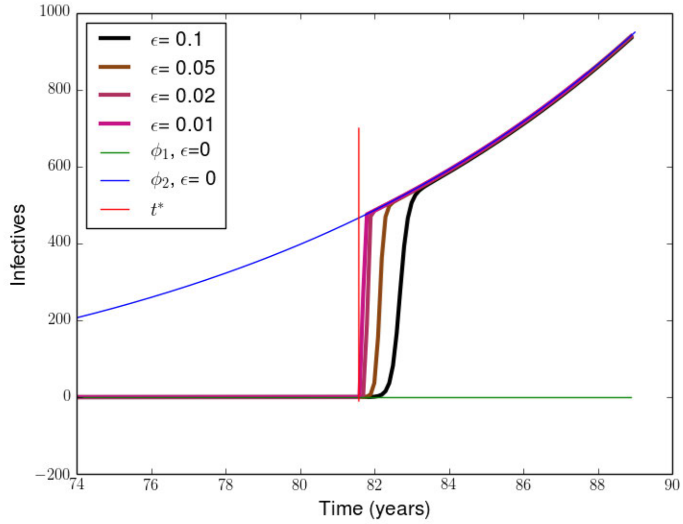

Figure 1.2: This figure shows the graph of the solution to the one dimensional SIS model obtained from (1.2.3) by setting $\mu_1 = \mu_2$. The solution plotted is that of the infective population for different value of ϵ. We used the analytical expression of the solution studied in Chapter 5. It can be observed that the solution leaves the first quasi steady state, ϕ_1, and tends to the second quasi steady state, ϕ_2, after $t^* = 81.7$. years.

necessary to understand the remaining chapters; in Chapter 4 we apply Tikhonov's theory to some epidemiological problems; in Chapter 5 we study the exchange of stability for a one dimensional model (case of a quick disease); in Chapter 6 we use the result developed in Chapter 5 and the method of upper and lower solutions [19] to analyse the two dimensional epidemiological model (1.2.3); in Chapter 7 we generalise the result found in Chapter 5 for non-autonomous one dimensional models; in Chapter 8 we generalise the result found in Chapter 6 for non-autonomous planar systems with QSS exhibiting a transcritical bifurcation; in Chapter 9 we apply the theory developed in chapter 8 to study a two dimensional ecological model with interactions given by the mass action law; Finally, Chapter 10 is the conclusion of the thesis.

2 Literature Review

2.1 Introduction

By multiple scale models we understand models of interlinked processes that occur at vastly different rates. As already mentioned earlier, in many cases the coexistence of such processes in the model is manifested by the presence of a small parameter ϵ that expresses the ratio of their characteristic times. The branch of mathematics that is devoted to studying multiple scale problems is called the *perturbation* theory. It focuses essentially on the study of the behaviour of the solutions of multi-scale systems when the small parameter ϵ tends to its critical value. It can be divided into two main parts: the *regular* and the *singular* perturbation theories. The regular perturbation theory studies the multiple scale problems where the solution of the original system converges uniformly to the solution of the degenerate system obtained by setting $\epsilon = 0$ in the original system. Conversely, the singular perturbation theory is devoted to the study of multiple scale problems, where the solution of the limit system, obtained by setting $\epsilon = 0$ in the original problem, is very different from that of the original system, [57]. The pertubation theory is found in the part of applied mathematics called *asymptotic analysis* whose purpose is primarily, for a given phenomenon, to approximate the solution to the original problem that described the phenomenon by the solution to another simpler problem without loosing the accuracy of the main description of the phenomenon; that is, it provides more effective computational methods, [7]. The second purpose of the asymptotic analysis is to help to validate phenomenological equations by rigorously showing links between solutions in various regimes, [7].

As already mentioned earlier, many works have been done in this domain of mathematics, leading to the development of different approaches to the analysis of singularly perturbed problems. There are for instance the geometric and the analytic approaches. The analytical asymptotic aproaches include the work of A.N. Tikhonov, [69], the work of Vasil'eva [73], the works of F. Hoppenstead, [39], the work of A.I. Neishtadt, [53, 54], the work of Butuzov et al. [19] and so on. The geometric techniques are used in the series of works by Fenichel, [31, 38], the exchange lemma [42, 71], the blow up technique [28], etc.

In order to give an overview of some of these methods, let us consider a singularly perturbed problem of the form (1.2.1). The model (1.2.1) can be reformulated as

$$\begin{cases} \frac{dx}{d\tau} = \epsilon f(x,z,\epsilon), & x(0,\epsilon) = x_0, \\ \frac{dz}{d\tau} = g(x,z,\epsilon), & z(0,\epsilon) = z_0, \end{cases} \tag{2.1.1}$$

by considering the time scale $\tau = \frac{t}{\epsilon}$. The time $t \in [0,T],\ 0 < T < \infty$, is called the slow time, while τ is called the fast time. Thus, the system (1.2.1) is said to be the *slow system* and (2.1.1) is the *fast system*. Both systems are equivalent except for $\epsilon = 0$. Letting ϵ tends to zero, we obtain, respectively, the degenerate system (1.2.2) and the limit system

$$\begin{cases} \frac{dx}{d\tau} = 0, & x(0) = x_0, \\ \frac{dz}{d\tau} = g(x,z,0), & z(0) = z_0. \end{cases} \tag{2.1.2}$$

The degenerate system is defined on the *critical manifold,* $\mathcal{M}_0 = \{(x,z) \in \mathbb{R}^n \times \mathbb{R}^m, g(x,z,0) = 0\}$. The critical manifold can also be viewed as a manifold of equilibria to the fast system. The critical manifold is said to by *normally hyperbolic* when all the eigenvalues of the Jacobian $\frac{\partial g}{\partial x}(x,z,0)\big|_{\mathcal{M}_0}$ have non-zero real part. Furthermore, if they all have positive (respectively, negative) real part, $\mathcal{M}_0$ is said to be repelling (respectively, attractive), [38].

2.2 Review of Some Geometric Approaches

The geometric singular perturbation approach was initiated by F. Fenichel [31] at the end of the 1970s. He developed a theory based on the center manifold theory. Fenichel's theory is concerned with the analysis of hyperbolic fast dynamics and focuses on the perturbation of the normally hyperbolic critical manifold. It states, for instance, that for a given compact subset $M_0 \subset \mathcal{M}_0$ and having the eigenvalues of the Jacobian $\frac{\partial g}{\partial x}(x,z,0)$ with negative real part at M_0, for any $\epsilon > 0$, there exists a manifold M_ϵ, $\mathcal{O}(\epsilon)-$close to M_0 and locally invariant under the flow of (1.2.1). In order words, the trajectories can enter or leave the perturbed manifold M_ϵ (also called *slow manifold*) only through its boundary. Many other results have also been developed in this theory which can be found in [13, 31, 38].

The next geometric approach that we will review is the Exchange Lemma. Let us consider the standard set-up of singular perturbation as described previously. For arbitrary positive integer k_1 and k_2, let us consider a normally hyperbolic invariant manifold of dimension k_1, and an unstable slow manifold of dimension k_1+k_2. The aim of this approach is to track a (locally) invariant manifold during its passage near the slow manifold, [24, 41]. Providing a precise estimate of the position of both manifolds and that of their tangent planes is one of the way to achieve such a task. This theory states that the tracked manifold will be C^1 exponentially close to a certain submanifold of the local unstable manifold on exit from its neighborhood if it transversely intersects the local stable manifold of the normally hyperbolic locally invariant manifold, [24, 41]. At first, the Exchange Lemma theory was developed for singularly perturbed systems with strictly slow dynamics on the normally hyperbolic and locally invariant manifold, [43, 44, 45]. It led to the analysis of the singularly perturbed Fitzhugh-Nagumo model, where the unstable manifold of the critical point was tracked and the homoclinic orbit to it was described, [44]. The most general version of the Exchange Lemma was presented by Tin, [71]. The advantages of the Tin's Exchange Lemma is that firstly, it helps to track manifolds whose dimensions are any integer in $[k_1+1, k_1+k_2]$, secondly, it can be applied to systems in which there are both fast and slow dynamics on the normally hyperbolic locally invariant manifold, and thirdly, it significantly weakened the assumption imposed in [43, 44, 45] which states that orbits spend asymptotically long times $(\mathcal{O}(\frac{1}{\epsilon}))$ on the tracked manifold around the slow manifold, [24]. The Exchange Lemma has found applications in multiple scale problem in many areas, [16, 22, 27, 36].

2.3 Review of Some Analytical Approaches

The analytical asymptotic methods consist in describing the qualitative behaviour of the solution of (1.2.1) on some interval of the value of the independent variable, [69], by determining the solution of (1.2.1) as a series expansion in term of the small parameter ϵ. The first systematic analysis for finite diminsional models using an analytical asymptotic method was presented by A.N. Tikhonov in the 40s. As already mentioned, according to the Tikhonov theorem, the solution to the slow system (1.2.1) tends to that of the degenerate system as ϵ tends to zero provided the solution to the degenerate equation, also called the *quasi steady state (QSS),* is isolated in a closed domain, say $\bar{\mathcal{X}}$, and, when considering the *auxiliary* equation

$$\frac{d\,\hat{z}}{d\,\tau} = g(x, \hat{z}, 0), \qquad (2.3.1)$$

the QSS is uniformly asymptotically stable with respect to x on $\bar{\mathcal{X}}$. We will fully describe this result in the next chapter. It is important to emphasize that the Tikhonov theorem focuses its analysis on a finite interval of time and it proves the uniform convergence in time of the solution of the $x-$ component only. An extension of this result to be global in time was proposed by F. Hoppenstead [39] and the asymptotic expansions uniformly approximating the solution of the problem in the $y-$ component was added by A.B. Vasil'eva using the concept of the initial layer [7, 74, 75]. This theory found its application in many fields including population modelling, neurophysiology, biochemistry...,[19, 7, 57]. One of our result in this thesis demonstrates how this theory facilitates the analysis of some complex epidemiological models evolving in two time scales.

In applications, however, we often encounter the situation where either the quasi steady state ceases to be hyperbolic along some submanifold (a fold singularity), or two (or more) quasi steady states intersect. The latter typically involves the so called "exchange of stabilities" as in the transcritical bifurcation; that is, when the branches of the quasi steady states change from being attractive to being repelling (or vice versa) across the intersection. The assumptions of the Tikhonov theorem fail to hold in the neighbourhood of the intersection. It is natural to expect that any solution that passes close to it follows the attractive branches of the quasi steady states on either side of the intersection. Such a behaviour is, indeed, often observed, see e.g. [19, 48, 49, 50]. However, in many cases, an unexpected behaviour of the solution is observed — it follows the attracting part of the quasi steady state and, having passed the intersection, it continues along the now repelling branch of the former quasi steady state for some prescribed time and only then jumps to the attracting part of the other quasi steady state. Such a behaviour, called the *delayed switch of stability*, [19], was first observed in [64] (and explained in [53]) in the case of a pitchfork bifurcation, where an attracting quasi steady state produces two new attracting branches, while continuing as a repelling one itself. The delayed switch of stabilities in the case of a fold singularity was observed in the van der Pol equation and have received explanations based on methods ranging from nonstandard analysis [12] to classical asymptotic analysis [29]. Solutions displaying such a behaviour were named *canard solutions*. In this thesis we shall mainly focus on the so called transcritical bifurcation. The delayed switch of stability in such a situation possibly was first observed in [35] and analysed in [62].

The interest in such problems stems from numerous applications in which the existence of the so-called slow-fast oscillations [29, 38, 41, 52, 58] is deduced on the basis of the existence of intersecting quasi steady states interchanging their stabilities. Another field of applications is in the dynamical bifurcation theory, where the bifurcation parameter is driven by another, slowly varying, equation coupled to the original system [17, 18]. In both cases failure to take into account the possibility of the stability switch delay may result in erroneous conclusions about the behaviour of the solutions, see e.g. [17, 18, 58].

2.4 Our Contributions

As we pointed earlier, there is a rich literature concerning these topics and, in particular, one of our problems is similar to that considered in [48]. We mainly focus on the case, where there is a delayed stability switch that is just briefly mentioned in [48] and we allow the system to be non-autonomous. One of the main contribution of this work is to offer a new approach to the analysis of the stability switch. By employing a monotonic structure of the equations and combining it with the method of upper and lower solutions of [19], we have managed to give a constructive and relatively elementary proof of the existence of a delayed stability switch for a large class of planar systems including the predator-prey models and some epidemiological models. Also, in contrast to the papers based on phase plane analysis, e.g. [58], we are able to give the precise time at which the stability switch occurs. Here we can also mention the recent paper [63], where the results of [19] have been employed to general predator-prey models to prove the existence of canard orbits, but in a different way that requires the system to be autonomous.

As a by-product of the method, we also provide some results on an immediate stability switch. Our results pertain to a slightly different class of problems than that considered in e.g. [48, 49, 50] but, when applied to the predator-prey and epidemiological systems, they give an analogous outcome.

3 Mathematical Preliminaries

In this chapter we provide some background important to the understanding of the subsequent chapters. We first present some basic results on systems of differential equations and follow with some important results in perturbation theory.

3.1 Notation

Unless stated otherwise, we denote by

- $\mathbb{R}^n_+$ the set of the n dimensional vectors with non- negative entries.
- $T \in (0, +\infty)$ an arbitrary chosen real number.
- $\mathbb{I}_\alpha = (0, \alpha)$ and $\bar{\mathbb{I}}_\alpha = [0, \alpha]$, where $\alpha > 0$.
- $B(a, r)$ a ball centered at $a \in \mathbb{R}^n$ with a radius $r \in \mathbb{R}_+$.
- $' = \frac{d}{dt}$.
- $|x|$ the norm of the vector $x \in \mathbb{R}^n$.

3.2 System of Differential Equations

Let us consider the following system of differential equations

$$\frac{dz}{dt} = g(t, z) \tag{3.2.1}$$

with initial condition $z(t_0) = z_0 \in \mathbb{R}^n$, and g a regular function acting from a subset of $\mathbb{R}_+ \times \mathbb{R}^n$ to $\mathbb{R}^n$.

3.2.1 Existence and Uniqueness of the Solution to a System of Differential Equations

3.2.1.1 Theorem (The Picard theorem). *If g is continuous with respect to $(t,z) \in U = \{(t,z), |t_0 - t| \leq r_1,\ |z_0 - z| \leq r_2\}$, with $r_1, r_2 \in \mathbb{R}_+$, and Lipschitz continuous with respect to z, then there is a unique solution to the problem* (3.2.1) *for the given initial condition at least on $[t_0 - \ell, t_0 + \ell]$ with $\ell = \min\{r_1, r_2/M\}$ and $M = \max_{(t,z)\in U} |g(t,z)|$, [6, 32].*

3.2.1.2 Definition. The interval $[t_1, t_2)$ is the *maximal forward interval of existence* of (3.2.1) if the solutions of (3.2.1) exist on $[t_1, t_2)$ and no solution exists on $[t_1, t_2 + \epsilon)$ for any $\epsilon > 0$, [6].

3.2.1.3 Theorem. *Let us denote $z = (z_i)_{i=1,..,n}$ and consider the function $g : \mathbb{R}_+ \times \mathbb{R}^n_+ \to \mathbb{R}^n$ continuous with respect to (t,z) and Lipschitz continuous with respect to z. If $g_i(t,z) \geq 0$ for $(t,z) \in \mathbb{R}_+ \times \mathbb{R}^n$, with $z_i = 0$ then for every $z_0 \in \mathbb{R}^n_+$ there exists $T > 0$ such that the solution to* (3.2.1) *exists on some interval $[0,T)$, is unique and positive. If $T < \infty$, then*

$$\limsup_{t\to T} \sum_{i=1}^{n} z_i = +\infty,$$

[57, 67].

3.2.1.4 Remark. Theorem 3.2.1.3 also means that if the right hand side of a differential equation is sufficiently regular, then its solution either exists for all time or blows up/becomes infinite in a finite time, [6]. Furthermore, the solution $z(t)$ to (3.2.1) with $z_0 > 0$ exists on $\mathbb{R}_+$ if for all $\tilde{T} > 0$,

$$\limsup_{t\to \tilde{T}} \sum_{i=1}^{n} z_i \in \mathbb{R},$$

[6].

3.2.2 Stability Results for Systems of Differential Equations

3.2.2.1 Definition. A solution $z(t)$ of (3.2.1) is *stable* if any other solution $y(t)$ with initial condition $y(0) = y_0$ sufficiently close to $z(t)$ will remain close to it for all times.

In other words, if for any $\epsilon_1 > 0$ there exists $\epsilon_2 > 0$ such that for any solution $y(t)$ of (3.2.1) we have

$$|z_0 - y_0| < \epsilon_2 \Rightarrow |z(t) - y(t)| < \epsilon_1,$$

then $z(t)$ is said to be *stable.* Additionally, if

$$|z_0 - y_0| < \epsilon_2 \Rightarrow \lim_{t\to\infty} |z(t) - y(t)| = 0,$$

then $z(t)$ is said to be *locally asymptotically stable.*

3.2.2.2 Definition. The *flow* of (3.2.1) is the map

$$\phi : \mathbb{R}_+ \times \mathbb{R}^n \ni (t, z_0) \mapsto \phi(t, z_0) = z(t) \in \mathbb{R}^n,$$

where $z(t)$ is the solution to (3.2.1).

3.2.2.3 Definition. Let Z be an asymptotically stable equilibrium point of (3.2.1). The *basin of attraction* of Z is the set $\mathcal{B}_Z = \{z; \lim_{t\to\infty} \phi(t, z) = Z\}$, where ϕ is the flow generated by (3.2.1). The equilibrium point Z is said to be *globally asymptotically stable,* if $\mathcal{B}_Z = \mathbb{R}^n$.

3.2.2.4 Definition. The *invariant* domain under the flow $(t, z) \mapsto \phi(t, z)$ generated by the system (3.2.1) is a set $\mathbf{S} \subset \mathbb{R}^n$ such that $\phi(t, z) \in \mathbf{S}$ for all $z \in \mathbf{S}$ and $t \in \mathbb{R}_+$.

3.2.2.5 Definition. Let us consider a function $g = (g_i)_{i=1,..,n} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, and let $\mathbb{S} \subset \mathbb{R}^n$ be open. The function g is said to be of type K in $\mathbb{S}$ if for each component i and for all $t \in \mathbb{I}_T$, $g_i(t, x) \leq g_i(t, y)$ for any two points $x, y \in \mathbb{S}$ such as $x \leq y$ and $x_i = y_i$, where $x = (x_i)_{i=1,..,n}$, and $y = (y_i)_{i=1,..,n}$, [65].

3.2.2.6 Theorem (Comparison theorem)**.** *Let us consider the system of differential equations* (3.2.1), *and assume that g is a continuous function of type K and $z(t)$ is a solution of* (3.2.1) *defined on $[t_0, t_3]$, with $t_0, t_3 \in \mathbb{R}$. If u is a function satisfying for all $t \in [t_0, t_3]$*

$$\frac{du}{dt} \geq g(t, u), \qquad u(0) \geq z_0,$$

then $u(t) \geq z(t)$ for $t \in [t_0, t_3]$. If $v(t)$ is a function satisfying on $[t_0, t_3]$

$$\frac{dv}{dt} \leq g(t, v), \qquad v(0) \leq z_0,$$

then $v(t) \leq z(t)$, $t \in [t_0, t_3]$, [65].

3.2.3 Implicit Function Theorem

3.2.3.1 Theorem. *Let us consider three Banach spaces* $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$*, an open subset* $S \subset \mathcal{S}_1 \times \mathcal{S}_2$ *and a function* $G : S \to \mathcal{S}_3$ *continuously differentiable in* S*. Let us assume that there exists a root* $(x_0, z_0) \in S$ *of the function* G *such that the map*

$$\frac{\partial G}{\partial z}(x_0, z_0) : \mathcal{S}_2 \to \mathcal{S}_3$$

is a (continuous) linear and invertible and let its inverse be likewise continuous. Then, there exist open sets: S_1 *containing* x_0 *and* S_2 *containing* z_0 *such that* $S_1 \times S_2 \subset S$*, and a differentiable function* $g : S_1 \to S_2$ *such that*

$$G(x, g(x)) = 0, \tag{3.2.2}$$

and

$$\frac{dg(x)}{dx} = -\Big(\frac{\partial G(x, g(x))}{\partial z}\Big)^{-1} \circ \frac{dG}{dx}(x, g(x))$$

for all $x \in S_1$*. Furthermore, for every* $x \in S_1,\ g(x)$ *is the only solution of* (3.2.2) *in* S_2*, [46].*

3.3 Perturbation Theories for Systems of Differential Equations

These theories can be divided into two groups: the *regular* perturbation theories and the *singular* perturbation theories. However, we will only present the ones we will need later.

3.3.1 Regular Perturbation Theory

Let us consider the following system of ordinary differential equations

$$\frac{dz}{dt} = g(t, z, \epsilon), \qquad z(0) = z_0, \tag{3.3.1}$$

where $g : M \times [-\epsilon_0, \epsilon_0] \mapsto \mathbb{R}^n$ is a continuous function,

$$M = [0, \alpha] \times \bar{\mathcal{Z}} = [0, \alpha] \times \{z \in \mathbb{R}^n;\ |z - z_0| \le b\},$$

and the parameters α, b, ϵ_0 are positive.

3.3.1.1 Theorem. *Let us assume that*

1. *the function* g *is a globally Lipschitz continuous function with respect to* z*, uniformly in* $\bar{\mathcal{Z}}$*,*
2. *the equation*
$$\frac{dz}{dt} = g(t, z, 0), \qquad z(0) = z_0,$$
with $(t, z) \in M$*, has a solution,* $\bar{z}$*, on* M*;*
3. $t_0(\epsilon)$ *and* $\ell(\epsilon)$ *are continuous functions satisfying*
$$t_0(\epsilon) \in [0, \alpha], \qquad t_0(0) = 0,\ \ell(0) = 0,$$
for $\epsilon \in [-\tilde{\epsilon}, \tilde{\epsilon}]$.

Then there exists $0 \leq \epsilon_0 \leq \tilde{\epsilon}$ *such that for* $\epsilon \in [-\epsilon_0, \epsilon_0]$*, the differential equation*
$$\frac{dz}{dt} = g(t, z, \epsilon), \qquad z(t_0(\epsilon)) = z_0 + \ell(\epsilon),\ t \in [0, \alpha]$$
has a solution $z(t, \epsilon)$ *satisfying*
$$\lim_{\epsilon \to 0} z(t, \epsilon) = \bar{z}(t),$$
uniformly on $[0, \alpha]$*, [7].*

3.3.2 Singular Perturbation Theories: The Tikhonov Theorem

Let us consider the following singularly perturbed problem
$$\begin{cases} \frac{dx}{dt} = f(t, x, z, \epsilon), \\ \epsilon \frac{dz}{dt} = g(t, x, z, \epsilon), \end{cases} \tag{3.3.2}$$
with initial condition $(x(0), z(0)) = (x_0, z_0)$, $t \in \bar{\mathbb{I}}_T$, $x \in \bar{\mathbb{M}}_x \subset \mathbb{R}^n$, $z \in \mathbb{M}_z \subset \mathbb{R}^m$, and $\epsilon \in [0, \epsilon_0]$, where $\epsilon_0 > 0$ is a small parameter.

3.3.2.1 Theorem (The Tikhonov theorem)**.** *Let us consider the following list of assumptions.*

T_1- *The functions* f *and* g *are defined on a subset* $\bar{\mathbb{I}}_T \times \bar{\mathbb{M}}_x \times \mathbb{M}_z \times [0, \epsilon_0]$ *with values, respectively, in* $\mathbb{R}^n$ *and* $\mathbb{R}^m$*. They are continuous with respect* t, x, z *and Lipschitz continuous with respect to* x *and* z *in the subset* $\bar{\mathbb{I}}_T \times \bar{\mathbb{M}}_x \times \mathbb{M}_z$*, where* $\bar{\mathbb{M}}_x$ *is compact and* $\mathbb{M}_z$ *is open.*

T_2- *Let* $\bar{z} = \phi(t, x) \in \mathbb{M}_z$ *be the solution to the degenerate equation in* (3.3.2)*. Then,* $\bar{z}$ *is continuous and isolated for* $(t, x) \in \bar{\mathbb{I}}_T \times \bar{\mathbb{M}}_x$*; that is, there exists* $\beta > 0$ *such that* $g(t, x, z, 0) \neq 0$ *for* $|z - \phi(t, x)| < \beta, \ (t, x) \in \bar{\mathbb{I}}_T \times \bar{\mathbb{M}}_x$*.*

T_3- *For the auxiliary equation associated to* (3.3.2)*, defined by*

$$\frac{d\hat{z}}{d\tau} = g(t, x, \hat{z}, 0), \qquad\qquad \hat{z}(0) = z_0,$$

where t, x *are treated as parameters, the equilibrium solution* $\hat{z} = \phi(t, x)$ *is uniformly asymptotically stable on* $\bar{\mathbb{I}}_T \times \bar{\mathbb{M}}_x$*. That is, for all* $\epsilon_4 > 0$ *there exists* $\epsilon_5 > 0$ *such that for all* $(t, x) \in \bar{\mathbb{I}}_T \times \bar{\mathbb{M}}_x$*, if*

$$|\hat{z}(0) - \phi(t, x)| < \epsilon_5,$$

then, for all $\tau > 0$, $|\hat{z}(t, x, \tau) - \phi(t, x)| < \epsilon_4$ *and*

$$\lim_{\tau \to \infty} \hat{z}(t, x, \tau) = \phi(t, x),$$

uniformly on $\bar{\mathbb{I}}_T \times \bar{\mathbb{M}}_x$*.*

T_4- *The function* $(t, x) \mapsto f(t, x, \phi(t, x), 0)$ *is Lipschitz continuous with respect to* x *in* $\bar{\mathbb{I}}_T \times \bar{\mathbb{M}}_x$*, and the solution* $\bar{x}(t)$ *of*

$$\frac{d\bar{x}}{dt} = f(t, \bar{x}, \phi(t, \bar{x}), 0), \qquad\qquad \bar{x}(0) = x_0,$$

satisfies $\bar{x} \in Int\bar{\mathbb{M}}_x$ *for all* $t \in \mathbb{I}_T$*.*

T_5- *The solution* $\hat{z} = \hat{z}(\tau)$ *of*

$$\frac{d\hat{z}}{d\tau} = g(0, x_0, \hat{z}, 0), \qquad\qquad \hat{z}(0) = z_0,$$

exists in $\mathbb{M}_z$ *for* $\tau \geq 0$ *and satisfies*

$$\lim_{\tau \to \infty} \hat{z}(\tau) = \phi(0, x_0).$$

Then, there exists $\hat{\epsilon} > 0$ *such that for any* $\epsilon \in (0, \hat{\epsilon}]$ *there is a unique solution of* (3.3.2) *on* $\mathbb{I}_T$ *such that*

$$\lim_{\epsilon \to 0} x(t,\epsilon) = \bar{x}(t), \qquad t \in [0,T],$$
$$\lim_{\epsilon \to 0} z(t,\epsilon) = \bar{z}(t), \qquad t \in (0,T],$$

where $(\bar{x}, \bar{z})$ *is the solution of the degenerate system of* (3.3.2)*, [7, 69].*

3.3.2.2 Remark. It is important to notice that if $\left.\frac{\partial g(t,x,\hat{z},0)}{\partial \hat{z}}\right|_{\hat{z}=\phi(t,x)} < 0$ on $\bar{\mathbb{I}}_T \times \bar{\mathbb{M}}_x$, then $\hat{z} = \phi(t,x)$ is uniformly asymptotically stable on $\bar{\mathbb{I}}_T \times \bar{\mathbb{M}}_x$, see [69].

3.3.3 Singular Perturbation Theories: Method of Upper and Lower Solutions

3.3.3.1 Definition. Let us consider the pair of continuous functions $(\underline{x}(t,\epsilon), \underline{z}(t,\epsilon))$ and $(\bar{x}(t,\epsilon), \bar{z}(t,\epsilon))$ that are continuous and piecewise continuously differentiable with respect to $t \in \bar{\mathbb{I}}_T$. They are called *ordered lower* and *upper solutions* to the problem (3.3.2) for $\epsilon \in \mathbb{I}_{\epsilon_0}$, $\epsilon_0 > 0$, if they satisfy

1. $\underline{x}(t,\epsilon) \le \bar{x}(t,\epsilon)$ and $\underline{z}(t,\epsilon) \le \bar{z}(t,\epsilon)$,
2. $\frac{d\underline{x}}{dt} - f(t,\underline{x},z,\epsilon) \le 0 \le \frac{d\bar{x}}{dt} - f(t,\bar{x},z,\epsilon)$ for $z \in [\underline{z}, \bar{z}]$ and,
   $\epsilon\frac{d\underline{z}}{dt} - g(t,x,\underline{z},\epsilon) \le 0 \le \epsilon\frac{d\bar{z}}{dt} - g(t,x,\bar{z},\epsilon)$ for $x \in [\underline{x}, \bar{x}]$,
3. $\underline{x}(t_0,\epsilon) \le x_0 \le \bar{x}(t_0,\epsilon)$ and $\underline{z}(t_0,\epsilon) \le z_0 \le \bar{z}(t_0,\epsilon)$,

on $\bar{\mathbb{I}}_T$ and for $\epsilon \in \mathbb{I}_{\epsilon_0}$, [19].

3.3.3.2 Theorem. *Let us consider the problem* (3.3.2) *with the functions* f *and* g *sufficiently regular. If there exist ordered lower and upper solutions to* (3.3.2)*, then the solution to* (3.3.2) *exists, is unique and satisfies*

$$\underline{x}(t,\epsilon) \le x(t,\epsilon) \le \bar{x}(t,\epsilon), \tag{3.3.3}$$
$$\underline{z}(t,\epsilon) \le z(t,\epsilon) \le \bar{z}(t,\epsilon), \tag{3.3.4}$$

for $t \in \bar{\mathbb{I}}_T$ and $\epsilon \in \mathbb{I}_{\epsilon_0}$, [19, 65]. Furthermore, if the lower and the upper solutions have the same limit as ϵ tends to zero, then

$$\lim_{\epsilon\to 0} x(t,\epsilon) = \lim_{\epsilon\to 0} \underline{x}(t,\epsilon) = \lim_{\epsilon\to 0} \bar{x}(t,\epsilon),$$
$$\lim_{\epsilon\to 0} z(t,\epsilon) = \lim_{\epsilon\to 0} \underline{z}(t,\epsilon) = \lim_{\epsilon\to 0} \bar{z}(t,\epsilon),$$

for $t \in \bar{\mathbb{I}}_T$ and $\epsilon \in \mathbb{I}_{\epsilon_0}$, [19].

3.3.4 Singular Perturbation Theories: Butuzov Theorem for Transcritical Bifurcations

The original Butuzov theorems in [19] refer to two types of intersections of the quasi steady states: the transcritical and the pitchfork bifurcations. We will present the assumptions and the result for the case of the transcritical bifurcation and later we will recall the assumptions and the result for the case of the pitchfork bifurcation.

Let us consider the singularly perturbed scalar differential equation

$$\begin{cases} \epsilon\frac{dx}{dt} = g(t,x,\epsilon), & t \in \mathbb{I}_T := \{t : t_0 < t \leq T\}, \\ x(t_0,\epsilon) = x_0, \end{cases} \tag{3.3.5}$$

with $x \in M_x$ being an open bounded interval containing the origin, and $\mathbb{I}_{\epsilon_0} = \{\epsilon : 0 < \epsilon < \epsilon_0 << 1\}$.

3.3.4.1 Theorem (The Butuzov theorem)**.** *Let us consider the singularly perturbed problem (3.3.5) with the initial value $x(t_0,\epsilon) = x_0 > 0$ and let us denote $D = M_x \times \mathbb{I}_T \times \mathbb{I}_{\epsilon_0}$. Consider the following assumptions:*

(I) $g \in C^2(\bar{D}, \mathbb{R})$.

(II) *The solution of the degenerate equation $g(t,x,0) = 0$ consists of two roots $x \equiv 0$ and $x = \phi(t)$, also called quasi steady states (QSS), which intersect at $t = t_c \in (t_0, T)$ such that*

$$\phi(t) < 0 \text{ for } t_0 \leq t < t_c,$$
$$\phi(t) > 0 \text{ for } t_c < t \leq T.$$

(III) *The roots of the degenerate equation switch stability at their intersection in the following sense*

$$g_x(t,0,0) < 0,\ g_x(t,\phi(t),0) > 0 \text{ for } t \in [t_0, t_c),$$
$$g_x(t,0,0) > 0,\ g_x(t,\phi(t),0) < 0 \text{ for } t \in (t_c, T].$$

(IV) The solution $x \equiv 0$ *satisfies* $g(t,0,\epsilon) \equiv 0$ *for* $(t,\epsilon) \in \bar{\mathbb{I}}_T \times \bar{\mathbb{I}}_{\epsilon_0}$.

(V) The function G, *defined by*

$$G(t,\epsilon) = \int_0^t g_x(s,0,\epsilon)ds, \qquad (t,\epsilon) \in \bar{\mathbb{I}}_T \times \bar{\mathbb{I}}_{\epsilon_0}, \tag{3.3.6}$$

has a root $t^* \in (t_0, T)$ *for* $\epsilon = 0$.

(VI) The inequality $g(t,x,\epsilon) \leq g_x(t,0,\epsilon)x$ *for* $t \in [t_0, t^*],\ \epsilon \in \bar{\mathbb{I}}_{\epsilon_0}, 0 \leq x \leq c_0,$ *is satisfied for some* $c_0 \in M_x$.

Then, for sufficiently small ϵ, *there exists a unique solution* $x(t,\epsilon)$ *of* (3.3.5) *with*

$$\lim_{\epsilon \to 0} x(t,\epsilon) = 0 \quad \text{for} \quad t \in (t_0, t^*), \tag{3.3.7}$$

$$\lim_{\epsilon \to 0} x(t,\epsilon) = \phi(t) \quad \text{for} \quad t \in (t^*, T], \tag{3.3.8}$$

see [19].

4 Some Applications of the Tikhonov Theorem in Epidemiology

In this chapter, we study the dynamics of some diseases that involve two different time scales. This is the case, of instance, for some vector-borne diseases such as the dengue fever and malaria, where the life cycle of the vector is faster than that of the host [60]. The mosquito life cycle is of order of days, while the host life cycle is of order of decades. The recovery period of the host is similar in range to that of the mosquito life span [60]. Another example is that of the influenza (flu), where the demographic processes are much slower than that of the infection and recovery process of the disease [9]. Using singular perturbation theory, we aim to understand to what extent the dynamics of the disease can be analysed by a simplified model. By doing so, we illustrate the advantage that the Tikhonov theorem can bring into the analysis of such problems. We will start with analysing a problem model in one dimension and latter we will consider problems in higher dimensions.

4.1 One Dimensional Case: Case of a Quick Disease (Influenza)

The influenza is a contagious respiratory illness, also known as flu. It is caused by the flu viruses and can progress from mild to severe illness and even lead to death, [47]. The recovery process usually lasts several days. However, complications due to flu are likely to be observed in people with certain chronic medical conditions such as children, pregnant women and old people [47]. Actually, in South Africa there has been an outbreak of a pandemic of influenza in 2009 that caused a lot of deaths [3].

Let us start by considering an SIS model with vital dynamics and assume that the mortality rate is the same for both infective and susceptible individuals. It follows that the mathematical model is, [9, 57],

$$\begin{cases} \frac{dS}{dt} = \beta N - \mu S - \tilde{\lambda} IS + \tilde{\gamma} I, \\ \frac{dI}{dt} = -\mu I + \tilde{\lambda} IS - \tilde{\gamma} I, \end{cases} \tag{4.1.1}$$

where the initial condition is $(S(0), I(0)) = (S_0, I_0)$, $N = S + I$ is the size of the population, S is the number of susceptible individuals and I is the number of infective individuals in the population. The parameters $\beta, \mu, \tilde{\lambda}, \tilde{\gamma}$ are positive parameters and stand, respectively, for the birth rate, the death rate, the transmission rate and the recovery rate. Since typically the disease lasts a few days, it is natural to take 1 day as the unit of time for the disease related parameters. However, the demographic processes occur at the scale of years, [9]. So, rescaling (4.1.1) using 1 year as the time unit, gives

$$\begin{cases} \frac{dS}{dt} = \beta N - \mu S - 365(\lambda SI - \gamma I), \\ \frac{dI}{dt} = -\mu I + 365(\lambda SI - \gamma I), \end{cases} \tag{4.1.2}$$

where the initial condition is $(S(0), I(0)) = (S_0, I_0)$, and we denoted $\tilde{\lambda} = 365\lambda, \tilde{\gamma} = 365\gamma$.

We introduce a small parameter ϵ representing the ratio of time scales and consider the following class of equations

$$\begin{cases} \frac{dS}{dt} = \beta N - \mu S - \frac{1}{\epsilon}(\lambda SI - \gamma I), & S(0) = S_0, \\ \frac{dI}{dt} = -\mu I + \frac{1}{\epsilon}(\lambda SI - \gamma I), & I(0) = I_0, \end{cases} \tag{4.1.3}$$

where (4.1.2) corresponds to the case $\epsilon = \frac{1}{365}$.

It follows from (4.1.3) that the dynamics of the total population satisfies

$$\frac{dN}{dt} = (\beta - \mu)N, \tag{4.1.4}$$

with $N(0) = N_0 = I_0 + S_0$. Hence, since $S = N - I$, system (4.1.3) becomes

$$\begin{cases} N(t) &= N_0 e^{(\beta-\mu)t}, \\ \epsilon \frac{dI}{dt} &= -\mu\epsilon I + I(\lambda(N - I) - \gamma), \end{cases} \tag{4.1.5}$$

with the initial condition $I(0) = I_0$.

4.1.0.1 Theorem. *Let us denote $\nu = \frac{\gamma}{\lambda}$ and $r = \beta - \mu$.*

1. *If $N_0 = \nu$ and $r = 0$, then the solution to (4.1.3) satisfies*

$$\lim_{\epsilon \to 0} I(t, \epsilon) = 0, \qquad t \in (0, T],$$
$$\lim_{\epsilon \to 0} S(t, \epsilon) = \nu, \qquad t \in (0, T].$$

2. *If $N_0 < \nu$ and $r \leq 0$, then the solution to* (4.1.3) *satisfies*

$$\lim_{\epsilon \to 0} I(t, \epsilon) = 0, \qquad t \in (0, T],$$

$$\lim_{\epsilon \to 0} S(t, \epsilon) = N_0 e^{rt}, \qquad t \in (0, T].$$

3. *If $N_0 > \nu$ and $r \geq 0$, then the solution to* (4.1.3) *satisfies*

$$\lim_{\epsilon \to 0} I(t, \epsilon) = N_0 e^{rt} - \nu, \qquad t \in (0, T],$$

$$\lim_{\epsilon \to 0} S(t, \epsilon) = \nu, \qquad t \in (0, T].$$

Proof. Let us denote $g(N, I, \epsilon) = -\mu\epsilon I + I(\lambda(N - I) - \gamma)$. If $N_0 = \nu$ and $r = \beta - \mu = 0$ then, from the first equation of (4.1.5) and for all $t \in [0, T]$, $N(t) = \nu$. The second equation of (4.1.5) becomes

$$\epsilon \frac{dI}{dt} = -\epsilon\mu I - \lambda I^2, \tag{4.1.6}$$

with the initial condition $I(0) = I_0$. From (4.1.6) it follows that

$$\frac{1}{\mu}\frac{dI}{I} - \frac{\lambda}{\epsilon\mu}\frac{dI}{\mu + \frac{\lambda}{\epsilon} I} = -dt.$$

Thus,

$$\ln\left(\frac{I}{\mu + \frac{\lambda}{\epsilon} I}\right) = -\mu t + c \text{ with } c \in \mathbb{R}.$$

Therefore, for $\epsilon > 0$ and $t \in [0, T]$,

$$I(t, \epsilon) = \frac{\mu k e^{-\mu t}}{1 - \frac{\lambda}{\epsilon} k e^{-\mu t}},$$

where $k = \frac{I_0}{\mu + \frac{\lambda}{\epsilon} I_0}$. It follows that, since $S = \nu - I$,

$$\lim_{\epsilon \to 0} I(t, \epsilon) = \lim_{\epsilon \to 0} \frac{\mu I_0 e^{-\mu t}}{(\mu + \frac{\lambda}{\epsilon} I_0)(1 - \frac{\lambda I_0 e^{-\mu t}}{\epsilon\mu + \lambda I_0})} = 0, \qquad t \in (0, T],$$

$$\lim_{\epsilon \to 0} S(t, \epsilon) = \nu, \qquad t \in (0, T].$$

Now, let us assume that $N_0 \neq \nu$ and consider the equation $\bar{I}(\lambda(\bar{N} - \bar{I}) - \gamma) = 0$. We have two solutions $\bar{I}_1 = 0$ and $\bar{I}_2 = \bar{N} - \nu$. It is important to notice that the solutions intersect at $\bar{N} = \nu$ and therefore stop being isolated at that point. Thus it is important to find an invariant domain of (4.1.4) containing one isolated quasi steady state. Therefore, we need to restrict ourselves to one of these two cases:

1. $\bar{I}_1 = 0$ and $\Psi_- = \{N, N < \nu\}$,
2. $\bar{I}_2 = \bar{N} - \nu$ and $\Psi_+ = \{N, N > \nu\}$.

Considering the first case, we have for $N_0 < \nu$ and $r \leq 0$,

$$N_0 e^{rt} < \nu, \qquad t \in [0, T].$$

Thus,

$$N(t) \in \Psi_-, \qquad t \in [0, T].$$

Therefore Ψ_- is invariant under the flow defined by (4.1.4) and this case corresponds to that of a stable population. Similarly, considering the second case, we have for $N_0 > \nu$, and $r \geq 0$,

$$N_0 e^{rt} > \nu, \qquad t \in [0, T].$$

Thus,

$$N(t) \in \Psi_+, \qquad t \in [0, T].$$

Therefore Ψ_+ is invariant under the flow defined by (4.1.4) and this case corresponds to that of an unstable population.

Let us consider the auxiliary equation

$$\frac{d\hat{I}}{d\tau} = \hat{I}(\lambda(N - \hat{I}) - \gamma),$$

where $\tau = \frac{t}{\epsilon}$ and N is a parameter. It is easy to see that $\hat{I} = \bar{I}_1$ and $\hat{I} = \bar{I}_2$ are equilibrium solutions to the auxiliary equation. Let us denote $\hat{g}(N, \hat{I}, 0) = \hat{I}(\lambda(N - \hat{I}) - \gamma)$. We have

$$\hat{g}_{\hat{I}}(N, \hat{I}, 0) = \lambda N - 2\lambda \hat{I} - \gamma.$$

Thus, $\hat{g}_{\hat{I}}(N, \bar{I}_1, 0) = \lambda N - \gamma$ and $\hat{g}_{\hat{I}}(N, \bar{I}_2, 0) = -\lambda N + \gamma$. It follows that $\hat{g}_{\hat{I}}(N, \bar{I}_1, 0) < 0$ for $N < \nu$. This means that $\bar{I}_1$ is asymptotically stable for $N < \nu$. Conversely, for $N > \nu$, we have $\hat{g}_{\hat{I}}(N, \bar{I}_2, 0) < 0$, meaning $\bar{I}_2$ is asymptotically stable.

Hence, according to the Tikhonov theorem, for $N_0 < \nu$ and $r \leq 0$,

$$\lim_{\epsilon \to 0} I(t, \epsilon) = 0, \qquad t \in (0, T],$$

$$\lim_{\epsilon \to 0} S(t, \epsilon) = N_0 e^{rt}, \qquad t \in (0, T],$$

and, for $N_0 > \nu$ and $r \geq 0$,

$$\lim_{\epsilon \to 0} I(t, \epsilon) = \bar{I}_2 = N_0 e^{rt} - \nu, \qquad t \in (0, T],$$
$$\lim_{\epsilon \to 0} S(t, \epsilon) = N(t) - \bar{I}_2(t) = \nu, \qquad t \in (0, T].$$

□

4.1.1 Numerical Simulations

In order to illustrate the results obtained from the Tikhonov theorem, we used numerical simulations. We used the following values of the parameters, [9, 57], $\lambda = 0.0018,\ \gamma = 0.14,\ \beta = 0.001,\ \mu = 1/70.0,\ I_0 = 5,\ N_0 = 50$ for the stable case and $\mu = 0.029,\ \beta = 0.057,\ \gamma = 0.14,\ \lambda = 0.0018,\ I_0 = 5,\ N_0 = 100$ for the unstable case. Figure 4.1 presents the dynamics of the infected population in both cases: the unstable and stable populations. The first figure shows the case of a stable population, while the second figure shows the case of an unstable population. Figure 4.2 shows the dynamics of the infective population for the case $N_0 = \nu$ and $r = 0$. The parameters used are the same as previously; that is $\gamma = 0.14,\ \lambda = 0.0018,\ I_0 = 5,\ N_0 = 100$. In either cases we observe that the solution of the original system (4.1.3) tends to the quasi steady state as epsilon tends to zero. These simulation results illustrate the Tikhonov theorem.

4.2 Two Dimensional Case: Case of a Quick Disease (Influenza)

Let us consider again the previous model but this time with a non-zero disease induced death rate. The mathematical model becomes

$$\begin{cases} \frac{dS}{dt} = \beta N - \mu S - \frac{1}{\epsilon}(\lambda SI - \gamma I), \\ \frac{dI}{dt} = -(\mu + \mu^*)I + \frac{1}{\epsilon}(\lambda SI - \gamma I), \end{cases} \tag{4.2.1}$$

where the initial condition is $(S(0), I(0)) = (S_0, I_0)$, the variables S and I are, respectively, the numbers of the susceptible and infected individuals in the population. The total population is $N = S + I$, the transmission rate from an infected individual to a susceptible individual is λ, the parameters μ and μ^* are, respectively, the natural death rate and the

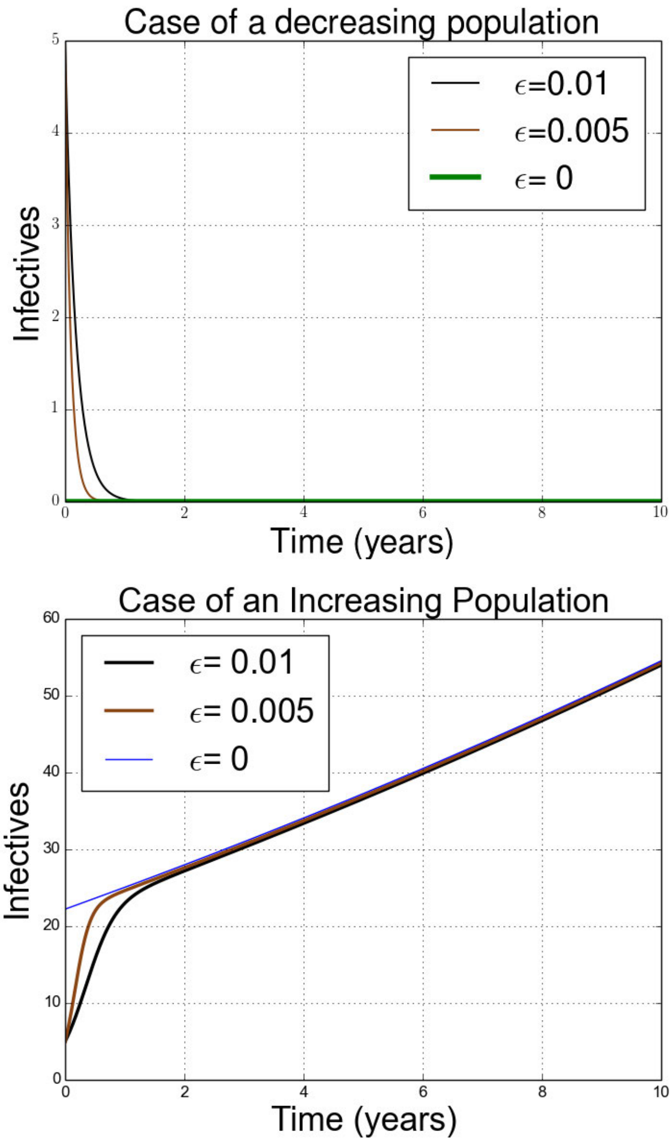

Figure 4.1: Dynamics of the infected population, solution to (4.1.1). The first figure presents the case of a stable population, while the second figure shows the case of an unstable population. In both cases, we observe that as ϵ tends to zero the solutions tend the QSS represented by the graph $\epsilon = 0$.

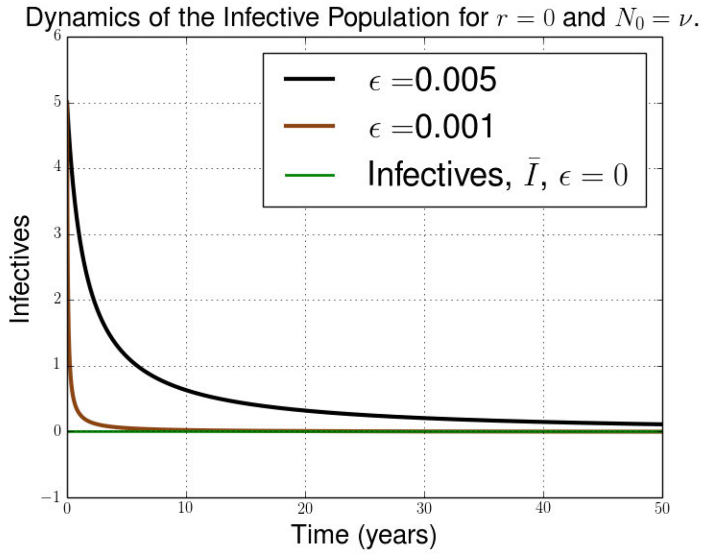

Figure 4.2: Dynamics of the infected population, solution to (4.1.1) for $N_0 = \nu$ and $r = 0$. We observe that the solution tends to the quasi steady state ($\bar{I}$) as ϵ tends to zero.

disease induced death rate, and the recovery rate from the disease is γ. The parameter $\epsilon > 0$ is a small parameter.

4.2.0.1 Theorem. *For* $(S_0, I_0) \in \mathbb{R}_+^2$, *the solution to* (4.1.3) *exists, is unique and non-negative on* $\mathbb{R}_+$.

Proof. Let us consider the function $f = (f_1, f_2)$ such that

$$f_1(S, I) = \beta N - \mu S - \frac{1}{\epsilon}(\lambda SI - \gamma I),$$
$$f_2(S, I) = -(\mu + \mu^*)I + \frac{1}{\epsilon}(\lambda SI - \gamma I).$$

It is easy to see that f is continuous and Lipschitz continuous with respect to S and I. Therefore, according to the Picard theorem, Theorem 3.2.1.1, there exists $T \in \mathbb{R}_+$ such that the solution to (4.2.1) exists and is unique on $[0, T]$. Further, for $(S, I) \in \mathbb{R}_+^2$,

$$f_1(0, I) = \beta I + \frac{\gamma I}{\epsilon} \geq 0,$$
$$f_2(S, 0) = 0.$$

It follows, according to Theorem 3.2.1.3, that S and I are non-negative on $[0, T)$. Finally, from (4.2.1),

$$\frac{dN}{dt} = (\beta - \mu)N - \mu^* I \leq (\beta - \mu)N.$$

It follows that

$$0 < N(t) < N_0 e^{(\beta-\mu)t}, \qquad t \in [0, T).$$

Hence, $N = S + I$ is bounded on $[0, T)$ for any $T > 0$. Therefore, according to Theorem 3.2.1.3, the solutions to (4.2.1) with initial condition in $\mathbb{R}^2_+$ exist on $\mathbb{R}_+$. □

Since, by (4.2.1), the dynamics of the total population is given by

$$\frac{dN}{dt} = (\beta - \mu)N - \mu^* I, \qquad N(0) = N_0, \tag{4.2.2}$$

system (4.2.1) is equivalent to

$$\begin{cases} \frac{dN}{dt} = (\beta - \mu)N - \mu^* I, \\ \epsilon \frac{dI}{dt} = -\epsilon(\mu + \mu^*)I + I(\lambda(N - I) - \gamma), \end{cases} \tag{4.2.3}$$

with initial condition $(N(0), I(0)) = (N_0, I_0)$, and the degenerate system is given by

$$\begin{cases} \frac{d\bar{N}}{dt} = (\beta - \mu)\bar{N} - \mu^* \bar{I}, \\ 0 = \bar{I}(\lambda(\bar{N} - \bar{I}) - \gamma), \end{cases} \tag{4.2.4}$$

with initial condition $\bar{N}(0) = N_0$.

4.2.0.2 Theorem. *1. If $N_0 < \nu$ and $\beta - \mu \leq 0$, then the solution to* (4.2.1) *satisfies*

$$\lim_{\epsilon \to 0} I(t, \epsilon) = 0, \qquad t \in (0, T],$$
$$\lim_{\epsilon \to 0} S(t, \epsilon) = N_0 e^{(\beta-\mu)t}, \qquad t \in (0, T].$$

2. If $N_0 > \nu$ and $\beta - \mu \geq 0$, then the solution to (4.2.1) *satisfies*

$$\lim_{\epsilon \to 0} I(t, \epsilon) = \bar{N}_2(t) - \nu, \qquad t \in (0, T],$$
$$\lim_{\epsilon \to 0} S(t, \epsilon) = \nu, \qquad t \in (0, T],$$

where $\bar{N}_2(t) = \frac{\left((\beta-\mu-\mu^*)N_0 + \mu^*\nu\right) e^{(\beta-\mu-\mu^*)t} - \mu^*\nu}{\beta-\mu-\mu^*}$.

Proof. Solving the degenerate system, we obtain two solutions: $(\bar{I}_1, \bar{N}_1)$ and $(\bar{I}_2, \bar{N}_2)$ where $\bar{N}_1(t) = N_0 e^{(\beta-\mu)t}$, $\bar{I}_1(t) = 0$, and $\bar{N}_2(t) = \frac{\left((\beta-\mu-\mu^*)N_0 + \mu^*\nu\right) e^{(\beta-\mu-\mu^*)t} - \mu^*\nu}{\beta-\mu-\mu^*}$, $\bar{I}_2(t) = \bar{N}_2(t) - \nu$, for $t \in \mathbb{I}_T$. It follows that there are two quasi steady states $\bar{I}_1 = 0$ and $\bar{I}_2 = \bar{N} - \nu$ which intersect at $\bar{N} = \nu$. Let us consider $\bar{I}_1 = 0$ as our isolated solution.

For $N_0 < \nu$ and $\beta - \mu \leq 0$, we have $\bar{N}_1(t) < \nu$ for all $t \in \bar{\mathbb{I}}_T$. That is, $\bar{N}_1(t) \in \Psi_-$ for $t \in \bar{\mathbb{I}}_T$. Since $\bar{N}_1$ satisfies the equation

$$\frac{d\bar{N}_1}{dt} = (\beta - \mu)\bar{N}_1, \qquad t \in \bar{\mathbb{I}}_T, \tag{4.2.5}$$

it follows that the domain $\Psi_- = \{N, N < \nu\}$ is invariant under the flow defined by (4.2.5). Similarly to the one dimensional case, it is easy to see, considering the auxiliary equation, that $\hat{I} = \bar{I}_1$ is uniformly asymptotically stable on Ψ_-. Therefore, according to the Tikhonov theorem, for $N_0 < \nu$ and $r \leq 0$,

$$\lim_{\epsilon \to 0} I(t, \epsilon) = 0, \qquad t \in (0, T],$$
$$\lim_{\epsilon \to 0} S(t, \epsilon) = \bar{N}_1(t), \qquad t \in (0, T].$$

This case corresponds to the case of a stable population.

Further, setting $I = N - S$, from (4.2.2) we have

$$\frac{dN}{dt} = (\beta - \mu)N - \mu^*(N - S), \qquad N(0) = N_0. \tag{4.2.6}$$

Thus (4.2.1) becomes

$$\begin{cases} \epsilon \frac{dS}{dt} = \epsilon(\beta N - \mu S) - (\lambda S - \gamma)(N - S), & S(0) = S_0, \\ \frac{dN}{dt} = (\beta - \mu)N - \mu^*(N - S), & N(0) = N_0. \end{cases} \tag{4.2.7}$$

Setting $\epsilon = 0$ in (4.2.7), we obtain the degenerate system

$$\begin{cases} 0 = -(\lambda \bar{S} - \gamma)(\bar{N} - \bar{S}), & \bar{S}(0) = S_0, \\ \frac{d\bar{N}}{dt} = (\beta - \mu)\bar{N} - \mu^*(\bar{N} - \bar{S}), & \bar{N}(0) = N_0. \end{cases} \tag{4.2.8}$$

Thus, there are two quasi steady states: $\bar{S}_1 = \bar{N}$ and $\bar{S}_2 = \nu$ where $\bar{S}_i = \bar{N} - \bar{I}_i$, $i = 1, 2$. Let us consider $\bar{S}_2 = \nu$ as our isolated quasi steady state. The solution of the degenerate system (4.2.8) for $\bar{S} = \nu$ is

$$\bar{N}_2(t) = -\frac{\mu^* \nu}{\beta - \mu - \mu^*} + (N_0 + \frac{\mu^* \nu}{\beta - \mu - \mu^*})e^{(\beta - \mu - \mu^*)t}.$$

Furthermore, let us consider the right hand side of the auxiliary equation of (4.2.7) denoted by

$$\hat{h} = (\lambda \hat{S} - \gamma)(\hat{S} - N)$$

We have,

$$\frac{\partial \hat{h}}{\partial \hat{S}} = 2\lambda \hat{S} - \gamma - \lambda N$$

It follows that $\frac{\partial \hat{h}}{\partial \hat{S}}|_{\hat{S}=\nu} = \gamma - \lambda N < 0$ for $N \in \Psi_+$. Therefore, $\hat{S} = \bar{S}_2$ is uniformly asymptotically stable on Ψ_+. Finally, let us find out whether or not Ψ_+ is invariant under the flow generated by the degenerated system. We distinguish three cases with respect to $\mathring{r} = \beta - \mu - \mu^*$; that is, either $\mathring{r} = 0$, $\mathring{r} < 0$, or $\mathring{r} > 0$.

- If $\mathring{r} = 0$ and $N_0 > \nu$ then $\bar{N}_2$ is solution to the differential equation

$$\frac{d\bar{N}_2}{dt} = \mu^* \nu, \tag{4.2.9}$$

with $\bar{N}_2(0) = N_0$. Thus,

$$\bar{N}_2(t) = \mu^* \nu t + N_0 \text{ for } t \in \mathbb{I}_T.$$

It follows that for $N_0 > \nu$,

$$\bar{N}_2(t) \geq \nu \text{ for } t \in \mathbb{I}_T,$$

since $\mu^* \nu t > 0$. Therefore,

$$\bar{N}_2(t) \in \Psi_+ \text{ for } t \in \mathbb{I}_T.$$

- For $\beta - \mu > 0$, $\beta - \mu - \mu^* > 0$ and $N_0 > \nu$, let us consider the equation satisfied by $\bar{N}_2$ and given by

$$\frac{d\bar{N}_2}{dt} = (\beta - \mu - \mu^*)\bar{N}_2 + \mu^* \nu, \tag{4.2.10}$$

with $\bar{N}_2(0) = N_0$. That is,

$$\bar{N}_2(t) = \frac{\mu^* \nu}{\beta - \mu - \mu^*}(e^{(\beta-\mu-\mu^*)t} - 1) + N_0 e^{(\beta-\mu-\mu^*)t}.$$

However, since $N_0 > \nu$,

$$N_0 e^{(\beta-\mu-\mu^*)t} > \nu \Rightarrow \frac{\mu^* \nu}{\beta - \mu - \mu^*}(e^{(\beta-\mu-\mu^*)t} - 1) + N_0 e^{(\beta-\mu-\mu^*)t} > \nu.$$

Thus,

$$\bar{N}_2(t) \in \Psi_+ \text{ for } t \in \mathbb{I}_T.$$

- For $\beta-\mu>0$, $\beta-\mu-\mu^*<0$ and $N_0>\nu$, let us consider again equation (4.2.10). Its equilibrium solution is

$$\bar{N}_2^* = -\frac{\mu^*\nu}{\mathring{r}} = (1-\frac{\beta-\mu}{\beta-\mu-\mu^*})\nu.$$

  Since $\beta-\mu>0$ and $\beta-\mu-\mu^*\leq 0$, it follows that

$$\bar{N}_2^* \geq \nu.$$

  Furthermore, let us denote the right hand side of (4.2.10) by

$$g_2(\bar{N}_2) = (\beta-\mu-\mu^*)\bar{N}_2 + \mu^*\nu.$$

  The derivative of g_2 with respect to $\bar{N}_2$ is

$$g_{2\bar{N}_2} = \beta-\mu-\mu^* < 0.$$

  Thus, the equilibrium $\bar{N}_2^*$ is attractive. Therefore, since the solutions are monotonic and $\bar{N}_2^*\geq\nu$, any solution to (4.2.10) with initial condition $N_0>\nu$ converges to the equilibrium while remaining greater than ν It follows that

$$\bar{N}_2(t)\in\Psi_+ \text{ for } t\in\mathbb{I}_T.$$

Hence, for $\beta-\mu>0$ and $N_0>\nu$,

$$\bar{N}_2(t)\in\Psi_+ \text{ for } t\in\bar{\mathbb{I}}_T.$$

In other words, Ψ_+ is invariant under the flow generated by (4.2.10). Therefore, according to the Tikhonov theorem, that for $N_0>\nu$ and $\beta-\mu\geq 0$,

$$\begin{aligned}
&\lim_{\epsilon\to 0} S(t,\epsilon) = \nu, && t\in(0,T],\\
&\lim_{\epsilon\to 0} I(t,\epsilon) = \bar{N}_2(t)-\nu, && t\in(0,T].
\end{aligned}$$

□

4.2.0.3 Remark (Some Comments on the Case of an Unstable-Stable Population)**.** Let us assume that

$$\beta-\mu-\mu^*<0 \text{ and } \beta-\mu>0. \tag{4.2.11}$$

We aim to analyse the existence and the stability of the equilibria of (4.2.7) under assumption (4.2.11). It is easy to see that $(N,S)=(0,0)$ is an equilibrium point of (4.2.7).

Let us determine the second equilibrium point, (N_E, S_E), of (4.2.7). From the second equation of (4.2.7), we have

$$N_E = \frac{-\mu^* S_E}{\beta - \mu - \mu^*}$$

and, since (4.2.1) and (4.2.7) are equivalent systems, we have from the second equation of (4.2.1)

$$S_E = \frac{\epsilon(\mu + \mu^*) + \gamma}{\lambda}.$$

It follows that $N_E > 0$ and $N_E > S_E$ if and only if $\beta - \mu - \mu^* < 0$ and $\beta - \mu > 0$. Therefore, under assumptions (4.2.11), there exists a relevant biological equilibrium point, (N_E, S_E), to (4.2.7). Moreover, we have

$$\lim_{\epsilon \to 0} S_E(\epsilon) = \nu,$$
$$\lim_{\epsilon \to 0} N_E(\epsilon) = \frac{-\mu^* \nu}{\beta - \mu - \mu^*},$$

where $S = \nu$ is the second quasi steady state of (4.2.7). It is asymptotically stable if $N \in \Psi_+$. The solution of the degenerate system of (4.2.7) for $S = \nu$ is

$$\bar{N}_2(t) = -\frac{\mu^* \nu}{\beta - \mu - \mu^*} + (N_0 + \frac{\mu^* \nu}{\beta - \mu - \mu^*})e^{(\beta - \mu - \mu^*)t}.$$

It can be noticed that

$$\lim_{t \to \infty} \bar{N}_2(t) = \frac{-\mu^* \nu}{\beta - \mu - \mu^*} = \lim_{\epsilon \to 0} N_E(\epsilon).$$

Let us now study the stability of the equilibria $(N, S) = (0, 0)$ and $(N, S) = (N_E(\epsilon), S_E(\epsilon))$. The matrix $\mathcal{B}_0$ of the linear system associated to (4.2.7) at $(N, S) = (0, 0)$ is given by

$$\mathcal{B}_0(\epsilon) = \begin{pmatrix} \beta - \mu - \mu^* & \mu^* \\ \beta + \frac{\gamma}{\epsilon} & -(\mu + \frac{\gamma}{\epsilon}) \end{pmatrix}.$$

If follows that the determinant is

$$\begin{aligned} det\mathcal{B}_0(\epsilon) &= -(\beta - \mu - \mu^*)(\mu + \frac{\gamma}{\epsilon}) - \mu^*(\beta + \frac{\gamma}{\epsilon}) \\ &= -\frac{(\beta - \mu)\gamma}{\epsilon} - \mu(\beta - \mu - \mu^*) - \mu^* \beta. \end{aligned}$$

Thus, since $\beta - \mu > 0$,

$$\lim_{\epsilon \to 0} det\mathcal{B}_0(\epsilon) = -\infty.$$

This means that $\mathcal{B}_0(\epsilon)$ has a positive eigenvalue for sufficiently small ϵ. Thus the equilibrium $(N,S)=(0,0)$ is unstable. Let us consider the equilibrium (N_E,S_E) and denote

$$\tilde{f}(N,S) = (\beta-\mu-\mu^*)N + \mu^* S,$$
$$\tilde{g}(N,S) = \beta N - \mu S - \frac{1}{\epsilon}(\lambda S-\gamma)(N-S).$$

We have at (N_E,S_E),

$$\frac{\partial \tilde{f}}{\partial N} = \beta-\mu-\mu^* \text{ and } \frac{\partial \tilde{f}}{\partial S} = \mu^*,$$
$$\frac{\partial \tilde{g}}{\partial N} = \beta - \frac{1}{\epsilon}(\lambda S_E-\gamma) = \beta-\mu-\mu^* \text{ and,}$$
$$\frac{\partial \tilde{g}}{\partial S} = -\mu - \frac{\lambda}{\epsilon}(N_E-S_E) + \frac{1}{\epsilon}(\lambda S_E-\gamma)$$
$$= \mu^* + \frac{\lambda}{\epsilon}\frac{1+\mu^*}{\beta-\mu-\mu^*}S_E = A + \frac{B}{\epsilon},$$

where $A = \mu^* + \frac{(1+\mu^*)(\mu^*+\mu)}{\beta-\mu-\mu^*}$ and $B = \frac{\gamma(1+\mu^*)}{\beta-\mu-\mu^*}$.

It follows that the Jacobian matrix at (N_E,S_E) is

$$\mathcal{B}_E(\epsilon) = \begin{pmatrix} \beta-\mu-\mu^* & \mu^* \\ \beta-\mu-\mu^* & A+\frac{B}{\epsilon} \end{pmatrix}.$$

The determinant of $\mathcal{B}_E$ is

$$det\mathcal{B}_E(\epsilon) = (\beta-\mu-\mu^*)(\mu^*-\mu) + (1+\mu^*)(\mu^*+\mu) + \frac{\gamma(1+\mu^*)}{\epsilon},$$

and the trace of $\mathcal{B}_E$ is

$$tr\mathcal{B}_E(\epsilon) = \beta-\mu + \frac{(1+\mu^*)(\mu^*+\mu)}{\beta-\mu-\mu^*} + \frac{\gamma(1+\mu^*)}{\beta-\mu-\mu^*}\frac{1}{\epsilon}.$$

Therefore,

$$\lim_{\epsilon\to 0} det\mathcal{B}_E(\epsilon) = +\infty,$$
$$\lim_{\epsilon\to 0} tr\mathcal{B}_E(\epsilon) = -\infty,$$

since, by assumption, $\beta-\mu-\mu^* < 0$. Thus, (N_E,S_E) is asymptotically stable for sufficiently small ϵ.

4.2.1 Numerical Simulations

Figure 4.3 shows the graphs of the dynamics of the infective population, solution to (4.2.3), for $\epsilon = 0.05,\ 0.01$ and that of the infective population, solution to the degenerate system

(4.2.4), represented by $\epsilon = 0$. The value of the parameters used for simulation are, [9, 57] $\lambda = 0.0018,\ \gamma = 0.14,\ \beta = 0.038,\ \mu = 0.013,\ \mu^* = 0.015$ with initial condition $(I_0, N_0) = (220, 270)$ for Case 2 in Theorem 4.2.0.2 (the unstable case). For Case 1 in Theorem 4.2.0.2 (the stable case), we considered $\lambda = 0.0018,\ \gamma = 0.14,\ \beta = 0.03,\ \mu = 0.038,\ \mu^* = 0.04,\ I_0 = 20$ and $N_0 = 70$. It can be observed that in both cases the solution to (4.2.3) tends to the solution of the degenerate system as epsilon tends to zero, as predicted by the Tikhonov theorem.

4.3 Two Dimensional Case: a Simple Model for the Dengue Fever

According to the World Health Organization (WHO), the annual number of dengue fever and dengue haemorrhagic fever cases has increased dramatically in recent years. In endemic countries the burden of dengue is nearly 1,300 disability-adjusted lives per million population [33]. The geographical areas with high risk factor have expanded in recent years and travellers from endemic areas can contribute to a further spread [34, 77, 40]. The dengue fever is a vector borne-disease mainly transmitted by *aedes aegypti* mosquitoes.

In order to capture the essence of the dynamics of the disease [60], let us consider the simplest model possible. It is obtained by considering only one strain and assuming that the long-life immunity is negligible. Let us consider the following model, [60],

$$\begin{cases} \frac{dS}{dt} &= \alpha(N-S) - \frac{\beta}{M}SV, \\ \frac{dI}{dt} &= \frac{\beta}{M}SV - \alpha I, \\ \frac{dU}{dt} &= \psi - \nu U - \frac{\vartheta}{N}UI, \\ \frac{dV}{dt} &= \frac{\vartheta}{N}UI - \nu V, \end{cases} \tag{4.3.1}$$

where the initial condition is $(S(0), I(0), U(0), V(0)) = (I_0, S_0, U_0, V_0)$, S and I represent, respectively, the numbers of susceptible and infected hosts; U and V are, respectively, the numbers of susceptible and infective vectors; N and M stand for the total host population and the total vector population, respectively and they are assumed to be constant over time. The vector's infection rate is ϑ and its death rate is ν. The recruitment into the vector's population is ψ. The force of infection in the host's population is $\frac{\beta}{M}$ and its recovery rate is α. The parameters $\alpha, \beta, \nu, \vartheta, \psi$ are positive parameters.

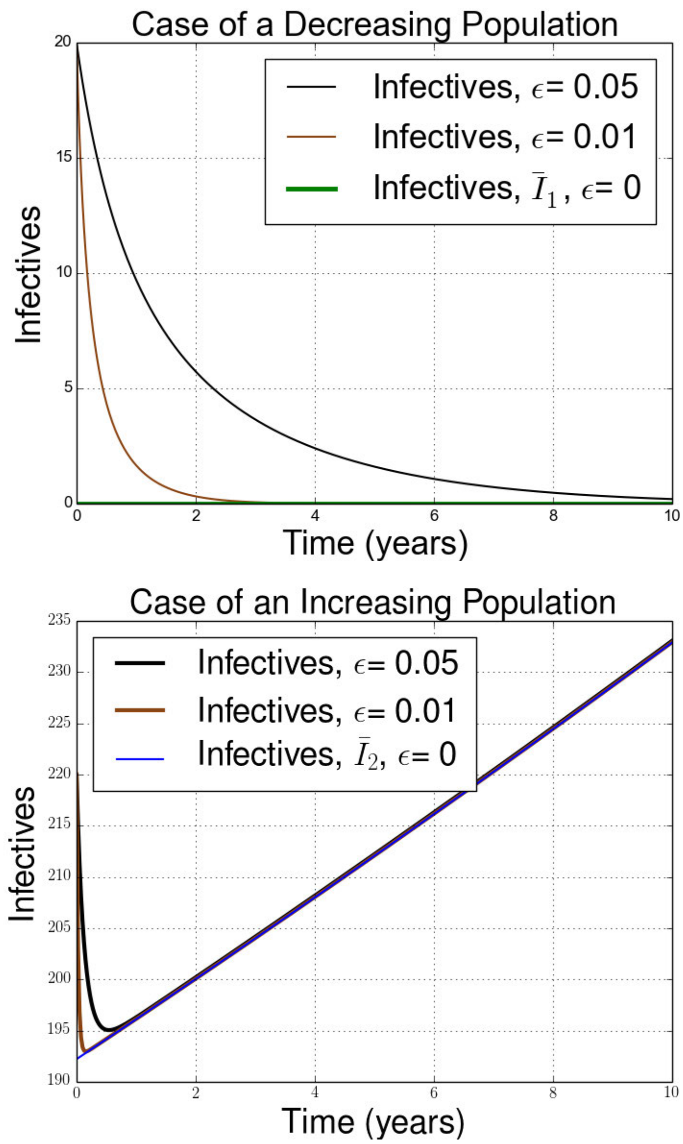

Figure 4.3: These figures show the graphs of the solutions to (4.2.3) for $\epsilon = 0.05,\ 0.01$ and the solution to the degenerate system (4.2.4) represented by $\epsilon = 0$ in both cases: stable (Case 1 in Theorem 4.2.0.2) and unstable (Case 2 in Theorem 4.2.0.2) population. The first figure shows a stable population, while the second presents an unstable population. One can easily notice that in both cases the solutions to (4.2.3) tend to that of the degenerate system (4.2.4) as ϵ tends to zero.

4.3.1 Analysis of the Model

4.3.1.1 Theorem. *For the initial condition* $(S_0, I_0, U_0, V_0) \in \mathbb{R}_+^4$ *at* $t_0 = 0$ *the solution to* (4.3.1) *exists, is unique and non-negative on* $\mathbb{R}_+$.

Proof. It is important to mention that this proof will be referred to as a generic example of a proof for similar problems. Let us consider model (4.3.1) and denote

$$\begin{aligned} f_1(S,I,U,V) &= \alpha I - \frac{\beta}{M}SV, \\ f_2(S,I,U,V) &= \frac{\beta}{M}SV - \alpha I, \\ f_3(S,I,U,V) &= \psi - \nu U - \frac{\vartheta}{N}UI, \\ f_4(S,I,U,V) &= \frac{\vartheta}{N}UI - \nu V, \end{aligned}$$

$f = (f_1, f_2, f_3, f_4)$, $X = (S, I, U, V)$, and $X_0 = (S_0, I_0, U_0, V_0)$.

Local existence and uniqueness of the solution. It can be noticed that the function f is continuous and Lipschitz continuous with respect to X. According to the Picard theorem, Theorem 3.2.1.1, there exists $\tau_0 > 0$ such that the solution to (4.3.1) exists and is defined locally at least on $[0, \tau_0]$. Again, let us consider the initial condition $X_1 = X(\tau_0)$ at $t_0 = \tau_0$. Using Theorem 3.2.1.1, it follows that there exist $\tau_0 \leq \tau_1 \in \mathbb{R}_+$ and a unique solution to (4.3.1) defined on $[\tau_0, \tau_1]$. By uniqueness of the solution of (4.3.1) with a given initial condition, it follows that the solutions of (4.3.1), obtained on $[0, \tau_0]$ and on $[\tau_0, \tau_1]$ form a unique solution of (4.3.1) on $[0, \tau_1]$ with the initial condition X_0 at $t_0 = 0$. Repeating this process many times, we obtain the maximal forward interval of a existence for the solutions of (4.3.1), say $[0, \hat{\tau})$ with $\hat{\tau} > 0$.

Positivity of the solution. For $X \in \mathbb{R}_+^4$,

$$\begin{aligned} f_1(0,I,U,V) &\geq 0, \\ f_2(S,0,U,V) &\geq 0, \\ f_3(S,I,0,V) &\geq 0, \\ f_4(S,I,U,0) &\geq 0. \end{aligned}$$

Therefore, according to Theorem 3.2.1.3, the unique solution of (4.3.1) is positive on $[0, \hat{\tau})$.

Global existence, uniqueness and positivity of the solution. Since S, I, U, V are positive on $[0, \hat{\tau})$, it follows from (4.3.1) that

$$\frac{dM}{dt} = \psi - \nu M \leq \psi \text{ and } N(t) = N_0,\ t \in [0, \hat{\tau}).$$

Thus $0 < U + V < \psi t + M_0$ and $S + I = N_0$ for $t \in [0, \hat{\tau})$. Hence the solution to (4.3.1) is bounded on $[0, \hat{\tau})$. In other words, it does not blow up on any finite interval of $\mathbb{R}_+$. Therefore, according to Theorem 3.2.1.3, the solution of (4.3.1) exists for all time.

Thus, for any initial condition in $\mathbb{R}^4_+$, the problem (4.3.1) possesses a unique and positive solution in $\mathbb{R}^4_+$. □

Since at any time t the total populations are equal to the respective sums of the numbers of susceptible and the infective individuals, we can simplify (4.3.1) to

$$\begin{cases} \frac{dI}{dt} = \frac{\beta}{M} V(N - I) - \alpha I, & I(0) = I_0, \\ \frac{dV}{dt} = \frac{\vartheta}{N} I(M - V) - \nu V, & V(0) = V_0, \end{cases} \tag{4.3.2}$$

with $S = N - I$, $U = M - V$. Furthermore, since the vector life cycle is faster than that of the host, the scaling of (4.3.2) leads to the following singularly perturbed model

$$\begin{cases} \frac{dI}{dt} = \frac{\beta}{M} V(N - I) - \alpha I, & I(0) = I_0, \\ \frac{1}{365}\frac{dV}{dt} = \frac{\tilde{\vartheta}}{N} I(M - V) - \tilde{\nu} V, & V(0) = V_0, \end{cases} \tag{4.3.3}$$

with $\vartheta = 365\tilde{\vartheta}$ and $\nu = 365\tilde{\nu}$.

As before, we introduce a small parameter ϵ and consider the following class of equations

$$\begin{cases} \frac{dI}{dt} = \frac{\beta}{M} V(N - I) - \alpha I, & I(0) = I_0, \\ \epsilon\frac{dV}{dt} = \frac{\tilde{\vartheta}}{N} I(M - V) - \tilde{\nu} V, & V(0) = V_0. \end{cases} \tag{4.3.4}$$

The problem (4.3.3) can be obtained from (4.3.4) with $\epsilon = \frac{1}{365}$.

The degenerate equation is $\frac{\tilde{\vartheta}}{N} I(M - \bar{V}) - \tilde{\nu}\bar{V} = 0$. It follows that its solution is unique (therefore isolated) and is given by

$$\bar{V} = \frac{\frac{\tilde{\vartheta}}{N} M I}{\frac{\tilde{\vartheta}}{N} I + \tilde{\nu}}.$$

The auxiliary equation is

$$\frac{d\hat{V}}{d\tau} = \hat{g}(\hat{V}, I) = \frac{\tilde{\vartheta}}{N} I(M - \hat{V}) - \tilde{\nu}\hat{V}, \qquad \hat{V}(0) = V_0,$$

where I and N are parameters. It is easy to notice that $\hat{V} = \bar{V}(I)$ is an equilibrium of the auxiliary equation. Differentiating the right hand side of the auxiliary equation with respect to $\hat{V}$, we obtain

$$\hat{g}_{\hat{V}}(\hat{V}, I) = -\frac{\tilde{\vartheta}}{N} I - \tilde{\nu}.$$

Since I is non-negative, it follows that $\hat{g}_{\hat{V}} < 0$. Therefore, $\hat{V} = \bar{V}(I)$ is asymptotically stable uniformly on $\mathbf{M} = \{I, I \geq 0\}$.

Thus, according to the Tikhonov theorem, the solution of (4.3.4) satisfies

$$\begin{aligned} &\lim_{\epsilon \to 0} I(t, \epsilon) = \bar{I}(t), && t \in [0, T], \\ &\lim_{\epsilon \to 0} V(t, \epsilon) = \bar{V}(t), && t \in (0, T], \end{aligned}$$

where $(\bar{I}, \bar{V})$ is the solution to the degenerate system

$$\begin{cases} \frac{d\bar{I}}{dt} = \frac{\beta}{M}(N - \bar{I}) \frac{\frac{\tilde{\vartheta}}{N} M \bar{I}}{\frac{\tilde{\vartheta}}{N}\bar{I} + \tilde{\nu}} - \alpha \bar{I}, & \bar{I}(0) = I_0, \\ \bar{V} = \frac{\frac{\tilde{\vartheta}}{N} M \bar{I}}{\frac{\tilde{\vartheta}}{N}\bar{I} + \tilde{\nu}}, & t \in \mathbb{I}_T. \end{cases} \tag{4.3.5}$$

4.3.2 Numerical Simulations

Figure 4.4 shows the graphs of the orbits $V(I)$ and $\bar{V}(\bar{I})$, respectively, for $\epsilon = 0.02,\ 0.005$ and $\epsilon = 0$. We considered the parameters, taken from [60], $\alpha = \frac{1}{10},\ \nu = \frac{1}{10},\ \tilde{\nu} = \alpha,\ \psi = \nu M,\ \vartheta = 2\nu,\ M = 10N,\ \beta = 2\alpha,\ \tilde{\vartheta} = \frac{\vartheta}{\nu}\alpha$ and the initial condition is $(I_0, V_0) = (100, 2500)$. The time unit is a year. It can be observed that V tends to $\bar{V}$ as ϵ tends to zero. Therefore, the quasi steady state is a good approximation of the solution of (4.3.4) when ϵ is small enough, on a finite interval $[0, T]$.

4.4 Three Dimensional Case: a Simple Model for Malaria

Malaria is among the most lethal and prevalent human infections worldwide. The *Plasmodium falciparum* is the parasite that causes the disease. It is transmitted by the female

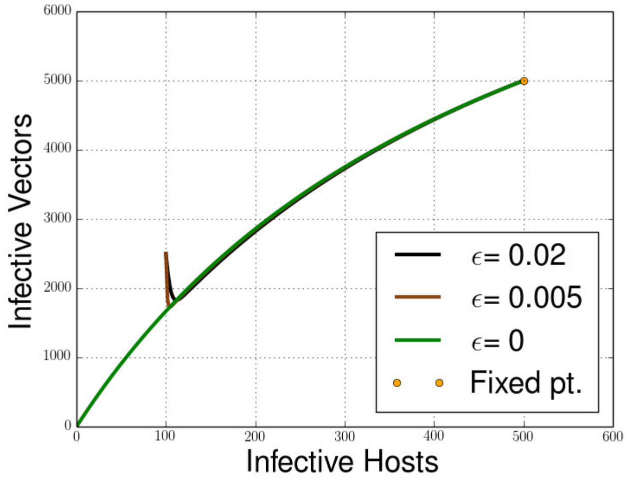

Figure 4.4: This figure presents the graph of $\bar{V}(\bar{I})$ represented by the line $\epsilon = 0$ and the orbits $(I(t,\epsilon), V(t,\epsilon))$ for $\epsilon = 0.02, 0.005$. It can be noticed that as ϵ tends to zero, V tends to $\bar{V}$.

vector of the *genus anopheles* mosquitoes who can get infected when feeding on an infected blood. This disease may be fatal. According to WHO, in 2015 almost half the global population lived in the areas, where the risk of getting an infection of malaria is high [30, 37]. Developing countries have a higher incidence of malaria [30, 72]. In particular, tropical and sub-tropical regions of the world each year have approximately 300-600 million cases of clinical malaria, of which 1-2 million result in death [30, 37], with 90% of malaria-associated deaths occurring in Africa.

In this study we look at a simplified model of malaria given by:

$$\begin{cases} \frac{dS_h}{dt} = \Psi_h N_h + \rho_h R_h - \lambda_h(t) S_h - \mu_{1h} S_h, \\ \frac{dI_h}{dt} = \lambda_h(t) S_h - \gamma_h I_h - \mu_{1h} I_h - \mu_{2h} I_h, \\ \frac{dR_h}{dt} = \gamma_h I_h - \rho_h R_h - \mu_{1h} R_h, \\ \frac{dS_v}{dt} = \Psi_v N_v - \lambda_v(t) S_v - \mu_{1v} S_v, \\ \frac{dI_v}{dt} = \lambda_v(t) S_v - \mu_{1v} I_v, \end{cases} \tag{4.4.1}$$

with initial condition $(S_h(0), I_h(0), R_h(0), S_v(0), I_v(0)) = (S_h^0, I_h^0, R_h^0, S_v^0, I_v^0)$ and where S_h, I_h and R_h denote, respectively, the susceptible, infective, and recovered class for the host. The variables S_v and I_v are, respectively, the numbers of susceptibles and infectives for the vector. The total population for the host (resp. vector) is $N_h = S_h + I_h + R_h$ (resp. $N_v = S_v + I_v$) and the birth rate for hosts (resp. vectors) is Ψ_h (resp. Ψ_v). The parameter γ_h is the recovery rate of the host from the disease and ρ_h is the rate of loss of immunity for hosts. The natural death rate in the host (resp. vector) population is μ_{1h}

(resp. μ_{1v}), the disease induced death rate is μ_{2h} in the host population and the force of infection of hosts (resp. vectors) is λ_h (resp. λ_v). According to Chitnis, [23], the force of infection of hosts (resp. vectors) is the product of the probability that a vector (resp. a host) is infectious, β_{vh} (resp. β_{hv}), the probability of transmission of the disease from a vector to a host (resp. from a host to a vector) and the number of vectors' bite a host can receive per unit of time, σ_h (resp. the number of bites a vector can make per unit of time, σ_v). The mathematical formulae are

$$\lambda_h = \ell(\sigma_v, \sigma_h)\beta_{hv}\frac{I_v}{N_h}$$

and

$$\lambda_v = \ell(\sigma_v, \sigma_h)\beta_{vh}\frac{I_h}{N_h},$$

where $\ell(\sigma_v, \sigma_h) = \frac{\sigma_v\sigma_h}{\sigma_v(N_v/N_h)+\sigma_h}$ represents the number of vector bites on the hosts, [23]. In particular, assuming the case of a very large host population and a very small vector population, we have the following approximations of the forces of infection, [23],

$$\lambda_h \approx \sigma_v\beta_{hv}\frac{I_v}{N_h} \text{ and } \lambda_v \approx \sigma_v\beta_{vh}\frac{I_h}{N_h}.$$

Substituting these values into (4.4.1) and assuming a stable vector population, that is, $\Psi_v = \mu_{1v}$, we obtain

$$\begin{cases} \frac{dS_h}{dt} = & \Psi_h N_h + \rho_h R_h - \sigma_v\beta_{hv}\frac{I_v S_h}{N_h} - \mu_{1h}S_h, \\ \frac{dI_h}{dt} = & \sigma_v\beta_{hv}\frac{I_v S_h}{N_h} - \gamma_h I_h - \mu_{1h}I_h - \mu_{2h}I_h, \\ \frac{dR_h}{dt} = & \gamma_h I_h - \rho_h R_h - \mu_{1h}R_h, \\ \frac{dS_v}{dt} = & \mu_{1v}N_v - \sigma_v\beta_{vh}\frac{I_h S_v}{N_h} - \mu_{1v}S_v, \\ \frac{dI_v}{dt} = & \sigma_v\beta_{vh}\frac{I_h S_v}{N_h} - \mu_{1v}I_v, \end{cases} \Rightarrow \begin{cases} \frac{dS_h}{dt} = & (\Psi_h - \mu_{1h})S_h + \Psi_h I_h + \\ & (\Psi_h + \rho_h)R_h - \sigma_v\beta_{hv}\frac{I_v S_h}{N_h}, \\ \frac{dI_h}{dt} = & \sigma_v\beta_{hv}\frac{I_v S_h}{N_h} - (\gamma_h + \mu_{1h} + \mu_{2h})I_h, \\ \frac{dR_h}{dt} = & \gamma_h I_h - (\rho_h + \mu_{1h})R_h, \\ \frac{dS_v}{dt} = & \mu_{1v}I_v - \sigma_v\beta_{vh}\frac{I_h S_v}{N_h}, \\ \frac{dI_v}{dt} = & -\mu_{1v}I_v + \sigma_v\beta_{vh}\frac{I_h S_v}{N_h}. \end{cases} \tag{4.4.2}$$

4.4.1 Analysis of the Model

4.4.1.1 Theorem. *Let us assume that $\Psi_h - \mu_{1h} - \mu_{2h} \geq 0$. For a positive initial condition $(S_h^0, I_h^0, R_h^0, S_v^0, I_v^0)$ the solution to (4.4.1) exists, is unique and positive on $\mathbb{R}_+$.*

Proof. The poof of the existence and the uniqueness of the solution to (4.4.1) is similar to that of Theorem 4.3.1.1. In order to prove that the solution exists globally, let us assume $\Psi_h - \mu_{1h} - \mu_{2h} \geq 0$. From (4.4.1) we have

$$\frac{dN_h}{dt} = (\Psi_h - \mu_{1h})N_h - \mu_{2h}I_h.$$

It follows that

$$(\Psi_h - \mu_{1h})N_h \geq \frac{dN_h}{dt} \geq (\Psi_h - \mu_{1h} - \mu_{2h})N_h. \tag{4.4.3}$$

From solving (4.4.3), we obtain

$$N_h^0 e^{(\Psi_h - \mu_{1h})t} \geq N_h(t) \geq N_h^0 e^{(\Psi_h - \mu_{1h} - \mu_{2h})t} \geq N_h^0, \quad t \geq 0.$$

Thus for $N_h^0 > 0$, $N_h(t) > 0$ and is bounded for all $t \in \mathbb{R}_+$. Therefore the solution exists globally. □

Substituting the parameter values shown in Table 4.4.1,[23], into system (4.4.2) we obtain

$$\begin{cases} \frac{dS_h}{dt} & = 1.2609 \times 10^{-2} S_h + 2.7981 \times 10^{-2} I_h + \\ & \quad 5.3569 \times 10^0 R_h - 4.3800 \times 10^0 \frac{I_v S_h}{N_h}, \\ \frac{dI_h}{dt} & = 4.3800 \times 10^0 \frac{I_v S_h}{N_h} - 1.3674 \times 10^0 I_h, \\ \frac{dR_h}{dt} & = 1.3520 \times 10^0 I_h - 5.3443 \times 10^0 R_h, \\ \frac{dS_v}{dt} & = 5.2159 \times 10^1 I_v - 1.8250 \times 10^2 \frac{I_h S_v}{N_h}, \\ \frac{dI_v}{dt} & = -5.2159 \times 10^1 I_v + 1.8250 \times 10^2 \frac{I_h S_v}{N_h}, \end{cases} \tag{4.4.4}$$

with $(S_h(0), I_h(0), R_h(0), S_v(0), I_v(0), R_v(0)) = (S_h^0, I_h^0, R_h^0, S_v^0, I_v^0, R_v^0)$. Considering the initial condition $S_h^0 = 1000.0,\ R_h^0 = 0,\ I_h^0 = 40.0,\ S_v^0 = 100.0,\ E_v^0 = 0,\ I_v^0 = 30.0$, [23], the numerical result of (4.4.4) is shown in Figure 4.5.

To ease the understanding of the model, we study the range of the coefficient values of the model (4.4.4) written in scientific notation. It can be observed that the coefficients are between factors of 10^{-2} and factors of 10^0 for the host and between factors of 10^1 and factors 10^2 for the vector. This significant difference in the ranges between the coefficients of the host and the vector reflects the fact that the vector population has a fast life cycle in comparison with the life cycle of the host. Let us merge these ranges by making sure that they have the lower bound of order 10^{-2}. The model (4.4.4) can be rewritten in the

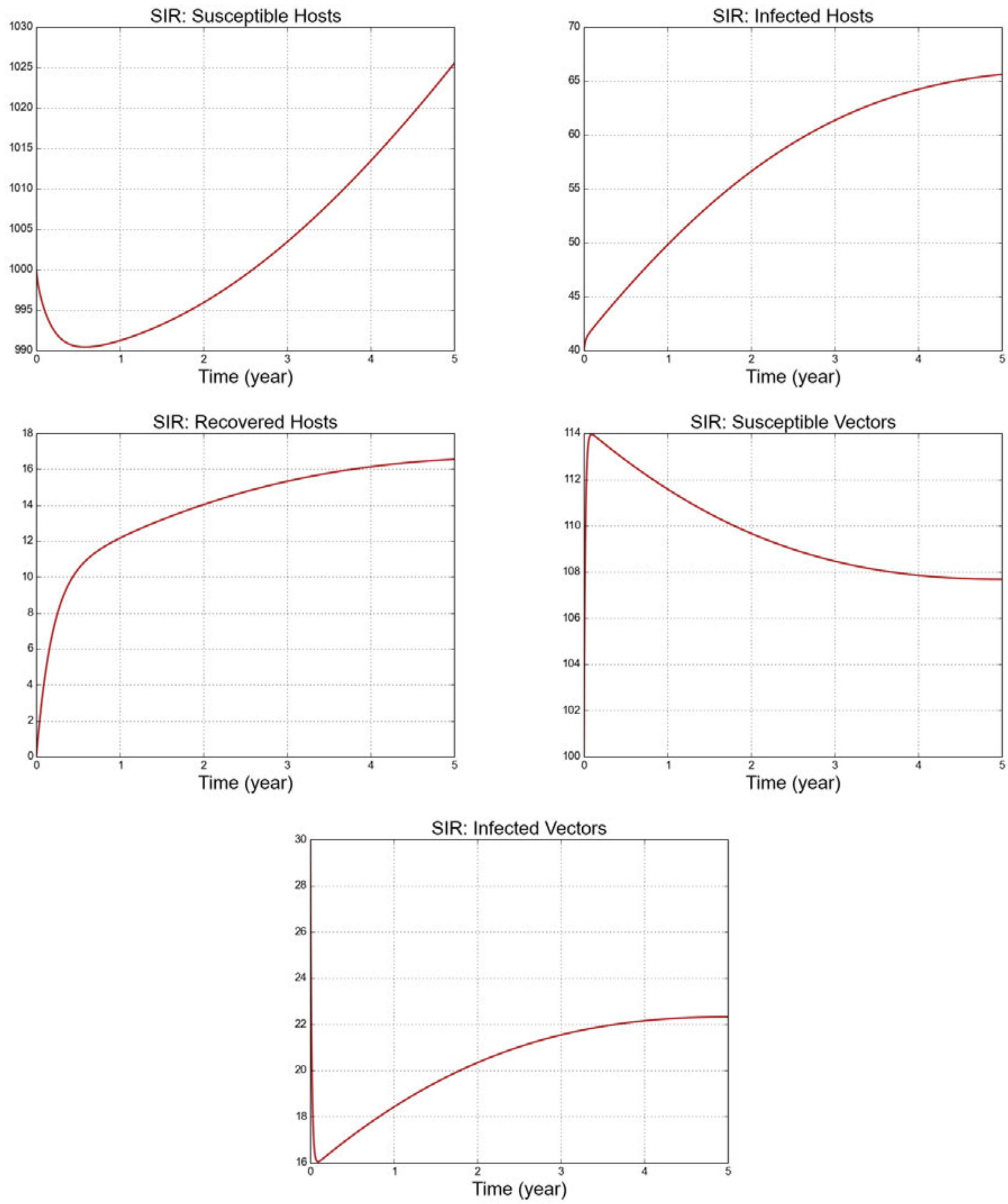

Figure 4.5: Solution to system (4.4.5).

Table 4.1: Parameter values

Parameter	Values per day	Values per year
Ψ_h	7.666×10^{-5}	2.7981×10^{-2}
μ_{1h}	4.212×10^{-5}	1.5372×10^{-2}
ρ_h	1.460×10^{-2}	5.3290×10^{0}
σ_v	0.6	2.1900×10^{2}
γ_h	3.704×10^{-3}	1.3520×10^{0}
μ_{2h}	10^{-7}	3.6500×10^{-5}
μ_{1v}	0.1429	5.2159×10^{1}
	Dimensionless Parameters	
β_{vh}	8.3333×10^{-1}	
β_{hv}	2.0000×10^{-2}	

form

$$\begin{cases} \frac{dS_h}{dt} & = 1.2609 \times 10^{-2} S_h + 2.7981 \times 10^{-2} I_h + \\ & \quad 5.3569 \times 10^{0} R_h - 4.3800 \times 10^{0} \frac{I_v S_h}{N_h}, \\ \frac{dI_h}{dt} & = 4.3800 \times 10^{0} \frac{I_v \bar{S}_h}{N_h} - 1.3674 \times 10^{0} I_h, \\ \frac{dR_h}{dt} & = 1.3520 \times 10^{0} I_h - 5.3443 \times 10^{0} R_h, \\ 10^{-3}\frac{dS_v}{dt} & = 5.2159 \times 10^{-2} I_v - 1.8250 \times 10^{-1} \frac{I_h S_v}{N_h}, \\ 10^{-3}\frac{dI_v}{dt} & = -5.2159 \times 10^{-2} I_v + 1.8250 \times 10^{-1} \frac{I_h S_v}{N_h}, \end{cases} \tag{4.4.5}$$

with $(S_h(0), I_h(0), R_h(0), S_v(0), I_v(0), R_v(0)) = (S_h^0, I_h^0, R_h^0, S_v^0, I_v^0, R_v^0)$. As before we introduce a small parameter ϵ and consider the following class of equations

$$\begin{cases} \frac{dS_h}{dt} & = 1.2609 \times 10^{-2} S_h + 2.7981 \times 10^{-2} I_h + \\ & \quad 5.3569 \times 10^{0} R_h - 4.3800 \times 10^{0} \frac{I_v S_h}{N_h}, \\ \frac{dI_h}{dt} & = 4.3800 \times 10^{0} \frac{I_v \bar{S}_h}{N_h} - 1.3674 \times 10^{0} I_h, \\ \frac{dR_h}{dt} & = 1.3520 \times 10^{0} I_h - 5.3443 \times 10^{0} R_h, \\ \epsilon\frac{dS_v}{dt} & = 5.2159 \times 10^{-2} I_v - 1.8250 \times 10^{-1} \frac{I_h S_v}{N_h}, \\ \epsilon\frac{dI_v}{dt} & = -5.2159 \times 10^{-2} I_v + 1.8250 \times 10^{-1} \frac{I_h S_v}{N_h}, \end{cases} \tag{4.4.6}$$

with $(S_h(0), I_h(0), R_h(0), S_v(0), I_v(0), R_v(0)) = (S_h^0, I_h^0, R_h^0, S_v^0, I_v^0, R_v^0)$. The system (4.4.5) is obtained for $\epsilon = 10^{-3}$ in (4.4.6). Since the total vector population is assumed to be stable, we have $I_v = N_v^0 - S_v$. Thus,

$$\begin{cases} \frac{dS_h}{dt} &= 1.2609 \times 10^{-2} S_h + 2.7981 \times 10^{-2} I_h + \\ & \quad 5.3569 \times 10^0 R_h - 4.3800 \times 10^0 \frac{(N_v^0 - S_v) S_h}{N_h}, \\ \frac{dI_h}{dt} &= 4.3800 \times 10^0 \frac{(N_v^0 - S_v) S_h}{N_h} - 1.3674 \times 10^0 I_h, \\ \frac{dR_h}{dt} &= 1.3520 \times 10^0 I_h - 5.3443 \times 10^0 R_h, \\ \epsilon \frac{dS_v}{dt} &= 5.2159 \times 10^{-2} (N_v^0 - S_v) - 1.8250 \times 10^{-1} \frac{I_h S_v}{N_h}, \end{cases} \tag{4.4.7}$$

with $(S_h(0), I_h(0), R_h(0), S_v(0), I_v(0), R_v(0)) = (S_h^0, I_h^0, R_h^0, S_v^0, I_v^0, R_v^0)$. It follows that the degenerate system of equations corresponding to (4.4.7) is given by

$$\begin{cases} \frac{d\bar{S}_h}{dt} &= 1.2609 \times 10^{-2} \bar{S}_h + 2.7981 \times 10^{-2} \bar{I}_h + \\ & \quad 5.3569 \times 10^0 \bar{R}_h - 4.3800 \times 10^0 \frac{\bar{I}_v \bar{S}_h}{N_h}, \\ \frac{d\bar{I}_h}{dt} &= 4.3800 \times 10^0 \frac{\bar{I}_v \bar{S}_h}{N_h} - 1.3674 \times 10^0 \bar{I}_h, \\ \frac{d\bar{R}_h}{dt} &= 1.3520 \times 10^0 \bar{I}_h - 5.3443 \times 10^0 \bar{R}_h, \\ 0 &= 5.2159 \times 10^{-2} (N_v^0 - \bar{S}_v) - 1.8250 \times 10^{-1} \frac{\bar{I}_h \bar{S}_v}{\bar{N}_h}, \end{cases} \tag{4.4.8}$$

with the initial condition $(\bar{S}_h(0), \bar{I}_h(0), \bar{R}_h(0), \bar{S}_v(0), \bar{I}_v(0)) = (S_h^0, I_h^0, R_h^0, S_v^0, I_v^0)$.

The solution to the degenerate equation is unique, and therefore isolated, and it is given by

$$\bar{S}_v = \frac{5.2159 \times 10^{-2} N_v^0}{1.8250 \times 10^{-1} \frac{\bar{I}_h}{\bar{N}_h} + 5.2159 \times 10^{-2} \bar{N}_v^0}.$$

Furthermore, setting $\tau = \frac{t}{\epsilon}$ in (4.4.7), we obtain the auxiliary equation

$$\frac{d\hat{S}_v}{d\tau} = 5.2159 \times 10^{-2} (N_v^0 - \hat{S}_v) - 1.8250 \times 10^{-1} \frac{I_h \hat{S}_v}{N_h},$$

where N_h and I_h are taken as parameters. It is easy to see that $\hat{S}_v = \bar{S}_v$ is an equilibrium to the auxiliary equation. Let us denote $\hat{g} = 5.2159 \times 10^{-2} (N_v^0 - \hat{S}_v) - 1.8250 \times 10^{-1} \frac{I_h \hat{S}_v}{N_h}$. It follows that

$$\frac{\partial \hat{g}}{\partial \hat{S}_v} = -5.2159 - 1.8250 \times 10^{-1} \frac{I_h}{N_h} < 0,$$

for $I_h, N_h > 0$. Since $\Psi_h - \mu_{1h} - \mu_{2h} \geq 0$, according to Theorem 4.4.1.1 we have $N_h(t) > 0$ for all $t \in \bar{\mathbb{I}}_T$. Therefore, $\hat{S}_v = \bar{S}_v(S_h, I_h, R_h)$ is uniformly asymptotically stable on $\bar{\mathbb{M}} = \{(S_h, I_h, R_h) \in \mathbb{R}^3_+, S_h + I_h + R_h \geq N_0 > 0\}$. Hence, according to the Tikhonov theorem, the solution to problem (4.4.6) satisfies

$$\begin{aligned}
&\lim_{\epsilon \to 0} S_h(t, \epsilon) = \bar{S}_h, && t \in [0, T],\\
&\lim_{\epsilon \to 0} I_h(t, \epsilon) = \bar{I}_h, && t \in [0, T],\\
&\lim_{\epsilon \to 0} R_h(t, \epsilon) = \bar{R}_h, && t \in [0, T],\\
&\lim_{\epsilon \to 0} S_v(t, \epsilon) = \bar{S}_v, && t \in (0, T],\\
&\lim_{\epsilon \to 0} I_v(t, \epsilon) = N_v^0 - \bar{S}_v, && t \in (0, T].
\end{aligned}$$

4.4.2 Numerical simulations

With the same initial conditions as on Figure 4.5, Figure 4.6 shows the graphs of the solutions to (4.4.5), (4.4.6) and (4.4.8). So, the solution to (4.4.5) is represented by the graph corresponding to $\epsilon = 10^{-3}$, the solution to (4.4.6) by the graphs with $\epsilon = 5 \times 10^{-4}$ and $\epsilon = 5 \times 10^{-3}$, and the solution to (4.4.8) by the graph $\epsilon = 0$. We can observe that as ϵ tends to zero, the solution of (4.4.6) tends to the solution to the degenerate system (4.4.8), agreeing with Tikhonov's approximation.

4.5 Conclusion

We observed that the Tikhonov theorem provides an easy way to approximate systems of differential equations. The two main assumptions are that the solution to the degenerate equation should be unique (or isolated) and uniformly asymptotically stable on a well defined domain. However, as in the case of influenza, we found that these conditions are not always satisfied. In the next chapters we will study the case where the solutions to the degenerate equation are not isolated but intersect each other and, at their intersection, the QSSs switch stability and the solution to the original problem exists.

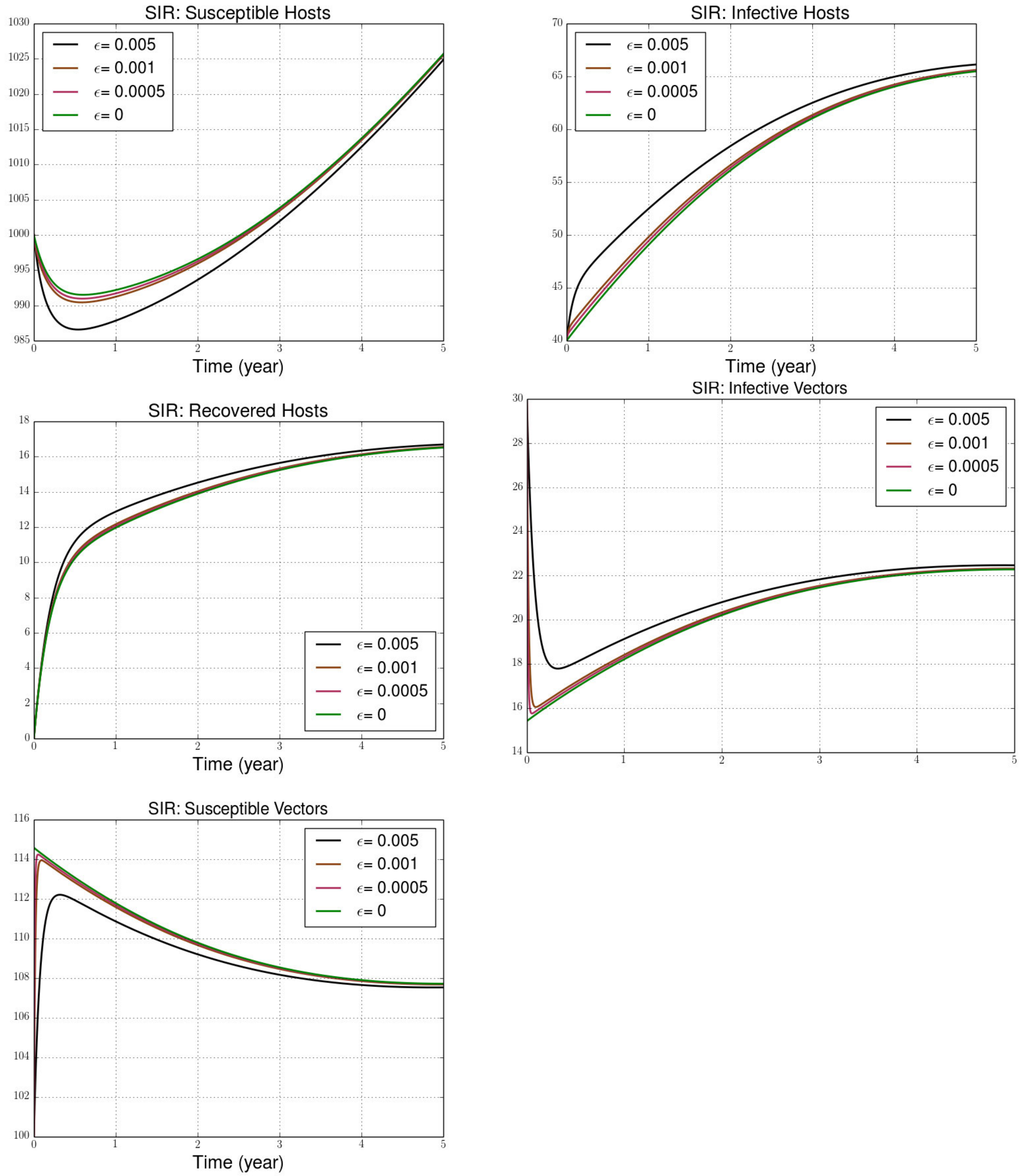

Figure 4.6: The solution to (4.4.5) is represented by the curve labelled $\epsilon = 10^{-3}$, the solution to (4.4.6) is represented by the graph $\epsilon = 5 \times 10^{-4},\ 5 \times 10^{-3}$ and the solution to (4.4.8) by $\epsilon = 0$. It is easy to see that as ϵ tends to 0, the solutions of (4.4.6) tend to the solution of (4.4.8).

5 Stability Switch in a One Dimensional Case Study: Influenza

In this chapter, we aim to go further in our analysis by considering the case where the solution passes through the intersection point of the quasi steady states. Even though the Butuzov theorem dealt with one dimensional problems of this type, we found it instructive to provide a direct study of the one dimensional SIS model. It is important to mention that this study has already been done in [57]. However, we decided to recall this result here because not only it constitutes the foundation of the extension of the method used in the Butuzov theorem to the two dimensional SIS model in the next chapter, but also details of the proof are needed and will be referred to later on.

5.1 Preliminary Results

Let us consider problem (4.1.1)

$$\begin{cases} \frac{dS}{dt} = \beta N - \mu S - \frac{1}{\epsilon}(\lambda IS - \gamma I), & S(0) = S_0, \\ \frac{dI}{dt} = -\mu I + \frac{1}{\epsilon}(\lambda IS - \gamma I), & I(0) = I_0, \end{cases} \tag{5.1.1}$$

where, as already mentioned, S and I describe, respectively, the size of the susceptible and the infective populations, λ is the transmission rate of the disease from an infected to a susceptible, γ is the recovery rate from the disease, μ is the death rate and N is the total size of the population ($N = S + I$). The parameter $\epsilon > 0$ is a small.

From (5.1.1) it follows that the dynamics of the total population is given by

$$\frac{dN}{dt} = rN,$$

where $r = \beta - \mu$ and the initial condition is $N(0) = N_0$. Therefore

$$N(t) = N_0 e^{rt},\ t \in [0, T].$$

Thus the problem (5.1.1) becomes:

$$\begin{cases} \frac{dI}{dt} = -\mu I + \frac{1}{\epsilon}(\lambda I(N - I) - \gamma I), \\ N = N_0 e^{rt} \text{ and } I(0) = I_0. \end{cases}$$

This implies,

$$\frac{dI}{dt} = -\mu I + \frac{1}{\epsilon}(\lambda I(N_0 e^{rt} - I) - \gamma I),\ I(0) = I_0. \quad (5.1.2)$$

1. **Solution of equation** (5.1.2)**.** Equation (5.1.2) can be written as

$$\frac{dI}{dt} = -\frac{\lambda I^2}{\epsilon} + I\Big(\frac{1}{\epsilon}(\lambda N_0 e^{rt} - \gamma) - \mu\Big).$$

Let us denote

$$\phi_\epsilon(t) = \frac{1}{\epsilon}(\lambda N_0 e^{rt} - \gamma) - \mu. \quad (5.1.3)$$

Then (5.1.2) becomes

$$\frac{dI}{dt} = -\frac{\lambda}{\epsilon} I^2 + \phi_\epsilon(t) I. \quad (5.1.4)$$

This is a Bernoulli equation. Let us change the variable I by setting

$$I = \frac{1}{z}.$$

This implies that

$$dI = -\frac{dz}{z^2}.$$

It follows from (5.1.4) that

$$\frac{dz}{dt} + \phi_\epsilon(t) z = \frac{\lambda}{\epsilon}, \quad (5.1.5)$$

with initial condition $z(0) = z_0 > 0$. This equation is a linear first order equation. Upon using $e^{\int_0^t \phi_\epsilon(s)ds}$ as an integrating factor, the equation (5.1.5) becomes

$$\frac{d}{dt}(z e^{\int_0^t \phi_\epsilon(s)ds}) = \frac{\lambda}{\epsilon} e^{\int_0^t \phi_\epsilon(s)ds}.$$

Integrating both sides, we get

$$z e^{\int_0^t \phi_\epsilon(s)ds} - z_0 = \frac{\lambda}{\epsilon} \int_0^t e^{\int_0^s \phi_\epsilon(x)dx} ds,$$

with $z_0 = I_0^{-1}$, and hence

$$z(t) = \left[z_0 + \frac{\lambda}{\epsilon} \int_0^t e^{\int_0^s \phi_\epsilon(x)dx} ds \right] e^{-\int_0^t \phi_\epsilon(s)ds}. \quad (5.1.6)$$

However, from (5.1.3) we have

$$\int_0^t \phi_\epsilon(t) = \frac{1}{\epsilon}\int_0^t (\lambda N_0 e^{rs} - \gamma)ds - \mu t = \frac{1}{\epsilon}\Big[\frac{N_0\lambda}{r}(e^{rt}-1) - \gamma t\Big] - \mu t.$$

Hence equation (5.1.6) becomes

$$z(t) = \Big[z_0 + \frac{\lambda}{\epsilon}\int_0^t e^{\frac{1}{\epsilon}\Big[\frac{N_0\lambda}{r}(e^{rs}-1)-\gamma s\Big]-\mu s} ds\Big] e^{-\frac{1}{\epsilon}\Big[\frac{N_0\lambda}{r}(e^{rt}-1)-\gamma t\Big]-\mu t}.$$

Therefore, the solution to (5.1.2) is

$$I(t,\epsilon) = \frac{1}{z(t)} = \frac{e^{\frac{1}{\epsilon}\Big[\frac{N_0\lambda}{r}(e^{rt}-1)-\gamma t\Big]-\mu t}}{\frac{1}{I_0} + \frac{\lambda}{\epsilon}\int_0^t e^{\frac{1}{\epsilon}\Big[\frac{N_0\lambda}{r}(e^{rs}-1)-\gamma s\Big]-\mu s} ds}. \tag{5.1.7}$$

2. **Quasi Steady States.** Let us consider equation (5.1.2) at $\epsilon = 0$. We have

$$I\bigg(\lambda(N_0 e^{rt} - I) - \gamma\bigg) = 0.$$

It follows that there are two solutions:

$$I_1(t) = 0 \text{ and } I_2(t) = N_0 e^{rt} - \nu, \qquad t \in [0,T], \tag{5.1.8}$$

They will be called, respectively, the *first and the second quasi steady state.*

3. **Determination of time t_c at which the quasi steady states intersect.** The time t_c is determined by $I_2(t_c) = I_1(t_c)$; that is, $N_0 e^{rt_c} - \nu = 0$. Solving, we obtain

$$t_c = \frac{1}{r}\log\Big(\frac{\nu}{N_0}\Big). \tag{5.1.9}$$

It follows that to ensure the time at which the intersection occurs is positive, we should have either $r > 0$ and $N_0 < \nu$ or $r < 0$ and $N_0 > \nu$. The former case corresponds to the unstable population and the latter corresponds to the stable population.

5.2 The Case of an Unstable Population

In this section, we consider the case of an unstable population. We will assume that the birth rate is bigger than the death rate and the initial total population is smaller than the threshold ν.

5.2.1 Characterisation of the Function G.

Consider again equation (5.1.2) and let $g(t, I, \epsilon) = I\Big(-\mu\epsilon - \gamma + \lambda(N_0 e^{rt} - I)\Big)$. We have

$$g_I(t, I, \epsilon) = -\mu\epsilon - \gamma + \lambda N_0 e^{rt} - 2\lambda I.$$

This implies that,

$$g_I(t, 0, \epsilon) = -\mu\epsilon - \gamma + \lambda N_0 e^{rt}.$$

Therefore, the function G of Theorem 3.3.4.1 is given by

$$G(t, \epsilon) = \int_0^t (-\mu\epsilon - \gamma + \lambda N_0 e^{rs}) ds = (-\mu\epsilon - \gamma)t + \frac{\lambda N_0}{r}(e^{rt} - 1), \qquad t \in \mathbb{I}_T,$$

and at $\epsilon = 0$ we have

$$G(t, 0) = -\gamma t + \frac{\lambda N_0}{r}(e^{rt} - 1), \qquad t \in \mathbb{I}_T. \tag{5.2.1}$$

Let us now study the properties of the function G. The derivative of the function G is given by

$$G^{'}(t, 0) = \lambda N_0 e^{rt} - \gamma.$$

It follows that the solution to

$$G^{'}(t, 0) = 0$$

is

$$t_c = \frac{1}{r}\log(\frac{\gamma}{\lambda N_0}).$$

So, $\frac{dG}{dt}(t_c, 0) = 0$. Since $\frac{dG}{dt}(0, 0) = \lambda N_0 - \gamma < 0$, we have $G^{'}(t, 0) < 0$ for $t < t_c$, and $G^{'}(t, 0) > 0$ for $t > t_c$. Consequently, the function $G(t, 0)$ is decreasing between 0 and t_c and increasing for $t > t_c$. In particular,

$$G(t_c, 0) < G(0, 0);$$

that is

$$G(t_c, 0) < 0. \tag{5.2.2}$$

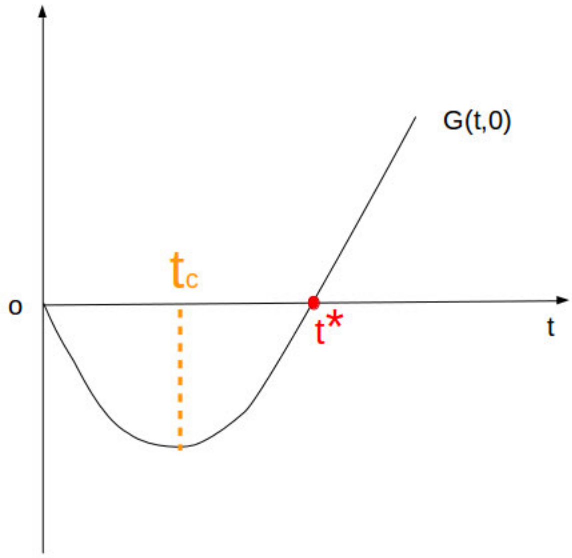

Figure 5.1: This figure illustrates the characteristics of the function G for an unstable population.

On the other hand, let us consider the limit of $G(t,0)$ at infinity. We have,

$$\begin{aligned}\lim_{t\to+\infty} G(t,0) &= \lim_{t\to+\infty} t\Big(-\gamma + \frac{\lambda N_0}{rt}(e^{rt}-1)\Big)\\ &= \lim_{t\to+\infty} t\Big(-\gamma + \frac{\lambda N_0}{r}(\frac{e^{rt}}{t} - \frac{1}{t})\Big).\end{aligned}$$

Since

$$\lim_{t\to+\infty} \frac{e^{rt}}{t} = +\infty \text{ and } \lim_{t\to+\infty} \frac{1}{t} = 0,$$

it follows that

$$\lim_{t\to+\infty} -\gamma + \frac{\lambda N_0}{rt}(e^{rt}-1) = \infty.$$

Thus

$$\lim_{t\to+\infty} G(t,0) = +\infty. \tag{5.2.3}$$

From equations (5.2.3) and (5.2.2), it follows, using the intermediate value theorem, that there exists a $t^* \in (t_c, +\infty)$ such that $G(t^*,0) = 0$. Since $G(.,0)$ is monotone on $(t_c, T]$, it follows that t^* is unique. Figure 5.1 shows the graph of a function G.

5.2.2 Dynamics of the Infected Population before t^*

From the definitions of the functions I and G given, respectively, by the equations (5.1.7) and (5.2.1), it follows that

$$0 \leq \lim_{\epsilon \to 0} I(t,\epsilon) = \lim_{\epsilon \to 0} \frac{I_0 e^{\frac{1}{\epsilon}G(t,0)-\mu t}}{1 + \frac{\lambda I_0}{\epsilon}\int_0^t e^{\frac{1}{\epsilon}G(s,0)-\mu s} ds} \leq \lim_{\epsilon \to 0} I_0 e^{\frac{1}{\epsilon}G(t,0)-\mu t},$$

since the term $\frac{\lambda I_0}{\epsilon}\int_0^t e^{\frac{1}{\epsilon}G(s,0)-\mu s} ds$ is positive for all $t > 0$. So,

$$\lim_{\epsilon \to 0} I(t,\epsilon) = 0$$

since $G(t,0) < 0$ on $(0,t^*)$. Therefore, as ϵ is getting smaller, the solution tends to the first QSS up to t^*. However, we found in the previous section that t^* is bigger than the time at which the intersection occurs. Therefore there is a delay in the stability switch.

5.2.3 Dynamics of the Infected Population after t^*

After proving that the solution to equation (5.1.2) first converges to the first quasi steady state up to t^*, in this section we will determine the behaviour to the solution after t^*. The solution (5.1.7) can also be rewritten as

$$I(t,\epsilon) = \frac{I_0}{e^{-\frac{1}{\epsilon}G(t,0)+\mu t} + \frac{\lambda}{\epsilon} I_0 e^{-\frac{1}{\epsilon}G(t,0)+\mu t}\int_0^{t^*} e^{\frac{1}{\epsilon}G(s,0)-\mu s} ds + \frac{\lambda}{\epsilon} I_0 e^{-\frac{1}{\epsilon}G(t,0)+\mu t}\int_{t^*}^{t} e^{\frac{1}{\epsilon}G(s,0)-\mu s} ds}.$$

Let us find the limit of each term of the denominator of I.

Since $G(t,0) > 0$ for $t > t^*$,

$$\lim_{\epsilon \to 0} e^{-\frac{1}{\epsilon}G(t,0)} = 0. \tag{5.2.4}$$

Thus,

$$\lim_{\epsilon \to 0} e^{-\frac{1}{\epsilon}G(t,0)+\mu t} = 0, \qquad t \in (t^*, T].$$

On the other hand, $G(t,0) \leq 0$ for $t \in [0,t^*]$, therefore

$$e^{\frac{1}{\epsilon}G(t,0)-\mu t} \leq 1,\ \forall \epsilon > 0,\ 0 \leq t \leq t^*.$$

It follows that $\int_0^{t^*} e^{\frac{1}{\epsilon}G(s,0)-\mu(s-t)} ds$ is bounded. In other words, there exists $M > 0$ such that

$$\int_0^{t^*} e^{\frac{1}{\epsilon}G(s,0)-\mu(s-t)} ds \leq M.$$

Therefore, since $G(t,0) > 0$ for $t > t^*$,

$$0 \leq \lim_{\epsilon\to 0} \frac{\lambda}{\epsilon} I_0 e^{-\frac{1}{\epsilon}G(t,0)+\mu t} \int_0^{t^*} e^{\frac{1}{\epsilon}G(s,0)-\mu s} ds \leq \lim_{\epsilon\to 0} \frac{\lambda}{\epsilon} I_0 M e^{-\frac{1}{\epsilon}G(t,0)} = 0,\ t > t^*.$$

Hence, it follows that

$$\lim_{\epsilon\to 0} \frac{\lambda}{\epsilon} I_0 e^{-\frac{1}{\epsilon}G(t,0)+\mu t} \int_0^{t^*} e^{\frac{1}{\epsilon}G(s,0)-\mu s} ds = 0,\ t > t^*. \tag{5.2.5}$$

Finally, let us determine

$$\lim_{\epsilon\to 0} \frac{\lambda}{\epsilon} I_0 e^{-\frac{1}{\epsilon}G(t,0)+\mu t} \int_{t^*}^{t} e^{\frac{1}{\epsilon}G(s,0)-\mu s} ds. \tag{5.2.6}$$

Notice that $G^{-1}(.,0)$ exists on (t^*,t) for $G(.,0)$ is a one-to-one function on that interval. Let us denote $z = G(s,0)$. Then $dz = G^{'}(s,0)ds$ and $s = G^{-1}(z,0)$. So, defining $a = G(t,0)$ and noticing that $G(t^*,0) = 0$, we have from (5.2.6),

$$\begin{aligned}
\lim_{\epsilon\to 0} \frac{\lambda}{\epsilon} I_0 e^{-\frac{1}{\epsilon}G(t,0)+\mu t} \int_{G(t^*,0)}^{G(t,0)} \frac{e^{\frac{1}{\epsilon}z} e^{-\mu G^{-1}(z,0)}}{G'\big(G^{-1}(z,0),0\big)} dz &= \lim_{\epsilon\to 0} \frac{\lambda}{\epsilon} I_0 e^{-\frac{a}{\epsilon}+\mu t} \int_0^a \frac{e^{\frac{1}{\epsilon}z} e^{-\mu G^{-1}(z,0)}}{G'\big(G^{-1}(z,0),0\big)} dz, \\
&= \lim_{\epsilon\to 0} \lambda I_0 e^{\mu t} \int_0^a R(z) \frac{e^{\frac{z-a}{\epsilon}}}{\epsilon} dz,
\end{aligned}$$

where $R(z) = \frac{e^{-\mu G^{-1}(z,0)}}{G'\Big(G^{-1}(z,0),0\Big)}$. Furthermore, let us define the function η_ϵ by

$$\eta_\epsilon(z) = \begin{cases} \frac{e^{\frac{z-a}{\epsilon}}}{\epsilon}, & 0 \leq z \leq a, \\ 0, & z > a \text{ and } z \leq 0. \end{cases}$$

It follows that $\lim_{\epsilon\to 0} \eta_\epsilon(z) = \infty$ if $z = a$ and $\lim_{\epsilon\to 0} \eta_\epsilon(z) = 0$ if $z \neq a$. Further,

$$\int_{-\infty}^{+\infty} \eta_\epsilon(z) dz = \int_0^a \frac{e^{\frac{z-a}{\epsilon}}}{\epsilon} dz = 1 - e^{-\frac{a}{\epsilon}}$$

which tends to 1 as ϵ tends to zero. Moreover for $0 \leq z < a$,

$$\lim_{\epsilon\to 0} \eta_\epsilon(z) = \lim_{\epsilon\to 0} \frac{e^{\frac{z-a}{\epsilon}}}{\epsilon} = 0.$$

Therefore, η_ϵ is a delta sequence, [57]. So,

$$\begin{aligned}
\lim_{\epsilon\to 0} \int_0^a R(z) \frac{e^{\frac{z-a}{\epsilon}}}{\epsilon} dz = \lim_{\epsilon\to 0} \int_{-\infty}^{+\infty} R(z) \frac{e^{\frac{z-a}{\epsilon}}}{\epsilon} dz = R(a) &= \frac{e^{-\mu G^{-1}(z,0)}}{G'\Big(G^{-1}(z,0),0\Big)}, \\
&= \frac{e^{-\mu t}}{G'(t,0)} = \frac{e^{-\mu t}}{\lambda N_0 e^{rt} - \gamma}.
\end{aligned}$$

Thus,

$$\lim_{\epsilon\to 0}\lambda I_0 e^{\mu t}\int_0^a R(z)\frac{e^{\frac{z-a}{\epsilon}}}{\epsilon}dz = \frac{\lambda I_0}{\lambda N_0 e^{rt}-\gamma}.$$

Therefore,

$$\lim_{\epsilon\to 0} I(t,\epsilon) = \frac{I_0}{\lambda\frac{I_0}{\lambda N_0 e^{rt}-\gamma}} = N_0 e^{rt} - \nu,\ t > t^*. \tag{5.2.7}$$

Hence, the solution converges to the second quasi steady state for $t > t^*$. In the next section we will prove that this convergence is almost uniform.

5.2.3.1 Proof that the Convergence to the Second Quasi Steady State is almost Uniform on any Interval $[\tilde{t}, T]$, $\tilde{t} > t^*$

According to the result of the previous section, in order to prove that the convergence of the solution to the second quasi steady state on $(t^*, T]$ is almost uniform, it is enough to show that the convergence to zero as ϵ tends to zero of

$$\int_0^a R(z)\frac{e^{\frac{z-a}{\epsilon}}}{\epsilon}dz - R(a)$$

where $a = G(t,0)$, is uniform for $t \in [\tilde{t}, T]$ with $\tilde{t} > \bar{t}^*$. This is equivalent to prove that for any $\delta > 0$ there exists $\epsilon_0 > 0$ such that for $\epsilon \in \mathbb{I}_{\epsilon_0}$, $a \in [\tilde{a}, A] = [G(\tilde{t},0), G(T,0)]$, we have

$$\Big|\int_0^a R(z)\frac{e^{\frac{z-a}{\epsilon}}}{\epsilon}dz - R(a)\Big| < \delta.$$

However,

$$\begin{aligned}\Big|\int_0^a R(z)\frac{e^{\frac{z-a}{\epsilon}}}{\epsilon}dz - R(a)\Big| &= \Big|\int_0^a R(z)\frac{e^{\frac{z-a}{\epsilon}}}{\epsilon}dz - R(a) - R(a)e^{-\frac{a}{\epsilon}} + R(a)e^{-\frac{a}{\epsilon}}\Big| \\ &= \Big|\int_0^a R(z)\frac{e^{\frac{z-a}{\epsilon}}}{\epsilon}dz \\ &\quad + R(a)(1-e^{-\frac{a}{\epsilon}}) - e^{-\frac{a}{\epsilon}}R(a)\Big| \\ &= \Big|\int_0^a (R(z)-R(a))\frac{e^{\frac{z-a}{\epsilon}}}{\epsilon}dz - e^{-\frac{a}{\epsilon}}R(a)\Big| \\ &= \Big|\int_0^{b_\epsilon} (R(z)-R(a))\frac{e^{\frac{z-a}{\epsilon}}}{\epsilon}dz \\ &\quad + \int_{b_\epsilon}^a (R(z)-R(a))\frac{e^{\frac{z-a}{\epsilon}}}{\epsilon}dz - e^{-\frac{a}{\epsilon}}R(a)\Big|,\end{aligned}$$

where $b_\epsilon \in [\tilde{a}, A]$ and $\lim_{\epsilon\to 0} b_\epsilon = a$. Let us denote $M = \sup_{z\in[0,b_\epsilon]} |R(z) - R(a)|$.

It follows that

$$\begin{aligned}
\Big|\int_0^a R(z)\frac{e^{\frac{z-a}{\epsilon}}}{\epsilon}dz - R(a)\Big| &\leq \Big| M\int_0^{b_\epsilon} \frac{e^{\frac{z-a}{\epsilon}}}{\epsilon}dz \\
&+ \sup_{z\in[b_\epsilon,a]} |R(z)-R(a)| \int_{b_\epsilon}^a \frac{e^{\frac{z-a}{\epsilon}}}{\epsilon}dz - e^{-\frac{a}{\epsilon}}R(a)\Big| \\
&\leq \Big| M(e^{\frac{-a+b_\epsilon}{\epsilon}} - e^{\frac{-a}{\epsilon}}) \\
&+ \sup_{z\in[b_\epsilon,a]} |R(z)-R(a)|(1-e^{-\frac{a}{\epsilon}}) - e^{-\frac{a}{\epsilon}}R(a)\Big| \\
&\leq \Big| M(e^{\frac{-a+b_\epsilon}{\epsilon}} - e^{\frac{-a}{\epsilon}}) \\
&+ \sup_{z\in[b_\epsilon,a]} |R(z)-R(a)| - e^{-\frac{a}{\epsilon}}R(a)\Big| \\
&\leq M\Big| e^{\frac{-a+b_\epsilon}{\epsilon}} - e^{\frac{-a}{\epsilon}}\Big| \\
&+ \sup_{z\in[b_\epsilon,a]} |R(z)-R(a)| + \Big|e^{-\frac{a}{\epsilon}}R(a)\Big|,
\end{aligned}$$

since $1 - e^{-\frac{a}{\epsilon}} \leq 1$. Let us show that $e^{-\frac{a}{\epsilon}}R(a)$ converges uniformly to zero as ϵ tends to zero for $a \in [\tilde{a}, A]$. For $a \in [\tilde{a}, A]$,

$$e^{-\frac{a}{\epsilon}} \leq e^{-\frac{\tilde{a}}{\epsilon}} \Rightarrow |e^{-\frac{a}{\epsilon}}R(a)| \leq |e^{-\frac{\tilde{a}}{\epsilon}}R(a)|.$$

Since R is continuous on $[\tilde{a}, A]$ there exists $P > 0$ such that $|R(z)| \leq P$. Thus for $a \in [\tilde{a}, A]$ we have $|e^{-\frac{a}{\epsilon}}R(a)| \leq |e^{-\frac{\tilde{a}}{\epsilon}}R(a)| \leq Pe^{-\frac{\tilde{a}}{\epsilon}} \to 0$ uniformly as ϵ tends to zero. Finally, since R is a continuous function defined on a compact interval $[\tilde{a}, a]$, then R is uniformly continuous on that interval. In other words, for any $\epsilon > 0$, there exists $\delta > 0$ such that for all $z_1, z_2 \in [\tilde{a}, A]$,

$$|z_1 - z_2| < \delta \Rightarrow |R(z_1) - R(z_2)| < \epsilon.$$

Thus $\sup_{z\in[b_\epsilon,a]} |R(z) - R(a)|$ tends to zero uniformly as ϵ tends to 0. On the other hand, for $e^{\frac{-a+b_\epsilon}{\epsilon}} - e^{\frac{-a}{\epsilon}}$ to converge uniformly to zero, it is enough to take $b_\epsilon = a - \sqrt{\epsilon}$. In fact, setting $b_\epsilon = a - \sqrt{\epsilon}$, for any $\delta > 0$ and $0 < \epsilon < \min\left(\frac{1}{ln^2(\frac{2}{\delta})}, \frac{a}{ln(\frac{2}{\delta})}\right)$, we have

$$e^{-\frac{a}{\epsilon}} < \delta/2, \text{ and } e^{-1/\sqrt{\epsilon}} < \delta/2.$$

It follows that

$$\begin{aligned}
|e^{\frac{-a+b_\epsilon}{\epsilon}} - e^{\frac{-a}{\epsilon}}| &< |e^{\frac{-1}{\sqrt{\epsilon}}}| + |e^{\frac{-a}{\epsilon}}|, \\
&< \frac{\delta}{2} + \frac{\delta}{2} = \delta.
\end{aligned}$$

Hence $M\left|e^{\frac{-a+b_\epsilon}{\epsilon}} - e^{\frac{-a}{\epsilon}}\right| + \sup_{z\in[b_\epsilon,a]}|R(z)-R(a)| + \left|e^{-\frac{a}{\epsilon}}R(a)\right|$ converges uniformly to zero as ϵ tends to zero. Thus, $\int_0^a R(z)\frac{e^{\frac{z-a}{\epsilon}}}{\epsilon}dz - R(a)$ converges uniformly to zero on $[\tilde{a}, A]$ as ϵ tends to zero.

5.3 The Case of a Stable Population

5.3.1 Preliminary Results

In this section we consider a stable population characterised by a birth rate smaller than the death rate $(r < 0)$ and we assume that $\nu < N_0$ so that according to equation (5.1.9), the time at which the intersection of the quasi steady states occurs is positive.

(a) **The Second Quasi Steady State.** According to (5.1.8), the second quasi state is $I_2(t) = N_0e^{rt} - \nu$, $t \in \bar{\mathbb{I}}_T$. However, from the definition (5.1.9) of t_c, we have for $0 \le t < t_c$,

$$0 \le t < \frac{1}{r}\log\frac{\gamma}{N_0\lambda}.$$

This implies that

$$N_0 - \nu > N_0e^{rt} - \nu \ge 0.$$

Therefore, for $0 \le t < t_c$, $I_2(t) > 0$, and similarly, we show that for $t > t_c$, $I_2(t) < 0$.

(b) **Stability of the Quasi Steady States.** Denote the right hand side of equation (5.1.2) by

$$g(t, I, \epsilon) = I\Big(- \mu\epsilon + \lambda(N_0e^{rt} - I) - \gamma\Big).$$

Its derivative with respect to I is $g_{_I}(t, I, \epsilon) = -2\lambda I - \mu\epsilon - \gamma + \lambda N_0e^{rt}$. Then, evaluating the derivative of the function g at the quasi steady states gives, for $\epsilon = 0$,

$$g_{_I}(t, I_1, 0) = -\gamma + \lambda N_0e^{rt} = \lambda I_2(t),$$

and

$$g_{_I}(t, I_2, 0) = -2\lambda I_2 - \mu\epsilon - \gamma + \lambda N_0e^{rt} = -\lambda I_2.$$

From the result found in part (a), it follows that for $0 < t < t_c$, $g_I(t, I_1, 0) > 0$ and $g_I(t, I_2, 0) < 0$, and for $t > t_c$, $g_I(t, I_1, 0) < 0$ and $g_I(t, I_2, 0) > 0$. Hence, according to Remark 3.3.2.2, I_2 is asymptotically stable for $t < t_c$ and unstable for $t > t_c$, while I_1 is unstable for $t < t_c$ and asymptotically stable for $t > t_c$. Therefore, there is a stability switch at the intersection of the QSSs.

(c) **Characterisation of the function G.** As before, let us consider the function G given by

$$G(t, \epsilon) = \int_0^t g_I(s, I, \epsilon) ds = (-\mu\epsilon - \gamma)t + \frac{N_0\lambda}{r}(e^{rt} - 1),$$

and denote by t^* the root of $G(t, 0)$; that is,

$$-\gamma t^* + \frac{\lambda N_0}{r}(e^{rt^*} - 1) = 0,$$

if it exists. In order to prove the existence of t^*, let us study the shape of function $G(., 0)$. Its derivative is $\frac{dG}{dt}(t, 0) = \lambda N_0 e^{rt} - \gamma = I_2(t),\ t \in [0, T]$, and

$$\frac{dG}{dt}(t, 0) = 0 \Rightarrow t = \frac{1}{r} \log \frac{\gamma}{N_0\lambda} = t_c.$$

It follows, according to the result in part (a), that $\frac{dG}{dt}(t, 0) > 0$ for $0 < t < t_c$, and $\frac{dG}{dt}(t, 0) < 0$ for $t \in (t_c, T]$. Therefore,

$$G(0, 0) < G(t_c, 0);$$

that is

$$G(t_c, 0) > 0. \tag{5.3.1}$$

Furthermore,

$$\lim_{t \to +\infty} G(t, 0) = -\gamma t + \frac{\lambda N_0}{r}(e^{rt} - 1) = -\infty, \tag{5.3.2}$$

since $r < 0$. From (5.3.1) and (5.3.2) it follows, using the intermediate value theorem, that there exists $t^* > t_c$ such that $G(0, t^*) = 0$. Furthermore, since $\frac{dG}{dt}(t, 0) = I_2(t) < 0$ for $t > t_c$, G is monotone on (t_c, T). Therefore, t^* is unique. Figure 5.2 shows the graph of a function G for a stable population.

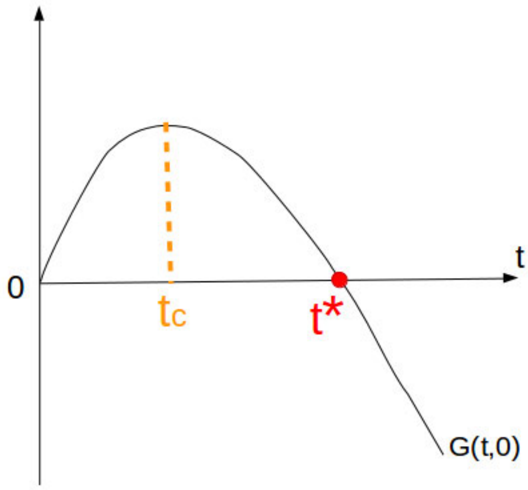

Figure 5.2: Illustration of the characteristics of the function G for a stable population.

5.3.2 Dynamics of the Infected Population before t_c

The dynamics of the behaviour of the infected population before the intersection of the quasi steady states can be described by its limit as ϵ tends to zero. For $t < t_c$,

$$\lim_{\epsilon\to 0} I(t,\epsilon) = \lim_{\epsilon\to 0} \frac{e^{\frac{1}{\epsilon}\left[\frac{N_0\lambda}{r}(e^{rt}-1)-\gamma t\right]-\mu t}}{\frac{1}{I_0} + \frac{\lambda}{\epsilon}\int_0^t e^{\frac{1}{\epsilon}\left[\frac{N_0\lambda}{r}(e^{rs}-1)-\gamma s\right]-\mu s} ds}$$

$$= \lim_{\epsilon\to 0} \frac{1}{\frac{1}{I_0}e^{-\frac{1}{\epsilon}G(t,0)+\mu t} + \frac{\lambda}{\epsilon}e^{-\frac{1}{\epsilon}G(t,0)+\mu t}\int_0^t e^{\frac{1}{\epsilon}G(s,0)-\mu s}ds}.$$

Let us set $z = G(t,0)$, then $dz = G^{'}(t,0)dt$ and we can define $s = G^{-1}(z,0)$ for the function G is a one-to-one function on $(0,t_c)$. So,

$$\lim_{\epsilon\to 0}\frac{\lambda}{\epsilon}e^{-\frac{1}{\epsilon}G(t,0)}\int_0^t e^{\frac{1}{\epsilon}G(s,0)}\frac{dz}{G'(G^{-1}(z,0),0)} = \lim_{\epsilon\to 0}\frac{\lambda}{\epsilon}e^{-\frac{1}{\epsilon}G(t,0)}\int_0^{G(t,0)} e^{\frac{1}{\epsilon}z}R(z)dz$$

with $R(z) = \frac{1}{G'(G^{-1}(z,0),0)}$. Thus,

$$\lim_{\epsilon\to 0}\frac{\lambda}{\epsilon}e^{-\frac{1}{\epsilon}G(t,0)}\int_0^t e^{\frac{1}{\epsilon}G(s,0)}\frac{dz}{G'(G^{-1}(z,0),0)} = \lim_{\epsilon\to 0}\frac{\lambda}{\epsilon}e^{-\frac{1}{\epsilon}G(t,0)}\int_0^{G(t,0)} e^{\frac{1}{\epsilon}z}R(z)dz,$$
$$= 1/I_2(t),$$

see Section 5.2.3. Moreover

$$\lim_{\epsilon\to 0}\frac{1}{I_0}e^{-\frac{1}{\epsilon}G(t,0)}=0,$$

since $G(t,0)>0$, for $0<t<t_c$. Hence

$$\lim_{\epsilon\to 0} I(t,\epsilon)=I_2(t),\quad t<t_c.$$

Therefore before reaching the point of intersection of the two quasi steady states, the solution converges to the second quasi steady state uniformly in $t\in(t_c,T]$ as ϵ tends to zero.

5.3.3 Dynamics of the Infected Population for the Time between t_c and t^*

Let us determine the limit of the infected population for the time between t_c and t^*.

$$\begin{aligned}\lim_{\epsilon\to 0} I(t,\epsilon) &= \lim_{\epsilon\to 0}\frac{e^{\frac{1}{\epsilon}\left[\frac{N_0\lambda}{r}(e^{rt}-1)-\gamma t\right]-\mu t}}{\frac{1}{I_0}+\frac{\lambda}{\epsilon}\int_0^t e^{\frac{1}{\epsilon}\left[\frac{N_0\lambda}{r}(e^{rs}-1)-\gamma s\right]-\mu s}ds}\\ &= \lim_{\epsilon\to 0}\frac{e^{\frac{1}{\epsilon}G(t,0)-\mu t}}{\frac{1}{I_0}+\frac{\lambda}{\epsilon}\int_0^t e^{\frac{1}{\epsilon}G(s,0)-\mu s}ds}\\ &= e^{-\mu t}\lim_{\epsilon\to 0}\frac{1}{\frac{1}{I_0}e^{-\frac{1}{\epsilon}G(t,0)}+\frac{\lambda e^{-\frac{1}{\epsilon}G(t,0)}}{\epsilon}\left(\int_0^{t_c} e^{\frac{1}{\epsilon}G(s,0)-\mu s}ds+\int_{t_c}^{t} e^{\frac{1}{\epsilon}G(s,0)-\mu s}ds\right)}.\end{aligned}$$

We have

$$\lim_{\epsilon\to 0}\frac{1}{I_0}e^{-\frac{1}{\epsilon}G(t,0)}=0,$$

since $G(t,0)>0$, for $0<t<t^*$, and

$$\lim_{\epsilon\to 0}\frac{\lambda e^{-\frac{1}{\epsilon}G(t,0)}}{\epsilon}\int_0^{t_c} e^{\frac{1}{\epsilon}G(s,0)-\mu s}ds=\frac{1}{I_2},$$

according to the result found in Section 5.2.3. Therefore,

$$\begin{aligned}\lim_{\epsilon\to 0} I(t,\epsilon) &= e^{-\mu t}\lim_{\epsilon\to 0}\frac{1}{(I_2)^{-1}+\frac{\lambda e^{-\frac{1}{\epsilon}G(t,0)}}{\epsilon}\int_{t_c}^{t} e^{\frac{1}{\epsilon}G(s,0)-\mu s}ds}\\ &= e^{-\mu t}\lim_{\epsilon\to 0}\frac{1}{(I_2)^{-1}+\lambda\int_{t_c}^{t}\frac{1}{\epsilon}e^{\frac{1}{\epsilon}(G(s,0)-G(t,0))-\mu s}ds}.\end{aligned}$$

Since G is a decreasing function between t_c and t^*, $G(s,0) \geq G(t,0)$ for $s \in (t_c, t)$ and hence

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} e^{\frac{1}{\epsilon}(G(s,0)-G(t,0))} = +\infty \text{ for } s \leq t.$$

Therefore,

$$\lim_{\epsilon \to 0} I(t,\epsilon) = 0 \text{ for } t \in (t_c, t^*).$$

In other words, as ϵ tends to zero, the solution converges to the second quasi steady state, passes close to the intersection of QSS and immediately switches to the first quasi steady state. Therefore, conversely to the unstable case (see Section 5.2.2), there is no delay in the stability switch in this case.

5.3.4 Dynamics of the Infected Population after t^*

Similarly, we will determine the asymptotic behaviour of the infected population by finding its limit for $t > t^*$ as ϵ tends to zero. For $t \in (t^*, T]$,

$$\begin{aligned}
\lim_{\epsilon \to 0} I(t,\epsilon) &= \lim_{\epsilon \to 0} \frac{e^{\frac{1}{\epsilon}G(t,0)-\mu t}}{\frac{1}{I_0} + \frac{\lambda}{\epsilon}\int_0^t e^{\frac{1}{\epsilon}G(s,0)-\mu s} ds} \\
&= \lim_{\epsilon \to 0} \frac{e^{\frac{1}{\epsilon}G(t,0)-\mu t}}{\frac{1}{I_0} + \frac{\lambda}{\epsilon}\int_0^{t^*} e^{\frac{1}{\epsilon}G(s,0)-\mu s} ds + \frac{\lambda}{\epsilon}\int_{t^*}^t e^{\frac{1}{\epsilon}G(s,0)-\mu s} ds} \\
&\leq \lim_{\epsilon \to 0} \frac{e^{\frac{1}{\epsilon}G(t,0)-\mu t}}{\frac{1}{I_0}}.
\end{aligned}$$

Since $G(t,0) < 0$ for $t > t^*$, we have $\lim_{\epsilon \to 0} e^{\frac{1}{\epsilon}G(t,0)-\mu t} = 0$. Therefore,

$$\lim_{\epsilon \to 0} I(t,\epsilon) = 0 \text{ for } t > t^*.$$

It follows that after reaching t^* the solution continues converging to the first quasi state as ϵ tends to zero.

5.4 Numerical Simulations

Figure 5.3 is made of two pictures. The first picture is for the stable case, while the second picture is for the unstable case. The values of the parameters used for the stable case are $\lambda = 0.0018,\ N_0 = 100,\ \gamma = 0.14,\ \beta = 0.001,\ I_0 = 100,\ \mu = 1/70.0$ and the values of the parameters for the unstable population are $\lambda = 0.0018,\ N_0 = 0.5,\ \gamma = 0.14,\ \beta = 0.1,\ I_0 = 0.4,\ \mu = 1/70.0$, [9, 57]. In both cases, the time is in years. It can be observed that the solution tends to the second QSS and then tends to the first QSS without delay in the case of a stable population. However, in the case of an unstable population, the solution tends to the first QSS and remains on the first QSS after reaching the intersection point. It will move to the second QSS only after the time t^*. Thus, we observe the delay in the switch of stability which, in this case, is approximatively 20 years.

5.5 Conclusion

As already found in [57], we realised that there is a delay of stability switch in the case of an unstable population, while in the case of a stable population we have no delay of stability switch. The solution moves immediately from the second quasi steady state to the first quasi state for a stable population. However, in the case of an unstable population, the solution converges to the first quasi steady state and, after crossing the intersection point, the solution remains for a while on the first quasi steady state, which became repulsive before moving to the second quasi steady state. The switch that was expected to occur at t_c occurs at t^*. Since $t^* > t_c$, there is a delay in stability switch. It is also important to mention that though the function G also has a unique root t^* in the stable case, the delay in stability switch is not observed. Also, we proved that the convergence to the second quasi steady state as ϵ tends to zero in the unstable case is almost uniform for $t > t^*$.

The next chapter consists in studying the two dimensional case of the flu model, when the total population is allowed to reach the threshold value ν.

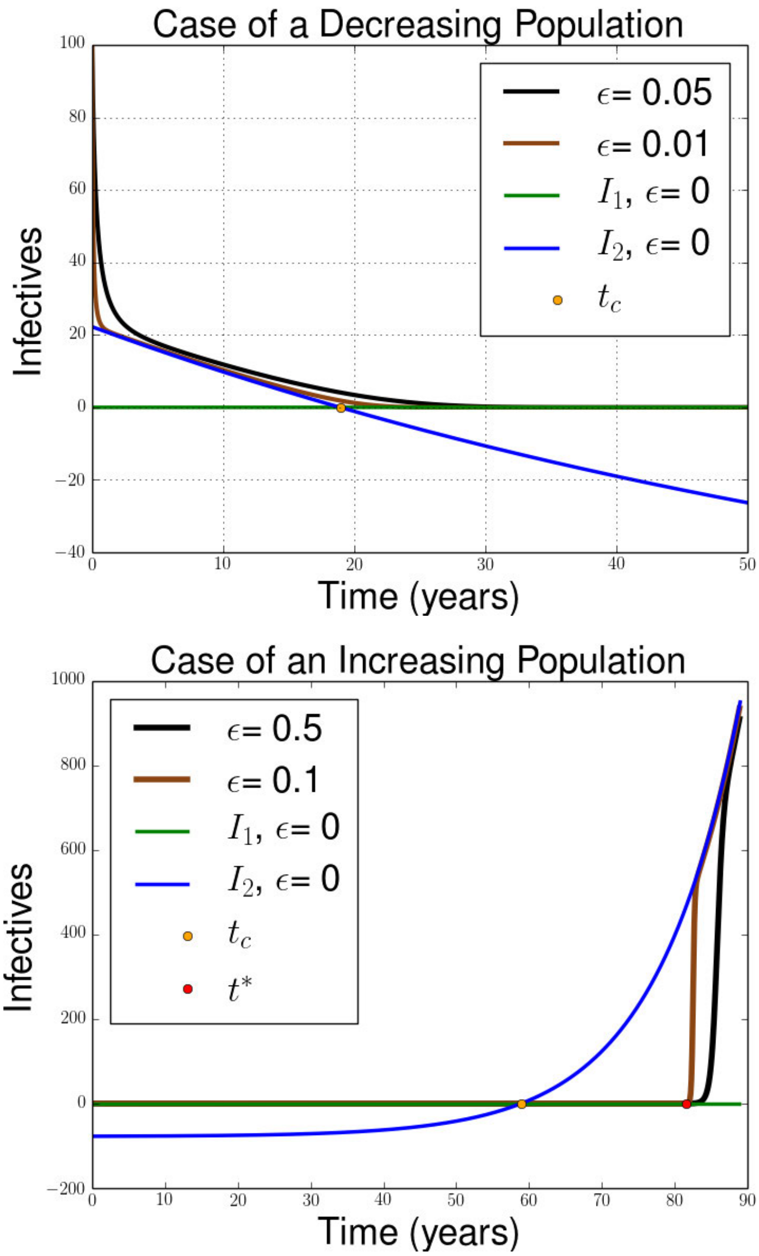

Figure 5.3: Dynamics of the infective population in the one dimensional case of the flu model for different values of epsilon (eps). The first picture presents the case of a stable population and the second picture presents the case of an unstable population. In general, in both cases the solution tends to the asymptotically stable part of the QSSs. There is no delay in the stability switch for the stable population, while a significant delay of approximatively 20 years is observed in the case of the unstable population.

6 Stability Switch for a Two Dimensional Case: Influenza

In this chapter, we study a two dimensional case of the influenza model obtained from [9, 57] by considering a non-zero disease induced death rate and we assume that the total population can reach the threshold value ν at which the two quasi steady states intersect. Similarly to the previous chapters, we consider two cases: the case of an unstable population and the case of a stable population. The aim of this chapter is to find the asymptotic behaviour of the solution. In particular, we seek to determine conditions under which a delay in stability switch and an immediate stability switch are observed. We use the method of upper and lower bounds. This method consists in finding two functions called the *lower bound* $(\underline{I}, \underline{N})$ and the *upper bound* $(\overline{I}, \overline{N})$ such that

$$\underline{I}(t,\epsilon) \le I(t,\epsilon) \le \overline{I}(t,\epsilon), \tag{6.0.1}$$

$$\underline{N}(t,\epsilon) \le N(t,\epsilon) \le \overline{N}(t,\epsilon), \tag{6.0.2}$$

for $t \in \mathbb{I}_T,\ \epsilon \in \mathbb{I}_{\epsilon_0}$, and having the same limit as ϵ tends to zero. Thus, using the squeeze theorem, respectively, on (6.0.1) and on (6.0.2), it follows that

$$\lim_{\epsilon\to 0}\underline{I}(t,\epsilon) = \lim_{\epsilon\to 0} I(t,\epsilon) = \lim_{\epsilon\to 0}\overline{I}(t,\epsilon),$$
$$\lim_{\epsilon\to 0}\underline{N}(t,\epsilon) = \lim_{\epsilon\to 0} N(t,\epsilon) = \lim_{\epsilon\to 0}\overline{N}(t,\epsilon),$$

for $t \in \mathbb{I}_T$.

6.1 Preliminary results

Consider the system (4.2.1) with $\mathring{\mu} = \mu + \mu^*$. The system (4.2.1) becomes

$$\begin{cases} \frac{dS}{dt} = \beta N - \mu S - \frac{1}{\epsilon}(\lambda I S - \gamma I), \\ \frac{dI}{dt} = -\mathring{\mu} I + \frac{1}{\epsilon}(\lambda I S - \gamma I), \end{cases} \tag{6.1.1}$$

with initial condition $(S(0), I(0)) = (S_0, I_0)$ and $\epsilon \in \mathbb{I}_{\epsilon_0},\ \epsilon_0 > 0$, a small parameter. Summing both equations of this system, we obtain

$$\frac{dN}{dt} = \beta N - \mu S - \mathring{\mu} I, \tag{6.1.2}$$

with initial condition $N(0) = N_0 = S_0 + I_0$. Using $N = S + I$, it follows that N satisfies both equations

$$\frac{dN}{dt} = (\beta - \mu)N - (\mathring{\mu} - \mu)I, \qquad N(0) = N_0, \tag{6.1.3}$$

and

$$\frac{dN}{dt} = (\beta - \mathring{\mu})N + (\mathring{\mu} - \mu)S, \qquad N(0) = N_0. \tag{6.1.4}$$

So, considering (6.1.3), system (6.1.1) becomes

$$\begin{cases} \epsilon\frac{dI}{dt} = -\epsilon\mathring{\mu}I + I(\lambda(N - I) - \gamma), \\ \frac{dN}{dt} = (\beta - \mu)N - (\mathring{\mu} - \mu)I, \end{cases} \tag{6.1.5}$$

with initial condition $(I(0), N(0)) = (I_0, N_0)$. In this chapter, we denote

$$g(I, N, \epsilon) = -\epsilon\mathring{\mu}I + I(\lambda(N - I) - \gamma)$$

and its derivative with respect to I by

$$g_I(I, N, \epsilon) = -\epsilon\mathring{\mu} - 2\lambda I + \lambda N - \gamma.$$

Our objective is to understand the asymptotic behaviour of the solution of the problem (6.1.5) using the method of upper and lower bounds to system (6.1.5). We will restrict our study to $\mathbb{R}^2_+$ since, according to Theorem 4.2.0.1, the solution to (6.1.5) exists, is unique and non-negative for non-negative initial condition.

6.1.1 Quasi Steady States

Setting $\epsilon = 0$ in the first equation of (6.1.5) we obtain two quasi steady states: $\phi_1(N) = 0$ and $\phi_2(N) = N - \nu$. Similarly to the one dimensional case, the quasi steady states intersect each other at $N = \nu$. Moreover, we have for $N \in \mathbb{R}_+$,

$$\phi_2(N) \le 0 \text{ for } N \le \nu, \tag{6.1.6}$$

$$\phi_2(N) \ge 0 \text{ for } N \ge \nu. \tag{6.1.7}$$

The derivative of g with respect to I at $\epsilon = 0$ is

$$g_I(I, N, 0) = -2\lambda I + \lambda N - \gamma.$$

Thus

$$g_I(\phi_1, N, 0) = \lambda N - \gamma,$$
$$g_I(\phi_2, N, 0) = -\lambda N + \gamma.$$

Hence, for $N < \nu$,

$$g_I(\phi_1, N, 0) < 0,$$
$$g_I(\phi_2, N, 0) > 0,$$

and, for $N > \nu$,

$$g_I(\phi_1, N, 0) > 0,$$
$$g_I(\phi_2, N, 0) < 0.$$

In other words, for $N < \nu$, ϕ_1 is asymptotically stable, while ϕ_2 is unstable. Conversely, for $N > \nu$, ϕ_1 is unstable, while ϕ_2 is asymptotically stable.

6.2 The Case of an Unstable Population

6.2.1 Determination of an Upper Bound for the Infective and Total Population

Let us consider the following ordinary differential equation

$$\frac{d\overline{N}}{dt} = r\overline{N}, \tag{6.2.1}$$

with initial condition $\overline{N}(0) = N_0 \geq 0$, $r = \beta - \mu$, and $t \in \bar{\mathbb{I}}_T$. The solution of (6.2.1) is $\overline{N}(t) = N_0 e^{rt},\ t \in \bar{\mathbb{I}}_T$. Let us assume $r \geq 0$. It follows that $\overline{N}(t) \geq 0$ for $t \in \bar{\mathbb{I}}_T$ since $N_0 \geq 0$. Moreover, we have $\mathring{\mu} - \mu > 0$ and, according to Theorem 4.2.0.1, $I \geq 0$. Therefore, according to the comparison theorem, Theorem 3.2.2.6, applied to (6.1.3) and (6.2.1), we have

$$\overline{N}(t) \geq N(t, \epsilon), \tag{6.2.2}$$

for $t \in \bar{\mathbb{I}}_T,\ \epsilon \in \mathbb{I}_{\epsilon_0}$. Furthermore, using (6.2.2), it follows that

$$-\mathring{\mu} I + \frac{I}{\epsilon}(\lambda(\overline{N} - I) - \gamma) \geq -\mathring{\mu} I + \frac{I}{\epsilon}(\lambda(N - I) - \gamma). \tag{6.2.3}$$

Hence, using the comparison theorem, Theorem 3.2.2.6, we obtain

$$\overline{I} \geq I,$$

where $\overline{I}$ is the solution to the equation

$$\frac{dI}{dt} = -\mathring{\mu} I + \frac{I}{\epsilon}(\lambda(\overline{N} - I) - \gamma),$$

with initial condition $I(0) = I_0$. The pair $(\overline{I}, \overline{N})$ is therefore an upper bound to the system (6.1.5).

6.2.1.1 Quasi Steady State Corresponding to the Upper Bound

By definition, $\overline{I}$ satisfies

$$\epsilon \frac{d\overline{I}}{dt} = \overline{I}(-\mathring{\mu}\epsilon + \lambda(\overline{N} - \overline{I}) - \gamma), \tag{6.2.4}$$

with initial condition $\overline{I}(0) = I_0$. Then, the quasi steady states to the upper bound are defined by $g(\overline{I}, \overline{N}, 0) = 0$. Hence, there are two steady states:

$$\overline{I}_1 = 0 \text{ and } \overline{I}_2 = N_0 e^{rt} - \nu,$$

for $t \in \bar{\mathbb{I}}_T$.

6.2.1.2 Intersection of Quasi Steady States for the Upper Bound.

Let us denote by $\bar{t}_c$ the intersection point of the two steady states of the upper bound. At the time $\bar{t}_c$,

$$\overline{I}_1(\bar{t}_c) = \overline{I}_2(\bar{t}_c).$$

It follows that

$$N_0 e^{r\bar{t}_c} - \nu = 0.$$

Therefore,

$$\bar{t}_c = \frac{1}{r} \log\left(\frac{\nu}{N_0}\right).$$

6.2.1.3 Convergence of the Infective Population to the First Quasi Steady State

Since $\overline{I}$ is the solution to (6.2.4), we see that all the one dimensional calculations in Chapter 5 can be applied. So, let us consider the function $\bar{G}$ defined by the equation (5.2.1) with $r > 0$ and let $\overline{t}^*$ be its root. It is easy to prove that

$$\lim_{\epsilon \to 0} \overline{I}(t, \epsilon) = 0 \text{ for } t \in (0, \overline{t}^*).$$

The proof is similar to the one in Section 5.2.2.

Thus, since $0 \le I(t, \epsilon) \le \overline{I}(t, \epsilon)$, we have

$$0 \le \lim_{\epsilon \to 0} I(t, \epsilon) \le \lim_{\epsilon \to 0} \overline{I}(t, \epsilon) = 0 \text{ for } t \in (0, \overline{t}^*).$$

Therefore,

$$\lim_{\epsilon \to 0} I(t, \epsilon) = 0 \text{ for } t \in (0, \overline{t}^*). \tag{6.2.5}$$

6.2.2 Maximum Interval on Which the Convergence to the First Quasi Steady State Holds Almost Uniformly

In this section, we prove that the interval $(0, \overline{t}^*)$ is the biggest interval, where the convergence of the solution $I(t, \epsilon)$ to the first QSS as ϵ tends to zero is almost uniform. We use the contradiction method. Therefore, let us assume that there exists $\hat{t} > \overline{t}^*$ such that

$$\lim_{\epsilon \to 0} I(t, \epsilon) = 0, \qquad\qquad t \in (0, \hat{t}), \tag{6.2.6}$$

almost uniformly.

6.2.2.1 Definition of the Lower Bounds

Relation (6.2.6) means that for any $\eta > 0$ and any $\delta > 0$, there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$, we have

$$0 \le I(t, \epsilon) \le \delta \text{ for } t \in [\eta, \hat{t} - \eta].$$

Let us consider the function $\underline{N}_{\eta\delta}$ such that

$$\frac{d\underline{N}_{\eta\delta}}{dt} = (\beta-\mu)\underline{N}_{\eta\delta} + (\mu-\mathring{\mu})\delta, \qquad t \in [\eta, \hat{t}-\eta], \tag{6.2.7}$$

with $\underline{N}_{\eta\delta}(\eta) = N_\eta$ which will be determined later. It follows that on the interval $[\eta, \hat{t}-\eta]$

$$0 \le \frac{d\underline{N}_{\eta\delta}}{dt} = (\beta-\mu)\underline{N}_{\eta\delta} + (\mu-\mathring{\mu})\delta \le (\beta-\mu)\underline{N}_{\eta\delta} + (\mu-\mathring{\mu})I,$$

since $\mu - \mathring{\mu} \le 0$ and $0 \le I \le \delta$. Thus

$$0 \le \frac{d\underline{N}_{\eta\delta}}{dt} \le (\beta-\mu)\underline{N}_{\eta\delta} + (\mu-\mathring{\mu})I.$$

Therefore, since the right hand side $(\beta-\mu)N + (\mu-\mathring{\mu})I$ is a continuous function of type K (for it is a scalar function), according to Theorem 3.2.2.6, and provided $N_\eta \le N(\eta)$, we have

$$\underline{N}_{\eta\delta}(t) \le N(t) \text{ for } t \in [\eta, \hat{t}-\eta].$$

In other words, $\underline{N}_{\eta\delta}$ is a lower bound on $[\eta, \hat{t}-\eta]$. Furthermore, let us define $\underline{N}_{\eta\delta}$ on $[0,\eta]$ by

$$\frac{d\underline{N}_{\eta\delta}}{dt} = (\beta-\mathring{\mu})\underline{N}_{\eta\delta}, \qquad \underline{N}_{\eta\delta}(0) = N_0. \tag{6.2.8}$$

Therefore

$$\underline{N}_{\eta\delta}(t) = N_0 e^{\mathring{r}t}, \qquad t \in [0,\eta],$$

with $\mathring{r} = \beta - \mathring{\mu}$. Since $\mathring{\mu} - \mu \ge 0$ and $S \ge 0$, it follows, from the comparison theorem applied to (6.1.4) and (6.2.8), that

$$N(t) \ge \underline{N}_{\eta\delta}(t), \qquad t \in [0,\eta]. \tag{6.2.9}$$

Solving (6.2.8) and (6.2.7) with $\underline{N}_{\eta\delta}(\eta) = N_0 e^{\mathring{r}\eta}$, we obtain

$$\underline{N}_{\eta\delta}(t) = \begin{cases} N_0 e^{\mathring{r}t}, & 0 \le t \le \eta, \\ (N_\eta - \frac{\delta\mu^*}{r})e^{r(t-\eta)} + \frac{\delta\tilde{\mu}}{r}, & \eta \le t \le \hat{t}-\eta, \end{cases}$$

where $r = \beta - \mu$, $N_\eta = N_0 e^{\mathring{r}\eta}$, $\mu^* = \mathring{\mu} - \mu$. By construction,

$$\underline{N}_{\eta\delta}(t) \le N(t), \tag{6.2.10}$$

on $[0, \hat{t}-\eta]$.

Note that for $\underline{N}_{\eta\delta}$ to be a meaningful lower bound, it has to be positive on $[0, \hat{t}-\eta]$. A sufficient condition is

$$N_\eta - \frac{\delta\mu^*}{r} > 0.$$

This can be achieved as N_η and δ are chosen independently. Going back to (6.2.7), we can select arbitrary $\eta > 0$ which gives a fixed value to N_η. Thus, for any $\delta < \frac{N_\eta r}{\mu^*}$, there exists ϵ_0 such that for all $\epsilon \in \mathbb{I}_{\epsilon_0}$,

$$0 < I(t,\epsilon) < \delta \text{ on } [\eta, \hat{t} - \eta].$$

Consequently, we define $\underline{I}_{\eta\delta}$ by

$$\begin{cases} \frac{d\underline{I}_{\eta\delta}}{dt} = -\mathring{\mu}\underline{I}_{\eta\delta} + \frac{\underline{I}_{\eta\delta}}{\epsilon}(\lambda(\underline{N}_{\eta\delta} - \underline{I}_{\eta\delta}) - \gamma), \\ \underline{I}_{\eta\delta}(0) = I_0. \end{cases} \tag{6.2.11}$$

Applying the comparison theorem to the definition of I and $\underline{I}_{\eta\delta}$, it follows that $\underline{I}_{\eta\delta} \leq I$. Therefore, $(\underline{I}_{\eta\delta}, \underline{N}_{\eta\delta})$ is a lower bound to (6.1.5).

6.2.2.2 Quasi Steady States for the Lower Bound to the Infective Population

From the definition of the lower bound given in (6.2.11), there are two quasi steady states: $\underline{I}_1 = 0$ or $\underline{I}_2 = \underline{N}_{\eta\delta} - \nu$. In full, the two quasi steady states are

$$\underline{I}_1 = 0 \text{ and } \underline{I}_2 = \begin{cases} N_0 e^{\mathring{r}t} - \nu, & 0 \leq t \leq \eta, \\ N_\eta e^{r(t-\eta)} + \frac{\delta\mu^*}{r}(1 - e^{r(t-\eta)}) - \nu, & \eta \leq t \leq \hat{t} - \nu. \end{cases}$$

6.2.2.3 Intersection of the Quasi Steady States of the Lower Bound to the Infective Population

Denote by $\underline{t_c}$ the time at which the quasi steady states of the lower bound intersect each other. Therefore, it satisfies the equality

$$\underline{I}_1(\underline{t}_c) = \underline{I}_2(\underline{t}_c). \tag{6.2.12}$$

Since the lower bound is a continuous function, we can choose η (for $N_\eta = N_0 e^{\mathring{r}\eta}$) small enough for $\underline{t_c} \geq \eta$. Then (6.2.12) can be written as

$$N_\eta e^{r(\underline{t}_c - \eta)} + \frac{\delta\mu}{r}(1 - e^{r(\underline{t}_c - \mu)}) - \nu = 0;$$

that is,

$$\underline{t}_c = \frac{1}{r}\log\Big(\frac{r\nu - \delta\mu}{rN_\eta - \delta\mu}\Big) + \eta. \tag{6.2.13}$$

We observe that

$$\lim_{\substack{\eta\to 0\\ \delta\to 0}} \underline{t}_c = \frac{1}{r}\log\Big(\frac{\nu}{N_0}\Big) = \bar{t}_c.$$

Since for $\epsilon \in \mathbb{I}_{\epsilon_0}$,

$$\underline{I}_{\eta\delta}(t,\epsilon) \leq I(t,\epsilon) \leq \overline{I}(t,\epsilon) \text{ for } t \in [\eta, \hat{t} - \eta],$$

$\bar{t}_c \geq \underline{t}_c$. These two points can be made as close to each other as we wish by choosing δ and η small enough.

6.2.2.4 Conclusion

From the definition of the function g, we have

$$g_{\underline{I}}(0, \underline{N}_{\eta\delta}, \epsilon) = -\mathring{\mu}\epsilon - \gamma + \lambda\underline{N}_{\eta\delta}.$$

Thus, for $t \in \bar{\mathbb{I}}_T$, and $\epsilon \in \mathbb{I}_{\epsilon_0}$,

$$G(t,\epsilon,\eta,\delta) = \int_0^t g_{\underline{I}}(0, \underline{N}_{\eta\delta}(s), \epsilon)ds = (-\mathring{\mu}\epsilon - \gamma)t + \lambda\int_0^t \underline{N}_{\eta\delta}(s)ds.$$

So, replacing $\underline{N}_{\eta\delta}$ by its value, we obtain

$$G(t,\epsilon,\eta,\delta) = \begin{cases} (-\mathring{\mu}\epsilon - \gamma)t + \frac{N_0\lambda}{\mathring{r}}(e^{\mathring{r}t} - 1), & 0 \leq t \leq \eta, \\ (-\mathring{\mu}\epsilon - \gamma)t + (-\mathring{\mu}\epsilon - \gamma)\lambda\eta + \frac{N_0\lambda}{\mathring{r}}(e^{\mathring{r}\eta} - 1) + \lambda\Big[\frac{\delta\mu^*}{r}(t-\eta) + \\ \frac{1}{r}(N_\eta - \frac{\delta\mu^*}{r})e^{-r\eta}(e^{rt} - e^{r\eta})\Big], & \eta \leq t \leq t^* - \eta. \end{cases} \tag{6.2.14}$$

We observe that $G(t,0,0,0) = -\gamma t + \frac{\lambda}{r}N_0(e^{rt} - 1)$ is the function defined by (5.2.1). Therefore, according to the result obtained in Section 5.2, there exists $\bar{t}^*$ such that $G(\bar{t}^*, 0, 0, 0) = 0$. In order to prove the existence of the root of $G(t,\epsilon,\eta,\delta)$, we use the implicit function theorem, Theorem 3.2.3.

The function

$$\begin{aligned} G(t,\epsilon,\eta,\delta) &= (-\mathring{\mu}\epsilon - \gamma)\lambda\eta + \frac{N_0\lambda}{\mathring{r}}(e^{\mathring{r}\eta} - 1) - \frac{\eta\lambda\delta\mu^*}{r} \\ &+ (-\gamma - \mathring{\mu}\epsilon + \frac{\lambda\delta\mu^*}{r})t + \frac{\lambda}{r}(N_\eta - \frac{\delta\mu^*}{r})e^{-r\eta}(e^{rt} - e^{r\eta}) \end{aligned}$$

is continuous and has continuous partial derivatives on $\mathbb{R}^4_+$. Let $P = (\bar{t}^*, 0, 0, 0) \in \mathbb{R}^4_+$. We have $G(\bar{t}^*, 0, 0, 0) = 0$ and

$$\frac{\partial G(t)}{\partial t} = -\gamma - \mathring{\mu}\epsilon + \frac{\lambda\delta\mu^*}{r} + \lambda(N_\eta - \frac{\delta\mu^*}{r})e^{r(t-\eta)};$$

that is

$$\frac{\partial G(P)}{\partial t} = -\gamma + \lambda N_0 e^{r\bar{t}^*}.$$

So,

$$\frac{\partial G(P)}{\partial t} \neq 0 \Leftrightarrow \bar{t}^* \neq \frac{1}{r}\log(\frac{\nu}{N_0}) \Leftrightarrow \bar{t}^* \neq \bar{t}_c,$$

which is true. Therefore, according to the implicit function theorem, Theorem 3.2.3, there exist positive parameters $\delta_0,\ \epsilon_0,\ \eta_0$ such that for all $\delta \in \mathbb{I}_{\delta_0},\ \epsilon \in \mathbb{I}_{\epsilon_0},$ and $\eta \in \mathbb{I}_{\eta_0}$, there exists a unique $\underline{t}^* = t(\epsilon, \eta, \delta)$ such that

$$G(\underline{t}^*, \epsilon, \eta, \delta) = 0.$$

Moreover,

$$\lim_{\epsilon,\eta,\delta\to 0} \underline{t}^* = \lim_{\epsilon,\eta,\delta\to 0} t(\eta, \delta, \epsilon) = \bar{t}^*. \tag{6.2.15}$$

Using the one dimensional theory developed in Section 5.2, we see that $\underline{I}_{\eta\delta}(t,\epsilon)$ does not converge to zero for any $t > \bar{t}^*$. Since $I(t,\epsilon) \geq \underline{I}_{\eta\delta}(t,\epsilon)$, $I(t,\epsilon)$ cannot converge to zero as ϵ tends to zero for any $t > \bar{t}^*$. Thus the assumption that there exists $\hat{t} > \bar{t}^*$ such that (6.2.6) holds is false. Therefore, the convergence of I to the first quasi steady state is uniform on any interval with right end point smaller than $\bar{t}^*$.

6.2.3 Convergence to the Second Quasi Steady State

6.2.3.1 Construction of a Lower Bound on $\mathbb{I}_T$

6.2.3.2 Theorem. *Let us consider $\tilde{t} \in (\bar{t}_c, \bar{t}^*)$ and define the pair of functions $(\underline{\underline{N}}_{\eta\delta}, \underline{\underline{I}}_{\eta\delta})$ by*

$$\underline{\underline{N}}_{\eta\delta}(t) = \begin{cases} \underline{N}_{\eta\delta}(t), & t \leq \tilde{t}, \\ \underline{\underline{N}}^{(2)}_{\eta\delta}(t), & t \geq \tilde{t}, \end{cases}$$

where $\underline{\underline{N}}^{(2)}_{\eta\delta}$ *satisfies*

$$\begin{cases} \frac{d\underline{\underline{N}}^{(2)}_{\eta\delta}}{dt} = (\beta - \mathring{\mu})\underline{\underline{N}}^{(2)}_{\eta\delta} + \nu(\mathring{\mu} - \mu), \\ \underline{\underline{N}}^{(2)}_{\eta\delta}(\tilde{t}) = \underline{N}_{\eta\delta}(\tilde{t}) > \nu, \end{cases} \tag{6.2.16}$$

and $\underline{\underline{I}}_{\eta\delta}$ *satisfies*

$$\begin{cases} \frac{d\underline{\underline{I}}_{\eta\delta}}{dt} = -\frac{\lambda}{\epsilon}\underline{\underline{I}}^2 + \left[\frac{1}{\epsilon}(\lambda\underline{\underline{N}}_{\eta\delta} - \gamma) - \mathring{\mu}\right]\underline{\underline{I}}_{\eta\delta}, \\ \underline{\underline{I}}_{\eta\delta}(0) = I_0. \end{cases}$$

Then the pair of functions $(\underline{\underline{N}}_{\eta\delta}, \underline{\underline{I}}_{\eta\delta})$ *is a lower bound to* (6.1.5) *on* $\mathbb{I}_T$.

Proof. In the previous section, we have determined $\bar{t}_c$ and $\bar{t}^*$ for the upper bound $\overline{N} = N_0e^{rt}$, and we found that the infective population I tends to zero as ϵ is getting smaller on $(0, \bar{t}^*)$. Applying the regular perturbation theorem to (6.1.3), we find that N tends to $\overline{N}$ on $(0, \bar{t}^*)$ as ϵ tends to zero. Moreover, from (6.2.10) and (6.2.2) on $(0, \bar{t}^*)$,

$$\underline{N}_{\eta\delta}(t) \le N(t, \epsilon) \le \overline{N}(t).$$

Furthermore, we have $\overline{N}(\bar{t}^*) > \nu$, since $\overline{N}$ is an increasing function, $\overline{N}(\bar{t}_c) = \nu$ and $\bar{t}^* > \bar{t}_c$. This means that for some $\tilde{t} \in (\bar{t}_c, \bar{t}^*)$, $\overline{N}(\tilde{t}) > \nu$ and hence $N(\tilde{t}, \epsilon) > \nu$ for sufficiently small ϵ. However, for $t \in \mathbb{I}_T,\ \epsilon \in \mathbb{I}_{\epsilon_0},\ S(t, \epsilon) = N(t, \epsilon) - I(t, \epsilon)$. Thus, considering sufficiently small ϵ, we have

$$S(\tilde{t}, \epsilon) > \nu$$

since $I(\tilde{t}, \epsilon)$ converges to zero as ϵ tends to zero. So $N(\tilde{t}, \epsilon)$ is approximately equal to $S(\tilde{t}, \epsilon)$ for ϵ small enough.

Consider the system (6.1.5) and let us determine the behaviour of the solution at $S = \nu$. Let us denote the normal vector $\omega = (1, -1)$. For $S = \nu$, the infective population satisfies $I = N - \nu$. It follows that

$$\omega.\Big(\frac{dN}{dt}, \frac{dI}{dt}\Big)\Big|_{I=N-\nu} = (1, -1).\big(r(I + \nu) - I\mu, -\mathring{\mu}I\big) = \beta I + r\nu > 0.$$

It follows that the infective population decreases with time. Since $S = N - I$, it follows that the susceptible population increases at $S = \nu$. Since $S(\tilde{t}, \epsilon) > \nu$ for ϵ sufficiently small, we have

$$S(t, \epsilon) \ge \nu \text{ for } t > \tilde{t}.$$

Hence

$$(\beta - \mathring{\mu})N + S(\mathring{\mu} - \mu) \geq (\beta - \mathring{\mu})N + \nu(\mathring{\mu} - \mu).$$

Thus, according to the comparison theorem, $N(t) \geq \underline{\underline{N}}^{(2)}_{\eta\delta}(t)$ for $t \geq \tilde{t}$.

Hence, $\underline{\underline{N}}_{\eta\delta} \leq N(t)$ for all $t \in \mathbb{I}_T$.

Furthermore, from the definition of $\underline{\underline{I}}_{\eta\delta}$ and of I,

$$-\mathring{\mu}\epsilon I + I\Big[\lambda(\underline{\underline{N}}_{\eta\delta} - I) - \gamma\Big] \leq -\mathring{\mu}\epsilon I + I\Big[\lambda(N(t) - I) - \gamma\Big].$$

Therefore

$$\underline{\underline{I}}_{\eta\delta} \leq I \text{ for } t \in \mathbb{I}_T,\ \epsilon \in \mathbb{I}_{\epsilon_0}.$$

The pair of functions $(\underline{\underline{N}}_{\eta\delta}, \underline{\underline{I}}_{\eta\delta})$ is therefore a lower bound to (6.1.5) on $\mathbb{I}_T$. □

6.2.3.3 Characterisation of the Function $\underline{\underline{G}}$

By solving (6.2.18), we obtain

$$\underline{\underline{N}}^{(2)}_{\eta\delta}(t) = -\frac{\nu\mu^*}{\mathring{r}} + \Big(\underline{N}_{\eta\delta}(\tilde{t}) + \frac{\nu\mu^*}{\mathring{r}}\Big)e^{\mathring{r}(t-\tilde{t})},\ t > \tilde{t}. \tag{6.2.17}$$

with $\mathring{r} = \beta - \mathring{\mu} \neq 0$. Therefore,

$$\underline{\underline{N}}_{\eta\delta}(t) = \begin{cases} N_0 e^{\mathring{r}t}, & t \leq \eta, \\ N_\eta e^{r(t-\eta)} + \frac{\delta\mu^*}{r}(1 - e^{r(t-\eta)}), & \eta \leq t \leq \tilde{t}, \\ -\frac{\nu\mu^*}{\mathring{r}} + Ke^{\mathring{r}(t-\tilde{t})}, & t \geq \tilde{t}, \end{cases}$$

where $K = \underline{N}_{\eta\delta}(\tilde{t}) + \frac{\nu\mu^*}{\mathring{r}} > 0$. Similarly to equation (5.1.3), let

$$\varphi(t, \epsilon, \eta, \delta) = \frac{1}{\epsilon}(\lambda\underline{\underline{N}}_{\eta\delta}(t) - \gamma) - \mathring{\mu},$$

denote $\Upsilon_\varphi = \int_0^t \varphi(s, \epsilon, \eta, \delta)ds$ and define

$$\underline{\underline{I}}_{\eta\delta}(t, \epsilon) = \frac{e^{\Upsilon_\varphi(t,\epsilon,\eta,\delta)}}{\frac{1}{\underline{\underline{I}}_0} + \frac{\lambda}{\epsilon}\int_0^t e^{\Upsilon_\varphi(s,\epsilon,\eta,\delta)}ds},$$

for $t \in \mathbb{I}_T$ and $\epsilon \in \mathbb{I}_{\epsilon_0}$. Let us denote

$$\underline{\underline{G}}(t, \epsilon, \eta, \delta) = \int_0^t \underline{\underline{g}}_I(s, 0, \epsilon, \eta, \delta)ds, \qquad t \in \mathbb{I}_T,\ \epsilon \in \mathbb{I}_{\epsilon_0},$$

with $\underline{\underline{g}}(t, I, \epsilon, \eta, \delta) = -\lambda I^2 + I(\lambda \underline{\underline{N}}_{\eta\delta}(t) - \gamma - \mathring{\mu}\epsilon) = -\lambda I^2 + \epsilon\varphi I$, and

$\underline{\underline{g}}_I(t, 0, \epsilon, \eta, \delta) = \epsilon\varphi$. Thus, $\underline{\underline{G}}(t, \epsilon, \eta, \delta) = \epsilon\Upsilon_{\varphi}$; that is,

$$\underline{\underline{G}}(t,0,\eta,\delta) = \begin{cases} \underline{\underline{G}}_{00} := (\frac{\lambda}{\mathring{r}} N_0 e^{\mathring{r}t} - \gamma t), & t \leq \eta, \\ \underline{\underline{G}}_{10} := \underline{\underline{G}}_{00}(\eta) + \frac{\lambda}{r}(N_\eta - \frac{\delta\mu}{r})(e^{r(t-\eta)} - 1) - (\gamma - \frac{\delta\mu\lambda}{r_1})(t-\eta), & \eta \leq t \leq \tilde{t}, \\ \underline{\underline{G}}_{20} := \underline{\underline{G}}_{10}(\tilde{t}) + \frac{\lambda}{\mathring{r}}\Big(\underline{N}_{\eta\delta}(\tilde{t}) + \frac{\nu(\mathring{\mu}-\mu)}{\mathring{r}}\Big)(e^{\mathring{r}(t-\tilde{t})} - 1), & \\ \qquad -(\frac{\nu\lambda(\mathring{\mu}-\mu)}{\mathring{r}} + \gamma)(t-\tilde{t}), & t \geq \tilde{t}. \end{cases}$$

Note that $\underline{\underline{G}}$ coincides on $[0, \tilde{t}]$ with the function studied in (6.2.14). So, for η small enough, similarly as in (6.2.13), it is easy to show that there exists $\underline{t}_c(\eta, \delta)$ such that

$$\frac{\partial \underline{\underline{G}}}{\partial t}(\underline{t}_c(\eta,\delta)) = 0 \text{ and } \lim_{\eta,\delta\to 0} \underline{t}_c(\eta,\delta) = \bar{t}_c,$$

and, according to (5.2.2),

$$\underline{\underline{G}}(\bar{t}_c, 0, 0) < 0.$$

So, by the continuity of $\underline{\underline{G}}$ around $\underline{t}_c(\eta,\delta)$, $\underline{\underline{G}}(\bar{t}_c, \eta, \delta) < 0$. Since $\tilde{t}$ is chosen to be smaller than $\bar{t}^*$, we use the function $\underline{\underline{G}}_{20}$ to prove the existence of $t^*(\eta,\delta)$, the root of $\underline{\underline{G}}(t,0,\eta,\delta)$. We know that $\underline{\underline{G}}_{20}(\tilde{t}) = \underline{\underline{G}}_{10}(\tilde{t}) < 0$ by the continuity of $\underline{\underline{G}}_{10}$. Also,

$$\lim_{t\to+\infty} \underline{\underline{G}}_{20}(t) = \lim_{t\to+\infty} \underline{\underline{G}}_{10}(\tilde{t}) + \frac{\lambda}{\mathring{r}}\Big(\underline{N}_{\eta\delta}(\tilde{t}) + \frac{\nu(\mathring{\mu}-\mu)}{\mathring{r}}\Big)(e^{\mathring{r}(t-\tilde{t})} - 1) - (\frac{\nu\lambda(\mathring{\mu}-\mu)}{\mathring{r}} + \gamma)(t-\tilde{t}).$$

In order to determine $\lim_{t\to+\infty} \underline{\underline{G}}_{20}(t)$, we have to study the sign of

$$\frac{\nu\lambda(\mathring{\mu}-\mu)}{\mathring{r}} + \gamma$$

for different values of $\mathring{r}$ (except for $\mathring{r} = 0$ which will be studied later).

Case 1: If $\mathring{r} = \beta - \mu - \mu^* < 0$ then, since by assumption $r = \beta - \mu \geq 0$,

$$\beta - \mathring{\mu} \geq \mu - \mathring{\mu} \Rightarrow \gamma < \frac{\mu - \mathring{\mu}}{\mathring{r}}\nu\lambda.$$

Thus,

$$\frac{\nu\lambda(\mathring{\mu}-\mu)}{\mathring{r}} + \gamma \leq 0.$$

Therefore

$$\lim_{t\to+\infty} \underline{\underline{G}}_{20}(t) = \lim_{t\to+\infty} -(\frac{\nu\lambda(\mathring{\mu}-\mu)}{\mathring{r}} + \gamma)(t-\tilde{t}) = +\infty.$$

Case 2: If $\mathring{r} = \beta - \mu - \mu^* > 0$, then

$$\lim_{t\to+\infty} \underline{\underline{G}}_{20}(t) = \lim_{t\to+\infty} \frac{\lambda}{\mathring{r}}\Big(\underline{N}_{\eta\delta}(\tilde{t}) + \frac{\nu(\mathring{\mu}-\mu)}{\mathring{r}}\Big)(e^{\mathring{r}(t-\tilde{t})} - 1) = +\infty.$$

It follows that there exists $t^*(\eta,\delta)$ such that $\underline{\underline{G}}(t^*, 0, \eta, \delta) = 0$. The uniqueness of $t^*(\eta,\delta)$ follows from the monotonicity of $\underline{\underline{G}}_{20}$ on $(\tilde{t}, T)$. Let us now consider the function $\underline{\underline{G}}(t, 0, \eta, \delta)$ and the point $P = (\bar{t}^*, 0, 0, 0)$. From the result found in Section 6.2.2.4, we have $\underline{\underline{G}}(P) = 0$ and $\frac{\partial \underline{\underline{G}}}{\partial t}(P) \neq 0$. So, according to the implicit function theorem, Theorem 3.2.3, there exists a neighbourhood of P such that $\underline{\underline{G}}(t, \epsilon, \eta, \delta) = 0$. In other words, there exist positive numbers $\delta_0, \epsilon_0, \eta_0$ such that for all $\delta \in \mathbb{I}_{\delta_0}, \epsilon \in \mathbb{I}_{\epsilon_0}, \eta \in \mathbb{I}_{\eta_0}$, there is a unique $t^* = t^*(\epsilon, \eta, \delta)$ such that $\underline{\underline{G}}(t^*, \epsilon, \eta, \delta) = 0$. Moreover,

$$\lim_{\eta,\epsilon,\delta\to 0} t^*(\epsilon, \eta, \delta) = \bar{t}^*.$$

6.2.3.4 Proof of the Convergence of the Lower Bound of I to its Second Quasi Steady State

For $\mathring{r} \neq 0$, the proof is similar to that of the one dimensional problem developed in Section 5.2.3. It can be obtained by replacing G by $\underline{\underline{G}}_{20}$ and N by $\underline{\underline{N}}^{(2)}_{\eta\delta}$. Moreover, similarly to the one dimensional case we prove that this convergence is uniform (see Section 5.2.3.1). Let us assume that $\mathring{r} = 0$. From equation (6.2.18) we have

$$\begin{cases} \frac{d\underline{\underline{N}}^{(2)}_{\eta\delta}}{dt} = \nu\mu^*, \\ \underline{\underline{N}}^{(2)}_{\eta\delta}(\tilde{t}) = \underline{N}_{\eta\delta}(\tilde{t}) > \nu. \end{cases} \tag{6.2.18}$$

It follows that, according to the Theorem 4.2.0.2, there is uniform convergence to the second quasi steady state.

6.2.3.5 Construction of an Upper bound on $\mathbb{I}_T$

Since $\underline{\underline{I}}_{\eta\delta}$ tends almost uniformly to the second quasi steady state $\underline{\underline{i}}(t) = \underline{\underline{N}}^{(2)}_{\eta\delta}(t) - \nu$ on $(\bar{t}^*, T]$, then for all small values $\theta > 0$ and $\delta_1 > 0$, there exists $\epsilon_0 > 0$ such that for all $\epsilon \in \mathbb{I}_{\epsilon_0}$,

$$|\underline{\underline{I}}_{\eta\delta}(t, \epsilon) - \underline{\underline{i}}(t)| < \delta_1,$$

for $t \in [\bar{t^*} + \theta, T]$. It follows that

$$-\delta_1 + \underline{\underline{i}}(t) \leq \underline{\underline{I}}_{\eta\delta}(t,\epsilon) \leq I(t,\epsilon),$$

for $\epsilon \in \mathbb{I}_{\epsilon_0},\ t \in [\bar{t^*} + \theta, T]$. Therefore,

$$\underline{\underline{i}}(t) - \delta_1 = -\delta_1 + \underline{\underline{N}}^{(2)}_{\eta\delta}(t) - \nu \leq I(t,\epsilon), \tag{6.2.19}$$

for $\epsilon \in \mathbb{I}_{\epsilon_0},\ t \in [\bar{t^*} + \theta, T]$. Now, let us use this lower bound of I to construct an upper bound for N which takes into account the fact that I is no longer small. Replacing I in (6.1.3) by $\underline{\underline{i}}(t) - \delta_1$, we obtain

$$(\beta - \mu)N(t,\epsilon) - \mu^* I(t,\epsilon) \leq (\beta - \mu)N(t,\epsilon) - \mu^*(\underline{\underline{i}}(t) - \delta_1), \tag{6.2.20}$$

for $\epsilon \in \mathbb{I}_{\epsilon_0},\ t \in [\bar{t^*} + \theta, T]$. Now, define $\overline{\overline{N}}^{(2)}_{\eta\delta\delta_1}$ on $[\bar{t^*} + \theta, T]$ as follows

$$\begin{cases} \frac{d\overline{\overline{N}}^{(2)}_{\eta\delta\delta_1}}{dt} = r\overline{\overline{N}}^{(2)}_{\eta\delta\delta_1} - \mu^*(\underline{\underline{i}}_{\epsilon} - \delta_1), \\ \overline{\overline{N}}^{(2)}_{\eta\delta\delta_1}(t_\theta) = N_0 e^{rt_\theta} \geq N(t_\theta), \end{cases} \tag{6.2.21}$$

where $t_\theta = \bar{t^*} + \theta$. It is easy to prove, using the comparison theorem, that

$$N(t) \leq \overline{\overline{N}}^{(2)}_{\eta\delta\delta_1}(t),$$

for $t \in [t_\theta, T]$. Solving (6.2.21) for $\overline{\overline{N}}^{(2)}_{\eta\delta\delta_1}$, we obtain

$$\begin{aligned} \overline{\overline{N}}^{(2)}_{\eta\delta\delta_1}(t) = &-\frac{\mu^*}{r}(\nu + \delta_1 + \frac{\mu^*\nu}{\mathring{r}})(1 - e^{r(t-t_\theta)}) + \Big(\underline{N}_{\eta\delta}(\tilde{t}) + \frac{\nu\mu^*}{\mathring{r}}\Big)\Big(e^{\mathring{r}(t-\tilde{t})} - e^{rt - \mathring{r}\tilde{t} - \mu^* t_\theta}\Big) \\ &+ \overline{\overline{N}}^{(2)}_{\eta\delta\delta_1}(t_\theta)e^{r(t-t_\theta)}, \end{aligned}$$

for $t \in [t_\theta, T]$. Therefore, we consider the upper bound for N

$$\overline{\overline{N}}_{\eta\delta\delta_1}(t) = \begin{cases} \overline{\overline{N}}_1 \quad = N_0 e^{rt}, & t \in [0, t_\theta], \\ \overline{\overline{N}}^{(2)}_{\eta\delta\delta_1} = -\frac{\mu^*}{r}(\nu + \delta_1 + \frac{\mu^*\nu}{\mathring{r}})(1 - e^{r(t-t_\theta)}) & \\ \qquad +\Big(\underline{N}_{\eta\delta}(\tilde{t}) + \frac{\nu\mu^*}{\mathring{r}}\Big)\Big(e^{\mathring{r}(t-\tilde{t})} - e^{rt-\mathring{r}\tilde{t}-\mu^* t_\theta}\Big) + N_0 e^{rt}, & t \in (t_\theta, T]. \end{cases}$$

The corresponding upper bound for I is the solution $\overline{\overline{I}}_{\eta\delta\delta_1}$, of

$$\begin{cases} \frac{dI}{dt} = -\mathring{\mu} I + \frac{1}{\epsilon} I\Big[\lambda(\overline{\overline{N}}_{\eta\delta\delta_1}(t) - I) - \gamma\Big], \\ I(0,\epsilon) = I_0, \end{cases} \tag{6.2.22}$$

for $\epsilon \in \mathbb{I}_{\epsilon_0}$ and $t \in \mathbb{I}_T$. Thus, according to (5.1.7), we have

$$\overline{\overline{I}}_{\eta\delta\delta_1}(t,\epsilon) = \frac{e^{\frac{1}{\epsilon}\overline{\overline{G}}(t,0,\eta,\delta,\delta_1)-\mathring{\mu}t}}{\frac{1}{I_0} + \frac{\lambda}{\epsilon}\int_0^t e^{\frac{1}{\epsilon}\overline{\overline{G}}(s,0,\eta,\delta,\delta_1)-\mathring{\mu}s}ds},$$

with

$$\begin{aligned}\overline{\overline{G}}(t,\epsilon,\eta,\delta,\delta_1) &= \int_0^t g_I(s,0,\overline{\overline{N}}_{\eta\delta\delta_1},\epsilon)ds, \\ &= (-\mathring{\mu}\epsilon - \gamma)t + \lambda\int_0^t \overline{\overline{N}}_{\eta\delta\delta_1}(s)ds,\end{aligned}$$

for $\epsilon \in \mathbb{I}_{\epsilon_0}$ and $t \in \mathbb{I}_T$.

6.2.3.6 Determination of the Limit of the Infective Population as ϵ Tends to Zero

From the definition (6.2.21), we have

$$\frac{d\overline{\overline{N}}^{(2)}_{\eta\delta\delta_1}}{dt} = r\overline{\overline{N}}^{(2)}_{\eta\delta\delta_1} - \mu^*(\underline{\underline{N}}^{(2)}_{\eta\delta} - \nu - \delta_1) \tag{6.2.23}$$

for $t \geq t_\theta$, while, from (6.2.18),

$$\frac{d\underline{\underline{N}}^{(2)}_{\eta\delta}}{dt} = r\underline{\underline{N}}^{(2)}_{\eta\delta} - (\underline{\underline{N}}^{(2)}_{\eta\delta} - \nu)\mu^*, \tag{6.2.24}$$

for $t \geq \tilde{t}$. It can be noticed that (6.2.23) is a regular perturbation of (6.2.24). Thus, according to the regular perturbation theory, Theorem 3.3.1.1,

$$\Delta N(t) = O(\delta_1)$$

uniformly on $[t_\theta, T]$, where $\Delta N = \overline{\overline{N}}^{(2)}_{\eta\delta\delta_1} - \underline{\underline{N}}^{(2)}_{\eta\delta}$.

Further, from equation (6.2.22), we have

$$\begin{cases} \frac{d\overline{\overline{I}}_{\eta\delta\delta_1}}{dt} = -\mathring{\mu}\overline{\overline{I}}_{\eta\delta\delta_1} + \frac{1}{\epsilon}\overline{\overline{I}}_{\eta\delta\delta_1}\Big[\lambda(\overline{\overline{N}}_{\eta\delta\delta_1} - \overline{\overline{I}}_{\eta\delta\delta_1}) - \gamma\Big], \\ \overline{\overline{I}}_{\eta\delta\delta_1}(0) = I_0,\ t \in \mathbb{I}_T. \end{cases}$$

So, according to the convergence proved in Chapter 5 (see (5.2.7)), it is enough to prove that $\overline{\overline{G}}(t,0) > 0$ for $t \geq t_\theta$ to conclude that

$$\overline{\overline{I}}_{\eta\delta\delta_1}(t,\epsilon) \to \overline{\overline{N}}^{(2)}_{\eta\delta\delta_1} - \nu \text{ for } t \in [t_\theta, T].$$

The derivative of $\overline{\overline{G}}$ on $[t_\theta, T]$ is given by $\frac{d\overline{\overline{G}}_2}{dt}(t,0) = \lambda \overline{\overline{N}}^{(2)}_{\eta\delta\delta_1}(t) - \gamma$. This implies that it is enough to show that

$$\frac{d\overline{\overline{G}}_2}{dt}(t,0) > 0;$$

that is,

$$\overline{\overline{N}}^{(2)}_{\eta\delta\delta_1}(t) > \nu, \tag{6.2.25}$$

for $t \in [t_\theta, T]$. But, since $\Delta N = O(\delta_1)$, proving (6.2.25) is equivalent to prove $\underline{\underline{N}}^{(2)}_{\eta\delta}(t) > \nu$. Let us consider the right hand side of equation (6.2.18),

$$f(\underline{\underline{N}}^{(2)}_{\eta\delta}) = \mathring{r}\underline{\underline{N}}^{(2)}_{\eta\delta} + \nu\mu^*.$$

The equilibrium point of (6.2.18) is

$$\underline{\underline{N}}^* = -\frac{\nu\mu^*}{\mathring{r}}.$$

We consider three cases here:

case 0 If $\mathring{r} = 0$ then $f(\underline{\underline{N}}^{(2)}_{\eta\delta}) = \nu\mu^* > 0$. This implies that solutions are monotone and increasing. Consequently, any solution of (6.2.18) with initial condition $\underline{\underline{N}}_{2\theta} = \underline{\underline{N}}^{(2)}_{\eta\delta}(t_\theta) > \nu$ remains greater than ν all the time.

Case 1 If $\mathring{r} = \beta - \mu - \mu^* > 0$, then $\underline{\underline{N}}^* < 0 < \nu$ and the derivative of f with respect to $\underline{\underline{N}}^{(2)}_{\eta\delta}$, $f_{\underline{\underline{N}}^{(2)}_{\eta\delta}}(\underline{\underline{N}}^{(2)}_{\eta\delta}) = \mathring{r}$, is positive. The equilibrium is repelling. This implies that, since we have monotonic solutions to (6.2.18), those with initial condition $\underline{\underline{N}}_{2\theta} = \underline{\underline{N}}^{(2)}_{\eta\delta}(t_\theta) > \nu$ remains greater than ν as time goes on.

Case 2 If $\mathring{r} = \beta - \mu - \mu^* < 0$, then

$$\underline{\underline{N}}^* = -\frac{\nu\mu^*}{\mathring{r}} = \nu\frac{-\mathring{r} + r}{-\mathring{r}} = \nu(1 + \frac{r}{-\mathring{r}}) > \nu$$

and

$$f_{\underline{\underline{N}}^{(2)}_{\eta\delta}}(\underline{\underline{N}}^{(2)}_{\eta\delta}) = \mathring{r} < 0.$$

Since $\underline{\underline{N}}^*$ is asymptotically stable, any initial condition $\underline{\underline{N}}_{2\theta} = \underline{\underline{N}}^{(2)}_{\eta\delta}(t_\theta) > \nu$ is in its domain of attraction. Therefore the solution tends to the equilibrium and remains above ν since solutions are monotonic.

In each case, the solution remains greater than ν. Hence, $\overline{\overline{I}}_{\eta\delta\delta_1}$ converges uniformly to $\overline{\overline{N}}_{\eta\delta\delta_1} - \nu$. However, by their definitions, $\underline{\underline{I}}_{\eta\delta}$ and $\overline{\overline{I}}_{\eta\delta\delta_1}$ satisfy, respectively,

$$\frac{d\underline{\underline{I}}_{\eta\delta}}{dt} = g(\underline{\underline{I}}_{\eta\delta}, \underline{\underline{N}}^{(2)}_{\eta\delta}, \epsilon)$$

and

$$\frac{d\overline{\overline{I}}_{\eta\delta\delta_1}}{dt} = g(\overline{\overline{I}}_{\eta\delta\delta_1}, \overline{\overline{N}}^{(2)}_{\eta\delta\delta_1}, \epsilon)$$

on $[t_\theta, T]$. Since $\overline{\overline{N}}^{(2)}_{\eta\delta\delta_1} - \underline{\underline{N}}^{(2)}_{\eta\delta} = O(\delta_1)$, $\overline{\overline{I}}_{\eta\delta\delta_1}$ is close to the solution $\tilde{I}_\epsilon$ of

$$\frac{d\tilde{I}_\epsilon}{dt} = g(\tilde{I}_\epsilon, \underline{\underline{N}}^{(2)}_{\eta\delta} + k\delta_1, \epsilon)$$

for some k. Furthermore, on one hand, we have

$$\lim_{\epsilon,\eta,\delta\to 0} \underline{\underline{I}}_{\eta\delta}(t,\epsilon) = \underline{\underline{N}}^{(2)}(t) - \nu,$$

where $\underline{\underline{N}}^{(2)}$ satisfies

$$\frac{d\underline{\underline{N}}^{(2)}}{dt} = \mathring{r}\underline{\underline{N}}^{(2)} + \nu(\mathring{\mu} - \mu), \qquad \underline{\underline{N}}^{(2)}(t_\theta) = N_0 e^{rt_\theta}$$

and on the other hand,

$$\lim_{\substack{\epsilon\to 0\\ \eta,\delta,\delta_1\to 0}} \tilde{I}_\epsilon(t,\epsilon) = \underline{\underline{N}}^{(2)}(t) - \nu.$$

This implies

$$\lim_{\eta,\delta,\delta_1,\epsilon\to 0} \overline{\overline{I}}_{\eta\delta\delta_1}(t,\epsilon) = \underline{\underline{N}}^{(2)}(t) - \nu.$$

Hence, since for $t > \bar{t}^*$, $\epsilon \in \mathbb{I}_{\epsilon_0}$,

$$\underline{\underline{I}}_{\eta\delta}(t,\epsilon) \le I(t,\epsilon) \le \overline{\overline{I}}_{\eta\delta\delta_1}(t,\epsilon),$$

it follows using the squeeze theorem that

$$\lim_{\substack{\eta,\delta,\epsilon\to 0\\ \theta\to 0}} \underline{\underline{I}}_{\eta\delta}(t,\epsilon) = \lim_{\substack{\eta,\delta,\delta',\epsilon\to 0\\ \theta\to 0}} \overline{\overline{I}}_{\eta\delta\delta_1}(t,\epsilon) = \lim_{\epsilon\to 0} I(t,\epsilon) = \underline{\underline{N}}^{(2)}(t) - \nu, \;\; \bar{t}^* < t.$$

6.3 The Case of a Stable Population

Let us consider system (6.1.5) and assume that

$$\beta - \mu < 0 \text{ and } N_0 > \nu. \tag{6.3.1}$$

6.3.1 Dynamics of the Infective Population before the Intersection of the QSSs

Consider $\delta > 0$, and define the domain $\mathcal{N}_\delta = \{N,\ N \geq \nu + \delta\}$. From (6.1.6), it follows that

$$\inf_{N \in \mathcal{N}_\delta} \phi_2(N) = \delta > 0.$$

Therefore, ϕ_2 is an isolated quasi steady state on $\mathcal{N}_\delta$. Further, the degenerate system associated to (6.1.5) is given by

$$\begin{cases} \bar{I}(\lambda(\bar{N} - \bar{I}) - \gamma) = 0, & \\ \frac{d\bar{N}}{dt} = (\beta - \mu)\bar{N} - (\mathring{\mu} - \mu)\bar{I}, & \bar{N}(0) = N_0. \end{cases} \tag{6.3.2}$$

Solving 6.3.2 for $\bar{I}(t) = \phi_2(N_2(t)) = N_2(t) - \nu$, we obtain

$$N_2(t) = \frac{\nu(\mathring{\mu} - \mu)}{\mathring{\mu} - \beta} + ke^{(\beta - \mathring{\mu})t}, \tag{6.3.3}$$

where $k = N_0 - \frac{\nu(\mathring{\mu} - \mu)}{\mathring{\mu} - \beta}$ and $t \in \mathbb{I}_T$. It follows that, for $t \in \mathbb{I}_T$,

$$N_2(t) = \nu\frac{\mathring{\mu} - \mu}{\mathring{\mu} - \beta}(1 - e^{(\beta - \mathring{\mu})t}) + N_0 e^{(\beta - \mathring{\mu})t} \geq 0,$$

since $\beta - \mu < 0$. Let us determine the time t_c at which $N_2 = \nu$. From 6.3.3, it follows that N_2 can be rewritten as

$$N_2(t) = -\frac{\nu\mu^*}{\mathring{r}} + \big(N_0 + \frac{\nu\mu^*}{\mathring{r}}\big)e^{\mathring{r}t}$$

with $\mathring{r} = \beta - \mu - \mu^*$ and $t \in \mathbb{I}_T$. Thus

$$t_c = \frac{1}{\mathring{r}}\ln\big(\frac{\nu + \frac{\nu\mu^*}{\mathring{r}}}{N_0 + \frac{\nu\mu^*}{\mathring{r}}}\big)$$

From condition (6.3.1), it follows that $t_c > 0$. Hence, applying the Tikhonov theorem on $\mathcal{N}_\delta$ and letting δ tend to zero, it follows, under condition (6.3.1), that

$$\lim_{\epsilon \to 0} I(t,\epsilon) = \phi_2(N_2(t)) = N_2(t) - \nu, \qquad t \in (0, t_c),$$
$$\lim_{\epsilon \to 0} N(t,\epsilon) = N_2(t), \qquad t \in [0, t_c).$$

6.3.2 Dynamics of the Infective Population after the Intersection of the QSSs

From the previous result, under the assumptions (6.3.1), and by the continuity of the solution of (6.1.5), for any $\rho_0 > 0$ there exist $t^1 < t_c$ and $\epsilon_0 > 0$ such that for $\epsilon \in \mathbb{I}_{\epsilon_0}$,

$$0 < \nu - \rho_0 < N(t^1, \epsilon) \le \nu + \rho_0 \text{ and } I(t^1, \epsilon) \le \rho_0. \tag{6.3.4}$$

Let us denote $f(I, N) = (\beta - \mu)N - \mu^* I$. Since $\beta - \mu < 0$, from (6.3.4), it follows that

$$f(I, N) < (\beta - \mu)N < (\beta - \mu)(\nu - \rho_0) < 0,$$

for all positive variables I, N. Therefore, the trajectory of the solution to (6.1.5) enters the domain of attraction of ϕ_1 and no trajectory can leave through the wall

$$R_{\rho_0} = \{(I, N), N = \nu - \rho_0, 0 < I < \rho_0\}$$

from $N < \nu - \rho_0$. Further, consider the composite function

$$\Phi(N) = \begin{cases} \phi_2(N), & N \ge \nu, \\ \phi_1(N), & N \le \nu. \end{cases}$$

For $I > \Phi$, we have $I > N - \nu$ for all $N > 0$. It follows that

$$g(I, N, 0) = I(\lambda(N - I) - \gamma) < 0,$$

for all $N, I > 0$ and $I > \Phi$. This implies that, considering $I \ge \Phi + \alpha$ with $\alpha > 0$, for $\epsilon \in \mathbb{I}_{\epsilon_0}$ there exists $\beta_1 > 0$ such that

$$g(I, N, \epsilon) = -\epsilon\mathring{\mu}I + I(\lambda(N - I) - \gamma) < -\beta_1,$$

for $N > 0$. That is

$$g(I, N, \epsilon) \le 0,$$

for $\epsilon \in \mathbb{I}_{\epsilon_0}$, $I > \Phi$, and $N > 0$. Therefore, for $\epsilon \in \mathbb{I}_{\epsilon_0}$, there exists $w_\epsilon > 0$ such that

$$0 < I(t, \epsilon) < w_\epsilon,$$

for $t > t^1$ with $w_\epsilon \to 0$ as $\epsilon \to 0$. In other words, the solution is bounded.

Hence letting t^1 tends to t_c we obtain, since the solution is in the region where $I'(t, \epsilon) < 0$,

$$\lim_{\epsilon \to 0} I(t, \epsilon) = 0,$$

uniformly on $[t_c, T]$. Therefore, by the regular perturbation theory,

$$\lim_{\epsilon \to 0} N(t, \epsilon) = N_0 e^{rt},$$

uniformly on $[t_c, T]$.

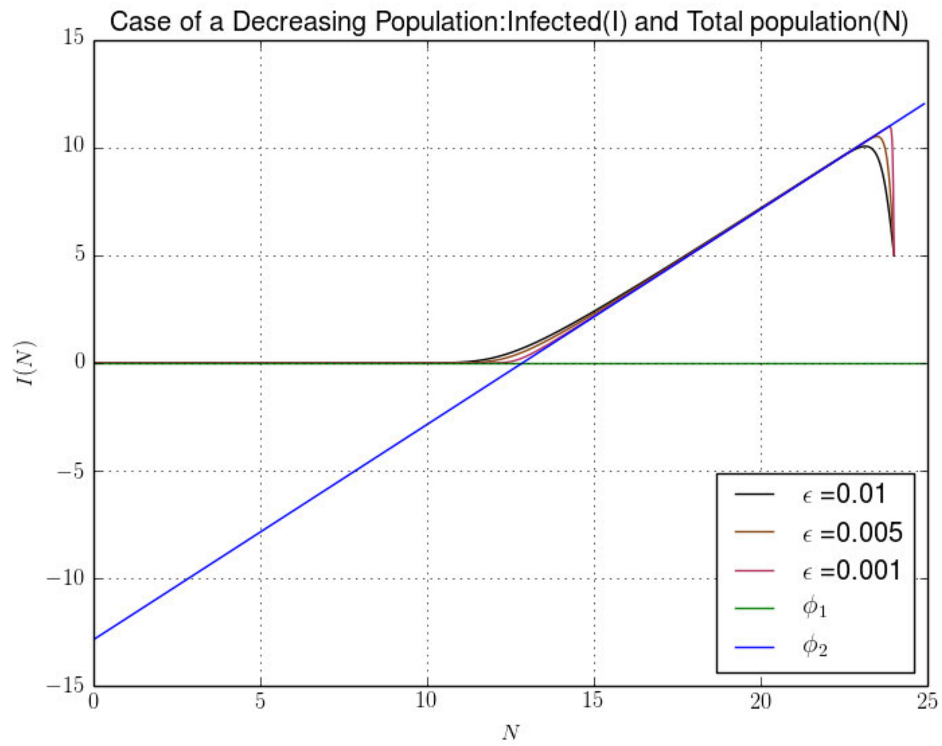

Figure 6.1: Case of a two dimensional stable population. The orbits are traversed from the right to the left. We observe the convergence to the second quasi steady state and to the first quasi steady state, without delay after passing the intersection point.

6.3.3 Numerical Analysis

Figure 6.1 shows the orbits of the system (6.1.5) for $\epsilon = 0.01,\ 0.005,\ 0.001$. It presents the dynamics of the infectives population in term of the total population. The parameters considered in this case are $\mu = 0.2,\ \mathring{\mu} = 0.3,\ \beta = 0.1,\ \gamma = 0.18,\ \lambda = 0.014$. The initial condition is $(N_0, I_0) = (24, 5)$. We can observe that there is no delay in the stability switch. The solutions tend to the second quasi steady state and then, after passing the intersection point, they tend to the first quasi steady state without delay.

6.4 Conclusion

In this chapter, we used the method of upper and lower bounds to study the two dimensional problem (6.1.1). We found similar results to the one dimensional case. We proved that there is a delay of stability switch in the case of an unstable population, while in the case of a stable population, the switch between the second quasi steady state and the first one is without delay.

The next chapter consists mainly in the generalisation of the method used in chapter 5 for general one dimensional non-autonomous models.

7 General Theories for One Dimensional Problems

In this chapter we aim to extend the Butuzov theorems,[19], in the light of the work done in Chapter 5. The Butuzov theorems concern two types of bifurcation (the transcritical and the pitchfork bifurcations) for the unstable one dimensional case and study the asymptotic behaviour of solutions of singularly perturbed scalar differential equations of the form

$$\begin{cases} \epsilon \frac{dx}{dt} = g(t, x, \epsilon), & t \in \mathbb{I}_T,\ x \in M_x, \\ x(t_0, \epsilon) = x_0, & \epsilon \in \mathbb{I}_{\epsilon_0}, \end{cases} \tag{7.0.1}$$

with $M_x \subset \mathbb{R}$ an open bounded interval containing the origin and

$$\epsilon \in \mathbb{I}_{\epsilon_0} = \{\epsilon : 0 < \epsilon < \epsilon_0 << 1\}.$$

We proceed first by making some remarks on the original proof of the Butuzov theorem with positive initial conditions for the case of transcritical bifurcation; second we provide a proof of the Butuzov theorems with both initial conditions (negative and positive) for the case of a pitchfork bifurcation; and finally we state and prove the theory for the one dimensional stable case with positive initial conditions.

7.1 Some Remarks on the Butuzov Theorem for the Case of a Transcritical Bifurcation

In the following remarks, we are going to present and prove some facts used in the proof of the Butuzov theorem for the case of a transcritical bifurcation without a full justification.

7.1.0.1 Remark. Let us consider the problem (7.0.1) and let us denote by ϕ the second quasi steady state satisfying the assumptions of the Butuzov theorem, Theorem 3.3.4.1. If a solution of (7.0.1) is in the basin of attraction of ϕ from a time $\bar{t}$ to a time $\hat{t} \leq T$, then it converges to the second quasi steady state for $t \in (\bar{t}, \hat{t}]$ as ϵ tends to zero. It is important to note that here the solution at $\bar{t}$ is depending on ϵ. This case is different from the original problem (7.0.1) where the initial condition is a constant.

Proof. Let us consider problem (7.0.1) but with initial time at $\bar{t}$. We obtain the following problem

$$\epsilon \frac{dx}{dt} = g(t, x, \epsilon),$$

where $t \in [\bar{t}, \hat{t}]$, $\epsilon \in \mathbb{I}_{\epsilon_0}$ and with the initial condition $x(\bar{t}, \epsilon)$. We want to prove that

$$\lim_{\epsilon \to 0} x(t, \epsilon) = \phi(t)$$

for $t \in (\bar{t}, \hat{t}]$. We know, by hypothesis, that the solutions of (7.0.1) are in the basin of attraction of ϕ only from a time $\bar{t}$ to a time $\hat{t}$. This means that there exist a and b in the basin of attraction of ϕ such that

$$a < x(\bar{t}, \epsilon) < b.$$

Let x_i be the solution to (7.0.1) with the initial condition $x_i(t_0, \epsilon) = i, \; i \in \{a, b\}$. According to the Tikhonov theorem,

$$\lim_{\epsilon \to 0} x_i(t, \epsilon) = \phi(t),$$

for $t \in (\bar{t}, \hat{t}]$. Since solutions cannot intersect, we must have

$$x_a(t, \epsilon) < x(t, \epsilon) < x_b(t, \epsilon), \; t \in (\bar{t}, \hat{t}].$$

It follows, using the squeeze theorem, that

$$\lim_{\epsilon \to 0} x(t, \epsilon) = \phi(t),$$

for $t \in (\bar{t}, \hat{t}]$. □

7.1.0.2 Remark. Let $x = \phi(t)$ be the second quasi steady state in problem (7.0.1) satisfying the assumptions of the Butuzov theorem, Theorem 3.3.4.1. Then, for any $\mathring{t} \in (t_c, T)$, there exist an $\hat{\epsilon} \in \mathbb{I}_{\epsilon_0}$ and $r > 0$ such that if a solution x to (7.0.1) satisfies for all $\epsilon \in \mathbb{I}_{\hat{\epsilon}}$, $s \in B(\phi(t), r)$,

$$\phi(\check{t}) - s < x(\check{t}, \epsilon) < \phi(\check{t}) + s,$$

for some $\check{t} \in [\mathring{t}, T]$, then

$$\phi(t) - s < x(t, \epsilon) < \phi(t) + s,$$

for all $t \in [\check{t}, T]$.

Proof. Since $\phi(t) > 0$ for all $t \in (t_c, T)$, then for $\mathring{t} \in (t_c, T)$, there exists $r > 0$ such that for $t \in (\mathring{t}, T)$, $s \in B(\phi(t), r)$, $\phi(t) + s > 0$ and $\phi(t) - s > 0$. Since $g \in \mathcal{C}^2(\bar{D})$, it follows that for r small enough $M_x \cap B(\phi(t), r) = B(\phi(t), r)$. Thus, from assumptions (II) and (III) of Theorem 3.3.4.1, $g(t, \phi(t) - s, 0) > 0$ and $g(t, \phi(t) + s, 0) < 0$ on $[\mathring{t}, T]$. Since the functions

$$(t, \epsilon) \to g(t, \phi(t) \pm s, \epsilon), \quad s \in B(\phi(t), r),$$

are continuous on $\mathbb{I}_T \times \mathbb{I}_{\epsilon_0}$ and $[\mathring{t}, T] \times \{0\}$ is a compact in $\mathbb{I}_T \times \mathbb{I}_{\epsilon_0}$, there is an open set containing $[\mathring{t}, T] \times \{0\}$ such that

$$g(t, \phi(t) + s, \epsilon) < 0 \text{ and } g(t, \phi(t) - s, \epsilon) > 0.$$

In particular there is $\epsilon_{\mathring{t}} > 0$ such that

$$\inf_{t \in [\mathring{t}, T],\ \epsilon \in [0, \epsilon_{\mathring{t}}]} g(t, \phi(t) - s, \epsilon) := d_- > 0 \text{ and } \sup_{t \in [\mathring{t}, T],\ \epsilon \in [0, \epsilon_{\mathring{t}}]} g(t, \phi(t) + s, \epsilon) := d_+ < 0. \tag{7.1.1}$$

Now, let us consider the following function

$$D(t) = (x(t, \epsilon) - \phi(t))^2.$$

It is a Lyapunov type function and represents the square of the distance between $x(t, \epsilon)$ and $\phi(t)$. Let us assume that the solution leaves the region $\{(t, x), \phi(t) - s \le x \le \phi(t) + s\}$ at a time τ. This means that D increases at τ and consequently, $\frac{dD}{dt}(\tau) > 0$. On the other hand, if the solution to (7.0.1) leaves the region $\{(t, x), \phi(t) - s \le x \le \phi(t) + s\}$ at time τ and if $x(\tau, \epsilon) = \phi(\tau) - s$, then

$$\frac{dD}{dt}(\tau) = -2s(\frac{1}{\epsilon} g(\tau, x(\tau, \epsilon), \epsilon) - \frac{d\phi}{dt}(\tau)).$$

From (7.1.1), it follows that

$$\frac{dD}{dt}(\tau) < 2s\Big(- \frac{d_-}{\epsilon} + \sup_{t \in [\mathring{t}, T]} \Big| \frac{d\phi}{dt}(t) \Big| \Big) < 0,$$

for ϵ small enough; that is, for $\epsilon \in (0, \hat{\epsilon})$, where $\hat{\epsilon} = \min(\epsilon_0, \frac{d_-}{\sup_{t \in [\mathring{t}, T]} |\phi(t)|})$. Thus a contradiction. In the same way, if the solution leaves the region $\{(t, x), \phi(t) - s \le x \le \phi(t) + s\}$ at $x(\tau, \epsilon) = \phi(\tau) + s$ at time τ, then

$$\frac{dD}{dt}(\tau) = 2s(\frac{1}{\epsilon} g(\tau, x(\tau, \epsilon), \epsilon) - \frac{d\phi}{dt}(\tau)) < 2s\Big(\frac{d_+}{\epsilon} + \sup_{t \in [\mathring{t}, T]} | \frac{d\phi}{dt}(t) | \Big) < 0,$$

for ϵ small enough. It is enough to take $\epsilon \in (0, \hat{\epsilon})$ with $\hat{\epsilon} = \min(\epsilon_0, \frac{-d_+}{\sup_{t \in [\mathring{t}, T]} |\phi(t)|})$. Thus a contradiction. □

7.1.0.3 Remark. Referring to the proof of the Butuzov theorem in [19], we noticed that the assumption that the function g is two times differentiable with respect to all variables is too strong. Indeed, for $\frac{G(t,\epsilon)}{\epsilon} \leq \frac{G(t,0)}{\epsilon} + k$ to be true, it is only needed that $g_x(t, 0, \epsilon)$ be a Lipschitz continuous function in $\mathbb{I}_{\epsilon_0}$, uniformly in $t \in [t_0, t^*]$. Furthermore, the relation $g_x(t, 0, \epsilon) - g_x(t, x, \epsilon) = -\frac{1}{2} g_{xx}(t, x^*, \epsilon)x^2$, together with earlier calculations, require g to be a C^2-function with respect to x. Finally, for ϵ small enough, it is sufficient for G to be a C^1-function in some neighbourhood of (t^*, ϵ) in which $g_x(t, 0, \epsilon)$ is C^1 which respect to ϵ and uniformly in t in a neighbourhood of t^*. Summarising, it is enough to assume that g is a Lipschitz function with respect to all variables, on $\bar{D}$, a C^2-function with respect to x, uniformly on M_x and finally, there is a neighbourhood of (t^*, ϵ) where $g_x(t, 0, \epsilon)$ is differentiable with respect to ϵ, uniformly in t.

7.1.0.4 Remark. Let us simplify condition (VI) of the Butuzov theorem, Theorem 3.3.4.1, which assumes the existence of a positive constant c_0 such that $\pm c_0 \in M_x$ and

$$g(t, x, \epsilon) \leq g_x(t, 0, \epsilon)x \tag{7.1.2}$$

for $t \in [t_0, t^*]$, $\epsilon \in \bar{\mathbb{I}}_{\epsilon_0}$, and $|x| \leq c_0$. According to the Taylor theorem, any continuous function S differentiable $n + 1$ times and with its $n - th$ derivative $\frac{d^n S}{dt^n}$ continuous on the closed interval between $a \in \mathbb{R}$ and x can be decomposed as $S(x) = T_n(x) + R_n(x)$ with

$$T_n(x) = S(a) + (x - a)\frac{dS}{dx}\Big|_{x=a} + \frac{(x - a)^2}{2!}\frac{d^2S}{dx^2}\Big|_{x=a} + \dots + \frac{(x - a)^n}{n!}\frac{d^nS}{dx^n}\Big|_{x=a},$$

and

$$R_n(x) = \frac{1}{n!}\int_a^x (x - s)^n \frac{d^{n+1}S}{dt^{n+1}}(s)ds.$$

So, for $S = g$, $a = 0$ and $n = 1$, we have

$$g(t, x, \epsilon) = T_1(t, x, \epsilon) + R_1(t, x, \epsilon)$$

with

$$T_1(t, x, \epsilon) = g(t, 0, \epsilon) + xg_x(t, 0, \epsilon) = xg_x(t, 0, \epsilon)$$

and

$$R_1(t, x, \epsilon) = \int_0^x (x - s)g_{xx}(t, s, \epsilon)ds.$$

It can be noted that $s \in (0, x)$; that is $x - s > 0$. Thus, if $g_{xx}(t, s, \epsilon) < 0$ for all $s \in [0, x]$, then

$$R_1(t, x, \epsilon) \leq 0,$$

and hence

$$g(t,x,\epsilon) \le x g_x(t,0,\epsilon).$$

In conclusion, for condition (7.1.2) to be satisfied, it is enough that g be differentiable, g_x continuous on a closed interval containing 0 and $g_{xx} \le 0$ on that interval.

7.2 Proof of the Butuzov Theorem for Pitchfork Bifurcation

Let us consider the following assumptions, [19].

$(II)^*$ The degenerate equation $g(t,x,0) = 0$ has a set of three quasi steady states: $x \equiv 0$ for $t \in [t_0, T]$, and $x = \psi_\pm(t)$ defined for $t \in (t_c, T]$ such that $\psi_\pm(t_c) = 0$, $\psi_+(t) > 0$ and $\psi_-(t) < 0$ for $t \in (t_c, T]$.

$(III)^*$ There is a stability switch at t_c; that is,

$$g_x(t,0,0) < 0, \quad t \in (t_0, t_c),$$
$$g_x(t,0,0) > 0 \text{ and } \; g_x(t,\psi_\pm(t),0) < 0, \; t \in (t_c, T].$$

$(VI)^*$ The following set of inequalities are satisfied

$$g(t,x,\epsilon) \le g_x(t,0,\epsilon)x \text{ for } t \in [t_0, t^*], \; x \in [0, c_0], \; \epsilon \in \bar{\mathbb{I}}_{\epsilon_0},$$
$$g(t,x,\epsilon) \ge g_x(t,0,\epsilon)x \text{ for } t \in [t_0, t^*], \; x \in [-c_0, 0], \; \epsilon \in \bar{\mathbb{I}}_{\epsilon_0},$$

for some c_0 such that $\pm c_0 \in M_x$, and where t^* is the root of the function G defined by (3.3.6).

7.2.0.1 Theorem. *Let us assume that the conditions (I), (IV), (V) of Theorem 3.3.4.1 and the conditions $(II)^*$, $(III)^*$, and $(VI)^*$ hold. Then, for sufficiently small $\epsilon > 0$, there exists a unique solution to problem* (7.0.1) *such that,*

$$\lim_{\epsilon \to 0} x(t,\epsilon) = \begin{cases} 0 & \text{for } t \in (t_0, t^*) \\ \psi_+(t) & \text{for } t \in (t^*, T] \end{cases} \qquad \text{if } x_0 > 0,$$

$$\lim_{\epsilon \to 0} x(t,\epsilon) = \begin{cases} 0 & \text{for } t \in (t_0, t^*) \\ \psi_-(t) & \text{for } t \in (t^*, T] \end{cases} \qquad \text{if } x_0 < 0,$$

see [19].

Proof. 1. Let us assume $x_0 > 0$. As in the proof of the transcritical bifurcation, [19], we use the method of lower and upper solutions knowing that the existence of the ordered upper and lower solutions to the problem (7.0.1) implies the existence of a unique solution $x(t,\epsilon)$. Under the assumptions $(I), (II)^*, (III)^*$, the Tikhonov theorem implies that for any given $\delta > 0$ with $t_0 + \delta < t_c$, there exists $\epsilon(\delta)$ such that for $\epsilon \in (0, \epsilon(\delta))$ the solution of (7.0.1) exists and it satisfies the inequality $x(t_0 + \delta, \epsilon) \leq c_0$ for $t \in [t_0, t_0 + \delta]$, where c_0 was introduced in $(VI)^*$. Thus, we can assume that $x_0 < c_0$ without loss of generality. Let us consider the following function

$$\bar{X}_1(t,\epsilon) = x_0 e^{\frac{G(t,\epsilon)}{\epsilon}}, \qquad t \in [t_0, t^* - \alpha] \tag{7.2.1}$$

where α is any small number such that $t^* - \alpha > t_c$ and the function G is defined in (3.3.6). Clearly,

$$\epsilon \frac{d\bar{X}_1}{dt} = g_x(t, 0, \epsilon)\bar{X}_1, \qquad \bar{X}_1(t_0, \epsilon) = x_0.$$

By the assumption (I), there exists $k \in \mathbb{R}_+$ such that $G(t,\epsilon) - G(t,0) < \epsilon k$ for $t \in [t_0, t^*]$, $\epsilon \in \mathbb{I}_{\epsilon_0}$. Let us consider α such that $t_0 + \alpha < t_c$. Since from $(III)^*$,

$$G(t,0) < 0 \text{ for } t_0 < t < t^*, \tag{7.2.2}$$

then, from (7.2.2), there exists $\epsilon(\alpha) \in \mathbb{I}_{\epsilon_0}$ such that for $\epsilon \in \mathbb{I}_{\epsilon(\alpha)}$,

$$G(t,\epsilon) \leq 0, \qquad t \in [t_0, t_0 + \alpha], \tag{7.2.3}$$

$$G(t,\epsilon)/\epsilon < 0, \qquad t \in [t_0 + \alpha, t^* - \alpha]. \tag{7.2.4}$$

From (7.2.1) and (7.2.4), it follows that

$$\bar{X}_1(t,\epsilon) \leq c_0$$

for $t \in [t_0 + \alpha, t^* - \alpha]$ and $\epsilon \in \mathbb{I}_{\epsilon(\alpha)}$. Hence, from condition $(VI)^*$,

$$\epsilon \frac{d\bar{X}_1}{dt} - g(t, \bar{X}_1, \epsilon) = g_x(t,0,\epsilon)\bar{X}_1 - g(t, \bar{X}_1, \epsilon) \geq 0 \text{ for } t \in [t_0 + \alpha, t^* - \alpha), \epsilon \in \mathbb{I}_{\epsilon(\alpha)}.$$

Thus, $\bar{X}_1$ is an upper solution to the problem (7.0.1) for $t \in [t_0 + \alpha, t^* - \alpha]$ and for $\epsilon \in \mathbb{I}_{\epsilon(\alpha)}$. Since α is an arbitrarily small positive number, it follows from (7.2.1) and (7.2.4) that

$$\lim_{\epsilon \to 0} \bar{X}_1(t,\epsilon) = 0 \text{ for } t \in (t_0, t^*).$$

By assumption (IV), since $x_0 > 0$, a trivial lower solution of the problem (7.0.1) is $\underline{X} \equiv 0$. It follows, by the squeeze theorem, that

$$\lim_{\epsilon \to 0} x(t,\epsilon) = 0 \text{ for } t \in (t_0, t^*),\ x_0 > 0.$$

On the other hand, in order to prove that

$$\lim_{\epsilon \to 0} x(t,\epsilon) = \psi_+(t),\ t > t^*,$$

let us consider the following function

$$\underline{X}(t,\epsilon) = \eta e^{\frac{G(t,\epsilon)-\delta(t-t_0)}{\epsilon}}, \qquad t \in (t_0, \bar{t}),$$

where $\bar{t} \in (t^*, T)$, η and δ are the small positive numbers, independent of ϵ that will be determined later. Let us also assume that

$$\eta \le \min(\min_{t^* \le t \le T} \psi_+(t), x_0).$$

It is easy to prove that $\underline{X}$ satisfies

$$\epsilon \frac{d\underline{X}}{dt} = (g_x(t,0,\epsilon) - \delta)\underline{X}, \ \ \underline{X}(t_0,\epsilon) = \eta \le x_0,$$

on $[t_0, \bar{t}]$. Thus,

$$\epsilon \frac{d\underline{X}}{dt} - g(t,\underline{X},\epsilon) = g_x(t,0,\epsilon)\underline{X} - g(t,\underline{X},\epsilon) - \delta\underline{X}. \tag{7.2.5}$$

By assumptions $(III)^*$ and (V), for $t \in (t_0, t^*)$, $G(t,0)$ is negative and has a simple zero at $t = t^*$. Hence, $G(t,0) - \delta(t - t_0)$ has a simple zero at $t = t^* + \Delta(\delta) > 0$ for sufficiently small positive number δ and where $\Delta > 0$ is a positive function such that $\Delta(\delta) \to 0$ as $\delta \to 0$. Moreover $G(t,0) - \delta(t-t_0)$ is negative on $(t_0, t^* + \Delta(\delta))$. It follows that there exists $\epsilon_1(\delta) \le \epsilon(\alpha)$, such that $G(t,\epsilon) - \delta(t - t_0)$ has a unique simple zero for $\epsilon \in \mathbb{I}_{\epsilon_1}$ at $t = \hat{t}_{\delta\epsilon} = t^* + \Delta(\delta) + \omega(\epsilon)$ with $G(t,\epsilon) - \delta(t-t_0)$ negative for $t \in (t_0, \hat{t}_{\delta\epsilon})$, where $\omega(\epsilon) \to 0$ as $\epsilon \to 0$, $\epsilon_1(\delta)$ is taken sufficiently small and $\Delta(\delta) + \omega(\epsilon) > 0$. Hence

$$\underline{X}(t,\epsilon) \le \eta,\ t \in (t_0, \hat{t}_{\delta\epsilon}), \tag{7.2.6}$$

where $\underline{X}(\hat{t}_{\delta\epsilon}, \epsilon) = \eta$.

From assumption (IV) it follows that

$$g(t,x,\epsilon) = g_x(t,0,\epsilon)x + \frac{1}{2} g_{xx}(t,x^*,\epsilon)x^2,\ x \in M_x.$$

It follows, for $0 < x < c_0$ and for some constant k_1, that

$$g_x(t, 0, \epsilon)x - g(t, x, \epsilon) \leq k_1 x^2.$$

Thus, from (7.2.5),

$$\epsilon \frac{d\underline{X}}{dt} - g(t, \underline{X}, \epsilon) \leq k_1 \underline{X}^2 - \delta \underline{X}.$$

For $\eta \leq \frac{\delta}{k_1}$, it follows, from (7.2.6), that

$$\epsilon \frac{d\underline{X}}{dt} - g(t, \underline{X}, \epsilon) \leq 0.$$

Thus, for $t \in (t_0, \hat{t}_{\delta\epsilon})$, $\underline{X}(t, \epsilon)$ is a lower solution to problem (7.0.1) and

$$x(\hat{t}_{\delta\epsilon}, \epsilon) \geq \underline{X}(\hat{t}_{\delta\epsilon}, \epsilon) = \eta.$$

Let us define $\bar{\omega}(\delta)$ to be the maximum of $|\omega(\epsilon)|$ on $I_{\epsilon_1(\delta)}$ and $\bar{t}_\delta = t^* + \Delta(\delta) + \bar{\omega}(\delta)$. We have $\bar{\omega}(\delta) \to 0$ as $\delta \to 0$ and thus $\bar{t}_\delta \to t^*$ as $\delta \to 0$. Now let us construct a lower solution on $[\hat{t}_{\delta\epsilon}, \bar{t}_\delta]$. Let us set ϵ_1 sufficiently small such that, according to (I), $(II)^*$ and (IV), for $\epsilon \in \mathbb{I}_{\epsilon_1}$,

$$g(t, \eta, \epsilon) > 0 \text{ for } t^* \leq t \leq T.$$

It is easy to see that $\underline{X} = \eta$ satisfies

$$\epsilon \frac{d\underline{X}}{dt} - g(t, \underline{X}, \epsilon) \leq 0.$$

Thus, the function $\underline{X}(t, \epsilon) = \eta$ is a lower bound on $[\hat{t}_{\delta\epsilon}, \bar{t}_\delta]$. Therefore

$$x(\bar{t}_\delta, \epsilon) \geq \eta > 0. \tag{7.2.7}$$

Since η lies in the basin of attraction of the stable root $\psi_+(t)$ on the interval $[\hat{t}_{\delta\epsilon}, \bar{t}_\delta]$, according to Remark 7.1.0.1, the solution x converges to the second quasi steady state on $[\hat{t}_{\delta\epsilon}, \bar{t}_\delta]$. It follows that there exist $s, \hat{\epsilon} > 0$ such that for all $\epsilon \in \mathbb{I}_{\hat{\epsilon}}$,

$$\psi_+(\check{t}_{\delta\epsilon}) - s < x(\check{t}_{\delta\epsilon}, \epsilon) < \psi_+(\check{t}_{\delta\epsilon}) + s$$

for some $\check{t}_{\delta\epsilon} \in [\hat{t}_{\delta\epsilon}, \bar{t}_\delta]$. Therefore, according to Remark 7.1.0.2,

$$\psi_+(t) - s < x(t, \epsilon) < \psi_+(t) + s,$$

for $t \in [\check{t}_{\delta\epsilon}, T]$. Since $\check{t}_{\delta\epsilon}$ tends to t^* as ϵ, δ tend to zero, it follows, according to Remark 7.1.0.1, that

$$\lim_{\epsilon \to 0} x(t, \epsilon) = \psi_+(t) \text{ for } t \in (t^*, T]. \tag{7.2.8}$$

2. Let us assume $x_0 < 0$ and let us now prove the theorem with a negative initial condition $x(t_0, \epsilon) = x_0$. Similarly to the previous case, we apply the same method of lower and upper solutions knowing that the existence of ordered upper and lower solutions of the problem (7.0.1) implies the existence of the unique solution $x(t, \epsilon)$. Under the assumptions $(I), (II)^*, (III)^*$, the Tikhonov theorem implies that for any given $\delta > 0$ with $t_0 + \delta < t_c$ there exists $\epsilon(\delta)$ such that for $\epsilon \in (0, \epsilon(\delta))$, the solution of (7.0.1) exists and, for $t \in [t_0, t_0 + \delta]$, satisfies the inequality $x(t_0 + \delta, \epsilon) \geq -c_0$, where c_0 is defined in $(VI)^*$. Thus we can assume that $x_0 > -c_0$ without loss of generality.

   Let us consider the following lower bound

$$\underline{X}_1(t, \epsilon) = x_0 e^{\frac{G(t,\epsilon)}{\epsilon}} \text{ for } t \in [t_0, t^* - \alpha],\ \epsilon \in \mathbb{I}_{\epsilon(\alpha)}, \tag{7.2.9}$$

   where $x_0 < 0$, α is any small number such that $t^* - \alpha > t_c$ and the function G is defined in (3.3.6). Clearly,

$$\epsilon \frac{d\underline{X}_1}{dt} = g_x(t, 0, \epsilon)\underline{X}_1 \text{ for } \underline{X}_1(t_0, \epsilon) = x_0$$

   is satisfied. As previously, according to (I) and $(III)^*$,

$$G(t, 0) < 0 \text{ for } t_0 < t < t^*, \tag{7.2.10}$$

   and there exists $k \in \mathbb{R}_+$ such that

$$|G(t, \epsilon) - G(t, 0)| < \epsilon k \text{ for } t \in [t_0, t^*],\ \epsilon \in \mathbb{I}_{\epsilon_0}. \tag{7.2.11}$$

   Thus, from (7.2.10), it follows that for a given α such that $t_0 + \alpha < t_c$ there exists $\epsilon(\alpha) \in (0, \epsilon_0)$ such that for $\epsilon \in \mathbb{I}_{\epsilon(\alpha)}$,

$$G(t, \epsilon) \leq 0, \quad t \in [t_0, t_0 + \alpha], \tag{7.2.12}$$

$$G(t, \epsilon)/\epsilon < 0, \quad t \in [t_0 + \alpha, t^* - \alpha]. \tag{7.2.13}$$

   From (7.2.9) and (7.2.13) it follows that

$$\underline{X}_1(t, \epsilon) \geq -c_0 \text{ for } t \in [t_0 + \alpha, t^* - \alpha],\ \epsilon \in \mathbb{I}_{\epsilon(\alpha)}.$$

   Hence, from condition $(VI)^*$, we obtain the following relation

$$\epsilon \frac{d\underline{X}_1}{dt} - g(t, \underline{X}_1, \epsilon) \leq g_x(t, 0, \epsilon)\underline{X}_1 - g(t, \underline{X}_1, \epsilon) \leq 0,$$

for $t \in [t_0 + \alpha, t^* - \alpha]$ and $\epsilon \in \mathbb{I}_{\epsilon(\alpha)}$. Thus, $\underline{X}_1$ is a lower solution of the problem (7.0.1) for $t \in [t_0 + \alpha, t^* - \alpha]$ and $\epsilon \in \mathbb{I}_{\epsilon(\alpha)}$. Since α is an arbitrarily small positive number, it follows from (7.2.9) and (7.2.13) that

$$\lim_{\epsilon \to 0} \underline{X}_1(t, \epsilon) = 0,$$

for $t \in (t_0, t^*)$.

Since $x_0 < 0$, by assumption (IV) a trivial upper solution of the problem (7.0.1) is $\underline{X} \equiv 0$.

Therefore, according to the squeeze theorem,

$$\lim_{\epsilon \to 0} x(t, \epsilon) = 0 \text{ for } t \in (t_0, t^*),\ x_0 < 0.$$

To prove the convergence of x to ψ_- after t^*, let us consider the following function defined on $(t_0, \bar{t})$, with $\bar{t} \in (t^*, T)$, by

$$\bar{X}(t, \epsilon) = -\eta e^{\frac{G(t,\epsilon) - \delta(t - t_0)}{\epsilon}},$$

where the small and positive numbers η and δ are independent of ϵ and will be chosen later. Also, let us assume

$$\eta \leq \min(\min_{t^* < t < T} |\psi_-(t)|, -x_0). \tag{7.2.14}$$

It is easy to prove that the following differential equation is satisfied:

$$\epsilon \frac{d\bar{X}}{dt} = (g_x(t, 0, \epsilon) - \delta)\bar{X}, \quad \bar{X}(t_0, \epsilon) = -\eta \geq x_0.$$

Thus,

$$\epsilon \frac{d\bar{X}}{dt} - g(t, \bar{X}, \epsilon) = g_x(t, 0, \epsilon)\bar{X} - g(t, \bar{X}, \epsilon) - \delta \bar{X}. \tag{7.2.15}$$

By assumptions $(III)^*$ and (V), for $t \in (t_0, t^*)$, $G(t, 0)$ is negative and has a simple zero at $t = t^*$. Hence, $G(t, 0) - \delta(t - t_0)$ has a simple zero at $t = t^* + \Delta(\delta) > 0$ for sufficiently small positive δ and where $\Delta > 0$ is a positive function such that $\Delta(\delta) \to 0$ as $\delta \to 0$. Moreover, $G(t, 0) - \delta(t - t_0)$ is negative on $(t_0, t^* + \Delta(\delta))$. It follows that there exists $\epsilon_1(\delta) \leq \epsilon(\alpha)$ such that $G(t, \epsilon) - \delta(t - t_0)$ has a unique simple zero for $\epsilon \in \mathbb{I}_{\epsilon_1}$ at $\hat{t}_{\delta\epsilon} = t^* + \Delta(\delta) + \omega(\epsilon)$ with $G(t, \epsilon) - \delta(t - t_0)$ negative on $(t_0, \hat{t}_{\delta\epsilon})$ with $\omega(\epsilon) \to 0$ as $\epsilon \to 0$ and $\epsilon_1(\delta)$ sufficiently small such that $\Delta(\delta) + \omega(\epsilon) > 0$. Hence

$$\bar{X}(t, \epsilon) \geq -\eta,\ t \in (t_0, \hat{t}_{\delta\epsilon}), \tag{7.2.16}$$

where $\bar{X}(\hat{t}_{\delta\epsilon}, \epsilon) = -\eta$. From assumption (IV) it follows that

$$g(t, x, \epsilon) = g_x(t, 0, \epsilon)x + \frac{1}{2}g_{xx}(t, x^*, \epsilon)x^2.$$

Therefore, for $-c_0 < x < 0$ and for some constant k_1,

$$g_x(t, 0, \epsilon)x - g(t, x, \epsilon) \geq k_1 x^2.$$

Hence, from (7.2.15),

$$\epsilon\frac{d\bar{X}}{dt} - g(t, \bar{X}, \epsilon) \geq k_1\bar{X}^2 - \delta\bar{X}.$$

For $\eta \geq \frac{\delta}{k_1}$, it follows, from (7.2.16), that

$$\epsilon\frac{d\bar{X}}{dt} - g(t, \bar{X}, \epsilon) \geq 0.$$

Thus, for $t \in (t_0, \hat{t}_{\delta\epsilon})$, $\bar{X}(t, \epsilon)$ is a upper solution of problem (7.0.1) and

$$x(\hat{t}_{\delta\epsilon}, \epsilon) \leq \bar{X}(\hat{t}_{\delta\epsilon}, \epsilon) = -\eta.$$

Let us define $\bar{\omega}(\delta)$ to be the maximum of $|\omega(\epsilon)|$ for $\epsilon \in I_{\epsilon 1(\delta)}$ and $\bar{t}_\delta = t^* + \Delta(\delta) + \bar{\omega}(\delta)$. We have $\bar{\omega}(\delta) \to 0$ as $\delta \to 0$ and thus $\bar{t}_\delta \to t^*$ as $\delta \to 0$. Now, let us construct an upper solution on $[\hat{t}_{\delta\epsilon}, \bar{t}_\delta]$. Let us set ϵ_1 sufficiently small such that, according to (7.2.14),(I), $(III)^*$ and (IV), for $\epsilon \in \mathbb{I}_{\epsilon_1}$,

$$g(t, -\eta, \epsilon) < 0 \text{ for } t^* \leq t \leq T.$$

It is easy to see that $\bar{X} = -\eta$ satisfies

$$\epsilon\frac{d\bar{X}}{dt} - g(t, \bar{X}, \epsilon) \geq 0.$$

Thus, the function $\bar{X}(t, \epsilon) = -\eta$ is an upper bound on $[\check{t}_{\delta\epsilon}, \bar{t}_\delta]$. Therefore

$$x(\bar{t}_\delta, \epsilon) \leq -\eta < 0.$$

Since $-\eta$ lies in the basin of attraction of the stable root $\psi_-(t)$ on the interval $[\hat{t}_{\delta\epsilon}, \bar{t}_\delta]$, according to Remark 7.1.0.1, the solution tends to ψ_- as ϵ tends to zero. It follows that there exist $s, \hat{\epsilon} > 0$, such that for all $\epsilon \in \mathbb{I}_{\hat{\epsilon}}$,

$$\psi_-(\check{t}_{\delta\epsilon}) - s < x(\check{t}_{\delta\epsilon}, \epsilon) < \psi_-(\check{t}_{\delta\epsilon}) + s$$

for some $\check{t}_{\delta\epsilon} \in [\hat{t}_{\delta\epsilon}, \bar{t}_\delta]$. Therefore, according to Remark 7.1.0.2,

$$\psi_-(t) - s < x(t, \epsilon) < \psi_-(t) + s \text{ for } t \in [\check{t}_{\delta\epsilon}, T].$$

Since $\check{t}_{\delta\epsilon}$ tends to t^* as ϵ, δ tend to zero, it follows from Remark 7.1.0.1 that

$$\lim_{\epsilon\to 0} x(t,\epsilon) = \psi_-(t) \text{ for } t \in (t^*, T].$$

□

7.3 One Dimensional Stable Case with a Positive Initial Condition

It is interesting to observe that, as in the one and two dimensional epidemiological cases studied respectively in Chapter 5 and Chapter 6 and illustrated in Figure 5.3, if the roles of the quasi steady states are reversed, the phenomenon of the delay exchange of stability, described in Theorem 3.3.4.1, does not occur even though the root of $G(t,0)$ exists.

7.3.0.1 Theorem. *Consider the problem* (7.0.1) *and assume that* $g(t,x,0) = 0$ *has two roots* $x \equiv 0$ *and* $x = \phi(t) \in C^2(\bar{\mathbb{I}}_T)$ *in* $M_x \times \bar{\mathbb{I}}_T$, *which intersect at* $t = t_c \in (0,T)$ *with* $\phi(t) > 0$ *for* $t \in (0,t_c)$, *and* $\phi(t) < 0$ *for* $t \in (t_c,T)$. *Furthermore, they switch stability at their intersection in the following sense*

$$g_x(t,0,0) > 0 \text{ and } g_x(t,\phi(t),0) < 0 \text{ for } 0 < t < t_c, \tag{7.3.1}$$

$$g_x(t,0,0) < 0 \text{ and } g_x(t,\phi(t),0) > 0 \text{ for } t > t_c. \tag{7.3.2}$$

Additionally, let us assume that condition (IV) *of Theorem 3.3.4.1 holds. Then for positive initial condition* x_0, *the solution to* (7.0.1) *exists and satisfies*

$$\lim_{\epsilon\to 0} x(t,\epsilon) = \phi(t), \qquad t \in (0,t_c), \tag{7.3.3}$$

$$\lim_{\epsilon\to 0} x(t,\epsilon) = 0, \qquad t \in [t_c,T]. \tag{7.3.4}$$

Proof. According to the Picard theorem, Theorem 3.2.1.1, the solution to (7.0.1) exists and is unique on $\bar{\mathbb{I}}_T$. In what follows, we prove the asymptotic behaviour of the solution on the intervals $[0,t_c)$ and $[t_c,T]$. Let us consider the domain $\mathcal{D} = [0, t_c - \alpha] \times [b, b_0]$ where b_0, b, α are arbitrary positive numbers such that $b_0 < \inf_{t\in[0,t_c-\alpha]} \phi(t)$ and $0 < b < x_0$. The quasi steady state $x = \phi(t)$ is an isolated and attracting quasi steady state in $\mathcal{D}$. So, according to the Tikhonov Theorem,

$$\lim_{\epsilon\to 0} x(t,\epsilon) = \phi(t),$$

for $t \in (0, t_c - \alpha]$. Since α is chosen arbitrarily, letting α tends to 0, we obtain (7.3.3).

On the other hand, let us consider a composite quasi steady state defined as follows:

$$\tilde{\phi}(t) = \begin{cases} \phi(t) & \text{for } 0 < t < t_c, \\ 0 & \text{for } t \geq t_c. \end{cases}$$

By the hypothesis, the quasi steady state $x = \phi(t)$ is attractive on $[0, t_c)$, while $x \equiv 0$ is attractive on $(t_c, T]$. It follows, from (7.3.1) and (7.3.2), that $g(t, x, 0) < 0$ for $x > \tilde{\phi}$ and for any $\epsilon \in \mathbb{I}_{\epsilon_0}$, there exists $w_\epsilon > 0$ such that $g(t, x, \epsilon) < 0$ for $x > \tilde{\phi} + \omega_\epsilon$ with $\omega_\epsilon \to 0$ as $\epsilon \to 0$. Thus for all $\epsilon \in \mathbb{I}_{\epsilon_0}$ there exists $\rho_{\omega_\epsilon} > 0$ such that for any $t_c - \alpha < t$, we have $x(t, \epsilon) \leq \rho_{\omega_\epsilon}$. In other words, the solution is bounded. The solution, $x(t, \epsilon)$, being in the region where $x'(t, \epsilon) < 0$ converges to zero on $[t_c, T]$ as ϵ and α tend to zero. Thus (7.3.4) is satisfied. □

7.4 Conclusion

In this chapter we discussed the proof of the Butuzov theorem by providing a proof to some facts used while demonstrating this theorem. Also, we gave a proof of the Butuzov theorem for pitchfork bifurcations with a positive and a negative initial condition and we generalized the study of immediate stability switches discussed in Chapter 5 for dimension one problems. The developed theory for immediate stability switch focuses particularly on non-autonomous systems having the first quadrant invariant under their flow. As already underlined, one of the most striking difference between the stable and the unstable case is that the positions of the quasi steady states are "flipped" around. Also, it is possible to generate a similar theory for one dimensional stable case with a negative initial condition by making obvious changes.

In the next chapter we will generalise the study done in Chapter 6 for general two dimensional non-autonomous problems.

8 General Theories for Two Dimensional Problems

After studying the two dimensional influenza problem using the method of upper and lower bounds in Chapter 6, we aim to generalize the method to more general two dimensional problems. In particular we will study the asymptotic behaviour of the solution to non-autonomous planar systems of the form

$$\begin{cases} \frac{dx}{dt} = f(t,x,y,\epsilon), \\ \epsilon\frac{dy}{dt} = g(t,x,y,\epsilon), \\ x(0) = x_0 > 0, y(0) = y_0 > 0, \end{cases} \tag{8.0.1}$$

on $\bar{V} := \bar{\mathbb{I}}_T \times \bar{\mathbb{I}}_M \times \bar{\mathbb{I}}_N \times \bar{\mathbb{I}}_{\epsilon_0}$ with $M, N, T \in (0, +\infty)$ and $\epsilon_0 > 0$ a small parameter. We proceed by stating and proving the theories for the two dimensional unstable and stable cases for a transcritical bifurcation. We assume that the stable parts of the quasi steady states of the transcritical bifurcation are non-negative.

8.1 Two Dimensional Unstable Case with a Positive Initial Condition

In this section, we analyse the case of a delay in stability switch. We assume that (8.0.1) has a transcritical bifurcation as described in Figure 8.1 where the stable parts of the QSSs are non-negative.

8.1.0.1 Theorem. *Let us consider the following general assumptions concerning the structure of the system:*

A_1- *Functions* f, g *are* $C^2(\bar{V})$.

A_2- $g(t,x,0,\epsilon) = 0$ *for all* $(t,x,\epsilon) \in \bar{\mathbb{I}}_T \times \bar{\mathbb{I}}_M \times \bar{\mathbb{I}}_{\epsilon_0}$.

A_3- *For all* $(t,x,y_1,\epsilon) \in \bar{V}, (t,x,y_2,\epsilon) \in \bar{V}$ *with* $y_2 \le y_1$, $f(t,x,y_1,\epsilon) \le f(t,x,y_2,\epsilon)$.

A_4- *For all* $(t,x_1,y,\epsilon) \in \bar{V}, (t,x_2,y,\epsilon) \in \bar{V}$ *with* $x_1 \le x_2$, $g(t,x_1,y,\epsilon) \le g(t,x_2,y,\epsilon)$.

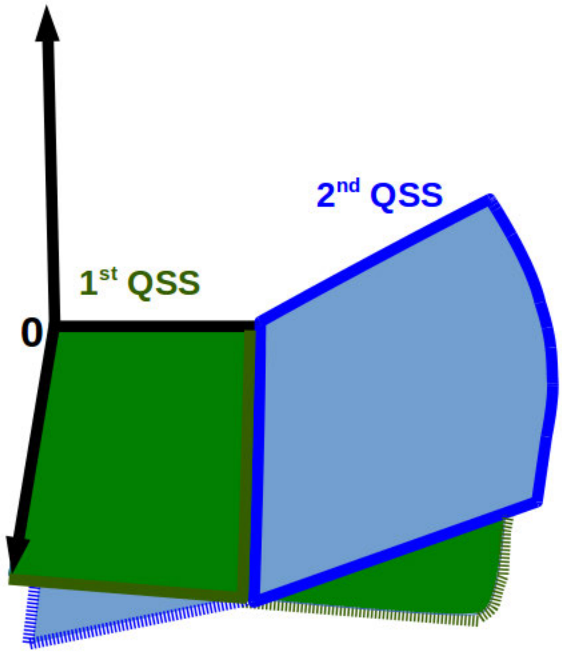

Figure 8.1: Transcritical Bifurcation considered to prove the delayed stability switch. The unstable parts of the QSSs are the parts with dotted contour and the stable parts have continous contours. We observed that the QSSs intersect each other, switch stability at their intersection and their stable parts are non-negative.

Furthermore, consider the following assumptions on the structure of the quasi steady states of (8.0.1).

A_5- *There are two quasi steady states,* $y = 0$ *and* $y = \phi(t, x)$*, that are the solutions of the equation*

$$g(t, x, y, 0) = 0 \tag{8.1.1}$$

in $\bar{\mathbb{I}}_T \times \bar{\mathbb{I}}_M \times \bar{\mathbb{I}}_N$*. There is a unique solution* $y = \psi(t)$ *to the equation*

$$\phi(t, x) = 0 \tag{8.1.2}$$

for $t \in \bar{\mathbb{I}}_T$, $x \in \bar{\mathbb{I}}_M$ *and* $\psi \in C^2(\bar{\mathbb{I}}_T)$*. Further, we assume that*

$$\phi(t, x) < 0 \text{ for } x - \psi(t) < 0,$$
$$\phi(t, x) > 0 \text{ for } x - \psi(t) > 0.$$

A_6- *There is a stability switch at the intersection of QSSs; that is,*

$$g_y(t, x, 0, 0) < 0 \text{ and } g_y(t, x, \phi(t, x), 0) > 0 \text{ for } x - \psi(t) < 0,$$
$$g_y(t, x, 0, 0) > 0 \text{ and } g_y(t, x, \phi(t, x), 0) < 0 \text{ for } x - \psi(t) > 0.$$

Denote by $\bar{x}(t,\epsilon)$ *the solution of*

$$\frac{dx}{dt} = f(t,x,0,\epsilon),\ x(0,\epsilon) = x_0. \tag{8.1.3}$$

Then,

A_7- *any solution* $\bar{x} = \bar{x}(t)$ *to the problem* (8.1.3) *with* $\epsilon = 0$,

$$\frac{d\bar{x}}{dt} = f(t,\bar{x},0,0),\ \bar{x}(0) = x_0, \tag{8.1.4}$$

where $0 < x_0 < \psi(0)$ *satisfies* $\bar{x}(T) > \psi(T)$ *and there is exactly one* $0 < \bar{t}_c < T$ *such that* $\bar{x}(\bar{t}_c) = \psi(\bar{t}_c)$.

Further, we define the function $\bar{G}$ *by*

$$\bar{G}(t,\epsilon) = \int_0^t g_y(s,\bar{x}(s,\epsilon),0,\epsilon)ds, \tag{8.1.5}$$

for $t \in \mathbb{I}_T,\ \epsilon \in \mathbb{I}_{\epsilon_0}$.

A_8- *the function* $\bar{G}(.,0)$ *has a root* $\bar{t}^* \in \mathbb{I}_T$.

Finally,

A_9- *there is* $c_0 \in \mathbb{I}_N$ *such as for all* $\epsilon \in \bar{\mathbb{I}}_{\epsilon_0},\ y \in \bar{\mathbb{I}}_{c_0}$,

$$g(t,\bar{x}(t,\epsilon),y,\epsilon) \leq g_y(t,\bar{x}(t,\epsilon),0,\epsilon)y$$

for all $t \in \bar{\mathbb{I}}_T$.

Then there exists a unique solution $\big(x(t,\epsilon),y(t,\epsilon)\big)$ *to the problem* (8.0.1) *such that*

$$\lim_{\epsilon\to 0} y(t,\epsilon) = 0 \quad \text{for} \ \ t \in (0,\bar{t}^*),$$
$$\lim_{\epsilon\to 0} x(t,\epsilon) = \bar{x}(t) \quad \text{for} \ \ t \in [0,\bar{t}^*).$$

The convergence is almost uniform on each interval and $(0,\bar{t}^*)$ *is the biggest interval on which the convergence of* y *to* 0 *as* ϵ *tends to zero is almost uniform .*

8.1.0.2 Remark. As in the one dimensional case we observe that, by assumption A_6, $\bar{G}$ reaches its unique negative minimum at $\bar{t}_c$ and is strictly increasing for $t > t_c$. Therefore the root $\bar{t}^*$, defined in assumption A_8, is unique on $\mathbb{I}_T$.

8.1.0.3 Remark. The assumption A_7 is simplified if we deal with autonomous problemes then $\psi(t)$ is constant and hence it is satisfied if f is separated from 0 on $[x_0, \infty)$ and T is large enough.

8.1.0.4 Remark. We will repeatedly use the following argument based on [[65], Appendix B]. Consider a system of differential equation

$$\begin{cases} \frac{dx}{dt} = F(t,x,y), & x(0) = x_0, \\ \frac{dy}{dt} = G(t,x,y), & y(0) = y_0, \end{cases} \tag{8.1.6}$$

with F, G satisfying the Lipschitz conditions with respect to x, y in some domain of $\mathbb{R}^2$, uniformly in $t \in \bar{\mathbb{I}}_T$. Assume that F satisfies

$$F(t,x,y_1) \leq F(t,x,y_2) \text{ for } y_1 \geq y_2, \tag{8.1.7}$$

and a solution $(x(t), y(t))$ of (8.1.6) satisfies

$$\phi_1(t, x(t)) \leq y(t) \leq \phi_2(t, x(t)) \tag{8.1.8}$$

on $\bar{\mathbb{I}}_T$ for some Lipschitz functions ϕ_1 and ϕ_2. Then for $t \in \bar{\mathbb{I}}_T$,

$$z_2(t) \leq x(t) \leq z_1(t),$$

where z_i satisfies

$$\frac{dz_i}{dt} = F(t, z_i, \phi_i(t, z_i)), \ z_i(0) = x_0, \tag{8.1.9}$$

$i = 1, 2$. Indeed, consider z_1 satisfying

$$\frac{dz_1}{dt}(t) \equiv F(t, z_1(t), \phi_1(t, z_1)), \ z_1(0) = x_0.$$

Then, from (8.1.7) and (8.1.8), we have

$$\frac{dx}{dt}(t) \equiv F(t, x(t), y(t)) \leq F(t, x(t), \phi_1(t, x(t))$$

and from the comparison theorem, Theorem 3.2.2.6, it follows that $x(t) \leq z_1(t)$ on $\bar{\mathbb{I}}_T$. We prove similarly that $x(t) \geq z_2(t)$ for all $t \in \bar{\mathbb{I}}_T$.

We also note that if F satisfies

$$F(t,x,y_1) \leq F(t,x,y_2)$$

for $y_1 \leq y_2$ and a solution $(x(t), y(t))$ of (8.1.6) satisfies

$$\phi_2(t, x(t)) \leq y(t) \leq \phi_1(t, x(t))$$

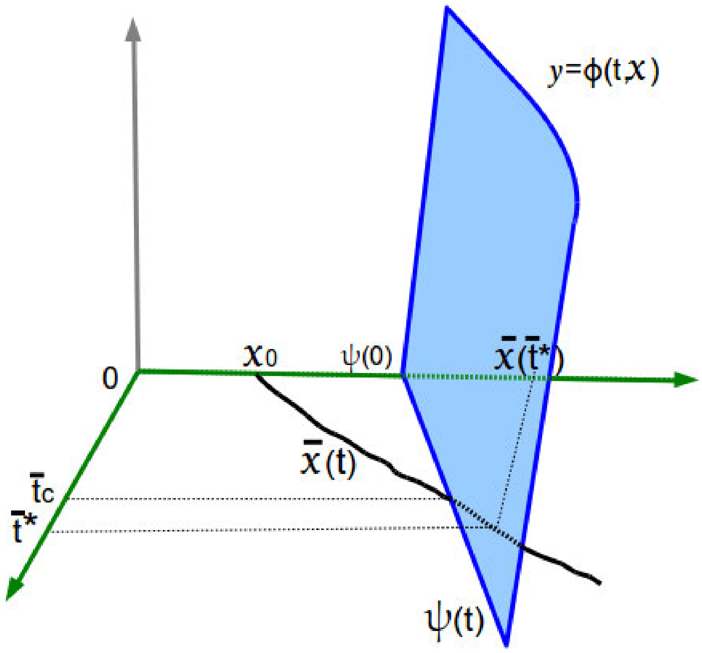

Figure 8.2: Illustration of the assumptions of Theorem 8.1.0.1.

on $\bar{\mathbb{I}}_T$ for some Lipschitz functions ϕ_1 and ϕ_2, then for $t \in \bar{\mathbb{I}}_T$,

$$z_2(t) \leq x(t) \leq z_1(t),$$

where z_i, $i = 1, 2$, is a solution of (8.1.9).

Proof. We proceed in three steps. Firstly, we prove the existence and the uniqueness of the solution to the problem (8.0.1); secondly, we show the convergence of the solution as ϵ tends to zero on the interval $(0, \bar{t}^*)$ and then we will prove that this interval is actually the biggest interval where the uniform convergence is observed.

Step 1- Since f, g are sufficiently regular, according to the Picard theorem (Theorem 3.2.1.1), for a given initial condition, the solution to (8.0.1) exists and is unique on $[0, T]$ for all $T > 0$.

Step 2- Let us prove the convergence of y to zero on $(0, \bar{t}^*)$ as $\epsilon \to 0$. Consider the following initial condition (x_0, y_0) with $0 < x_0 < \psi(0)$ and $y_0 > 0$. Let $\big(x, y\big)$ be the corresponding solution to problem (8.0.1). To simplify the notation, we will not mention the initial conditions in this part since they are fixed. According to A_2, for all $t \in \mathbb{I}_T$, $\epsilon \in \mathbb{I}_{\epsilon_0}$, $y(t, \epsilon) > 0$ and, from assumption A_3, $f(t, x, y, \epsilon) < f(t, x, 0, \epsilon)$.

Therefore, from the comparison theorem, Theorem 3.2.2.6,

$$x(t,\epsilon) \leq \bar{x}(t,\epsilon),$$

where $\bar{x}$ is the solution to (8.1.3). Similarly, let $\bar{y}$ be the solution to

$$\epsilon\frac{dy}{dt} = g(t,\bar{x},y,\epsilon),\ y(0,\epsilon) = y_0. \tag{8.1.10}$$

Thus from Remark 8.1.0.4 it follows that

$$0 \leq y(t,\epsilon) \leq \bar{y}(t,\epsilon). \tag{8.1.11}$$

To shorten the notation, we denote

$$\bar{g}(t,y,\epsilon) := g(t,\bar{x}(.,\epsilon),y,\epsilon),$$

where, clearly, $\bar{g}(t,0,0) = g(t,\bar{x}(t),0,0)$. Hence, we consider the shortened version of (8.1.10),

$$\epsilon\frac{dy}{dt} = \bar{g}(t,y,\epsilon),\ y(0,\epsilon) = y_0. \tag{8.1.12}$$

Then, from A_5, the only solutions to $\bar{g}(t,y,0) = 0$ are $y = 0$ and $y = \phi(t,\bar{x}(t))$. Let $\varphi(t) = \phi(t,\bar{x}(t))$. From (8.1.2), $\phi(t,x) = 0$ if and only if $x = \psi(t)$ and thus $\varphi(t) = 0$ if and only if $\bar{x}(t) = \psi(t)$; that is, by A_7, for $t = \bar{t}_c : \varphi(\bar{t}_c) = \phi(\bar{t}_c,\psi(\bar{t}_c)) = 0$ with $\varphi(t) < 0$, for $t < \bar{t}_c$ and $\varphi(t) > 0$ for $t > \bar{t}_c$ (by assumption A_5). Hence, assumption (II) of Theorem 3.3.4.1 is satisfied. Further, since $\bar{g}_y(t,y,\epsilon) = g_y(t,\bar{x}(t,\epsilon),y,\epsilon)$, from assumption A_6,

$$\bar{g}_y(t,0,0) < 0 \ \text{ and } \ \bar{g}_y(t,\varphi(t),0) > 0 \ \text{ for } \ t < t_c,$$

and

$$\bar{g}_y(t,0,0) > 0 \ \text{ and } \ \bar{g}_y(t,\varphi(t),0) < 0 \ \text{ for } \ t > t_c.$$

Therefore, assumption (III) of Theorem 3.3.4.1 is satisfied. In the same way, assumptions A_8 and A_9 show that assumptions (V) and (VI) of Theorem 3.3.4.1 are satisfied for (8.1.12) and thus $\bar{y}(t,\epsilon)$ satisfies the hypotheses of Theorem 3.3.4.1. Therefore

$$\lim_{\epsilon\to 0}\bar{y}(t,\epsilon) = 0 \text{ for } \ t \in (0,\bar{t}^*), \tag{8.1.13}$$

$$\lim_{\epsilon\to 0}\bar{y}(t,\epsilon) = \phi(t,\bar{x}(t)) \text{ for } \ t \in (\bar{t}^*,T]. \tag{8.1.14}$$

Together with (8.1.11), it yields

$$\lim_{\epsilon\to 0} y(t,\epsilon) = 0 \text{ for } t \in (0,\bar{t}^*). \tag{8.1.15}$$

Now, for any x_0 satisfying A_7, there is a neighbourhood U of x_0 and $\hat{t} \in (0,\bar{t}_c)$ such that $y = 0$ is an isolated quasi steady state on $[0,\hat{t}] \times \bar{U}$ so that (8.0.1) satisfies the assumptions of the Tikhonov theorem, Theorem 3.3.2.1. Thus,

$$\lim_{\epsilon\to 0} x(t,\epsilon) = \bar{x}(t)$$

for $t \in [0,\hat{t}]$ and, on the other hand, the problem

$$\frac{dx}{dt} = f(t,x,y(t,\epsilon),\epsilon) \tag{8.1.16}$$

with initial condition $x(\hat{t},\epsilon)$ is a regularly perturbed problem on $[\hat{t},\bar{t}^*)$. Therefore,

$$\lim_{\epsilon\to 0} x(t,\epsilon) = \bar{x}(t)$$

on $[\hat{t},\bar{t}^*)$. Combining the above observations, we have

$$\lim_{\epsilon\to 0} x(t,\epsilon) = \bar{x}(t), \tag{8.1.17}$$

uniformly on $[0,\bar{t}^*)$.

Step 3- We shall prove next that the interval $(0,\bar{t}^*)$ is the maximum interval on which $y(t,\epsilon)$ converges to zero almost uniformly. Let us assume, to the contrary, that $\lim_{\epsilon\to 0} y(t,\epsilon) = 0$ almost uniformly on $(0,t_1]$ for some $t_1 > \bar{t}^*$; that is, for any $\rho > 0$ and any $\theta > 0$ there is $\epsilon_1 = \epsilon_1(\rho,\theta)$ such that for any $t \in [\theta,t_1]$ and $\epsilon \in \mathbb{I}_{\epsilon_1}$,

$$0 \le y(t,\epsilon) \le \rho. \tag{8.1.18}$$

Let us fix $\rho > 0$ and $\theta > 0$ such that $0 \le y(t,\epsilon) < \rho$ on $[\theta,t_1]$. Then, by assumption A_3, on $[\theta,t_1]$,

$$f(t,x,\rho,\epsilon) \le f(t,x,y(t,\epsilon),\epsilon). \tag{8.1.19}$$

However, according to the initial layer proposition [[7], Proposition 3.4.1], there exist bounded functions $\bar{y}_1$ and $\tilde{y}_1$ such that for $\delta > 0$ there exists $\epsilon_2 \in \mathbb{I}_{\epsilon_1}$ such that for $t \in [0,t_1]$, $\epsilon \in \mathbb{I}_{\epsilon_2}$,

$$\left| y(t,\epsilon) - \bar{y}_1(t) + \tilde{y}_1\left(\frac{t}{\epsilon}\right) \right| \le \delta.$$

This implies

$$y(t,\epsilon) \leq C,\ t \in [0,t_1],\ \epsilon \in \mathbb{I}_{\epsilon_2},$$

where $C = \max_{t\in[0,t_1],\epsilon\in\mathbb{I}_{\epsilon_2}} |\bar{y}_1(t) - \tilde{y}_1(\frac{t}{\epsilon})| + \delta$. Thus, by A_3, on $[0,\theta]$,

$$f(t,x,C,\epsilon) \leq f(t,x,y(t,\epsilon),\epsilon). \tag{8.1.20}$$

Let us define the function

$$\underline{x}^1(t) = \begin{cases} x_1(t,\epsilon), & t \in [0,\theta) \\ x_2(t,\epsilon), & t \in [\theta, t_1], \end{cases} \tag{8.1.21}$$

where x_1 and x_2 are, respectively, solutions to

$$\frac{dx_1}{dt} = f(t,x_1,C,\epsilon), \qquad x_1(0) = x_0$$

and

$$\frac{dx_2}{dt} = f(t,x_2,\rho,\epsilon), \qquad x_2(\theta,\epsilon) = x_1(\theta,\epsilon).$$

It follows that

$$\underline{x}^1(t,\epsilon) \leq x(t,\epsilon),\ t \in [t_0,t_1],\ \epsilon \in (0,\epsilon_2).$$

However, we cannot use $\underline{x}^1$ to construct a lower bound for $y(t,\epsilon)$ since it is not continuous. To fix this problem, let us consider x_3 such that

$$\frac{dx_3}{dt} = f(t,x_3,\rho,0), \qquad x_3(0) = x_0,$$

on $[0,t_1]$. Since f is Lipschitz continuous, it follows that

$$\begin{aligned} |\underline{x}^1(t,\epsilon) - x_3(t)| &= \Big| \int_0^\theta f(s,\underline{x}_1,C,\epsilon) - f(s,x_3,\rho,\epsilon)ds \\ &\qquad + \int_\theta^{t_1} f(s,\underline{x}^1,\rho,\epsilon) - f(s,x_3,\rho,0)ds \Big|, \\ &\leq L_1\theta + L_2(\epsilon)\int_\theta^{t_1} |\underline{x}^1(t,\epsilon) - x_3(t)|ds \leq L\theta, \end{aligned}$$

where L_i, $i = 1,2$ is a Lipschitz constant and $L = L_1 e^{L_2(\epsilon)t_1}$, according to Gronwall's lemma. It is possible to consider L independent of ϵ as f is C^2 with respect to all variables on $\bar{V}$. Thus, for $t \in [0,t_1]$,

$$|\underline{x}^1(t,\epsilon) - x_3(t)| \leq L\theta \Rightarrow -2L\theta \leq -\underline{x}^1(t,\epsilon) + x_3(t) - L\theta \leq 0.$$

This implies

$$-2L\theta \leq \underline{x} := x_3(t) - L\theta \leq \underline{x}^1(t,\epsilon).$$

Hence

$$\underline{x}(t,\rho,\theta) \leq x(t,\epsilon),\ \epsilon \in \mathbb{I}_{\epsilon_2},\ t \in [0,t_1]. \tag{8.1.22}$$

Then, according to Remark 8.1.0.4, the solution $\underline{y} = \underline{y}(t,\rho,\theta,\epsilon)$ to

$$\epsilon\frac{d\underline{y}}{dt} = g(t,\underline{x}(t,\rho,\theta),\underline{y},\epsilon),\ \underline{y}(0,\rho,\theta,\epsilon) = y_0, \tag{8.1.23}$$

satisfies

$$\underline{y}(t,\rho,\theta,\epsilon) \leq y(t,\epsilon),\ t \in [0,t_1].$$

Let us denote

$$\underline{g}(t,y,\rho,\theta,\epsilon) = g(t,\underline{x}(t,\rho,\theta),y,\epsilon),\ \epsilon \in \mathbb{I}_{\epsilon_2},\ t \in [t_0,t_1].$$

As with g, we note that $\underline{g}$ is a C^2 function with respect to all variables. We define the function

$$\underline{G}(t,\rho,\theta,\epsilon) = \int_0^t \underline{g}_y(s,0,\rho,\theta,\epsilon)ds,\ \epsilon \in \mathbb{I}_{\epsilon_2},\ t \in [t_0,t_1]. \tag{8.1.24}$$

We observe that $\underline{g}(t,0,0,0,\epsilon) = \bar{g}(t,\epsilon) = g(t,\bar{x}(t,\epsilon),0,\epsilon)$ and also $\underline{g}_y(t,0,0,0,\epsilon) = \bar{g}_y(t,\epsilon) = g_y(t,\bar{x}(t,\epsilon),0,\epsilon)$. Clearly

$$\underline{G}(0,\rho,\theta,0) = \int_0^0 \underline{g}_y(s,0,\rho,\theta,0)ds = 0.$$

Since $\underline{G}(\bar{t}^*,0,0,0) = \bar{G}(\underline{t}^*,0) = 0$ and $\underline{G}^{'}(\bar{t}^*,0,0,0) = g_y(\bar{t}^*,0,0,0) > 0$, the implicit function theorem shows that for sufficiently small ρ,θ there is a C^2 function $\underline{t}^* = \underline{t}^*(\rho,\theta)$ such that $\underline{G}(\underline{t}^*,\rho,\theta,0) \equiv 0$ with $\underline{t}^*(\rho,\theta) \to \bar{t}^*$ as $\rho,\theta \to 0$. Furthermore, since by A_4 $g(t,x_1,y,0) \leq g(t,x_2,y,0)$ for $x_1 \leq x_2$ and $g(t,x,0) = 0$, we easily obtain

$$g_y(t,x_1,0,0) \leq g_y(t,x_2,0,0),\ x_1 \leq x_2. \tag{8.1.25}$$

Since

$$\underline{x}(t,\rho,\theta) \leq x(t,\epsilon) \leq \bar{x}(t),\ t \in [0,t_1],$$

it follows that $\underline{G}(t,\rho,\theta,0) \leq \bar{G}(t,0)$ and $\underline{t}^*(\rho,\theta) \geq \bar{t}^*$.

The problem (8.1.23) is, by construction, in the form allowing for the application of Theorem 3.3.4.1. However, we are not going to consider the full theorem but only the lower bound found in its proof, [19]. Let us denote by $\underline{Y}(t,\rho,\theta,\eta,\epsilon)$ the function defined by

$$\underline{Y}(t,\rho,\theta,\eta,\epsilon) = \eta e^{\frac{\underline{G}(t,\epsilon)-\delta t}{\epsilon}}, \tag{8.1.26}$$

with $\underline{G}$ defined by (8.1.24) and the parameter δ independent of ρ and η so that

$$\underline{G}(t(\rho,\theta,\delta,\epsilon),\rho,\theta,\epsilon) - \delta t \equiv 0$$

and

$$\underline{Y}(t(\rho,\theta,\delta,\epsilon),\rho,\theta,\delta,\epsilon) = \eta,$$

[19].

This function $\underline{Y}$ is a lower bound to (8.1.23) provided $\eta \leq \frac{\delta}{k}$, see [19], where k is independent of any other parameters. So, for some ρ_0, θ_0 such that

$$\sup_{\{\rho\in\bar{\mathbb{I}}_{\rho_0},\theta\in\bar{\mathbb{I}}_{\theta_0}\}} \underline{t}^*(\rho,\theta) < t_1, \tag{8.1.27}$$

and since $\bar{\mathbb{I}}_{\theta_0}$ and $\bar{\mathbb{I}}_{\rho_0}$ are compact intervals, there exists $\tilde{t}$ such that

$$\sup_{\{\rho\in\bar{\mathbb{I}}_{\rho_0},\theta\in\bar{\mathbb{I}}_{\theta_0}\}} \underline{t}^*(\rho,\theta) \leq \tilde{t} < t_1. \tag{8.1.28}$$

Thus, for the parameters ρ, θ satisfying the above, we have

$$t(\rho,\theta,\delta,\epsilon) = t^*(\rho,\theta) + w(\delta,\epsilon)$$

with δ and ϵ_1 such that $w(\delta,\epsilon) + \tilde{t} < t_1$ for all $\epsilon < \epsilon_1$. For such a δ, let us set $\rho < \eta < \delta/k$. Then,

$$y(t(\rho,\theta,\delta,\epsilon),\epsilon) \geq \underline{Y}(t(\rho,\theta,\delta,\epsilon),\rho,\theta,\delta,\eta,\epsilon) = \eta > \rho. \tag{8.1.29}$$

On the other hand, for sufficiently small ϵ, according to (8.1.18),

$$y(t,\epsilon) < \rho, \quad t > \theta.$$

Thus, the assumption that there is $t_1 > t^*$ such that $y(t,\epsilon)$ converges almost uniformly to zero on $(0,t_1]$ is false.

□

Our next step is to investigate the behaviour of the solution beyond $\bar{t}^*$. Clearly, we cannot use the lower bound $\underline{y}$ defined by (8.1.23) since it is a lower bound only for $y(t,\epsilon) \leq \rho$ which holds for $t \leq \bar{t}^*$. Thus, another lower bound for $y(t,\epsilon)$ has to be found. To achieve this, an additional assumption need to be adopted to ensure that the solution does not return to the region of attraction of $y = 0$. Thus, we assume

$$\frac{g'}{g_x} + f\Big|_{(t,x,y,\epsilon)=(t,\psi(t),0,0)} > 0, \quad t \in \bar{\mathbb{I}}_T. \tag{8.1.30}$$

8.1.0.5 Remark. Geometrically, the condition (8.1.30) has a clear interpretation since the normal to the curve $x = \psi(t)$ pointing towards the region $\{(t,x); x > \psi(t)\}$ is given by $(-\psi'(t), 1)$. On one hand, we have $0 \equiv \phi(t, \psi(t))$, hence by differentiating both sides we get

$$\phi'(t,\psi(t)) + \phi_x(t,\psi(t)).\psi'(t) = 0;$$

that is

$$\psi' = -\frac{\phi'}{\phi_x}\Big|_{(t,x)=(t,\psi(t))}.$$

On the other hand, we have

$$g(t,x,\phi(t,x)) = 0,$$

which gives by differentiation with respect to t and x, respectively,

$$g' + g_y\phi'(t,x) = 0, \tag{8.1.31}$$

and

$$g_x + g_y\phi_x(t,x) = 0. \tag{8.1.32}$$

It follows that

$$\frac{g'}{g_x} = -\frac{\phi'}{\phi_x}.$$

Hence

$$\psi'(t) = -\frac{g'}{g_x}\Big|_{(t,x,y,\epsilon)=(t,\psi(t),\phi(t,\psi(t)),0)=(t,\psi(t),0,0)}. \tag{8.1.33}$$

Thus, condition (8.1.30) is equivalent to

$$(-\psi',1).(1,x') = (-\psi',1).(1,f), \quad (t,x,y,\epsilon) = (t,\psi(t),0,0).$$

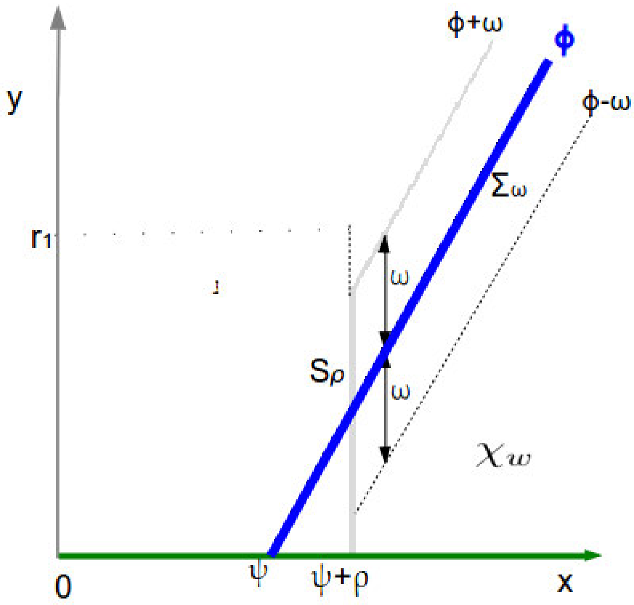

Figure 8.3: Construction used in the proof of Theorem 8.1.0.6.

This means that the solution x of (8.1.4) can only cross $x = \psi(t)$ from below.

In the case of an autonomous problem, condition (8.1.30) becomes

$$f\,|_{(x,y,\epsilon)=(a,0,0)} > 0, \quad t \in \bar{\mathbb{I}}_T.$$

where $x = a \equiv \psi(t)$. So, the graph of the solution x of (8.1.11) strictly increases when it crosses the line $x = a$.

8.1.0.6 Theorem. *Let us consider the assumptions A_1-A_9 and condition* (8.1.30)*. Then, for sufficiently small ϵ, there exists a unique solution $\big(x(t,\epsilon), y(t,\epsilon)\big)$ to the problem* (8.0.1) *on $[0,T]$ such that for $t \in (\bar{t}^*, T]$,*

$$\lim_{\epsilon\to 0} y(t,\epsilon) = \phi(t, x_*(t)),$$
$$\lim_{\epsilon\to 0} x(t,\epsilon) = x_*(t),$$

where the function x_* is the solution to

$$\frac{dx_*}{dt} = f(t, x_*, \phi(x_*, t), 0), \quad x_*(\bar{t^*}) = \bar{x}(\bar{t^*}), \ t \in [0,T].$$

Moreover, the convergence on $(\bar{t}^*, T]$ is almost uniform.

Proof. According to the Picard theorem,Theorem 3.2.1.1, the solution to (8.0.1) exists and is unique on $\mathbb{I}_T$. In order to make the proof more concrete, we will proceed by steps.

Step 1 - **Proof that the solution, having crossed ϕ, does not go back.** Let us consider arbitrary $\hat{t} \in (\bar{t}_c, \bar{t}^*)$. According to the assumption A_7, there exists $\rho_0 > 0$ such that $\bar{x}(\hat{t}) > \psi(\hat{t}) + \rho_0$. Additionally, as established in the previous proof, we have $x(\hat{t}, \epsilon) \to \bar{x}(t)$ and $y(\hat{t}, \epsilon) \to 0$, therefore there exists a positive number ϵ_0 such that for any $\epsilon \in (0, \epsilon_0)$ $x(\hat{t}, \epsilon) > \psi(\hat{t}) + \rho_0/2$ and $0 < y(\hat{t}, \epsilon) < \rho$. Let us define the function Ψ as follows:

$$\Psi(t, x, y, \epsilon) := \frac{g_t(t, x, y, \epsilon)}{g_x(t, x, y, \epsilon)} + f(t, x, y, \epsilon).$$

It follows that $\Psi(t, \psi(t), 0, 0) > 0$ for $t \in \bar{\mathbb{I}}_T$ (according to (8.1.30)) and hence there exist $\alpha_1, r_1, r_2, \epsilon_0$ such that for all $|y| \leq r_1$, $|\rho| < r_2, |\epsilon| < \epsilon_0$,

$$\Psi(t, \psi(t) + \rho, y, \epsilon) \geq \alpha_1.$$

Let $S_\rho = \{(t, x, y); t \in \bar{\mathbb{I}}_T, x = \psi(t) + \rho, y \in [0, r_1]\}$ and $S_\phi = \{(t, x, y); t \in \bar{\mathbb{I}}_T, \psi(t) \leq x \leq M, y = \phi(t, x)\}$. Since $\phi(t, \psi(t)) = 0$, by continuity, there exists sufficiently small $\rho_{{}_0}$ such that

$$\max_{t \in \bar{\mathbb{I}}_T} \{\phi(t, \psi(t) + \rho_{{}_0})\} < r_1.$$

Let us consider the number,

$$\alpha_{\rho_0} = \min_{t \in \bar{\mathbb{I}}_T, x \in [\psi(t) + \rho_0, M]} \phi(t, x) > 0,$$

the layer

$$\Sigma_w = \{(t, x, y); t \in \bar{\mathbb{I}}_T, \psi(t) + \rho_{{}_0} \leq x \leq M, y \in [\phi(t, x) - w, \phi(t, x) + w]\},$$

for $w \in (0, \min\{\alpha_{\rho_0}/2, r_1 - \max_{0 \leq t \leq T} \phi(t, \psi(t) + \rho_{{}_0})\})$ and also, the domain

$$\chi_w = \{(t, x, y); 0 \leq t \leq T, [\psi(t) + \rho_{{}_0}, M], y \in [0, \phi(t, x) + w]\}.$$

We can notice that the left wall of χ_w,

$$L_{\chi_w} = \{(t, x, y); t \in \bar{\mathbb{I}}_T, x = \psi(t) + \rho, y \in [0, \phi(t, x) + w]\},$$

satisfies condition (8.1.30) and therefore it is contained in the set

$$\{(t, x, y); \Psi(t, x, y, \epsilon) > 0\}.$$

Thus, according to Remark 8.1.0.5, no trajectory can leave χ_w across the left wall L_{χ_w} from the region $x > \psi(t) + \rho_0$.

Step 2 -**Proof that the solution, once in Σ_w, will not leave it.** Let us prove that if for some $\epsilon \leq \epsilon_{\rho_0}$, a solution $(x(t,\epsilon), y(t,\epsilon))$ satisfies

$$\phi(\mathring{t}, x(\mathring{t},\epsilon)) - w < y(\mathring{t},\epsilon) < \phi(\mathring{t}, x(\mathring{t},\epsilon)) + w \tag{8.1.34}$$

for some $\mathring{t} \in (\hat{t}, T]$, then

$$\phi(t, x(t,\epsilon)) - w < y(t,\epsilon) < \phi(t, x(t,\epsilon)) + w$$

for all $t \in [\mathring{t}, T]$. Similarly to the proof of Remark 7.1.0.2, there exists ϵ_{ρ_0} such that

$$\inf_{\hat{t}<t<T, \epsilon\in[0,\epsilon_{\rho_0}], \psi(t)+\rho_0<x} g(t,x,\phi(t,x)-w,\epsilon) := d_- > 0,$$

$$\sup_{\hat{t}<t<T, \epsilon\in[0,\epsilon_{\rho_0}], \psi(t)+\rho_0<x} g(t,x,\phi(t,x)+w,\epsilon) := -d_+ < 0.$$

Let us consider the Lyapunov type function

$$V(t) = (y(t,\epsilon) - \phi(t, x(t,\epsilon)))^2$$

which is the square of the difference between the y coordinates of the solution $(x(t,\epsilon), y(t,\epsilon))$ and $(x(t,\epsilon), \phi(t, x(t,\epsilon)))$. In order to prove that the solution cannot leave Σ_w, we will proceed by contradiction. Since the trajectory cannot leave Σ_w through the left wall $L_{\chi(w)}$, let us assume that it leaves it at some time $\tilde{t} \in [\mathring{t}, T]$ with $y = \phi(\tilde{t}, x) - w$ or $y = \phi(\tilde{t}, x) + w$. On the one hand, $V' \geq 0$. On the other hand, if $y(\tilde{t},\epsilon) = \phi(\tilde{t}, x(\tilde{t},\epsilon)) - w$, then

$$\begin{aligned} V'(\tilde{t}) &= 2(y(\tilde{t},\epsilon) - \phi(\tilde{t}, x(\tilde{t},\epsilon)))\Big(y'(\tilde{t},\epsilon) - \frac{d}{dt}\phi(t, x(t,\epsilon))\Big|_{t=\tilde{t}}\Big) \\ &\leq 2w\Big(\frac{1}{\epsilon}g(\tilde{t},x,y,\epsilon) - \frac{d}{dt}\phi(t,x(t,\epsilon))\Big|_{t=\tilde{t}}\Big) \leq 2w\big(-\frac{d_-}{\epsilon} + M\big) < 0 \end{aligned}$$

for $\epsilon \in \mathbb{I}_{\hat{\epsilon}}$ where $\hat{\epsilon} < \min\{\epsilon_{\rho 0, d_-/M}\}$ with $M = \max_{(t,x,y,\epsilon)\in\bar{V}}(|\phi_t| + |\phi_x||f|)$. Similar calculations apply for the case $y(\tilde{t},\epsilon) = \phi(\tilde{t}, x(\tilde{t},\epsilon)) + w$. Thus, the solution $(x(t,\epsilon), y(t,\epsilon)) \in \Sigma_w$ for $t \in [\hat{t}, T]$.

Step 3 - **Construction of a lower bound to x defined beyond $\bar{t}^*$.** In Theorem 8.1.0.1, we proved that for any $\rho > 0$ there exists $\epsilon < \epsilon_0$ such that $0 < y(t,\epsilon) < \rho$ on $[\theta, \bar{t}^* - \theta]$ with sufficiently small $\theta > 0$. Moreover, from (8.1.22), we have

$$\underline{x}(t,\theta,\rho) \leq x(t,\epsilon), t \in [0, \bar{t}^* - \theta]. \tag{8.1.35}$$

Then, let us denote by x_4 the solution to

$$\frac{dx_4}{dt} = f(t, x_4, \phi(t,x_4) + w, \epsilon), \quad x_4(\hat{t}) = \underline{x}(\hat{t},\theta,\rho), \quad t \in [\hat{t}, T].$$

From Step 2, $y(t,\epsilon) \le \phi(t,x_4) + w$. It follows that, according to A_3,

$$f(t,x_4,\phi(t,x_4)+w,\epsilon) \le f(t,x_4,y,\epsilon).$$

Thus, using Remark 8.1.0.4, we have $x_4(t,\theta,\rho,\epsilon) \le x(t,\epsilon)$ for all sufficiently small ϵ. On the other hand, let us define x_5 to be the solution to

$$\frac{dx_5}{dt} = f(t,x_5,\phi(t,x_5),0), \quad x_5(\hat{t}) = \underline{x}(\hat{t},\theta,\rho), \quad t \in [\hat{t},T]. \tag{8.1.36}$$

It follows, according to the regular perturbation theory, that for any $\wp > 0$, there exists $\epsilon_5 > 0$ such that for any $\epsilon \in \mathbb{I}_{\epsilon_5}$,

$$|x_5(t,\theta,\rho) - x_4(t,\theta,\rho,\epsilon)| < C\wp, \ t \in [\hat{t},T],$$

where C is independent of the parameters $\theta,\rho,\epsilon,\wp$. Let us now define the function $\underline{X}$ by

$$\underline{X}(t,\theta,\rho,\wp) = -C\wp + \begin{cases} \underline{x}(t,\theta,\rho) \text{ for } 0 \le t \le \hat{t}, \\ x_5(t,\theta,\rho) \text{ for } \hat{t} < t \le T. \end{cases} \tag{8.1.37}$$

It is clear that $\underline{X}$ satisfies

$$\underline{X}(t,\theta,\rho,\wp) \le x(t,\epsilon), \ t \in \mathbb{I}_T. \tag{8.1.38}$$

By assumption A_1, there exists $K \in \mathbb{R}$ such that $\inf_{\bar{V}} f \ge K$. Since $f = \frac{dx_5}{dt}$, it follows by integration that

$$x_5(t) - x_5(\hat{t}) \ge K(t-\hat{t}) \Rightarrow x_5(t) \ge x_5(\hat{t}) + K(t-\hat{t}). \tag{8.1.39}$$

By condition (8.1.30), there exists $t^1 < \bar{t}^*$ such that $\bar{x}(t) \ge \psi(t) + \Omega'$ for $t \in [t^1, \bar{t}^*]$ and for some $\Omega' > 0$. Further, according to the poof of Theorem 8.1.0.1, the lower bound $\underline{x}$ and the upper bound $\bar{x}$ can be made as close as one wishes for $t < \hat{t}$. Then, for small parameters θ,ρ, there is $0 < \Omega'' \le \Omega'$ such that $\underline{X}(t,\theta,\rho,\wp) \ge \psi(t) + \Omega''$ for $t \in [t^1,\bar{t}^*]$. Let $\Omega \in [0,\Omega'']$. Then, from (8.1.39), it follows that

$$x_5(t) \ge \psi(t) + \Omega'' + K(t-\hat{t});$$

that is,

$$x_5(t) \ge \psi(\hat{t}) - \psi(t) + K(t-\hat{t}) - C\wp + \Omega'' - \Omega + \Omega + \psi(t) + C\wp.$$

Since the constants Ω, Ω'' and C are taken to be independent of the choice of $\hat{t}$ and considering $\hat{t} \in [t^1, \bar{t}^*]$, it follows, by continuity, that there are $\tilde{t} > \bar{t}^*$, $\hat{t}$ sufficiently close to $\bar{t}^*$ and $\wp > 0$ such that

$$\underline{X}(t,\theta,\rho,\wp) \ge \psi(t) + \Omega, \ t \in [\hat{t},\tilde{t}]. \tag{8.1.40}$$

Step 4 - **Determination of a lower bound with respect to y and its characterisation.** Let us now consider the solution $\underline{Y}(t,\theta,\rho,\wp)$ of the Cauchy problem

$$\begin{cases} \epsilon \underline{Y}^{'} = g(t, \underline{X}(t,\theta,\rho,\wp), \underline{Y}, \epsilon), \\ \underline{Y}(0,\theta,\rho,\wp,\epsilon) = y_0, \end{cases} \tag{8.1.41}$$

at least on $[0,\tilde{t}]$. According to the assumption A_5, the above equation has two quasi-steady states $y = 0$ and $y = \phi(t, \underline{X}(t,\theta,\rho,\wp))$ which intersect at $\underline{t}_c$. Let us now define the function $\underline{\mathcal{G}}$ by

$$\underline{\mathcal{G}}(t,\rho,\theta,\wp,\epsilon) = \int_0^t g_y(s, \underline{X}(s,\theta,\rho,\wp), 0, \epsilon) ds, \ t \le \tilde{t}. \tag{8.1.42}$$

According to the Butuzov Theorem, Theorem 3.3.4.1, we have to consider

$$\underline{\mathcal{G}}(t,\rho,\theta,\wp,0) = \int_0^t g_y(s, \underline{X}(s,\theta,\rho,\wp), 0, 0) ds, \ t \le \tilde{t}. \tag{8.1.43}$$

By the definition of $\underline{X}$ and from (8.1.25) it follows that for $t < \hat{t}$,

$$\underline{\mathcal{G}}(t,\rho,\theta,\wp,0) = \int_0^t g_y(s, -C\wp + \underline{x}(s,\theta,\rho), 0, 0) ds \le \underline{G}(t,\rho,\theta,0).$$

Similarly, since $\underline{X} \le x \le \bar{x}$ on $\bar{\mathbb{I}}_T$,

$$\underline{\mathcal{G}}(t,\rho,\theta,\wp,\epsilon) \le \bar{G}(t,0). \tag{8.1.44}$$

Since $\bar{G}(t,0) \le 0$ for $t \in [0,\bar{t}^*]$, it follows that $\underline{\mathcal{G}} < 0$ for $t \in [0,\bar{t}^*]$, and $\underline{\mathcal{G}} \to 0$ as $\hat{t} \to \bar{t}^*$, and $\theta, \rho, \wp \to 0$. Now, from the definition of $\underline{X}$, for $t > \hat{t}$ we have

$$\underline{\mathcal{G}}(t,\rho,\theta,\wp,0) = \int_0^{\hat{t}} g_y(s, \underline{x}(s,\theta,\rho) - C\wp, 0, 0) ds + \int_{\hat{t}}^t g_y(s, x_5(t,\theta,\rho) - C\wp, 0, 0) ds.$$

Since $\underline{X} \ne \psi(t)$, from (8.1.40), it follows that, $g_y(t, \underline{X}, 0, 0) \ne 0$ for $t \in [\hat{t}, \tilde{t}]$. From A_6, it follows that $g_y(t, \underline{X}, 0, 0) > 0$ and thus

$$g_y(t, \underline{X}, 0, 0) \ge L, \tag{8.1.45}$$

for some $L > 0$. By integrating both sides of the equation (8.1.45) on $[\bar{t}^*, \tilde{t}]$, we have

$$\int_{\bar{t}^*}^{\tilde{t}} g_y(s, x_5(s,\theta,\rho) - C\wp, 0, 0) ds \ge L(\tilde{t} - \bar{t}^*).$$

On the other hand, from (8.1.44) and A_8,

$$\underline{\mathcal{G}}(t,\rho,\theta,\wp,\epsilon) \le \bar{G}(t,0) < 0, \ t \in (0,\bar{t}^*).$$

Thus,

$$\int_0^{\hat{t}} g_y(s, x_5(s,\theta,\rho) - C\wp, 0, 0)ds < \int_0^{\hat{t}} g_y(s, \bar{x}(s), 0, 0)ds < 0.$$

Hence

$$\int_{\bar{t}^*}^{\tilde{t}} g_y(s, x_5(s,\theta,\rho) - C\wp, 0, 0)ds \geq L(\tilde{t} - \bar{t}^*) > 0 > \int_0^{\hat{t}} g_y(s, x_5(s,\theta,\rho) - C\wp, 0, 0)ds.$$

Therefore, by continuity of the function $\underline{\mathcal{G}}$, there is a solution $\underline{t}^* = \underline{t}^*(\hat{t}, \rho, \theta, \wp) < \bar{t}^*$ to $\underline{\mathcal{G}}(t, \rho, \theta, \wp, 0) = 0$.

Moreover, from (8.1.44), it follows that $\underline{\mathcal{G}}$ is monotonic for $t > \hat{t}$. Thus its root $\underline{t}^*$ is unique and it tends to $\bar{t}^*$ of as $|\bar{t}^* - \hat{t}|, \theta, \rho, \wp$ tend to zero. Let us set $\wp, \theta, \rho, \bar{t}$. Then, for $(t, \epsilon) \in (\hat{t}, \bar{t}^*) \times (0, \bar{\epsilon})$, $\underline{\mathcal{G}}$ is C^2-function with $\bar{\epsilon}$ chosen such that the relation (8.1.38) is satisfied for all $\epsilon \in (0, \bar{\epsilon})$. Thus, by applying the Butuzov theorem and Remark 7.1.0.3,

$$\lim_{\epsilon \to 0} \underline{Y}(t, \theta, \rho, \wp, \epsilon) = \phi(t, x_5(t) - C\wp),$$

almost uniformly on $(\underline{t}^*, \tilde{t}]$.

Step 5 - **Determination of upper bounds for x and y and their characterisation.**

By the definition of the uniform convergence, for any $\underline{t}^* < \tau < \tilde{t}$ and any $\delta' > 0$, there is $\tilde{\epsilon} > 0, \tilde{\wp} > 0$, such that for any $\epsilon \in \mathbb{I}_{\tilde{\epsilon}}$, $\wp \in \mathbb{I}_{\tilde{\wp}}$ and $t \in [\tau, \tilde{t}]$,

$$|\underline{Y}(t, \theta, \rho, \wp, \epsilon) - \phi(t, x_5)| \leq \delta'.$$

It follows that

$$-\delta' \leq \underline{Y}(t, \theta, \rho, \wp, \epsilon) - \phi(t, x_5(t)) \leq y(t, \epsilon) - \phi(t, x_5) \Rightarrow y(t, \epsilon) \geq \phi(t, x_5) - \delta'. \tag{8.1.46}$$

Let us denote by x_6 the solution to the problem

$$\begin{cases} \frac{dx_6}{dt} = f(t, x_6, \phi(t, x_5) - \delta', \epsilon), \\ x_6(\tau, \epsilon) = \bar{x}(\tau, \epsilon), \end{cases}$$

where ϵ is sufficiently small. From (A_3) and (8.1.46) it follows that

$$x_6 \geq x(t, \epsilon), \ t \in [\tau, \tilde{t}].$$

Thus, the composite function defined by

$$\bar{X}(t,\epsilon) = \begin{cases} \bar{x}(t,\epsilon),\ t \in [0,\tau], \\ x_6(t,\epsilon),\ t \in (\tau,\tilde{t}], \end{cases}$$

is an upper bound for $x(t,\epsilon)$ on $[0,\tilde{t}]$ and the function $\bar{Y}$, defined as the solution of

$$\epsilon\bar{Y} = g(t,\bar{X}(t,\epsilon),\bar{Y},\epsilon),\ \bar{Y}(0,\epsilon) = y_0,$$

is an upper bound for $y(t,\epsilon)$. For $0 < t < \tau$, we have $g(t,\bar{X}(t,0),0,0) = g(t,\bar{x}(t),0,0)$. Hence

$$\bar{\mathcal{G}}(t,0) = \int_0^t g_y(s,\bar{X}(s),0,0)ds = \int_0^t g_y(s,\bar{x}(s),0,0)ds = \bar{G}(t,0).$$

Thus, $\bar{\mathcal{G}}(\bar{t},0) < 0$ for $t < \bar{t}^*$, $\bar{\mathcal{G}}(\bar{t}^*,0) = 0$ and $\bar{G}(\bar{t},0) > 0$ for $t \in (\bar{t}^*,\tilde{t})$. According to the Butuzov theorem, it follows that

$$\lim_{\epsilon\to 0}\bar{Y}(t,\epsilon) = \phi(t,x_6(t,0))$$

uniformly on $[\tau,\tilde{t}]$.

Step 6 - **Relationship between the lower bound, the upper bound and the original solution.**

Now, let us consider the following equations

$$\frac{dx_6}{dt} = f(t,x,\phi(t,x_5)-\delta^{'},0),\quad x_6(\tau,\epsilon) = \bar{x}(t,0) \tag{8.1.47}$$

and

$$\frac{dx_5}{dt} = f(t,x_5,\phi(t,x_5),0),\quad x_5(\hat{t},\epsilon) = \underline{x}(\hat{t},\theta,\rho),\ t > \hat{t}.$$

It can be noticed that the former equation is a regular perturbation of the latter one. Thus, for any $\delta^{"} > 0$ there exist $\hat{t},\tau,\theta,\rho,\wp,\delta^{'},\epsilon^{"}$ such that for all $\epsilon < \epsilon^{"}$,

$$|x_6(t,0) - x_5(t)| < \delta^{"},\ t \in [\hat{t},\tilde{t}].$$

Finally, let us consider the following equation

$$\frac{dx_*}{dt} = f(t,x_*,\phi(t,x_*),0),\quad x_*(\bar{t^*}) = \bar{x}(\bar{t^*}).$$

It can be noticed that this equation is a regular perturbation of both (8.1.36) and (8.1.47). Thus, for any $\alpha > 0$, for ϵ sufficiently small,

$$\phi(t,x_*(t)) - \alpha \le y(t,\epsilon) \le \phi(t,x_*(t)) + \alpha, t \in [\tau,\tilde{t}]. \tag{8.1.48}$$

This implies

$$\lim_{\epsilon\to 0} y(t,\epsilon) = \phi(t, x_*(t)),\ t \in [\tau, \tilde{t}].$$

By the regular perturbation theory, Theorem 3.3.1.1, it follows that

$$\lim_{\epsilon\to 0} x(t,\epsilon) = x_*(t) \tag{8.1.49}$$

uniformly on $t \in [\tau, \tilde{t}]$. Thus, by (8.1.49), relation (8.1.48) can be rewritten as follows

$$\phi(t, x(t,\epsilon)) - \tilde{\delta} \le y(t,\epsilon) \le \phi(t, x(t,\epsilon)) + \tilde{\delta},\ t \in [\tau, \tilde{t}],$$

with arbitrarily small $\tilde{\delta} > 0$.

Step 7 - **Conclusion.** We proved in Step 2 that the trajectory cannot leave the layer Σ_w; that this

$$\phi(t, x(t,\epsilon)) - \tilde{\delta} \le y(t,\epsilon) \le \phi(t, x(t,\epsilon)) + \tilde{\delta},\ t \in [\tau, T].$$

Since τ can be chosen arbitrarily close to $\bar{t^*}$, it follows that

$$\lim_{\epsilon\to 0} y(t,\epsilon) = \phi(t, x_*(t)),$$
$$\lim_{\epsilon\to 0} x(t,\epsilon) = x_*(t),$$

on $t \in (\bar{t^*}, T]$.

□

8.2 Two Dimensional Stable Case with a Positive Initial Condition

Let us study the case of immediate stability switch for two dimensional non-autonomous problemes with a transcritical bifurcation described in Figure 8.4. It can be observed that the stable parts of the quasi steady states are non-negative. Consider the following conditions.

B_5- **Existence and intersection of two quasi steady states.**

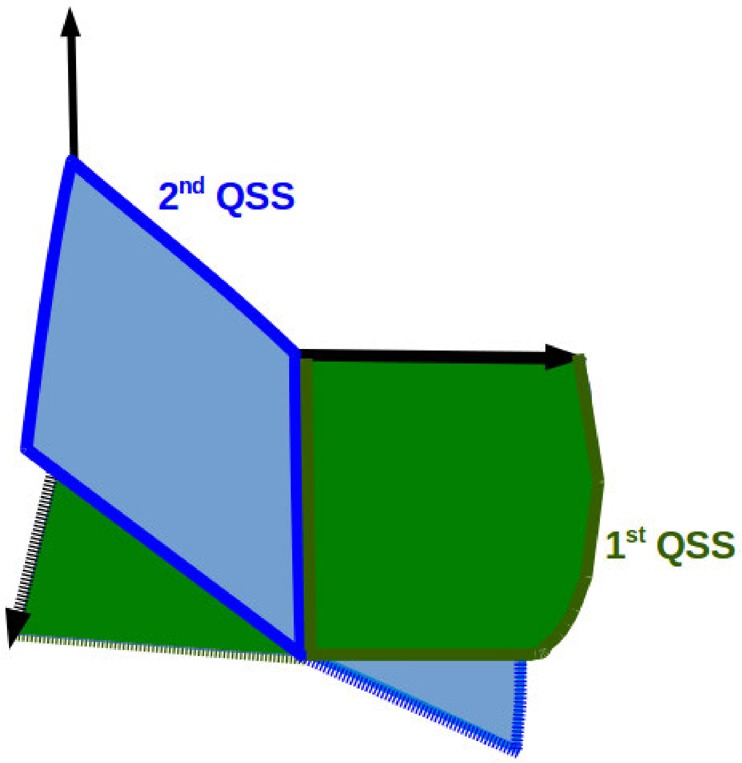

Figure 8.4: Transcritical Bifurcation considered to prove the immediate stablity. The unstable parts of the QSSs are the parts with dotted contour and the stable parts have continous contours. We observed that the QSSs intersect each other, switch stability at their intersection and their stable parts are non-negative.

Assume that there are two quasi steady states $y = 0$ and $y = \phi(t, x)$ that are solutions of the equation $g(t, x, y, 0) = 0$ in $\bar{\mathbb{I}}_T \times \bar{\mathbb{I}}_N \times \bar{\mathbb{I}}_M$. There is a unique solution $y = \psi(t)$ to the equation

$$\phi(t, x) = 0 \text{ for } t \in \bar{\mathbb{I}}_T,$$

$x \in \bar{\mathbb{I}}_M$ and $\psi \in C^2(\bar{\mathbb{I}}_T)$. Also, we assume

$$\phi(t, x) > 0 \text{ for } x - \psi(t) < 0,$$
$$\phi(t, x) < 0 \text{ for } x - \psi(t) > 0.$$

B_6- **Exchange of stability at the intersection of quasi steady states.** Assume that the quasi steady states switch stability at their intersection in the following way:

$g_y(t, x, 0, 0) > 0$ and $g_y(t, x, \phi(t, x), 0) < 0$ for $x - \psi(t) < 0$,

$g_y(t, x, 0, 0) < 0$ and $g_y(t, x, \phi(t, x), 0) > 0$ for $x - \psi(t) > 0$.

B_7- **Assumptions on the behaviour of the solution close to the quasi steady states.**

Since we are concerned with the behaviour of solutions close to the intersection of quasi steady states, we must assume that they actually pass close to it. Denote by

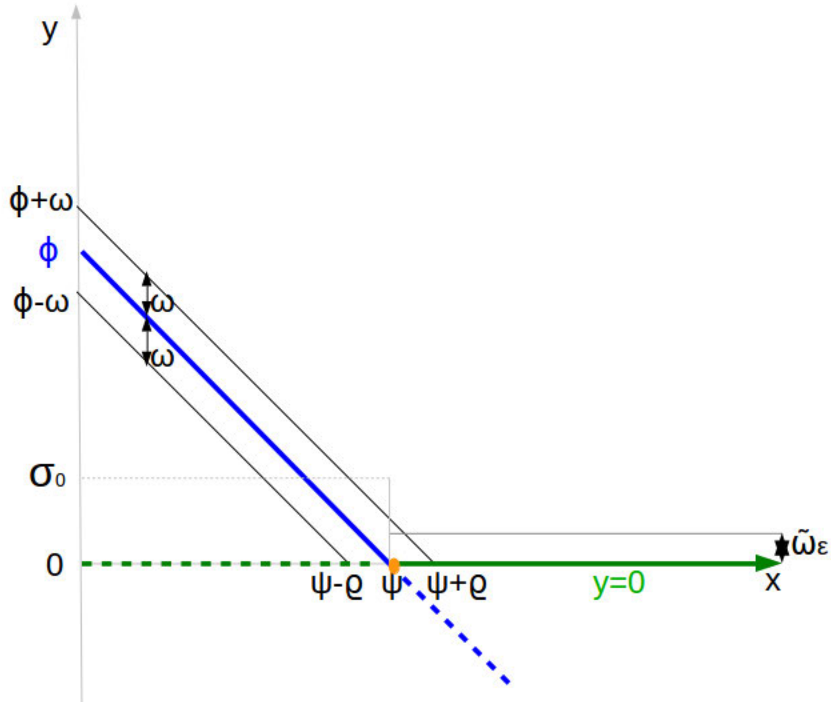

Figure 8.5: Construction used in the proofs of Theorem 8.2.0.1 and Theorem 8.2.0.2.

$x_\phi(t,\epsilon)$ the solution of

$$\frac{dx}{dt} = f(t, x, \phi(t, x), \epsilon),\ x(0, \epsilon) = x_0. \tag{8.2.1}$$

Then we assume that any solution $x_\phi = x_\phi(t)$ to problem (8.2.1) with $\epsilon = 0$,

$$\frac{dx}{dt} = f(t, x, \phi(t, x), 0),\ x(0) = x_0, \tag{8.2.2}$$

and $0 < x_0 < \psi(0)$ satisfies $x_\phi(T) > \psi(T)$ and there is exactly one $\tilde{t}_c \in \mathbb{I}_T$ such that $x_\phi(\tilde{t}_c) = \psi(\tilde{t}_c)$.

8.2.0.1 Theorem. *Let us assume that $A_1, A_2,\ B_5,\ B_6,\ B_7,\ A_8$ and A_9 hold. Then the solution to (8.0.1) exists on $[0, T]$, is unique and satisfies*

$$\lim_{\epsilon\to 0} y(t,\epsilon) = \phi(t, x_\phi(t)) \text{ for } t \in (0, \tilde{t}_c),$$
$$\lim_{\epsilon\to 0} x(t,\epsilon) = x_\phi(t) \text{ for } t \in [0, \tilde{t}_c),$$

where $x_\phi(t)$ is solution to

$$\frac{dx_\phi}{dt} = f(t, x_\phi, \phi(t, x_\phi), 0),\ x_\phi(0) = x_0.$$

On each interval of time, the convergence is almost uniform.

Proof. According to the Picard theorem, Theorem 3.2.1.1, the solution to (8.0.1) exists and is unique on $\bar{\mathbb{I}}_T$. From the assumption B_7, for any $t^1 < \tilde{t}_c$ there exists a constant c^1 such that

$$c^1 \leq \inf_{t\in[0,t^1]} \Big(\psi(t) - x_\phi(t)\Big).$$

Let us consider $0 < \varrho < c^1$, and $S_\varrho = \{(t,x), t \in (0,t^1), x \in (0, \psi(t) - \varrho)\}$. From B_5, it follows that

$$\kappa_\varrho = \inf_{(t,x)\in S_\varrho} \phi(t,x) > 0.$$

Therefore, ϕ is an isolated quasi steady state on S_ϱ. Thus, according to the Tikhonov theorem on S_ϱ and, by letting t^1 tends to $\tilde{t}_c$ and ϱ to 0, it follows that

$$\lim_{\epsilon\to 0} y(t,\epsilon) = \phi(t, x_\phi(t)) \text{ for } t \in (0, \tilde{t}_c),$$
$$\lim_{\epsilon\to 0} x(t,\epsilon) = x_\phi(t) \text{ for } t \in [0, \tilde{t}_c),$$

where $x_\phi(t)$ is the solution to

$$\frac{dx_\phi}{dt} = f(t, x_\phi, \phi(t, x_\phi), 0), \qquad x_\phi(0) = x_0.$$

□

8.2.0.2 Theorem. *Let the assumptions* A_1, A_2, B_5, B_6, A_7, A_8, A_9 *and condition* (8.1.30) *hold. Then, the solution to* (8.0.1) *exists on* $[0,T]$, *is unique and satisfies*

$$\lim_{\epsilon\to 0} y(t,\epsilon) = 0 \textit{ for } t \in (\tilde{t}_c, T], \tag{8.2.3}$$
$$\lim_{\epsilon\to 0} x(t,\epsilon) = \bar{x}(t) \textit{ for } t \in [\tilde{t}_c, T], \tag{8.2.4}$$

where $\bar{x}(t)$ *is the solution to*

$$\frac{d\bar{x}}{dt} = f(t, \bar{x}, 0, 0),\ \bar{x}(0) = x_0.$$

Proof. According to the Picard theorem, Theorem 3.2.1.1, the solution to (8.0.1) exists and is unique on $\bar{\mathbb{I}}_T$. We note that some technical steps of this proof are similar to those in the proofs of Theorem 8.1.0.1 and 8.1.0.6. Therefore only a sketch of them will be provided. From the continuity of the solution and the assumption B_7 it follows that for any $\varrho > 0$ there exists $t^1 < \tilde{t}_c$ such that $\psi(t^1) - \varrho < x(t^1, \epsilon) < \psi(t^1) + \varrho$ and $y(t^1, \epsilon) < \varrho$. Now, as in (8.1.30), there exist positive parameters $\alpha_1, \sigma_0, \epsilon_0$ such that

$$\Psi(t, \psi(t) + \sigma, y, \epsilon) := \frac{g_t(t, \psi(t) + \sigma, y, \epsilon)}{g_x(t, \psi(t) + \sigma, y, \epsilon)} + f(t, \psi(t) + \sigma, y, \epsilon) \geq \alpha_1 \tag{8.2.5}$$

for all $|y| \geq \sigma_0,\ |\sigma| < \sigma_0, |\epsilon| < \epsilon_0$. Furthermore, let us define the composite stable quasi steady state Φ by

$$\Phi(t, x) = \begin{cases} \phi(t, x), & t \in \mathbb{I}_T,\ x \in (0, \psi(t)), \\ 0, & t \in \mathbb{I}_T,\ x \in (\psi(t), M), \end{cases}$$

and consider $w > 0$ such that $w < \sigma < \sigma_0$ and $\phi(t, \psi(t) - \varrho) + w < \sigma$. Then it is clear that $y(t, \epsilon) < \sigma$. Arguing as in the Step 1 of Theorem 8.1.0.1, it follows from (8.2.5) that the trajectory cannot go back through the left wall $\{(t, x, y); t \in \mathbb{I}_T, x = \psi(t) - \varrho, y \in (0, \phi(t, \psi(t) - \varrho) + w)\}$. Using (8.2.5) we can get a more detailed picture of the solution. From (8.2.5), we see that

$$x(t, \epsilon) > \psi(t) + \varrho \ \text{ for } t < t^1 + \frac{2\varrho}{\alpha_1}$$

and, for sufficiently small ϵ, the solution $(x(t, \epsilon), y(t, \epsilon))$ cannot cross back through the wall $\{(t, x, y); t \in [0, T], x = \psi(t) + \varrho, y \geq 0\}$. So the only possibility of exit is through

$$y = \Phi(t, x) + w \text{ for } x > \psi(t) - \varrho.$$

It follows that, by the selection of constants, the trajectory enters the region where

$$g_y(t, x, y, 0) < 0.$$

Further, from assumption B_6 we have $g(t, x, y, 0) < 0$ for $t \in \mathbb{I}_T,\ x \in \mathbb{I}_M$ and $y \in (\Phi(t, x), N)$. Therefore, for any $\gamma > 0$ there exists $\tilde{\beta} > 0$ such that $g(t, x, y, 0) < -\tilde{\beta}$ for $y > \Phi + \gamma$. Hence $g(t, x, y, \epsilon) \leq 0$ for ϵ sufficiently small and for $y > \Phi + \gamma$. It follows that for all $\epsilon \in I_{\epsilon_0}$ there exists $\tilde{w}_\epsilon > 0$ such that we have $0 < y(t, \epsilon) < \tilde{w}_\epsilon$ for $t > t^1$. In other words, the solution y is bounded. Therefore, letting t^1 tends $\tilde{t}_c$ and since the solution is in the region where $y'(t, \epsilon) < 0$, we obtain

$$\lim_{\epsilon \to 0} y(t, \epsilon) = 0 \tag{8.2.6}$$

uniformly on $[\tilde{t}_c, T]$. Since the problem

$$\frac{dx}{dt} = f(t, x, y(t, \epsilon), \epsilon), \quad x(\tilde{t}_c) = x_\phi(\tilde{t}_c)$$

is a regular perturbation of

$$\frac{dx}{dt} = f(t, x, 0, 0),\ x(\tilde{t}_c) = x_\phi(\tilde{t}_c),$$

according to the regular perturbation theory, Theorem 3.3.1.1,

$$\lim_{\epsilon \to 0} x(t, \epsilon) = \bar{x}(t),\ t \in [\tilde{t}_c, T]. \tag{8.2.7}$$

It follows that (8.2.3) and (8.2.4) are satisfied. □

8.3 Conclusion

In this chapter we stated and proofed general theories on the detection of the delay and immediate stability switch for non-autonomous singularly perturbed planar systems with a positive initial condition and possessing the first quadrant invariant under their flow. The method used is that of the upper and lower solutions. Using the theories developed in this chapter, one will be able to easily test and detect the presence of a delay and an immediate stability switch in a large number of models. They can also serve in the development of more specialised numerical simulations to improve the accuracy of the numerical approximations of stiff equations. Similar theories for non-autonomous singular perturbed planar systems with a negative initial condition and having the fourth quadrant invariant can be obtained by performing obvious changes.

In the next chapter we study a classical prey-predator model by applying the theories found in this chapter.

9 Study of a Two Dimensional Ecological Model

The predator-prey systems study is one of the cornerstones in Mathematical Biology. The prey-predator theories in general involve the *logistic* equation, the Lotka-Volterra equations, the incorporation into the equation of the prey and the predator of the ratio-dependent or the Michaelis-Menten-Holling functional response [14]. The aim of this chapter is to apply the theorems for planar systems developed in the previous chapter to a two dimensional ecological model of two species, say prey and predator, with interactions governed by the mass action law. Let us consider the following model

$$\begin{cases} \frac{dx}{dt} = x(A + Bx + Cy), \\ \epsilon\frac{dy}{dt} = y(D + Fx + Ey), \end{cases} \tag{9.0.1}$$

where $(t, x, y, \epsilon) \in \mathbb{I}_T \times \mathbb{I}_M \times \mathbb{I}_N \times \mathbb{I}_{\epsilon_0}$ with $T, M, N \in \mathbb{R}_+$ and $\epsilon_0 > 0$ being a small parameter. The initial condition is $(x(0), y(0)) = (x_0, y_0)$, and A, B, C, D, E, F are parameters. As in [58], in order to have a slow-fast predator-prey system, we assume that the dynamics of both species differ; that is, one of the species has a fast dynamics, while the other has a slow dynamics. In this case, $\epsilon > 0$ represents the ratio of the time scales of the dynamics of the two species. We will denote by a, b, c, d, e, f, respectively, the absolute values of A, B, C, D, E, F. Our goal is to determine conditions under which solutions of (9.0.1) exhibit a delay in stability switch and an immediate stability switch as described in Sections 8.1 and 8.2.

9.1 Preliminary Study

9.1.0.1 Theorem. *Consider the system of equations* (9.0.1). *For* $(x_0, y_0) \in \mathbb{R}^2_+$, *there exists globally a unique and non-negative solution* (x, y) *to* (9.0.1).

Proof. The proof is similar to that of Theorem 4.3.1.1. □

It can be observed, after setting $\epsilon = 0$ in the second equation of (9.0.1), that there are two quasi steady states:

$$y_1 = 0 \text{ and } y_2 = -\frac{F}{E}x - \frac{D}{E}.$$

They intersect each other at $x = -\frac{D}{F} = x_c$. In order to determine their stabilities, let us denote $g(x,y) = y(D+Fx+Ey)$. It follows that $g_y(x,y) = D+Fx+2Ey$. Substituting y by y_1 and y_2, respectively, in the expression of g_y, we obtain

$$\begin{cases} g_y(x,y_1) = D + Fx \\ g_y(x,y_2) = D + Fx + 2E(-\frac{F}{E}x - \frac{D}{E}) = -D - Fx. \end{cases} \tag{9.1.1}$$

Therefore, the quasi steady states switch stability at $x_c = -\frac{D}{F}$ for any value of D and $F \neq 0$. In order to keep our study realistic, we will focus only on the cases where the stable parts of both quasi steady states are non-negative.

Let us denote by $\bar{x}$ the solution to

$$\frac{d\bar{x}}{dt} = \bar{x}(A + B\bar{x}), \qquad \bar{x}(0) = x_0; \tag{9.1.2}$$

that is,

$$\bar{x}(t) = \frac{x_0 A e^{At}}{A + Bx_0 - Bx_0 e^{At}}, \tag{9.1.3}$$

with $t \neq \frac{\ln\left(1+\frac{A}{x_0 B}\right)}{A}$. It is easy to prove that $\bar{x}$ is an upper solution of (9.0.1) by applying the comparison theorem, Theorem 3.2.2.6, to (9.0.1) and (9.1.3). From (9.1.3) and according to the assumption A_7 of Theorem 8.1.0.1,

$$\bar{t}_c = \frac{1}{A}\ln\Big(\frac{-\frac{D}{F}(A + Bx_0)}{x_0(A - B\frac{D}{F})}\Big) = \frac{1}{A}\ln\Big(\frac{-\frac{D}{F}(\frac{A}{B} + x_0)}{x_0(\frac{A}{B} - \frac{D}{F})}\Big). \tag{9.1.4}$$

It follows that $\bar{t}_c$ exists if $-\frac{D}{F}(\frac{A}{B} + x_0)$ and $x_0(\frac{A}{B} - \frac{D}{F}) \neq 0$ have same sign. It is positive if either

- $A > 0$ and $\big| - \frac{D}{F}(\frac{A}{B} + x_0)\big| > \big|x_0(\frac{A}{B} - \frac{D}{F})\big|$;
- or $A < 0$ and $\big| - \frac{D}{F}(\frac{A}{B} + x_0)\big| < \big|x_0(\frac{A}{B} - \frac{D}{F})\big|$.

Furthermore, the function $\bar{G}$ is given by

$$\bar{G}(t) = \int_0^t g_y(\bar{x}(s), 0)ds = \int_0^t D + F\bar{x}(s)ds \tag{9.1.5}$$

$$= Dt - \frac{F}{B}\ln|A + Bx_0(1 - e^{At})| + \frac{F}{B}\ln|A|, \; t \in \mathbb{I}_T, \tag{9.1.6}$$

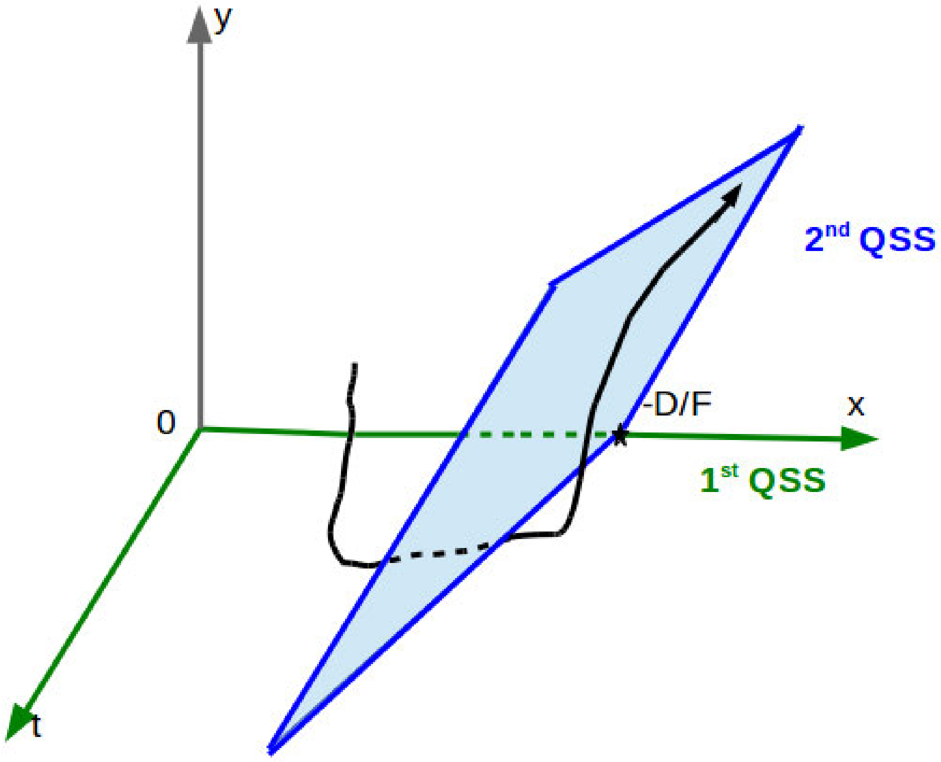

Figure 9.1: Case of an unstable population. It can be noticed that for the trajectory to cross the line $x = -\frac{D}{F}$, the initial condition should be $0 \leq x_0 < -\frac{D}{F}$.

with $t \neq \frac{\ln\left(1+\frac{A}{x_0 B}\right)}{A}$. In the discussion below we have distinguished two cases depending on the sign of the slope of the second quasi steady state.

9.2 Case of the Second Quasi Steady State Increasing with Respect to x

This case corresponds the unstable case considered in Section 8.1. The assumptions of this case are geometrically represented in Figure 9.1. It can be observed that for the trajectory to cross the plane $x = -\frac{D}{F}$ the initial condition should be such that $0 < x_0 < -\frac{D}{F}$. We assumed that the slope of the quasi steady state is positive; that is $m = -\frac{F}{E} > 0$. Therefore, F and E have opposite sign. There are two possible options: either $F < 0 < E$ or $E < 0 < F$.

1. If $F < 0 < E$, then $F = -f$ and $E = e$. We have, $x_c = -\frac{D}{F} = \frac{D}{f} > 0$ if and only if

$D = d$. The second equation of (9.0.1) becomes

$$\epsilon \frac{dy}{dt} = y(d - fx + ey).$$

Moreover, the system (9.1.1) becomes

$$g_y(x, y_1) = d - fx,$$
$$g_y(x, y_2) = -d + fx.$$

It follows that for $x \leq x_c$,

$$g_y(x, y_1) = d - fx \geq 0,$$
$$g_y(x, y_2) = -d + fx \leq 0.$$

Therefore, the quasi steady state y_1 is unstable for $x \leq x_c$, while y_2 is stable. Conversely, for $x \geq x_c$,

$$g_y(x, y_1) = d - fx \leq 0,$$
$$g_y(x, y_2) = -d + fx \geq 0;$$

that is, the quasi steady state y_2 is unstable for $x \geq x_c$, while y_1 is stable. Hence, there is a stability switch at x_c. However, we will not study this case since the stable parts of the quasi steady states are non-positive.

2. If $E < 0 < F$, then $E = -e$ and $F = f$. It follows that $x_c = -\frac{D}{F} = -\frac{D}{f} > 0$ if and only if $D = -d$. The second equation of (9.0.1) becomes

$$\epsilon \frac{dy}{dt} = y(-d + fx - ey). \tag{9.2.1}$$

From (9.1.1), we have

$$g_y(x, y_1) = -d + fx,$$
$$g_y(x, y_2) = d - fx.$$

Thus, for $x \leq x_c$,

$$g_y(x, y_1) = -d + fx \leq 0,$$
$$g_y(x, y_2) = d - fx \geq 0.$$

Therefore, the quasi steady state y_1 is stable, while y_2 is unstable for $x \leq x_c$. Conversely, for $x \geq x_c$,

$$g_y(x, y_1) = -d + fx \geq 0,$$
$$g_y(x, y_2) = d - fx \leq 0;$$

that is, the quasi steady state y_1 is unstable, while y_2 is stable. Therefore, there is stability switch at x_c with non-negative stable parts of the quasi steady states.

In order to study the case of the delay in stability switch, let us denote

$$f(x, y) = x(A + Bx + Cy).$$

According to the assumption A_2 of Theorem 8.1.0.1, f should be decreasing with respect to y. In other words,

$$\frac{\partial f}{\partial y} = Cx \leq 0;$$

that is

$$C = -c,$$

since x is non-negative. Furthermore, let us consider the function $\bar{G}$, as defined in assumption A_8 of Theorem 8.1.0.1. We have

$$\bar{G}(t) = \int_0^t g_y(\bar{x}(s), 0)ds = \int_0^t -d + f\bar{x}(s)ds, \qquad t \in \mathbb{I}_T.$$

Thus, its derivative with respect to time is

$$\bar{G}^{'}(t) = -d + f\bar{x}(t)$$

and its second derivative with respect to time is

$$\bar{G}^{''}(t) = f\bar{x}^{'}(t).$$

Since, by definition, $\bar{x}$ is solution to the equation $\frac{dx}{dt} = f(x, 0) = x(A + Bx)$, it follows that $\bar{G}^{''}(t) = f\bar{x}(A + B\bar{x})$. For the root $\bar{t}^*$ of $\bar{G}$ to exist, it is enough that the second derivative $\bar{G}^{''}(t) > 0$ for $t \in \mathbb{I}_T$; that is $A + B\bar{x} > 0$. Therefore, there are three possible options for A and B.

Case 1: $A = a$ and $B = b$; $\bar{t}_c > 0$ if and only if $x_0 < x(\bar{t}_c)$.

Case 2: $A = a$ and $B = -b$; $\bar{t}_c > 0$ if and only if $k > x(\bar{t}_c) > x_0$, where $k = \frac{a}{b}$.

Case 3: $A = -a$ and $B = b$; $\bar{t}_c > 0$ if and only if $x(\bar{t}_c) > x_0 > k$ with $k = \frac{a}{b}$.

Recapitulating, a delay in stability switch is observed in

$$\begin{cases} \frac{dx}{dt} = x(a + bx - cy), \\ \epsilon\frac{dy}{dt} = y(-d + fx - ey), \\ x_0 < \frac{d}{f}, \end{cases}$$

$$\begin{cases} \frac{dx}{dt} = x(a - bx - cy), \\ \epsilon\frac{dy}{dt} = y(-d + fx - ey), \\ x_0 < \frac{d}{f} < \frac{a}{b}, \end{cases}$$

$$\begin{cases} \frac{dx}{dt} = x(-a + bx - cy), \\ \epsilon\frac{dy}{dt} = y(-d + fx - ey), \\ \frac{a}{b} < x_0 < \frac{d}{f}. \end{cases}$$

9.2.0.1 Numerical Simulations

Let us consider the following problem

$$\begin{cases} \frac{dx}{dt} = x(1 + 0.1x - 10^{-2}y), \\ \epsilon\frac{dy}{dt} = y(-1 + 0.2x - 0.2y), \end{cases} \tag{9.2.2}$$

where the initial condition is $(2, 0.5)$. We considered $a = 1,\ b = 0.1,\ c = 0.01,\ d = 1,\ e = f = 0.2$. Therefore the condition $x_0 < \frac{d}{f}$ is satisfied. Moreover, after setting $\epsilon = 0$ in (9.2.2), we obtain two quasi steady states:

$$y_1(x) = 0 \text{ and } y_2(x) = x - 5.$$

Therefore the second quasi steady state is increasing with respect to x. The intersection of the quasi steady states occurs at $x_c = \frac{d}{f} = 5$. From (9.1.4), we have $\bar{t}_c = 0.7$ time units. According to (9.1.6), the function $\bar{G}$ is given by

$$\bar{G}(t) = -t - 2\ln|1.2 - 0.2e^t|,$$

with $t \in [0, ln(6))$. The numerical approximation of the root $\bar{t}^*$ of $\bar{G}$ is $\bar{t}^* = 1.17$ time units. Thus the delay of stability switch is approximately equal to 0.47 time units. Figure 9.2 shows the graph of the orbits of the solutions to the problem (9.2.2) for $\epsilon = 0.03,\ 0.02,\ 0.015,\ 0.011$. It can be noticed that as ϵ tends to 0, the orbits of the solution tend to the first quasi steady state and then switch to second quasi steady state with a delay.

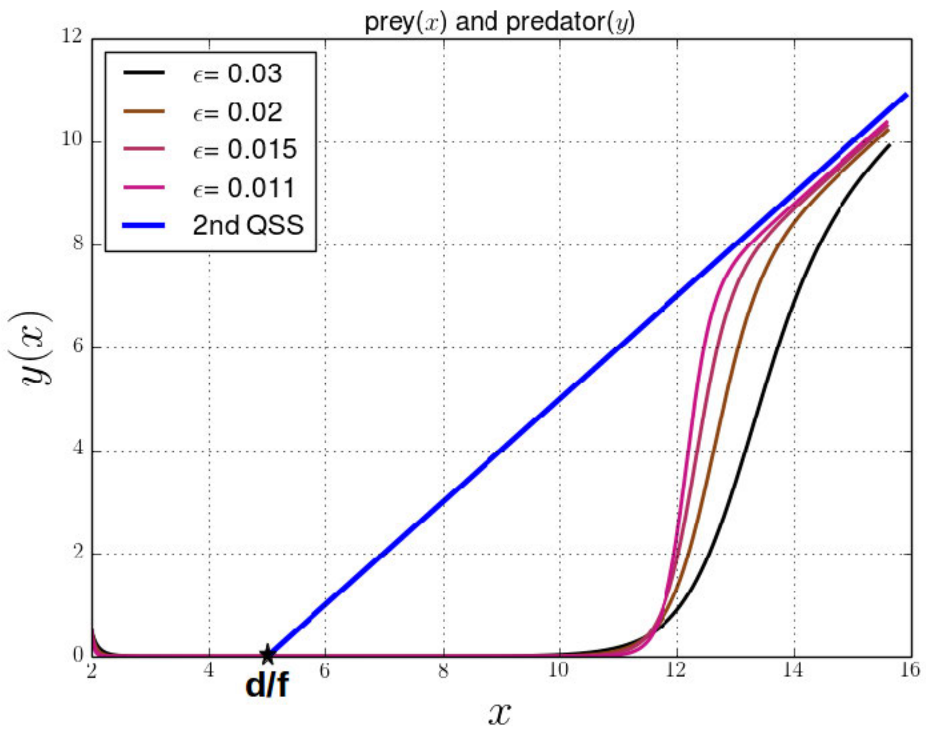

Figure 9.2: Illustration of the unstable case with a delay in stability switch.

9.3 Case of the Second Quasi Steady State Decreasing with Respect to x

This case corresponds to the situation described in Section 8.2. The assumptions of this case are geometrically represented in Figure 9.3. It can be observed that for the trajectory to cross the plane $x = -\frac{D}{F}$ the initial condition should be $x_0 < -\frac{D}{F}$. Since the second quasi steady state is assumed to be decreasing with respect to x; that is $m = -\frac{F}{E} < 0$, it follows that F and E must have the same sign. In other words, F and E are either both positive or they are both negative.

1. If $E = e$ and $F = f$ then $x(\bar{t}_c) = -\frac{D}{f} > 0$ if and only if $D = -d < 0$. Further, from (9.1.1), it follows that

$$g_y(x, y_1) = D + Fx = -d + fx,$$
$$g_y(x, y_2) = -D - Fx = +d - fx.$$

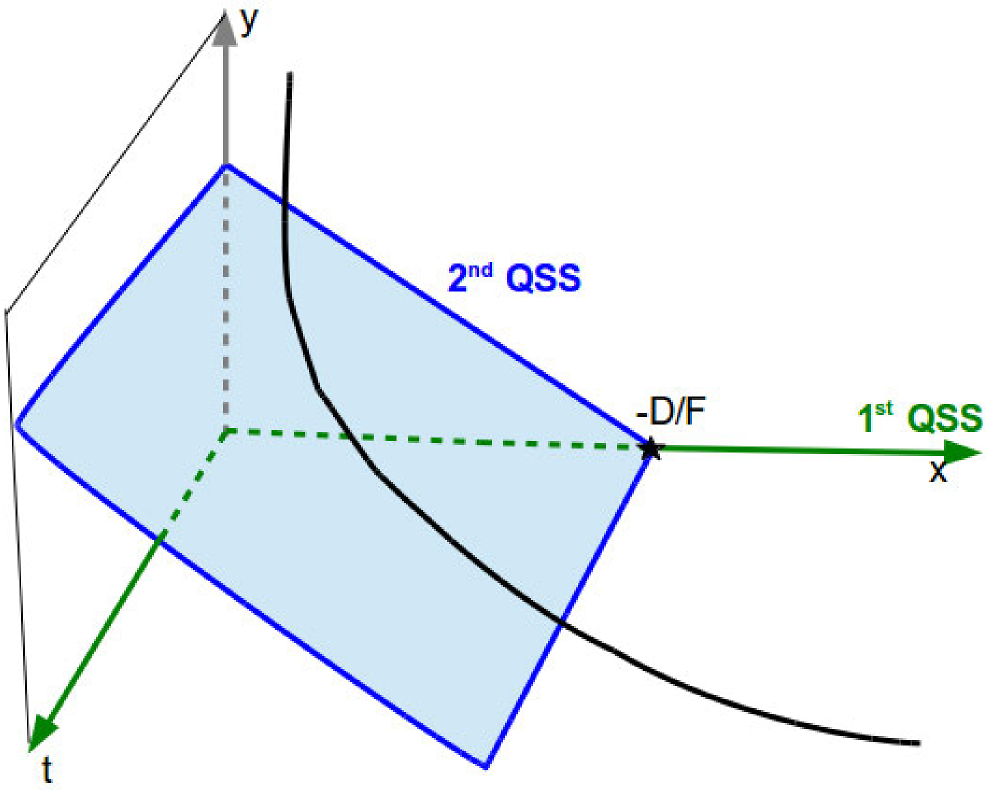

Figure 9.3: Case of a stable population. It can be noticed that for the trajectory to cross the plane $x = -\frac{D}{F}$, the initial condition should be $0 \leq x_0 < -\frac{D}{F}$.

Thus, for $x \leq -\frac{D}{F}$,

$$g_y(x, y_1) = -d + fx \leq 0,$$
$$g_y(x, y_2) = +d - fx \geq 0.$$

Therefore, y_1 is attracting and y_2 is repelling. On the other hand, for $x \geq -\frac{D}{F}$,

$$g_y(x, y_1) = -d + fx \geq 0,$$
$$g_y(x, y_2) = +d - fx \leq 0.$$

Therefore, y_1 is repelling, while y_2 is attracting. Since y_2 is considered to be decreasing with respect to x, it follows that the attracting parts of the quasi steady states are non-positive. Therefore, we will not further our study in this case.

2. If $E = -e < 0$ and $F = -f < 0$ then, $x_c = \frac{D}{f} > 0$ if and only if $D = d > 0$. From (9.1.1), we have

$$g_y(x, y_1) = D + Fx = d - fx,$$
$$g_y(x, y_2) = -D - Fx = -d + fx.$$

It follows that, for $x \leq x_c$,

$$g_y(x, y_1) = d - fx \geq 0,$$
$$g_y(x, y_2) = -d + fx \leq 0.$$

Therefore, y_2 is attracting and y_1 is repelling for $x \leq x_c$. However, for $x \geq x_c$,

$$g_y(x, y_1) = d - fx \leq 0,$$
$$g_y(x, y_2) = -d + fx \geq 0.$$

Therefore, for $x \geq x_c$, y_2 becomes repelling, while y_1 is attracting. Thus, there is a switch of stability at the intersection of the quasi steady states with non-negative stable parts.

From assumption B_7 of Theorem 8.2.0.1, let us determine the function x_ϕ. By definition, x_ϕ satisfies the differential equation

$$\frac{dx}{dt} = x(A + Bx + Cy_2) = \frac{x}{e}(Ae + Cd + (Be - Cf)x). \tag{9.3.1}$$

We will consider two cases: $Cf - Be = 0$ and $Cf - Be \neq 0$.

(a) If $Cf - Be = 0$, then equation (9.3.1) possesses only one equilibrium point $x = 0$ which is either repelling, or attractive depending on the sign of $Ae + Cd$. However, from B_7, we have $x_\phi(T) > \frac{f}{d}$. Therefore $x = 0$ should be repelling. Thus $Ae + Cd > 0$; that is, $-A < C\frac{d}{e}$. Since $C = \frac{Be}{f}$, it follows that

$$-A < \frac{Be}{f} \cdot \frac{d}{e};$$

that is

$$-A < B\frac{d}{f}.$$

(b) If $Cf - Be \neq 0$, then there is a critical point at $x_{eq} = \frac{Ae+Cd}{Cf-Be}$. For B_7 to be satisfied; that is, for $x_\phi(T) > \frac{d}{f}$, one of these cases should be satisfied.

i. Either the equilibrium point (x_{eq}, y_{eq}) is attracting and $x_{eq} > \frac{d}{f}$,

ii. or the equilibrium point (x_{eq}, y_{eq}) is repelling and $0 < x_{eq} < \frac{d}{f}$.

However, $Cf - Be \neq 0$ implies that either $Cf - Be > 0$ or $Cf - Be < 0$.

- If $Cf - Be < 0$, then $x_{eq} < \frac{d}{f}$ implies that

$$\frac{Ae + Cd}{Cf - Be} < \frac{d}{f};$$

that is, $-A < B\frac{d}{f}$. Similarly, $x_{eq} > \frac{d}{f}$ implies that

$$-A > B\frac{d}{f}.$$

- Conversely, if $Cf - Be > 0$, then $x_{eq} > \frac{d}{f}$ implies that

$$-A < B\frac{d}{f},$$

while $x_{eq} < \frac{d}{f}$ implies that

$$-A > B\frac{d}{f}.$$

Let assume that $-A < B\frac{d}{f}$, then the possible values for A and B are

$$A = a, B = b \text{ or } A = -a, B = b, \text{ with } \frac{a}{b} < \frac{d}{f} \text{ or } A = a, B = -b, \text{ with } \frac{d}{f} < \frac{a}{b}.$$

In summary we observe a switch of stability with no delay, as described in Section 8.2, in the following cases

$$\begin{cases} \frac{dx}{dt} = x(a + bx + Cy), \\ \epsilon\frac{dy}{dt} = y(d - fx - ey), \end{cases}$$
$$\begin{cases} \frac{dx}{dt} = x(-a + bx + Cy), \\ \epsilon\frac{dy}{dt} = y(d - fx - ey), \ \frac{a}{b} < \frac{d}{f}, \end{cases}$$
$$\begin{cases} \frac{dx}{dt} = x(a - bx + Cy), \\ \epsilon\frac{dy}{dt} = y(d - fx - ey), \ \frac{a}{b} > \frac{d}{f}, \end{cases}$$

with $C \in \mathbb{R}$ such that

- if $Cf - Be = 0$, then (x_{eq}, y_{eq}) is repelling;
- if $Cf - Be < 0$, then (x_{eq}, y_{eq}) repelling with $x_{eq} \in (0, \frac{d}{f})$;
- if $Cf - Be > 0$, then (x_{eq}, y_{eq}) attracting with $\frac{d}{f} < x_{eq}$.

Similar conditions with assumption $-A > B\frac{d}{f}$ can be obtained in the same way, except that the case (a) will not satisfy assumption B_7 of Theorem 8.2.0.1.

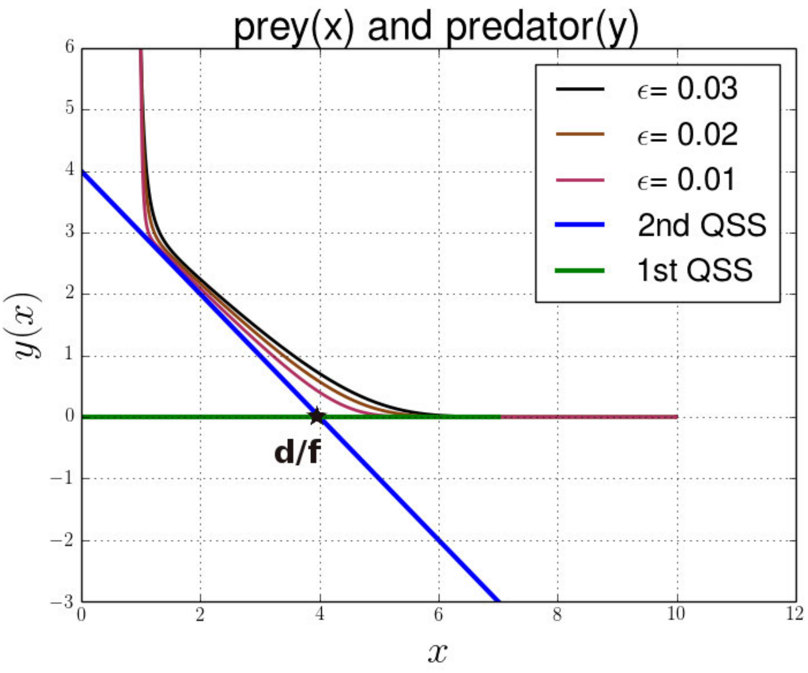

Figure 9.4: Case of the second quasi steady state decreasing with respect to x and an increasing prey population x. The orbit is traversed from the left to the right. It can be noticed that there is no delay in stability switch.

9.3.0.1 Numerical Simulations

Let us consider the following problem

$$\begin{cases} \frac{dx}{dt} = x(10 - x + y), \\ \epsilon \frac{dy}{dt} = y(4 - x - y), \end{cases} \tag{9.3.2}$$

with initial condition $(x_0, y_0) = (1, 6)$. This problem satisfies the condition $Cf - Be = 0$ with (x_{eq}, y_{eq}) repelling. The graph of the solutions to (9.3.2) is shown in Figure 9.4 for $\epsilon = 0.03,\ 0.02,\ 0.01$. We observe that as ϵ tends to zero, the solutions converge to the second quasi steady state and then to the first quasi steady state with no delay in the switch of stability.

9.4 Relationship Between Cases

In this section, we aim to determine additional assumptions under which some models satisfying the assumptions of Theorem 8.1.0.1 and Theorem 8.1.0.6 can be transformed into models satisfying the assumptions of Theorem 8.2.0.1 and Theorem 8.2.0.2 and vice

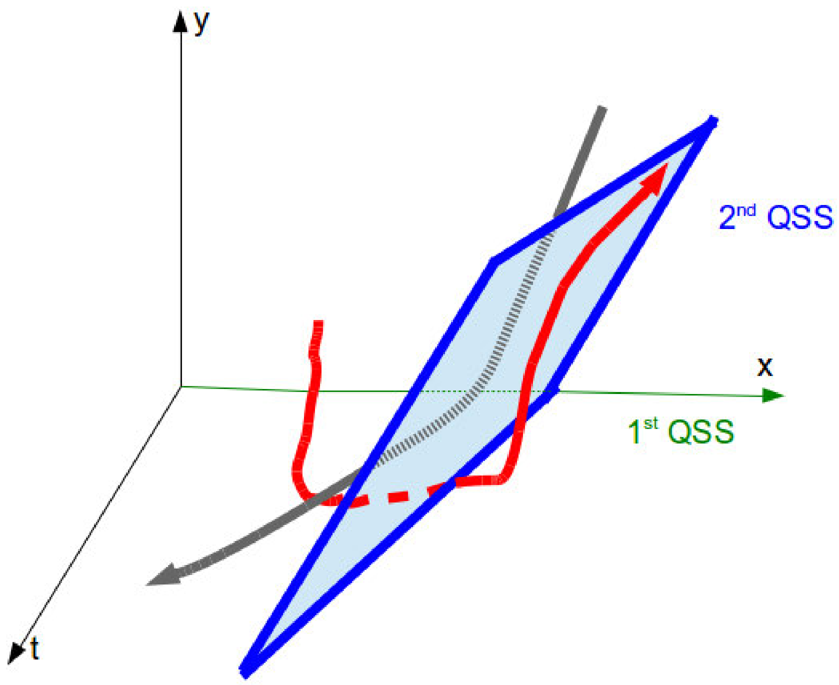

Figure 9.5: Case of the second quasi steady state increasing with respect to x. The red graph represents a graph of a solution in an unstable case while the gray graph is the graph of a solution in a stable one.

versa as indicated, respectively, in Figure 9.5 and 9.6 where the red graph is transformed into the gray graph. Let us consider a general model whose parameters satisfy the assumptions of Theorem 8.2.0.1 and Theorem 8.2.0.2 and let us determine under which conditions it can be transformed into a problem whose parameters satisfy the assumptions of Theorem 8.1.0.1 and Theorem 8.1.0.6. In other words, we want to study the possibility of having a delay in stability switch with the second quasi steady state decreasing with respect x. According to the previous study, a general model whose parameters satisfy the assumptions of Theorem 8.2.0.1 and Theorem 8.2.0.2 is given by

$$\begin{cases} \frac{dx}{dt} = x(A + Bx + Cy), \\ \epsilon\frac{dy}{dt} = y(d - ey - fx), \\ (x(0), y(0)) = (x_0, y_0), \end{cases} \tag{9.4.1}$$

where $A, B, C \in \mathbb{R}$ are parameters.

Let us set $x = \frac{2d}{f} - z$. The system (9.4.1) becomes

$$\begin{cases} \frac{dz}{dt} = -\frac{dx}{dt} = (z - \frac{2d}{f})(A + B(\frac{2d}{f} - z) + Cy), \\ \epsilon\frac{dy}{dt} = y(d - ey - f(\frac{2d}{f} - z)); \end{cases}$$

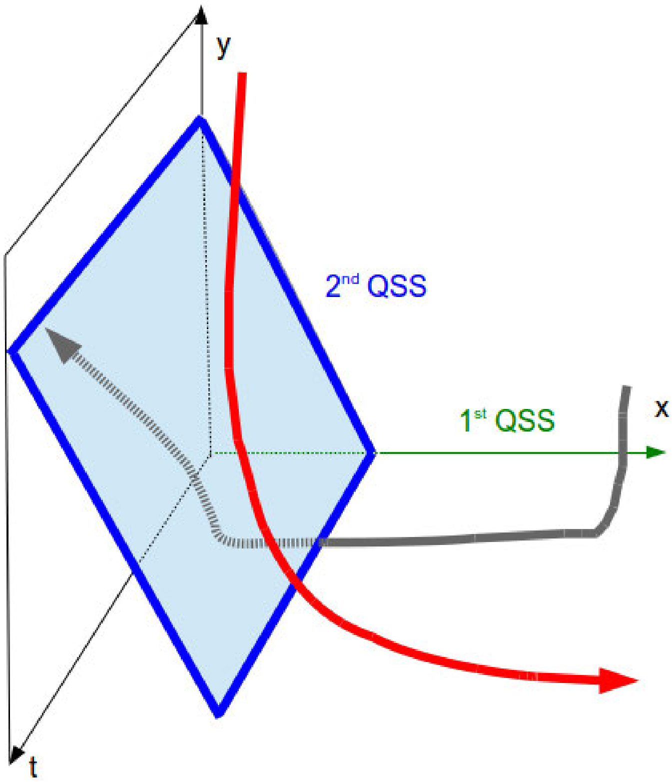

Figure 9.6: Case of a the second quasi steady state decreasing with respect to x. The graph in red is the graph of a solution without a delay in the switch of stability while the gray graph is the graph of a solution with a delay in stability switch.

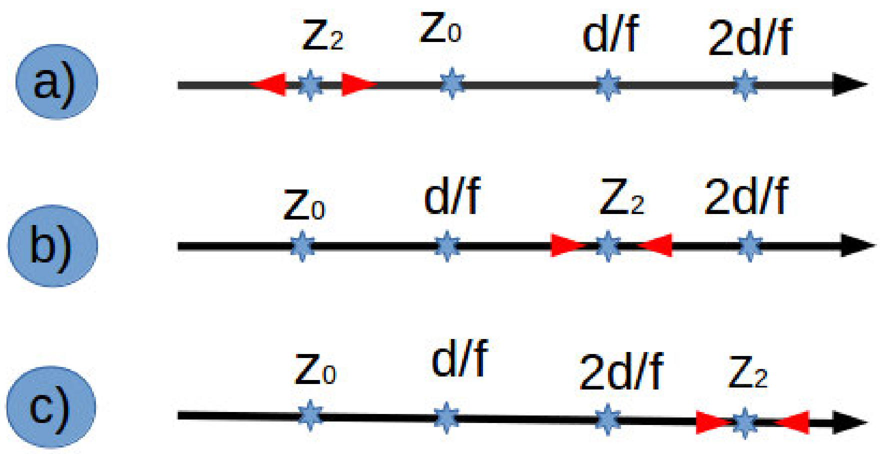

Figure 9.7: Illustrations of the different options that satisfy assumption A_7.

that is

$$\begin{cases} \frac{dz}{dt} = (z - \frac{2d}{f})(A + B(\frac{2d}{f} - z) + Cy), \\ \epsilon \frac{dy}{dt} = y(-d - ey + zf). \end{cases} \tag{9.4.2}$$

We observe that the last equation of (9.4.2) is the same as equation (9.2.1). Let us now determine the condition on A, B, and C for the delay to occur. Since x is positive, the factor $z - \frac{2d}{f} < 0$. Thus, according to the assumption A_3 of Theorem 8.1.0.1, $C = c > 0$. On the other hand, for $y = 0$, we have two equilibria $z_1 = \frac{2d}{f}$ and $z_2 = \frac{A}{B} + \frac{2d}{f}$. Assumption A_7 of Theorem 8.1.0.1 will be satisfied in the following cases:

1. $z_2 < \frac{d}{f}$ and it is repelling with $z_0 \in (z_2, \frac{d}{f})$. This case is shown in Figure 9.7 case a). However, substituting z_2 and z_0 by their values, we obtain the following implications:

$$z_2 < \frac{d}{f} \Rightarrow \frac{d}{f} < -\frac{A}{B},$$
$$z_2 < z_0 < \frac{d}{f} \Rightarrow -\frac{A}{B} > x_0 > \frac{d}{f},$$

and

$$x_2 = \frac{2d}{f} - z_2 = -\frac{A}{B}.$$

It follows that A, B should be of opposite sign with $\frac{d}{f} < x_0 < \frac{a}{b}$ and $x_2 = \frac{a}{b}$ being a repelling equilibrium.

2. $\frac{d}{f} < z_2 < \frac{2d}{f}$ and it is attracting with $z_0 < \frac{d}{f}$. This case is described in Figure 9.7

case b). Substituting z_2 and z_0 by their values, we obtain the following implications:

$$\frac{d}{f} < z_2 < \frac{2d}{f} \Rightarrow -\frac{A}{B} < \frac{d}{f},$$
$$z_0 < \frac{d}{f} \Rightarrow x_0 > \frac{d}{f},$$

and

$$x_2 = \frac{2d}{f} - z_2 = -\frac{A}{B}.$$

Thus, A, B should have opposite sign with $x_0 > \frac{d}{f} > \frac{a}{b}$, and $x_2 = \frac{a}{b}$ is attractive.

3. $z_2 > \frac{2d}{f}$ and z_1 attractive with $z_0 < \frac{d}{f}$ as described in Figure 9.7 case c). However, substituting z_2 and z_0 by their values, we obtain the following implications:

$$z_2 > \frac{2d}{f} \Rightarrow \frac{A}{B} > 0,$$
$$z_0 < \frac{d}{f} \Rightarrow x_0 > \frac{d}{f},$$

and

$$z_1 = \frac{2d}{f} - x_1 \Rightarrow x_1 = \frac{2d}{f} - z_1 = 0.$$

Therefore, we should have A, B with the same sign, $x_1 = 0$ attractive and $x_0 > \frac{d}{f}$.

In summary, there are three forms of the model that will exhibit a delay in stability switch with a decreasing quasi steady state. There are

$$\begin{cases} \frac{dx}{dt} = x(-a - bx + cy), \\ \epsilon\frac{dy}{dt} = y(d - ey - fx), \\ x_0 > \frac{d}{f},\ y_0 > 0, \end{cases}$$
$$\begin{cases} \frac{dx}{dt} = x(a - bx + cy), \\ \epsilon\frac{dy}{dt} = y(d - ey - fx), \\ x_0 > \frac{d}{f} > \frac{a}{b},\ y_0 > 0, \end{cases}$$
$$\begin{cases} \frac{dx}{dt} = x(a - bx + cy), \\ \epsilon\frac{dy}{dt} = y(d - ey - fx), \\ y_0 > 0,\ \frac{d}{f} < x_0 < \frac{a}{b}. \end{cases}$$

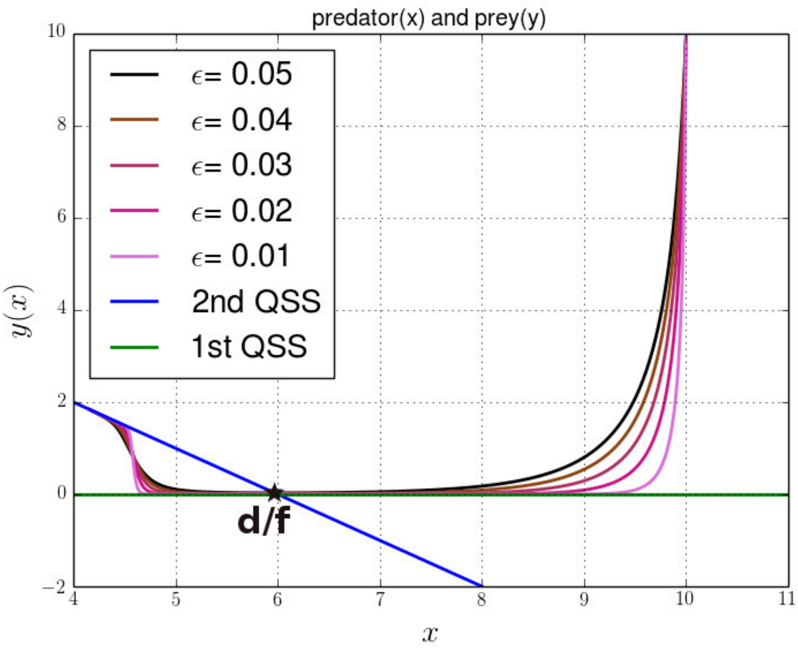

Figure 9.8: Case of the second quasi steady state decreasing with respect to x and a fast prey population y. As time go on, the orbit is traversed from the right to the left. We observe a delay in stability switch.

9.4.0.1 Numerical Simulation

Let us consider the following system of differential equation

$$\begin{cases} \frac{dx}{dt} = x(4 - x + 0.08y) \\ \epsilon \frac{dy}{dt} = y(6 - y - x) \\ (x_0, y_0) = (10, 10), \end{cases} \tag{9.4.3}$$

Setting $\epsilon = 0$ in the second equation of (9.4.3), we obtain two quasi steady states: $y_1 = 0$ and $y_2 = -x+6$. Therefore we have the second quasi steady state decreasing with respect to x and a fast prey population y. From equation (9.1.4), we have $\bar{t}_c = 0.14$ time units. From (9.1.6), the function $\bar{G}$ is given by

$$\bar{G}(t) = 6t + \ln|-6 + 10e^{4t}| + \ln(4),$$

for $t > 0$. The numerical computation of the root of $\bar{G}$ gives $\bar{t}^* = 0.39$ time units. Therefore, the delay is approximately equal to 0.25 time units. Figure 9.8 shows the graphs of the solutions to the system (9.4.3) for $\epsilon = 0.05,\ 0.04,\ 0.03,\ 0.02,\ 0.01$. It can be observed that the solutions tend first to the first quasi steady state and then, switch to the second quasi state as ϵ tends to 0 with a delay in the switch of stability.

9.5 Conclusion

In this chapter we have been able to study a general two dimensional prey and predator model governed by the mass action law using the theories developed in the previous chapter. We generated a number of systems of equations which exhibit a delay in stability switch and an immediate stability switch depending on the parameter values. Also we showed the relationship between both cases (stable and unstable) by presenting under some assumptions a transformation of an immediate stability switch into a delayed stability switch.

10 Conclusion

We introduced the thesis by a numerical example showing some dangers of using standard computer software to deal with singularly perturbed problems. Singularly perturbed problems belong to the class of stiff problems for which standard ODE solvers often produce erroneous results. We found that such erroneous but plausible results generated by standard packages occur to some of the discussed models. In this thesis we have presented some results concerning the application of the Tikhonov theorem to singularly perturbed problems in epidemiology for the cases of the delay and the immediate stability switch. Classical Tikhonov theory shows that if in a singularly perturbed problem with a small parameter there is an isolated attractive quasi steady state, then all solutions originating from initial conditions in the basin of attractiyon of this quasi steady state will converge to it when the small parameter tends to zero. This theory enabled us to understand the asymptotic dynamics of some complex epidemiological problems including the cases of dengue fever, malaria and of the influenza under some conditions. The stability switch occurs when the quasi steady states are non-isolated; that is, if they have intersection points at which there is an exchange of their attractiveness. In many cases, according to intuition, often the switch of stability is immediate; that is, a solution in the basin of attraction of one quasi steady state will converge to it and then, having passed close to the intersection of the quasi steady states, it will "switch" stability by immediately starting to follow the other quasi steady state, which is now attracting. Contrary to this intuition, a delay of switch of stability can be observed. In other words, what often happens is that the solution, having passed the intersection, continues along the now repelling branch of the first quasi steady state for a fixed time, and only later jumps towards the attracting branch of the new one. We studied the delayed stability switch and the immediate stability switch for one and two dimensional models with transcritical bifurcations. The developed theories target non-autonomous models having the first quadrant invariant under their flow and possessing a positive initial condition. Furthermore for the delay to occur we focused on models whose functions possess a specific form of monotonicity (see Assumptions A_3 and A_4 in Chapter 8). The presented analysis was mainly based on the method of upper and lower bounds which, for the purpose of the discussed models, was generalized to a two dimensional case. We distinguished two types of behaviour: the unstable and the stable cases. We found many applications to the theories among which we have: an ecological model between two species with interactions satisfying the mass action law, and an epidemiological model with vital dynamics and a disease of quick turnover. In both cases we found that, depending on the assumptions, the model can exhibit either a delay or an immediate stability switch which have been proved to be related in the case the ecological model with interactions satisfying the general mass action law. The presented theories allows not only to spot the erroneous numerical results and thus to apply more refined methods to provide a correct numerical picture of the singularly perturbed evolution, but also to characterise analytically models exhibiting a delay and an immediate stability switch. It is worth mentioning that even though we used models with positive initial conditions, it is possible to derive similar theories for models with a negative initial condition and having the fourth quadrant invariant under their flow by making obvious changes.

Acknowledgements

I would like to thank God for His grace, His protection and the knowledge He has given me.

I thank my supervisor Prof. Jacek Banasiak for his excellent supervision of the thesis, for his patience and his great advices. I particularly thank God for my supervisor for he has been to me more than a supervisor but also a mentor, a leader and a role-model. Not only has he equipped me with all the mathematical tools I needed to continuous in research but also by his example, he contributed greatly to my development into becoming a good researcher. May the good God bless him more in life and always.

I thank the DAAD-AIMS scholarship and IIASA-YSSP for the financial support.

I thank the Department of Mathematics, Statistic and Computer Science and its staffs, the college of Agriculture, Engineering and Sciences, and the University of Kwazulu Natal for making my stay at UKZN beautiful.

I thank all my lecturers of mathematics and specially Dr Ngakeu F. who saw my potential, believed in me and encouraged me.

I am grateful to God for the life of my father, the late Mr Tchamga Francois, who passed away during the period of my PhD thesis. He taught me mathematics from infant, instilled in me the passion for mathematics, he taught me discipline and did his best for me to achieve this far.

I thank my family and particularly my mother Mrs Djomkap Charlotte, my brothers and sisters for their unconditional love, prayers and supports.

I thank all my colleagues from the University of KwaZulu Natal and all my friends for their love and supports.

References

[1] Afrajmovich, V., Arnol'd, V., Yu, S., and Shil'nikov, L. (1993). *Encyclopaedia of mathematical sciences, Volume 5. Dynamical system V: bifurcation theory and catastrophe theory*. Spinger, Berlin.

[2] Amman, H. and Escher, J. (2008). *Analysis II*. Birkhauser, Boston.

[3] Archer, B., Cohen, C., Naidoo, D., Thomas, J., Makunga, C., Blumberg, Venter, M., Timothy, G., Puren, A., McAnerney, J., Cengimbo, A., and Schoub, B. (2009). Interim report on pandemic H1N1 influenza virus infections in South Africa, april to october 2009: epidemiology and factors associated with fatal cases. *Eurosurveillance*, 4(47):1–5.

[4] Baer, S. and Erneux, T. (1986). Singular Hopf bifurcation to relaxation oscillations I. *SIAM Journal on Applied Mathematics*, 46(5):721–739.

[5] Baer, S. and Erneux, T. (1992). Singular Hopf bifurcation to relaxation oscillations II. *SIAM Journal on Applied Mathematics*, 52(6):1651–1664.

[6] Banasiak, J. (2013). *Mathematical modelling in one dimension*. Cambridge University Press, AIMS Library Series.

[7] Banasiak, J. and Lachowicz, M. (2014). *Methods of small parameter in mathematical biology*. Springer, Heidelberg/New York.

[8] Banasiak, J. and Massoukou, R. M. (2014). A singularly perturbed age structured SIRS model with fast recovery. *Discrete Continue Dynamical System Ser B*, 19(8):2383–2399.

[9] Banasiak, J. and Phongi, E. (2015). Canard-type solution in epidemiological models. *Dynamical Systems, Differential Equations and Applications, AIMS Proceedings*, 10:85–93.

[10] Banasiak, J., Phongi, E., and Lachowicz, M. (2013). A singularly perturbed SIS model with age structure. *Mathematical Biosciences and Engineering*, 10(3):499–521.

[11] Benoît, E. (1991). *Lecture Notes Math, Volume 1493. Dynamic bifurcation*. Springer, New York.

[12] Benoît, E., Callot, J., Diener, F., and Diener, M. (1981). Chasse au canards. *Collectanea Mathematica*, 32:37–119.

[13] Berglund, N., Gentz, B., and Kuehn, C. (2012). Hunting french ducks in a noisy environment. *Journal of Differential Equations*, 252(9):4786–4841.

[14] Berryman, A. (1992). The origins and evolution of predator-prey theory. *The Ecological Society of America*, 73(5):1530–1535.

[15] Bohun, A. and Turaev, D. (2011). Stability loss delay in dynamical bifurcations. *Phd thesis, University of Basel*.

[16] Bose, A. (1995). Symmetric and antisymmetric pulses in parallel coupled nerve fibers. *SIAM Journal on Applied Mathematics*, 55:1650–1674.

[17] Boudjellaba, H. and Sari, T. (1999). Stability loss delay in harvesting competing populations. *Journal of Differential Equations*, 152:394–408.

[18] Boudjellaba, H. and Sari, T. (2009). Dynamic transcritical bifurcations in a class of slow-fast predator-prey models. *Journal of Differential Equations*, 246:2205–2225.

[19] Butuzov, V., Nefedov, N., and Schneider, D. (2004). Singularly perturbed problems in case of exchange of stabilities. *Mathematical sciences*, 121:1973–2020.

[20] Butuzov, V., Nefedov, N., and Schneider, K. (2000). Singularly perturbed reaction-diffusion systems in cases of exchange of stabilities. *Natural Resource Modelling*, 13:247–269.

[21] Callot, J., Diener, F., and Diener, M. (1978). "le problème de la chasse au canard". *Comptes rendus de l'Acadmie des sciences Paris.*, 286(22):Séries A–B.

[22] Camassa, R., Kovacic, G., and Tin, S. (1998). A melnikov method for homo clinic orbits with many pulses. *Archive for Rational Mechanic Analysis*, 143:105–193.

[23] Chitnis, N., Cushing, J., and Hyman, J. (2006). Bifurcation analysis of a mathematical model for malaria transmission. *SIAM Journal on Applied Mathematics*, 67:24–45.

[24] Christopher, K. and Khibnik, A. (2001). *The IMA volumes in mathematics and its applications, Volume 122. Multiple-Time-Scale Dynamical Systems.* Springer-Verlag, New York.

[25] Cronin, J. (1987). *Cambridge Studies in Mathematical Biology, Volume 7. Mathematical aspects of Hodgkin-Huxley neural theory*. Cambridge University Press, Cambridge.

[26] Cronin, J. and O'Malley, R. (1999). *Proceedings of symposia in applied mathematics, Volume 56. Analyzing multiscale phenomena using singular perturbation methods*. American Mathematical Society, Providence, Rhode Island.

[27] Doelman, A., Kaper, T., and Zegeling, P. (1997). Pattern formation in the one-dimensional gray-scott model. *Nonlinearity*, 10:523–563.

[28] Dumortier, F. and Roussarie, R. (1996). *Canard cycles and center manifolds*. American Mathematical Society, Providence.

[29] Eckhaus, W. (2006). *Series Lecture Notes in Mathematics, Volume 985. Relaxation oscillations including a standard chase on french ducks. Asymptotic Analysis II*. Springer, Berlin.

[30] Elsheikh, S., Patidar, K., and Ouifki, R. (2013). Analysis of a malaria model with a distributed delay. *IMA Journal of Applied Mathematics*, 79(6):1–22.

[31] Fenichel, N. (1979). Geometric singular perturbation theory for ordinary differential equations. *Journal of Differential Equations*, 31:53–98.

[32] Glendinning, P. (1994). *Stability, instability and chaos: an introduction to the theory of nonlinear differential equations*. Cambridge University Press, Cambridge.

[33] Gubler, D. and Meltzer, M. (1999). Impact of dengue/dengue hemorrhagic fever on the developing world. *Advances in Virus Research*, 53:35–70.

[34] Guzman, M., Halstead, S., Artsob, H., Buchy, P., Farrar, J., Gubler, D., Hunsperger, E., Kroeger, A., Margolis, H., Martnez, E., Nathan, M., Pelegrino, J., Simmons, C., Yoksan, S., and Peeling, R. (2010). Dengue: a continuing global threat. *Nature Reviews Microbiology*, 8(12):S7–S116.

[35] Haberman, R. (1979). Slowly varying jump and transition phenomena associated with algebraic bifurcation problems. *SIAM Journal on Applied Mathematics*, 37:69–106.

[36] Haller, G. (1999). *Volume 138,Applied Mathematical Sciences Series, Chaos near resonance*, volume 10. Springer-Verlag, New York.

[37] Hay, S., Guerra, C., Tatem, A., Noor, A., and Snow, R. (2004). The global distribution and population at risk of malaria: past, present, and future. *The Lancet Infectious Diseases*, 4:327–336.

[38] Hek, G. (2010). Geometric singular perturbation theory in biological practice. *Mathematical Biology*, 60:347–386.

[39] Hoppenstead, F. (1967). Stability in systems with parameter. *Journal of Mathematical Analysis and Applications*, 18:129–134.

[40] Jelinek, T. (2000). Dengue fever in international travelers. *Clinical Infectious Diseases*, 31:144–147.

[41] Jones, C. (1994). *Springer lecture notes Math, Volume 1609. Geometric singular perturbation theory.* Springer-Verlag, Berlin.

[42] Jones, C. and Kopell, N. (1994a). Tracking invariant manifolds with differential forms in singularly perturbed systems. *Journal of Differential Equations*, 108:64–88.

[43] Jones, C. and Kopell, N. (1994b). Tracking invariant manifolds with differential forms in singularly perturbed systems. *Journal of Differential equation*, 118:64–88.

[44] Jones, C., Kopell, N., and Langer, R. (1991). IMA volumes in mathematics and its applications, Volume 118. construction of the fitzhugh- nagumo pulse using differential forms, in patterns and dynamics in reactive media. *Journal of Differential equation*, 37:101–116.

[45] Jones, C., T.J.Kaper, and Kopell, N. (1996). Tracking invariant manifolds up to exponentially small errors. *SIAM Journal on Mathemaltical Analysis*, 119(27):558–577.

[46] Jost, J. (1998). *Postmodern analysis*. Springer, Berlin.

[47] Katz, M., Schoub, B., Heraud, J., Breiman, R., Njenga, M., and Widdowson, M. (2012). Influenza in Africa: uncovering the epidemiology of a long-overlooked disease. *Journal of Infectious Diseases*, 206:1–4.

[48] Krupa, M. and Szmolyan, P. (2000). Extending slow manifolds near transcritical and pitchfork singularities. *Nonlinearity*, 14(6):1473–1491.

[49] Lebovitz, N. and Schaar, R. (1975). Exchange of stabilities in autonomous systems. *Studies in Applied Mathematics*, 54:229–260.

[50] Lebovitz, N. and Schaar, R. (1977). Exchange of stabilities in autonomous systems II., vertical bifurcation. *Studies in Applied Mathematics*, 56:1–50.

[51] Miller, J. and Sons (2007). *Advances in Chemical Physics, Volume 97. Singular Perturbation Problems in Chemical Physics: Analytic and Computational Methods.* Advance in Chemical Physics, New York.

[52] Mishchenko, E. and Rozov, N. (1980). *Differential equations with small parameter and relaxation oscillations*. Plenum Press, New York.

[53] Neishtadt, A. (1987). Persistence of stability loss for dynamical bifurcations I. *Differentsial'nye Uravneniya*, 23:2060–2067.

[54] Neishtadt, A. (1988). Persistence of stability loss for dynamical bifurcations II. *Differentsial'nye Uravneniya*, 24:226–233.

[55] Panazzolo, D. (2000). On the existence of canard solutions. *Publication Mathemàtique*, 44(2):503–592.

[56] Pao, C. (1992). *Nonlinear parabolic and elliptic equations*. Plenum Press, New York.

[57] Phongi, E. (2015). *On singular perturbed problems and exchange of stabilities*. PHD thesis, UKZN.

[58] Rinaldi, S. and Muratori, S. (1992). Slow-fast limit cycles in predator-prey models. *Ecological Modelling*, 61:287–308.

[59] Robinson, J. (2001). *Infinite-dimensional dynamical system: an introduction to dissipative parabolic PDEs and Theory of Global Attractors*. Cambridge University Press, United States of America.

[60] Rocha, F., Aguiar, M., Souza, M., and Stollenwerk, N. (2013). Time-scale separation and centre manifold analysis describing vector-borne disease dynamics. *International Journal of Computer Mathematics*, 90(10):2105–2125.

[61] Sakamoto, K. (1990). Invariant manifolds in singular perturbation problems for ordinary differential equations. *Proceedings of the Royal Society of Edinburgh*, 116:45–78.

[62] Schecter, S. (1985). Persistent unstable equilibria and closed orbits of a singularly perturbed equation. *Journal of Differential Equation*, 60:131–141.

[63] Shen, J. (2016). Canard limit cycles and global dynamics in a singularly perturbed predator-prey system with non-monotonic functional response. *Nonlinear Analysis: Real World Applications*, 31:146–165.

[64] Shishkova, M. (1973). Investigation of a system of differential equations with a small parameter at the highest derivatives. *Dokl. Akad.Nauk SSSR*, 209:576–579.

[65] Smith, H. and Waltman, P. (1995). *The Theory of the chemostat*. Cambridge University Press, Cambridge.

[66] Szmolyan, P. and Wechselberger, M. (2001). Canards in R3. *Journal of Differential Equations*, 177:419–453.

[67] Thieme, H. (2003). *Mathematics in population biology*. Princeton University Press, Chicago.

[68] Tikhonov, A. (1948). On the dependence of solutions of differential equations on a small parameter. *Matematicheskii Sbornik*, 64:193–204.

[69] Tikhonov, A., Vasileva, A., and Sveshnikov, A. (1985). *Differential equations*. Springer,Berlin.

[70] Tikhonov, A. N. (1952). Systems of differential equations containing small parameters. *Matematicheskii Sbornik*, 73:575–586.

[71] Tin, S. (1994). *On the dynamics of tangent spaces near normally hyperbolic manifolds and singularly perturbed boundary value problems*. PhD thesis - Brown University.

[72] Tornatore, E. and Buccellato, S. (2009). Parasite population delay model of malaria type with stochastic perturbation and environmental criterion for limitation of disease. *Journal of Mathematical Analysis and Applications*, 360:624–630.

[73] Vasileva, A. (1962). The asymptotic formular for the solution of ordinary differential equations with a small parameter on the highest derivative, which are valid over a semi-infinite interval. *Dokl. Akad. Nauk SSSR*, 142.

[74] Vasileva, A. and Butuzov, V. (1973). *Asymptotic expansions of solutions of singularly perturbed equations*. Nauka, Moscow.

[75] Vasileva, A. and Butuzov, V. (1990). *Asymptotic expansion in the theory of singular perturbation*. Shkola, Moscow.

[76] Vasileva, A., Butuzov, V., and Kalachev, L. (1995). The boundary function method for singular perturbation problems. *SIAM, Studies in Applied Mathematics. Philadelphia*.

[77] Wilder-Smith, A. (2005). Dengue in travelers. *The New England Journal of Medicine*, 353:924–932.