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Sums and products of equivalence orbits of integral matrices

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Sums and products of equivalence orbits of integral matrices

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1. INTRODUCTION

If A and B are $n \times n$ matrices over $\mathbb{Z}$, we say they are **equivalent** if there exist invertible U and V such that $B = UAV$. It is well-known that A and B are equivalent if and only they have the same invariant factors. We briefly recall the definition of these.

Let A be $n \times n$. Then A is equivalent to

$$UAV = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix},$$

where $a_1 \,|\, a_2 \,|\, \ldots \,|\, a_n$. This diagonal matrix is called the Smith normal form of A and $a_1, a_2, \ldots, a_n$ are the **invariant factors** of A.

The invariant factors are (apart from units) uniquely determined by A, because

$$a_k = \frac{d_k(A)}{d_{k-1}(A)}, k = 1, \ldots, n$$

where $d_k(A)$ – the so-called k-th determinantal divisor of A – is the gcd of all $k \times k$ minors of A, with the convention that $d_0 = 1$.

All of the above (equivalence to Smith normal form, characterization of invariant factors, criterion for equivalence) holds for matrices over Euclidean domains or, more generally, principal ideal domains.

In this note we shall also consider **elementary divisor domains**, defined by Kaplansky in 1949 [10] as domains over which every matrix is equivalent to its Smith normal form. So PIDs $\subset$ EDDs and the inclusion is strict (the main counter-examples being rings of complex functions which are not unique factorization domains).

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2. ORBITS

Given an n-tuple $a = (a_1, a_2, \ldots, a_n)$ of elements of a domain R, with $a_1 \mid a_2 \mid \ldots \mid a_n$, denote by D_a the diagonal matrix diag $(a_1, a_2, \ldots, a_n)$. We shall be interested in the **equivalence orbit**

$$\mathcal{O}_a = \{UD_aV : U, V \text{ invertible}\}.$$

That is, $\mathcal{O}_a$ is the set of all matrices with invariant factors a.

Trivially we have

$$R^{n\times n} = \cup_a \mathcal{O}_a$$

Two fundamental questions arise: given two n-tuples a and b, what can we say about $\mathcal{O}_a + \mathcal{O}_b$ and $\mathcal{O}_a\mathcal{O}_b$? We easily see that both sets are unions of orbits, so the questions become

$$\mathcal{O}_a + \mathcal{O}_b = \cup_{c\in ?} \mathcal{O}_c$$

$$\mathcal{O}_a \mathcal{O}_b = \cup_{c\in ?} \mathcal{O}_c$$

3. THE PRODUCT PROBLEM

We want to know which orbits $\mathcal{O}_c$ occur in $\mathcal{O}_a \mathcal{O}_b$. In other words, given sequences $a_1 \mid a_2 \mid \ldots \mid a_n$ and $b_1 \mid b_2 \mid \ldots \mid b_n$, which sequences $c_1 \mid c_2 \mid \ldots \mid c_n$ can occur as the invariant factors of the product of a matrix with invariant factors $a_1, a_2, \ldots, a_n$ and a matrix with invariant factors $b_1, b_2, \ldots, b_n$?

This problem has a long history. It is related to module theory, combinatorics, group representations, algebraic geometry and other areas of Mathematics.

The problem was completely solved in 1968 when R is a principal ideal domain [11]. In this setting we can use a localization technique. Fix a prime $p \in R$ and work over the local ring

$$R_p = \left\{\frac{r}{s} : p \text{ does not divide } s\right\}$$

(*i.e.*, essentially work only with powers of p). Then, for each i, a_i is replaced by $p^{\alpha_{n-i+1}}$, b_i is replaced by $p^{\beta_{n-i+1}}$ and c_i is replaced by $p^{\gamma_{n-i+1}}$, where $\alpha_1 \geq \ldots \geq \alpha_n$, $\beta_1 \geq \ldots \geq \beta_n$, $\gamma_1 \geq \ldots \geq \gamma_n$, are nonnegative integers.

Denote by $IF(\alpha, \beta)$ the set of possible γ in the invariant factor product problem. Then Klein's theorem states that, for given n-tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$, one has

$$IF(\alpha, \beta) = LR(\alpha, \beta)$$

where $LR(\alpha, \beta)$ is the set of n-tuples γ which can be obtained from α and β using the Littlewood-Richardson rule [4].

The Klein solution is "algorithmic" but not very explicit. It shows there is a connection of the product problem over a PID to the representation theory of $GL_n(\mathbb{C})$: if V_α and V_β are irreducible representations of $GL_n(\mathbb{C})$, then $\gamma \in IF(\alpha, \beta)$ if and only if V_γ occurs in the decomposition into irreducible components of $V_\alpha \otimes V_\beta$.

Another approach to the product problem consists of searching for divisibility relations involving the a_i, the b_i and the c_k. Apart from the trivial equality $a_1a_2 \dots a_n b_1 b_2 \dots b_n = c_1 c_2 \dots c_n$, several relations were found in the 1970s and 80s of the form

$$a_{i_1} a_{i_2} \dots a_{i_r} b_{j_1} b_{j_2} \dots a_{j_r} \mid c_{k_1} c_{k_2} \dots c_{k_r}$$

where $r < n$, $1 \le i_1 < \dots < i_r \le n$, $1 \le j_1 < \dots < j_r \le n$, $1 \le k_1 < \dots < k_r \le n$.

Starting in the 1980s, this approach had, as a source of inspiration, a collection of analogies with another matrix problem: what are the possible eigenvalues $\gamma_1 \ge \dots \ge \gamma_r$ of a sum $A + B$ if A and B are $n \times n$ complex Hermitian matrices with eigenvalues $\alpha_1 \ge \dots \ge \alpha_r$ and $\beta_1 \ge \dots \ge \beta_r$, respectively? Here the α's, β's and γ's are real numbers.

Denote the set of possible γ by $E(\alpha, \beta)$. Apart from the trivial equality

$$\gamma_1 + \gamma_2 + \dots + \gamma_n = \alpha_1 + \alpha_2 + \dots + \alpha_n + \beta_1 + \beta_2 + \dots + \beta_n$$

many inequalities were found in the 1950s of the form

$$\gamma_{k_1} + \gamma_{k_2} + \dots + \gamma_{k_r} \le \alpha_{i_1} + \alpha_{i_2} + \dots + \alpha_{i_r} + \beta_{j_1} + \beta_{j_2} + \dots + \beta_{j_n}$$

for indices as before.

In 1962, an extraordinary conjecture about $E(\alpha, \beta)$ was made by Alfred Horn [9]. We proceed to describe it.

For $I = (i_1, \dots, i_r)$ with $1 \le i_1 < \dots < i_r \le n$ write

$$\lambda(I) = (i_r - r, \dots, i_2 - 2, i_1 - 1).$$

Abbreviate $\alpha_{i_1} + \alpha_{i_2} + \dots + \alpha_{i_r}$ to $\Sigma\alpha_I$ and the same in similar situations.

Then Horn's conjecture is:

$$\gamma \in E(\alpha,\beta) \iff \begin{cases} \Sigma\,\gamma = \Sigma\,\alpha + \Sigma\,\beta\,, \\ \Sigma\gamma_K \le \Sigma\alpha_I + \Sigma\beta_J \text{ whenever } \lambda(K) \in E[\lambda(I),\lambda(J)], 1 \le r < n \end{cases}$$

So the set E should be described by a recursion on itself.

The first condition is the trace equality. The others are a collection of inequalities whose number grows rapidly with n, the "Horn inequalities".

A general approach to obtain inequalities of the type

$$\gamma_{k_1} + \gamma_{k_2} + \dots + \gamma_{k_r} \le \alpha_{i_1} + \alpha_{i_2} + \dots + \alpha_{i_r} + \beta_{j_1} + \beta_{j_2} + \dots + \beta_{j_n}$$

uses a 1962 theorem by Hersch and Zwahlen [8], which states that

$$\alpha_{i_1} + \dots + \alpha_{i_r} = \min_{L \in \Omega_I(E)} \mathrm{tr}(A_{|L}) = \max_{L \in \Omega_{I'}(E')} \mathrm{tr}(A_{|L})$$

where $\mathrm{tr}(A_{|L})$ is the Rayleigh trace of A with respect to the subspace L and $\Omega_I(E)$ is the Schubert variety associated to $I = (i_1, \dots, i_r)$ and the sequence $E = (E_1, \dots, E_n)$ of subspaces $E_i = \text{span}\,\{v_1, \dots, v_i\}$ built from the eigenvectors of A associated to the α's. (I' and E' are "complements" to these: $I' = (n - i_r + 1, \ldots, n - i_1 + 1)$, $E' = (E'_n, \ldots, E'_1)$ with $E'_{n-i+1} = \operatorname{span}\{v_i, \ldots, v_n\}$.)

The so-called "Schubert calculus" describes conditions under which three Schubert varieties intersect non-trivially (see *e.g.* [4]) and using it we get that

$$\lambda(K) \in LR[\lambda(I), \lambda(J)] \Rightarrow \gamma_{k_1} + \ldots + \gamma_{k_r} \leq \alpha_{i_1} + \ldots + \alpha_{i_r} + \beta_{j_1} + \ldots \beta_{j_r}.$$

In other words, we have

$$\gamma \in E(\alpha,\beta) \implies \begin{cases} \Sigma\, \gamma = \Sigma\, \alpha + \Sigma\, \beta\,, \\ \\ \Sigma\, \gamma_K \leq \Sigma\, \alpha_I + \Sigma\, \beta_J \text{ whenever } \lambda(K) \in LR[\lambda(I), \lambda(J)],\ 1 \leq r < n \end{cases}$$

In a deep 1998 paper [12], Klyachko proved that the reverse implication is also true, so all the admissible inequalities describing $E(\alpha, \beta)$ come from the intersection geometry of Schubert varieties.

The above-mentioned analogies between the invariant factor product problem and the Hermitian sum eigenvalue problem manifest themselves in the indices of valid relations for the two problems:

$$a_{i_1} a_{i_2} \ldots a_{i_r} b_{j_1} b_{j_2} \ldots b_{j_r} \mid c_{k_1} c_{k_2} \ldots c_{k_r}$$

in the first, and

$$\gamma_{k_1} + \gamma_{k_2} + \ldots + \gamma_{k_r} \leq \alpha_{i_1} + \alpha_{i_2} + \ldots + \alpha_{i_r} + \beta_{j_1} + \beta_{j_2} + \ldots + \beta_{j_r}$$

in the second. The natural guess is that there should be a connection between the sets $LR(\alpha, \beta)$ and $E(\alpha, \beta)$.

In [16] it was shown that, for integral nonnegative ordered n-tuples α and β, one has $E(\alpha, \beta) \cap \mathbb{Z}^n \supseteq LR(\alpha, \beta)$. The proof uses a 1982 result by Heckman [6] stated in the spirit of Kirillov's method of orbits.

Using results of [12], Knutson and Tao [13] proved that one actually has $E(\alpha, \beta) \cap \mathbb{Z}^n = LR(\alpha, \beta)$.

So Horn's conjecture follows, as we can replace $LR[\lambda(I), \lambda(J)]$ by $E[\lambda(I), \lambda(J)]$ in Klyachko's result above. A good survey on all of this is [5]. A very readable account of the Hermitian problem is [1].

As a consequence, we see that the invariant factor product problem (local version) has a recursive solution in terms of inequalities. Therefore, the global version of the invariant factor product problem (**over a principal ideal domain**) has a solution in terms of divisibility relations of the type

$$a_{i_1} a_{i_2} \ldots a_{i_r} b_{j_1} b_{j_2} \ldots b_{j_r} \mid c_{k_1} c_{k_2} \ldots c_{k_r}.$$

But the proof is critically dependent on localization at primes.

What if R is an elementary divisor domain? In this case, the problem is open, as we no longer have the localization technique at our disposal. So, we have a new problem: describe invariant factors of products of matrices over an elementary divisor domain.

A different approach is needed. An idea is to try to replicate the Hersch-Zwahlen and Schubert calculus argument.

We shall omit a lot of details. Suppose that $A \in R^{n \times n}$ has invariant factors $a_1 \mid \ldots \mid a_n$, $UAV = \operatorname{diag}(a_1, \ldots, a_n)$ and $\nu_1, \ldots, \nu_n$ are the columns of V. Introduce the notations

$$V_i = \operatorname{span}_R\{\nu_1, \ldots, \nu_i\},\ V = (V_1, \ldots, V_n)$$

and

$I = (i_1, \ldots, i_r)$ with $1 \leq 1 < i_1 < \ldots < i_r \leq n$, I' and V' as before.

Then we have

Theorem. [2]

$$\alpha_{i_1} \alpha_{i_2} \ldots \alpha_{i_r} = \operatorname*{lcm}_{L \in \Omega_I(V)} \frac{d_r(AX)}{d_r(X)} = \operatorname*{gcd}_{L \in \Omega_{I'}(V')} \frac{d_r(AX)}{d_r(X)}$$

where L denotes a pure submodule of R^n, $x_1, \ldots, x_r \in L$ are linearly independent such that L is the pure closure of span $_R\{x_1, \ldots, x_r\}$, $X = [x_1, \ldots, x_r]$ and the d_r are the determinantal divisors defined before.

Again omitting details, from this theorem we get exactly the divisibility relations with indices as in Horn's conjecture (as in the PID case):

Theorem. [2] Let $r < n$. If $\lambda(K) \in LR\,[\lambda(I), \lambda(J)]$, then

$$a_{i_1} a_{i_2} \ldots a_{i_r} b_{j_1} b_{i_2} \ldots b_{j_r} \mid c_{k_1} c_{k_2} \ldots c_{k_r}.$$

The conclusion is that, over a PID, the families of indices appearing in the description of $\mathcal{O}_a \mathcal{O}_b$ are exactly the same as those appearing in the Horn equalities, say

$$\mathcal{O}_a \mathcal{O}_b = \cup_{c \in \mathcal{H}} \mathcal{O}_c$$

Over an EDD, we proved that

$$\mathcal{O}_a \mathcal{O}_b \subseteq \cup_{c \in \mathcal{H}} \mathcal{O}_c$$

It is natural to ask if the last theorem (plus the determinant equality) is the complete solution for the product problem over EDDs.

4. THE SUM PROBLEM

Here we want to know which orbits $\mathcal{O}_c$ occur in $\mathcal{O}_a + \mathcal{O}_b$. In other words, given sequences $a_1 \mid a_2 \mid \ldots \mid a_n$ and $b_1 \mid b_2 \mid \ldots \mid b_n$, which sequences $c_1 \mid c_2 \mid \ldots \mid c_n$ can occur as the invariant factors of the sum of a matrix with invariant factors $a_1, a_2, \ldots, a_n$ with a matrix with invariant factors $b_1, b_2, \ldots, b_n$?

This problem is open, even over the integers. What do we know about it?

The first result is classical:

Theorem. [17] Let R be a PID. If $\mathcal{O}_c \subset \mathcal{O}_a + \mathcal{O}_b$ then

$$\gcd\{a_i, b_j\} \mid c_{i+j-1}$$

for all indices i, j such that $i + j - 1 \leq n$.

This is still true if R is an EDD (see the proof in [14]).

Since, if $C = A + B$, we have $B = -A + C$ and $A = -B + C$, we see that there are actually three families of relations:

$$\gcd\{a_i, b_j\} \mid c_{i+j-1},\ \gcd\{a_i, c_j\} \mid b_{i+j-1} \text{ and } \gcd\{b_i, c_j\} \mid a_{i+j-1}$$

These relations are highly restrictive on the sequences a, b, c.

Let $A \in \mathcal{O}_a$, $B \in \mathcal{O}_b$ and $C \in \mathcal{O}_c$ such that $C = A + B$. Taking determinants (choosing, from this point on, the invariant factors of each matrix so that their product equals the determinant), we have

$$a_1 \dots a_n \equiv c_1 \dots c_n \pmod{b_1},$$

$$b_1 \dots b_n \equiv c_1 \dots c_n \pmod{a_1},$$

$$a_1 \dots a_n \equiv (-1)^n b_1 \dots b_n \pmod{c_1}.$$

Together with the three families of divisibility relations above, we have collected a set of necessary conditions for $\mathcal{O}_c \subset \mathcal{O}_a + \mathcal{O}_b$.

Em 1986, R. C. Thompson [18] conjectured that this set of conditions is the exact solution for the problem of describing $\mathcal{O}_a + \mathcal{O}_b$.

In 1990, E. Marques de Sá [15] showed that the conjecture is false, by finding additional necessary conditions about $\det(A + B)$.

This led us to the idea of trying to find the exact restrictions on $\det(A + B)$ for $A \in \mathcal{O}_a$ and $B \in \mathcal{O}_b$.

Theorem. [3] Let $A \in \mathcal{O}_a$ and $B \in \mathcal{O}_b$. Write

$$\delta = \gcd\{a_1 b_1 \dots b_{n-1}, a_1 a_2 b_1 \dots b_{n-2}, \dots, a_1 \dots a_{n-1} b_1\}.$$

Then

$$\det(A + B) \equiv a_1 \dots a_n + b_1 \dots b_n \pmod{\delta}.$$

Conversely, given sequences a and b and defining δ as above, if $x \equiv a_1 \dots a_n + b_1 \dots b_n \pmod{\delta}$ then there exist $A \in \mathcal{O}_a$ and $B \in \mathcal{O}_b$ such that $\det(A + B) = x$.

The necessity part of this statement follows from the explicit formula for the determinant of the sum of two matrices:

$$\det(A + B) = \textstyle\sum_{k=0}^{n} \sum_{\mu,\nu \in Q_{k,n}} (-1)^{\Sigma\mu + \Sigma\nu} \det A\,[\mu|\nu].\ \det B\,[\mu'|\nu']$$

where $Q_{k,n}$ is the set of strictly increasing sequences with k elements taken from $\{1, 2, \dots, n\}$, $A[\mu|\nu]$ is the submatrix of A with rows and columns indexed by μ and ν, and μ', ν' are the complementary sequences to μ, ν. For the converse, we construct by induction, under the hypothesis, matrices A and B satisfying the required conditions. The statement is valid for matrices over an elementary divisor domain.

So we have found the exact range of det($A + B$) when A and B have the prescribed invariant factors. Is is natural to add the corresponding restrictions (together with the analogous ones coming from $B = -A + C$ and $A = -B + C$) to the three families of divisibility relations described before and conjecture that we then obtain the full solution of the invariant factor sum problem. We have substantial computational evidence to support this.

We have proved the conjecture in some particular cases, like $n = 3$ for $a = (1,1, a_3)$ and $b = (1,1, b_3)$ (the case $n = 2$ was covered by Thompson's work). An interesting additional situation is n arbitrary, $a = b = (1,1, ..., 1)$. In this case, the conjecture reduces to the statement that any $n \times n$ matrix C over R is the sum of two invertible matrices. This is known [7]. For C = diag (c_1, ..., c_n), take

$$A = \begin{bmatrix} c_1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & c_2 & 0 & \cdots & 0 & 0 \\ 0 & 1 & c_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & c_n \end{bmatrix}$$

For general C, first diagonalize it by equivalence and then apply the diagonal case.

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