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On Gaugino Condensation in String Theory

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Abstract

We derive the precise form of the low-energy four-dimensional EFT for type IIB string theory compactified on the complex cone over Kähler-Einstein del Pezzo surfaces, including N spacetime-filling D3-branes and assuming Minkowski externally. We explicitly derive the theory for the Kähler modulus in the simplest case of a complex cone over the complex projective plane, with a stack of four D7-branes and one O7-plane wrapped around the base of the cone. An effective scalar potential appears in the theory, due to gaugino condensation taking place at low energies over the D7-branes stack, exhibiting a runaway direction and an unstable de Sitter vacuum. We find an explicit cosmological-like solution for the Kähler modulus, showing that the warped volume of the internal complex projective plane inflates with time in a runaway fashion. We conclude that type IIB string theory compactified on the complex cone over the complex projective plane, with four D7-branes and one O7-plane wrapped around the complex projective plane, is unstable. We explicitly derive the ten-dimensional equations of motion for maximally symmetric time-dependent metric perturbations by means of an *ad hoc* procedure, and we exhibit both stationary and time-dependent solutions, whose boundary conditions are imposed in part by the gaugino condensate stress-energy tensor. We partially fix the free parameters of the time-dependent solution using the results from the four-dimensional low-energy EFT. This thesis contains also an introduction to string compactifications, to the KKLT scenario and to the literature about ten-dimensional effects of gaugino condensation.

Contents

Introduction

In the last few decades, the standard model of particle physics has proved to describe consistently and remarkably well all known fundamental interactions, except gravity, at energy scales far below the Planck scale $M_p \sim 10^{19}$ GeV. In light of its many shortcomings, however, it is regarded today as a low-energy effective field theory (EFT), whose UV completion is still to be found. String theories are long-standing candidates for such UV-complete theory.

It has been twenty years since the proposal of the KKLT scenario [15], a putative realization of de Sitter vacua in string theory, and it remains unclear to these days whether it is a viable construction, or whether string theory could admit de Sitter vacua altogether. Gaugino condensation plays a central role in the KKLT construction. Its non-perturbative nature has proved itself rather difficult to capture using the ten-dimensional tools of critical superstring theories, sparking a remarkable scientific interest both for academic reasons in their own right, and in the hope of substantiating KKLT further. Among the research efforts on gaugino condensation found in the literature, the paper [34] published in 2011 studied a very explicit string theory model embedding gaugino condensation – a local type IIB compactification on the complex cone over the complex projective plane – and it exhibited a closed-form supergravity solution supposedly encoding the non-perturbative effect. This thesis studies that very same model, which could seem like the perfect playground for studying gaugino condensation in string theory given its extremely simple features, and it shows from a four-dimensional analysis that *it does not lead to a stable string theory vacuum*, precisely due to the effect of gaugino condensation. Therefore, the solution presented in [34] seems to be not as physically sensible as it was argued to be. To corroborate this conclusion further, we take on the study of this very same model from a ten-dimensional point of view. After dealing with a pathological equation we found along the way by means of an *ad hoc* trivialization procedure, we present the ten-dimensional equations of motion for the perturbation to the background geometry, and we solve them both in a stationary hypothesis and in a time-dependent regime. Some free parameters of these solutions are fixed imposing to reproduce the results of the four-dimensional EFT.

The material of this thesis is organized as follows. In chapter 1 we give a presentation of some aspects of the physics of string theory compactifications relevant to this work, with an emphasis on de Sitter vacua and gaugino condensation. In chapter 2 all major mathematical facts about the compactification space of the explicit model studied in this thesis are stated in a single place. This has the advantage of keeping things tidy for the rest of the discussion, but the drawback of testing one's patience. The uninterested reader can mostly skip this chapter, as we explain in its introduction. Chapter 3 constructs the low-energy four-dimensional EFT for type IIB string theory compactified over Kähler-Einstein del Pezzo surfaces, which we call *local del Pezzo models*, including N spacetime-filling D3-branes in the background. This is done adapting established results from the literature. In chapter 4 we exhibit the low-energy four-dimensional effective field theory for the moduli of the compactification considered in [34]. In particular, we compute the effective potential for its unique chiral field and we show that it has an unstable dS vacuum and it induces runaway dynamics, which implies instability of the compactification. These are all original results. Finally, in chapter 5 we compute the ten-dimensional equations of motion for time-dependent and leading order metric perturbations sourced by gaugino condensation in the same model as the previous chapter, in the hope of reproducing the results from the four-dimensional EFT analysis. We find two main classes of solutions depending on a number of parameters, one stationary and one time-dependent. We argue that the former might be associated with the unstable dS vacuum, and that a subclass of the latter might be a ten-dimensional description of the runaway dynamics found in the four-dimensional EFT.

Chapter 1

KKLT scenario and gaugino condensation

Remarkably, string theory imposes extremely strict constraints on the low-energy effective field theories it can generate. Among the constrained features string theory compactifications impose on its low-energy EFT, the cosmological constant is crucial from a phenomenological standpoint. While experimental evidence shows that our universe has a positive and exponentially suppressed cosmological constant

$$\Lambda \sim 10^{-122} M_p \qquad (\hbar = c = 1), \tag{1.1}$$

the Maldacena-Nuñez no-go theorem shows that string theory compactifications to de Sitter spaces necessarily involve singular internal manifolds, which makes them harder to deal with. It has been recently conjectured by Obied et al. [42] that (metastable) de Sitter vacua might not even belong to the string theory landscape, making them a part of the so-called *swampland*. This is known as de Sitter conjecture, and it is manifestly in tension with the validity of the KKLT scenario.

The goal of this chapter is to provide some context and motivation for the rest of this thesis, making explicit the relation between gaugino condensation and dS vacua in string theories. In §1.1 we introduce the basic ingredients of string theory compactifications and we state a number of facts concerning them which we make use of in this work, trying to be as little pleonastic as possible. In §1.2 we review the KKLT scenario, introducing all of its building blocks in a rather telegraphic fashion. Finally, in §1.3 we present a mildly discursive overview of the research on gaugino condensation during the last twenty years, also providing context for where this thesis places itself.

1.1 String compactification generalities

Consistent perturbative superstring theories only exist in $D = 10$ spacetime dimensions[1]. In order to have them generate a low-energy four-dimensional theory, they need to be *compactified*, essentially rendering the appearance of the extra six dimensions of string theory a high-energy effect, possibly close to the Planck scale. Thus, string compactifications are born out of the application to string theory of a very large class of techniques aimed at reproducing four-dimensional EFTs with the desired characteristics, and for each single string compactification there are in general more than one, possibly thousands or even infinite *vacua*, namely solutions of the classical ten-dimensional equations of motion (EOMs) which yield a four-dimensional theory. The features of the resulting low-energy theory, like its symmetries (e.g. supersymmetry, visible gauge symmetries like the Standard Model gauge group), its particle content and spectrum, the classic values of its couplings and its cosmological constant, are going to be determined dynamically by the specifics of the compactification.

1.1.1 Kaluza-Klein compactifications

The most immediate (geometric) compactification technique is to consider a string theory over a ten-dimensional manifold which is globally isomorphic to a direct product:

$$X_{10} = \mathcal{M}_4 \times X, \tag{1.2}$$

where $\mathcal{M}_4$ is the visible four-dimensional (external) spacetime, X is an internal six-dimensional manifold, and X_{10} makes up the *background*[2] of the string theory. This is called *Kaluza-Klein compactification*. The topological structure of (1.2) can be endowed with a metric of block diagonal form (in the Einstein frame)

$$\mathrm{d}s_{10}^2 = \mathrm{d}s_4^2(x) + \mathrm{d}s_X^2(y), \tag{1.3}$$

where $\mathrm{d}s_4^2(x)$ is maximally symmetric, i.e. Poincaré-invariant, and $\mathrm{d}s_X^2(y)$ is Kähler and Ricci-flat. By SUSY of the low-energy EFT, X is required to be Calabi-Yau (CY), and $\mathrm{d}s_X^2(y)$ is chosen to be Ricci-flat in order to satisfy the vacuum Einstein equations, up to a cosmological constant. It should be noted that *compact* CY manifolds have no continuous isometries, therefore maximal symmetry cannot be enforced on $\mathrm{d}s_X^2(y)$. Moreover, isometries of the internal space would introduce massless spin one particles to the low-energy EFT spectrum, like in the classic Kaluza-Klein theory, which are not phenomenologically welcome, therefore this is a desired byproduct of the internal CY structure. The length scale of X broadly defines the *compactification scale*, roughly

[1]Consistency of the theory is found requiring cancellation of the superconformal anomaly, i.e. the anomaly associated with the (gauged) Weyl rescaling of the metric $g \mapsto \mathrm{e}^{\omega(x)} g$, while perturbativity is understood with respect to the string coupling constant $g_s = \langle \mathrm{e}^{\phi} \rangle$.

[2]Often times, one refers to *background* quantities like fluxes or metrics in order to distinguish them from localized ones, as in living on extended solitonic objects like Dp-branes as excitations of those vacuum states.

identified with the (bulk) Kaluza-Klein (KK) scale L_{KK}, and we expect it to be finite and above the electroweak (EW) energy scale in realistic compactifications, in order not to clash with experimental data:

$$\frac{1}{L_{KK}} > E_{EW}. \tag{1.4}$$

In particular X is necessarily compact in realistic compactifications, and in this case the four-dimensional EFTs from (1.2) are going to be gravitational, namely with dynamical four-dimensional gravity. Contrary to general relativity, strings can propagate consistently even on singular backgrounds like orbifolds or in presence of D-branes and O-planes, therefore X is allowed to have singularities of various nature like D-brane stacks wrapping multiple submanifolds.

One could even relax the metric Ansatz (1.2), allowing for some dependence of the external metric $\mathrm{d}s_4^2$ on the internal coordinates. Imposing again Poincaré-invariance externally, this is achieved through so-called *warped compactifications*

$$X_{10} = \mathcal{M}_4 \times_A X, \tag{1.5}$$

where $A(y)$ is a function of the internal coordinates. For type IIB string theory, the background metric (1.3) is modified into

$$\mathrm{d}s_{10}^2 = \mathrm{e}^{2A(y)}\mathrm{d}s_4^2(x) + \mathrm{e}^{-2A(y)}\mathrm{d}s_X^2(y). \tag{1.6}$$

The function $\mathrm{e}^{2A(y)}$ is commonly known as *warp factor*, and the specific form of the Ansatz (1.6) has been chosen so that SUSY of the low-energy EFT constraints the internal manifold to be *conformally* Calabi-Yau (that is, the CY constraint is on $\mathrm{d}s_X^2$).

String theories have no dimensionless parameters, but they come with a fundamental scale related to the Regge slope parameter α', which has dimensions $[\alpha'] = \ell^2$, where ℓ denotes the length dimension. Throughout this thesis, we will work in units

$$\hbar = c = 1, \tag{1.7}$$

and we define the string length as

$$\ell_s = 2\pi\sqrt{\alpha'}, \tag{1.8}$$

which can be identified as the *fundamental scale* of the string theory. With these conventions, the tension of a Dp-brane in the Einstein frame is given by

$$T_{\mathrm{D}p} = \frac{2\pi}{\ell_s^{p+1}}, \tag{1.9}$$

which gets a further $1/g_s$ factor in the string frame, where g_s is the string coupling constant. On the other hand, the string tension in the string frame is

$$T = \frac{2\pi}{\ell_s^2} = \frac{1}{2\pi\alpha'}, \tag{1.10}$$

so that the $\alpha' \sim 0$ limit of the theory corresponds to the rigid string limit. More intuitively, the α' perturbative expansion controls the stringy corrections to point-particle quantities. On ten-dimensional Minkowski backgrounds, since there are no characteristic length scales, the only possible expansion parameter is $\alpha' E^2$, where E is the energy of the process, so that sending $\alpha' \ll 1/E^2$ corresponds to the low-energy supergravity limit of the theory. On compactification backgrounds like (1.6), another possible expansion parameter is $\alpha'/D(X)^2$, where $D(X)$ is the diameter of the internal manifold, so that the $\alpha' \sim 0$ limit is recovered in the limit of low-energy and large volume of the internal manifold. More generally, the characteristic lengths $\ell_c^{(a)}$ of any given non-trivial cycle Σ_a of the internal manifold define the *KK scales* of the compactification

$$\Lambda_{KK}^{(a)} = \frac{1}{\ell_c^{(a)}}, \tag{1.11}$$

which are distinguished from the bulk KK scale Λ_{KK} and dominated by it, and they can give rise to a dimensionless expansion parameter $\alpha'/\left(\ell_c^{(a)}\right)^2$. On general grounds, the low-energy EFTs of string theory are supergravity theories, and the former provide consistent curvature corrections controlled by α' to the Einstein-Hilbert action. By the above considerations, when dealing with string compactifications, the validity of the ten-dimensional supergravity approximation at low energies needs to be assessed, in particular making sure than none of the internal non-trivial cycles (including the internal manifold itself) is stabilized to a problematically small volume, so that the KK scale is kept sufficiently below the string scale and the higher-order corrections are under control.

In a supergravity approximation of string theories, the ten-dimensional Newton's constant is determined by the string length. In our units, this is given precisely by

$$\kappa_{10}^2 = \frac{\ell_s^8}{4\pi}. \tag{1.12}$$

The four-dimensional Newton's constant is determined by the (warped) volume of the internal manifold and by the string length,

$$\kappa_4^2 = \frac{\kappa_{10}^2}{\mathrm{Vol}_w(X)} = \frac{\ell_s^8}{4\pi\,\mathrm{Vol}_w(X)}, \tag{1.13}$$

where

$$\mathrm{Vol}_w(X) = \int_X \mathrm{d}^6 y \sqrt{g_6}\, \mathrm{e}^{-4A}. \tag{1.14}$$

The unwarped case can be easily retrieved setting the warp factor to one. This can be seen expressing the ten-dimensional Ricci tensor in terms of the four-dimensional one using the warped Ansatz (1.6). The Einstein-Hilbert action then takes the form

$$\begin{aligned} S_{\mathrm{EH}} &= \frac{1}{2\,\kappa_{10}^2}\int_{X_{10}} \mathrm{d}^{10}X \sqrt{-\hat{g}}\, R = \\ &= \frac{1}{2\,\kappa_{10}^2}\int_{X_{10}} \mathrm{d}^4x\, \mathrm{d}^6y \sqrt{-g_4 g_6}\left(\mathrm{e}^{-4A} R^{(4)} + \dots\right) = \\ &= \left(\frac{1}{2\,\kappa_{10}^2}\int_X \mathrm{d}^6y \sqrt{g_6}\,\mathrm{e}^{-4A}\right)\int_{\mathbb{R}^{1,3}} \mathrm{d}^4x \sqrt{-g_4}\, R^{(4)} + \dots. \end{aligned} \tag{1.15}$$

Therefore, if X is assumed non-compact, the four-dimensional Newton's constant goes to zero

$$\kappa_4^2 = \frac{\kappa_{10}^2}{\mathrm{Vol}(X)} \sim 0, \tag{1.16}$$

and the resulting EFT is going to be non-gravitational, with fixed external metric. While this would not make for a realistic compactification, it can be regarded as a sensible approximation when one is interested in non-gravitational aspects of the low-energy EFT, e.g. if one imagines X as obtained from some *decompactification limit* of a compact internal manifold $\tilde{X}$ around some local sector. String compactifications with non-compact internal spaces are called *local* models, as in local description of compact realistic models. While they have infinite compactification scale, local models still have a finite KK scale, which is less trivially identified with some combination of the sizes of the internal cycles.

Finally, let us briefly digress on SUSY. This thesis will deal with type IIB string compactifications, so that our focus is going to revolve around type II superstring theories and F-theory, as obtained from M-theory (see for instance [25, 23]). These theories have 32 (real) supercharges, which is the maximal number one can have while being compatible with locality, and which correspond to $\mathcal{N} = 2$ SUSY in $D = 10$ or $\mathcal{N} = 1$ in $D = 11$ dimensions[3]. Due to the approximately SUSY features of the Standard Model (SM), like its approximate gauge couplings unification at GUT energies or the hierarchy problem, it appears natural to ask that its UV completion is minimally SUSY ($\mathcal{N} = 1$ in four dimensions, i.e. with four supercharges) at its KK scale, with spontaneous symmetry breaking (SSB) of SUSY at some lower energy scale, but still above the EW scale. Therefore, a possible hierarchy of scales for a string compactification well described in the supergravity approximation is displayed in figure 1.1.

Type II string theories compactified on Calabi-Yau three-folds X give rise to $\mathcal{N} = 2$ four-dimensional EFTs. One can include spacetime-filling (BPS) D-branes in the compactification, so that they break half SUSY locally, preserving 16 supercharges on their worldvolume [22], and this also requires us to include orientifold planes in order to cancel the tadpole[4] that the branes introduce. This is called orientifold compactification of type II string theories, and it produces $\mathcal{N} = 1$ four-dimensional EFTs, which is what we are looking for. F-theory compactified on Calabi-Yau four-folds Z already encodes the D7-branes and O7-planes information at a geometric level[5], so that it also gives rise to $\mathcal{N} = 1$ four-dimensional EFTs. It has been argued [9] that if Z is also *elliptically fibered* with base of fibration[6] X then in some cases one can go to a region of the moduli space of Z around which the theory looks like an orientifold compactification of

[3] This is due to the dimension of irreducible spinor representations in D dimensions, see [4]. More precisely, in $D = 10$ there exist Majorana-Weyl spinors, while in $D = 11$ only Majorana spinors survive.

[4] Roughly speaking, tadpoles are charges associated with D-branes of different dimensions. Typically, one needs to cancel them all since the total flux of their corresponding density forms through a compact space should vanish.

[5] D5- and D3-branes can also be included in type IIB compactifications in a F-theory setup by means of M5- and M2-branes respectively.

[6] That is, if Z can be described as the total space of a fibration $\pi : Z \to X$ with fiber $\pi^{-1}(x)$ which is an elliptic curve, i.e. a two-torus.

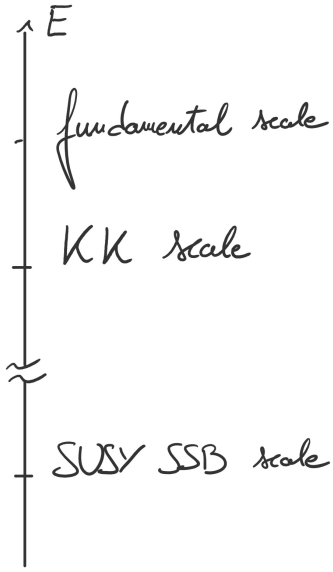


Figure 1.1: *Hierarchy of scales in string compactifications ensuring a controlled supergravity regime of the low-energy EFT. It is understood that all scales lie above the EW scale.*

type IIB string theory on X, and in this *orientifold limit* of F-theory one can actually make out *bulk* quantities from *brane* quantities, like Kähler potentials and fluxes. More precisely, D7-branes in F-theory are described as degeneration loci of the elliptic fibration within the elliptically fibered Calabi-Yau internal space, where the 1-cycle degenerating to a point is the one over which M-theory is compactified, when recovering F-theory from M-theory. Moreover, due to F-theory carrying all the brane information of the compactification in a geometric manner, it naturally brings forth a unification of complex structure moduli of X with brane deformation moduli, which will come in handy when dealing with their stabilization. For these reasons, it is often convenient to work directly with F-theory compactifications on Calabi-Yau four-folds instead of type IIB orientifold compactifications, and this is the approach usually chosen to illustrate the KKLT scenario. It should be noted that F-theory vacua are actually more general than type IIB orientifold vacua, in the sense that the former provide corrections to the latter, and it has been possible to construct F-theory vacua which cannot be described in a type IIB orientifold, see e.g. [33]. All of these compactifications can be achieved with Ricci-flat internal manifolds at leading order in the large internal volume limit.

Let us conclude this introduction to string compactifications introducing two objects that will play a central role throughout this thesis.

1.1.2 Orientifold planes

Orientifold planes are non-dynamical extended objects in string theories, acting both on the spectrum of the theory and on its geometric background. Notably, they can carry negative tension, which makes them a natural candidate to cancel the tadpoles generated by stacks of D7-branes, corresponding to gauge anomalies in the gauge theories supported on them. Including an orientifold plane[7] in a string theory CY compactification means modding out the UV spectrum by the simultaneous action of the worldsheet parity Ω_p and the appropriate lift of a holomorphic and isometric involution of the internal space σ, projecting out of the theory the non-invariant degrees of freedom. Type II string theory compactifications on Calabi-Yau 3-folds give rise to $\mathcal{N}=2$ four-dimensional effective field theories [23], however the inclusion of O7-planes in the compactification breaks half of the supersymmetry, yielding a $\mathcal{N}=1$ SUSY EFT [18]. Broadly speaking, this low energy effective field theory in the supergravity approximation is derived via Kaluza-Klein reduction, namely integrating out all of the massive modes out of the ten-dimensional action by expanding the type IIB supergravity potentials, the Kähler form and the complex structure in terms of their respective massless deformations. In the absence of D7-branes and O7-planes, these arrange themselves in $\mathcal{N}=2$ supermultiplets, while the inclusion of O7-planes projects out the theory the orientifold-odd part of the spectrum, in such a way that it rearranges itself into a collection of $\mathcal{N}=1$ supermultiplets. In the following, we will only refer to type IIB string theory orientifold compactifications, where the internal space is required by supersymmetry to be (conformally) Calabi-Yau, since this is the class of compactifications our model fits in.

By definition, the involution σ is an internal automorphism which squares to the identity. Moreover, it is a holomorphic isometry[8], it leaves the metric and the complex structure invariant, which implies that the Kähler form is also invariant under σ. The fixed-point locus of σ defines the cycle where the orientifold plane is wrapped. For instance, O7-planes wrap 4-cycles of the internal space (e.g. compact divisors) and fill the external space. Since σ is required to leave the external four-dimensional space invariant and to be a holomorphic involution of the internal space, orientifold planes are necessarily even-dimensional[9], namely in type IIB string theory compactifications one can have only O3, O5, O7 and O9-planes. However, the action of σ on the holomorphic 3-form Ω coming from the Calabi-Yau structure is left unconstrained. The involution requirement implies that $\sigma^{\star 2}\Omega=\Omega$, so that $\sigma^\star\Omega=\pm\Omega$, and choosing one of the two possibilities actually completely fixes the full orientifold action. This is given by [18]

$$\mathcal{O}=\begin{cases}\mathcal{O}_-=(-1)^{F_L}\circ\Omega_p\circ\sigma^\star & \text{if } \sigma^\star\Omega=-\Omega\\ \mathcal{O}_+=\Omega_p\circ\sigma^\star & \text{if } \sigma^\star\Omega=\Omega\end{cases} \tag{1.17}$$

where F_L is the number of spacetime fermions in the left-moving sector. The $\mathcal{O}_-$ action is associated to O3 and O7-planes, while $\mathcal{O}_+$ is associated to O5 and O9-planes. This is easily seen observing that locally one can always find complex coordinates (z^i) for the 3-fold such that the holomorphic 3-form takes the form

$$\Omega\propto \mathrm{d}z^1\wedge\mathrm{d}z^2\wedge\mathrm{d}z^3, \tag{1.18}$$

[7]The use of the word *plane* does not refer to the dimensionality of the object. Orientifold planes in string theory compactifications can have dimensions from four up to ten (including time), and further constraints on their dimension comes from consistency with the theory they are embedded in.

[8]Since Calabi-Yau only have discrete isometries, σ is necessarily a discrete symmetry.

[9]Indeed, σ is holomorphic iff it commutes with the complex structure.

which shows that the only possibilities for σ to change the sign of Ω are that it reverses the sign of either one or three complex coordinates (z^i). This corresponds respectively to O7 and O3-planes. An analogous argument shows that $\mathcal{O}_+$ is associated to O5 and O9-planes[10].

Let us consider the case of O7-planes. The orientifold action is given by $\mathcal{O}_-$, and we are interested in determining what part of the type IIB bosonic spectrum survives the orientifold projection. This can be easily seen recalling that the worldsheet parity acts on the type IIB bosons as follows [18],

$$\begin{aligned} \Omega_p \phi &= \phi & \Omega_p C_0 &= -C_0 \\ \Omega_p g &= g & \Omega_p C_2 &= C_2 \\ \Omega_p B_2 &= -B_2 & \Omega_p C_4 &= -C_4 \end{aligned} \tag{1.19}$$

while $(-1)^{F_L}$ changes sign to the RR fields C_0, C_2 and C_4, while leaving the NSNS fields B_2, g_{AB}, ϕ invariant. Thus, in the case of O7-planes, the orientifold-invariant type IIB spectrum has to satisfy

$$\begin{aligned} \sigma^\star \phi &= \phi & \sigma^\star C_0 &= C_0 \\ \sigma^\star g &= g & \sigma^\star C_2 &= -C_2 \\ \sigma^\star B_2 &= -B_2 & \sigma^\star C_4 &= C_4 \end{aligned} \tag{1.20}$$

Finally, let us comment on D7-brane and O7-plane stacks. When dealing with orientifold 7-planes one has to formally consider the double cover of the internal space Z, such that the internal space X is recovered as the quotient of Z by the geometric orientifold action $\sim$, i.e. $X \simeq Z/\sim$ (more precisely, $\sim$ is a representation of $\mathbb{Z}_2$). Thus, D7-branes in orientifold compactifications always come in mirrored pairs with respect to the orientifold geometric involution. When they do not coincide with the O7-plane, the mirrored pair corresponds to one D7-brane degree of freedom, while when the D7-branes coincide with the O7-plane, the mirrored pair naturally collapses to a single D7-brane. For concreteness, let us consider the model studied in this thesis, namely the complex cone over $\mathbb{P}^2$, with four D7-branes and one O7-plane wrapped around the $\mathbb{P}^2$ base. By this we mean that this type IIB orientifold compactification presents *eight* D7-branes organized into *four* mirrored pairs by orientifold projection. This can be described from an *upstairs* or a *downstairs* point of view, namely before and after taking into account the orientifold projection. In the upstairs geometry, one has eight D7-branes in $\mathcal{O}_{\mathbb{P}^2}(-3)$, whose dynamics is reduced to that of four by O7-plane projection. In the downstairs geometry, one has four D7-branes in $\mathcal{O}_{\mathbb{P}^2}(-3)$, coming from eight D7-branes in $\widetilde{\mathcal{O}_{\mathbb{P}^2}(-3)}$, the double cover of $\mathcal{O}_{\mathbb{P}^2}(-3)$ due to the O7-plane wrapped around the $\mathbb{P}^2$. The gauge group associated to the D7-brane stack is affected by the orientifold projection. From the upstairs perspective, N coincident D7-branes undergo a reduction of the gauge group by a proper lift of the orientifold projection

$$\pi^\star_{O7} : U(N) \to SO(N). \tag{1.21}$$

[10] O9-planes correspond to $\sigma = \mathrm{id}$.

From the downstairs perspective, N coincident D7-branes undergo a gauge group enhancement from $U(N)$ to $SO(2N)$. We present an introduction to gauge theories supported on the worldvolume of D-branes in §4.1.

1.1.3 D-branes in type II string theories

D-branes are dynamical extended objects of non-perturbative nature[11] in string theories. They can be described as submanifolds of the geometric background, while physically they are characterized as the locus where open strings of the theory can end. More precisely, they are associated with Dirichlet boundary conditions for open strings along a number of directions. Their dynamics is described by a gauge theory for the abelian field strenth F_2 supported on their worldvolume, which corresponds to the vibration of the open string ending on it.

D-branes in type IIB compactifications need to preserve the bulk supersymmetry[12], which requires them to wrap supersymmetric cycles of the background. D-branes couple (electrically or magnetically) to the RR supergravity potentials C_p [13], where

$$\begin{cases} p = 1,3 & \text{type IIA} \\ p = 0,2,4 & \text{type IIB} \end{cases} \tag{1.22}$$

This ultimately determines the supersymmetric Dp-branes in type IIA and IIB string theories.

Given a Dp-brane and a $(p+1)$-form potential C_{p+1} with field strength $F_{p+2} = \mathrm{d}C_{p+1}$, they can couple naturally through the action

$$S_{\text{int}} = e_p \int_{\text{D}p} C_{p+1}, \tag{1.23}$$

where e_p is the Dp-brane charge, and where we set $\kappa_{10}^2 = 1$. The action (1.23) shows explicitly how Dp-branes couple electrically to background fluxes. This also implies that D-branes source for background fluxes, analogously to Maxwell's electromagnetism. Indeed, in a D-dimensional spacetime, the EOMs for C_p show that e_p in (1.23) is really

[11]For instance, denoting by $g_s = \mathrm{e}^{\phi}$ the string coupling constant, one can see that the tension of a Dp-brane in the string frame has form $T_{Dp} \sim \frac{1}{g_s}$ [23]. This is non-perturbative, although it is also different from the dependence $1/g_s^2$ typical of solitonic objects. Recall that a Dp-brane is a $(p+1)$-dimensional object, with p spacial dimensions and one time dimension.

[12]This also implies that they locally preserve half of the supersymmetry generators, which makes many of their properties persist even at strong coupling. F-theory provides a natural playground to investigate type IIB D-branes properties at strong coupling thanks to the monodromies of the gauge coupling it embeds.

[13]In F-theory, (p,q) 7-branes can couple to both C_2 and B_2, with D7-branes to be identified with the $(1,0)$ 7-branes. The classification of 7-branes in F-theory proceeds through the associated monodromy of the $SL(2,\mathbb{Z})$ doublet (B_2, C_2) when circling around the considered D7-brane in the transverse plane. Notice that this is reminiscent of (p,q)-strings, with the caveat that consistency requires to identify the $(1,0)$-string with the non-perturbative excitation coupled to B_2 (i.e. the F-string), while $(0,1)$ couples to C_2 (the D-string).

the *electric flux* sourced by the Dp-brane through a $(D-p-2)$-sphere surrounding the brane:

$$e_p = \int_{S^{D-p-2}} \star F_{p+2}. \tag{1.24}$$

Then, one can also consider its magnetic dual brane, whose magnetic flux is computed by

$$\mu_{D-p-2} = \int_{S^{p+2}} F_{p+2}. \tag{1.25}$$

Since a $(p+2)$-sphere can enclose a $(D-p-3)$-dimensional submanifold, the magnetic dual to a Dp-brane is a D$(D-p-4)$-brane. The Dirac quantization condition generalizes to Dp-branes:

$$e_p \, \mu_{D-p-2} \in 2\pi\mathbb{Z}. \tag{1.26}$$

The fluxes (1.24) and (1.25) give us a pragmatic way to determine the stable D-branes of type II string theories. In D dimensions, a Dn-brane can couple to a p-form field strength F_p iff

$$\begin{cases} n = D-p-2 & \text{magnetically coupled} \\ n = p-2 & \text{electrically coupled} \end{cases} \tag{1.27}$$

Since type II superstring theories live on backgrounds of dimension $D = 10$, the magnetic dual to Dp-brane is a D$(6-p)$-brane. From (1.27) and (1.22) we find that

Type IIA

- D0-branes only couple electrically to F_2;
- D2-branes only couple electrically to F_4;
- D4-branes only couple magnetically to F_4;
- D6-branes only couple magnetically to F_2.

Type IIB

- D1-branes (D-strings) only couple electrically to F_1;
- D3-branes couple both electrically and magnetically to F_5;[14]
- D5-branes only couple magnetically to F_3;
- D7-branes only couple magnetically to F_1.

We conclude that Dp-branes of type IIA string theory have $p = 0, 2, 4, 6$ (even), while Dp-branes of type IIB string theory have $p = 1, 3, 5, 7$ (odd)[15].

[14] This is consistent with the self-duality of F_5.

[15] Actually, type IIA string theory can admit D8-branes, electrically coupled to a non-dinamical 10-form field strength, and euclideanized type IIB string theory admits D(-1)-branes, called D-instantons, electrically coupled to F_1.

1.2 KKLT scenario

The main phenomenological challenge with naive string compactifications of the form (1.3) is that they generally come with moduli. By definition, these are massless scalars of the four-dimensional EFT without any potential. They are an issue for the phenomenology of these theories for multiple reasons. For instance, it follows that their vacuum expectation values (VEVs) are left *unstabilized* (i.e. unfixed), so that the couplings of the EFT are undetermined as well, in a possibly chaotic manner[16]. This leaves the theory devoid of any predictive power. Even assuming a chaotic stabilization of the moduli, these massless scalars would couple at least with gravitational strength to matter. Whenever their masses are fixed far above the EW scale, this would not be phenomenologically problematic, but in a chaotic scenario one is bound to find patches of the universe where such scalars are light, and modern day cosmological data does not easily accommodate such particles, e.g. due to structure formation observational data. This is known as *moduli stabilization problem*, or *moduli-space problem*. It shows that we should look for string compactifications which are able to generate scalar potentials for the moduli[17] and such that their VEVs are fixed at a *finite* value (possibly zero, depending on the specific modulus). This operation is often called *moduli stabilization*. It should be stressed that once a modulus ρ is stabilized at a finite value $\langle\rho\rangle$ by a scalar potential $V(\rho)$, in general it also automatically receives a mass

$$m_\rho^2 = \frac{\partial^2 V}{\partial \rho^2}\Big|_{\rho=\langle\rho\rangle} > 0, \tag{1.28}$$

effectively *lifting* it from the moduli space of the theory.

The KKLT scenario is a class of type IIB warped compactifications which has been claimed to be able to stabilize all moduli, and to generate a de Sitter low-energy EFT. In order to explain how this works, we need to build some results and terminology.

1.2.1 Moduli space structure

By definition, moduli fields parametrize through their VEVs continuous deformations of string compactifications which are degenerate in energy, since they correspond to flat directions of the four-dimensional effective scalar potential. Moduli can have a number of different origins, but in this thesis we will be concerned with just the following kinds.

Dilaton ϕ It defines the string coupling constant

$$g_s = \langle \mathrm{e}^{\phi} \rangle = \mathrm{e}^{\langle\phi\rangle}, \tag{1.29}$$

for all string theories. This is already a modulus on ten-dimensional Minkowski spacetime, and it controls the worldsheet perturbative expansion of string theory,

[16] As in different patches of the universe get different VEVs.

[17] In string theory literature, it is common to refer to these scalar fields as moduli even after they received a scalar potential and a mass.

analogous to the loop expansion of Feynman diagrams in quantum field theories[18].

Axions In supergravity theories, they roughly correspond to changes in the cohomology class of their p-form potentials C_p. By means of the harmonic representatives, this translates to adding harmonic forms to C_p, which clearly does not affect the field strength $F_{p+1} = \mathrm{d}C_p$. A more thorough characterization is presented in §3.4.

Metric moduli String compactifications on (conformally) CY manifolds have zero modes δg_{MN} of the ten-dimensional metric g_{MN} given by the four-dimensional metric $g_{\mu\nu}$ and by the zero modes of the internal unwarped metric δg_{mn}, determined by requiring that they preserve SUSY and the internal topology. The latter in turn define the metric moduli, which come in two different classes.

Complex structure moduli They are associated to non-hermitian metric deformations of the form δg_{ij}, where i, j are holomorphic complex indices. They are the generalization of the complex structure modulus $\tau = \frac{\omega_2}{\omega_1}$ of the torus $T^2 = \mathbb{C}/(\omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z})$, and for this reason they can also be thought of as shape moduli.

Kähler moduli They are associated to (1,1) deformations of the Kähler form J on X. They parametrize the size of p-cycles of the internal manifold. For instance, in the absence of background fluxes[19], the volume of the compactification manifold X is always a Kähler modulus, thanks to the scale invariance $g_{mn} \mapsto r\, g_{mn}$ of the Einstein equations in the vacuum. More generally, the volume of the internal cycles are Kähler moduli. Thus, they are also known as size moduli.

D7-brane deformation moduli Stable D7-branes in string compactifications wrap internal holomorphic 4-cycles Σ_4 (that is, a divisor of X). If the line bundle $\mathcal{L}_{\Sigma_4}$ built on Σ_4 admits global holomorphic sections, these are by definition continuous deformations of the brane preserving SUSY. Therefore, they generate zero-modes of the brane, which correspond to massless scalar fields on the worldvolume theory of the brane. A sufficient condition for $\mathcal{L}_{\Sigma_4}$ to admit no global holomorphic sections is for it to be a negative line bundle.

D3-brane moduli At tree-level, spacetime-filling D3-branes do not feel any force[20]. Thus, the position on the internal space of spacetime-filling D3-branes become moduli of the four-dimensional EFT.

In orientifold type IIB compactifications, the spectrum of the four-dimensional EFT only comes from the ten-dimensional fields obeying the transformations rules under

[18] In this analogy, Feynman diagrams correspond to Riemann surfaces, and higher loop order corresponds to higher genus of the surface representing the string scattering.

[19] Namely, setting the VEVs of all the supergravity p-form field strengths to zero.

[20] It has been shown [28] that the D3-brane superpotential in type IIB warped compactifications is sourced by IASD fluxes, whose presence is a purely quantum effect.

orientifold involution (1.20) required to make the spectrum invariant. On the other hand, the orientifold action σ splits the cohomology groups $H^{p,q}(X)$ into orientifold-even and orientifold-odd subspaces, with dimensions $h^{p,q}_{\pm}(X)$. From the above definition, we expect $b_{p,\pm}(X)$ (real) axions coming from a respectively orientifold-even/odd p-form potential C_p. Recall that the only non-trivial Hodge numbers of a compact CY three-fold X with non-vanishing Euler characteristic are [23]

$$h^{1,1}(X) = h^{2,2}(X); \tag{1.30a}$$
$$h^{1,2}(X) = h^{2,1}(X); \tag{1.30b}$$
$$h^{0,0}(X) = h^{3,3}(X) = h^{3,0}(X) = h^{0,3}(X) = 1; \tag{1.30c}$$

while for a compact CY four-fold Z with non-vanishing Euler characteristic[21] they are [23]

$$h^{1,1}(Z) = h^{3,3}(Z); \tag{1.31a}$$
$$h^{1,2}(Z) = h^{2,1}(Z) = h^{1,4}(Z) = h^{4,1}(Z); \tag{1.31b}$$
$$h^{2,2}(Z) = 2\left(22 + 2h^{1,1}(Z) + 2h^{1,3}(Z) - h^{1,2}(Z)\right); \tag{1.31c}$$
$$h^{0,0}(Z) = h^{4,4}(Z) = h^{4,0}(Z) = h^{0,4}(Z) = 1. \tag{1.31d}$$

Therefore, in a type IIB compactification with O7/O3-planes we see from (1.20) that there are $h^{1,1}_{+}(X)$ C_4 axions, $h^{1,1}_{-}(X)$ B_2 axions and $h^{1,1}_{-}(X)$ C_2 axions, and the one axion C_0 surviving the orientifold projection. Similarly, the dilaton ϕ is kept in the spectrum of the low-energy EFT.

As for the metric moduli, complex structure deformations ψ^α on a CY 3-fold are actually in one-to-one correspondence with harmonic $(2,1)$-forms[22] χ_α through the holomorphic (3,0)-form Ω [18]:

$$\delta g_{ij} = \frac{i}{\|\Omega\|^2}\,\overline{\psi}^{\alpha}\,(\overline{\chi}_\alpha)_{i\bar{j}\bar{k}}\,\Omega^{\bar{j}\bar{k}}{}_{j}, \tag{1.32}$$

where $\|\Omega\|^2 = \frac{1}{3!}\,\Omega_{ijk}\overline{\Omega}^{ijk}$. Since the orientifold involution is an holomorphic isometry, (1.32) shows that only orientifold-odd complex structure deformations introduce moduli to the low-energy EFT. In particular, there are $2h^{1,2}_{-}(X)$ complex structure moduli. As for the Kähler moduli, thanks to the fact that the Kähler form is invariant under orientifold involution (thanks to the fact that σ is a holomorphic isometry), they are counted by orientifold-even harmonic $(1,1)$-forms, and there are $h^{1,1}_{+}(X)$ of them. Considering D7-branes deformation moduli, for each divisor Σ_4 they are spanned by holomorphic $(2,0)$-forms over Σ_4 defined by [25]

$$\omega_r = \iota_{\delta_r n}\Omega\big|_{\Sigma_4}, \tag{1.33}$$

[21] Actually, here we are also *assuming* $h^{2,0}(Z) = 0$.

[22] Analogously, on a CY n-fold they correspond to harmonic $(n-1,1)$-forms.

where $\delta_r n$ are the holomorphic deformation vector fields normal to Σ_4 (that is, a global holomorphic section of the line bundle $\mathcal{L}_{\Sigma_4}$), corresponding to a deformation $\delta\phi^r$ of the moduli of Σ_4, and $\iota_\alpha\beta$ denotes the interior product. Since σ inverts the sign of Ω, (1.33) shows that there are $h^{2,0}_-(\Sigma_4)$ D7-branes deformation moduli for each four-cycle Σ_4 of X wrapped by a D7-brane. Finally, since the internal space is six-dimensional, in type IIB compactification with N_{D3} D3-branes, there are $6N_{D3}$ D3-brane moduli given by the coordinates of the D3-brane positions. Notice that there is an equal number of C_4 axions and of Kähler moduli. In fact, by SUSY they combine together to form a *complexified* Kähler modulus[23]

$$T_A \simeq \frac{1}{\ell_s^4} \int_{D_{4,\,A}} \left(C_4 + i\,\mathrm{dVol}(D_{4,\,A})\right), \tag{1.34}$$

where $D_{4,\,A}$ is a basis of divisors of X. This is the quantity entering the low-energy EFT. Analogously, the dilaton ϕ and the axion C_0 combine into the complexified axio-dilaton

$$\tau = C_0 + i\,\mathrm{e}^{-\phi}, \tag{1.35}$$

which transforms like the modulus of a two-torus under the symmetry $SL(2,\mathbb{Z})$ of type IIB string theory.

On general grounds, while SUSY ensures that the full moduli space $\mathcal{M}$ is Kähler, its global structure is complicated, possibly involving multiple non-trivial fibrations thereof. As a result, even locally $\mathcal{M}$ cannot be written as a direct product of Kähler manifolds. In particular, this is due to the fact that D3-branes and Kähler moduli mix in a non-trivial way in the Kähler potential. While we will actually exhibit the full Kähler potential in presence of D3-branes in the case of the local model studied in this thesis, for the sake of an overview of the KKLT model it will not be necessary to include them. Moreover, for now we will also be scantly concerned with the B_2 and C_2 axions. This is due to the fact that their dynamics cannot destabilize the compactification, so that one does not need to compute the value at which they are stabilized, nor their masses, in order to assess the validity of a specific string vacuum.

Assuming no B_2 and C_2 axions are present in the compactification, and that no D3-branes are included, the full moduli space locally splits (at least at tree-level) into a complex structure moduli subspace, a Kähler moduli subspace, a D7-branes deformations moduli space and an axio-dilaton moduli space,

$$\mathcal{M} = \mathcal{M}_K \times \mathcal{M}_c \times \mathcal{M}_\tau \times \mathcal{M}_{D7}. \tag{1.36}$$

Neglecting the warping coming from curvature corrections on D7-branes, discussed below (1.61), the full Kähler potential then splits as well into a sum of the Kähler potentials for

[23] In presence of D3-brane moduli, this definition actually gets modified by the appearance of the warp factor, which is sourced by the D3-branes, in a non-trivial fashion.

each sector, and at tree-level it is given by[24]

$$K = K_K(T,\overline{T}) + K_c(\psi,\overline{\psi}) + K_\tau(\tau,\overline{\tau}) + g_s K_{D7}(\psi,\phi,\overline{\psi},\overline{\phi}); \tag{1.37a}$$

$$K_K(T,\overline{T}) = -2\log \mathrm{Vol}(X) = -2\log \int \frac{1}{3!} J\wedge J\wedge J; \tag{1.37b}$$

$$K_c(\psi,\overline{\psi}) = -\log 2i \int_X \Omega\wedge\overline{\Omega}; \tag{1.37c}$$

$$K_\tau = -\log \mathrm{Im}\,\tau. \tag{1.37d}$$

Here ϕ denotes D7-branes deformation moduli, and it is understood that we are in the weak coupling limit $g_s \sim 0$. Notice that the D7-brane deformation component enters the Kähler potential at order g_s, since this is quantum (non-perturbative) effect. We omit its specific form since it will have no relevance in our treatment of the KKLT scenario, nor in the analysis of this thesis. Moreover, here and in the following we set

$$\kappa_4^2 = 1. \tag{1.38}$$

The complex structure Kähler potential actually provides a very convenient basis for $H^{1,2}(X)$, labeled by complex structure deformations. Indeed, a complex structure deformation for a CY n-fold can be viewed as a deformation of the complex coordinates

$$\delta z^i = \epsilon\, f_a^i(z,\overline{z})\delta\psi^\alpha, \tag{1.39}$$

where f_a, $\alpha = 1,\dots,h^{1,2}(X)$, is a non-holomorphic function (if it were holomorphic, the complex structure would not be affected). The corresponding deformation of the holomorphic 3-form $\delta_a\Omega$ is going to be of rank $(3,0)+(2,1)$,

$$\begin{aligned}\delta\Omega &= \frac{\epsilon}{2!}\,\Omega_{ijk}\mathrm{d}z^i\wedge\mathrm{d}z^j\wedge\left(\partial_l f_a^k \mathrm{d}z^l + \partial_{\bar{l}} f_a^k \mathrm{d}\overline{z}^{\bar{l}}\right)\delta\psi^\alpha \\ &=: \partial_a\Omega\,\delta\psi^a.\end{aligned} \tag{1.40}$$

Then, if we define a covariant derivative over the complex structure moduli space

$$D_a W := (\partial_a + \partial_a K_c)\, W, \tag{1.41}$$

using (1.37c) it is straightforward to see that $D_a\Omega$ is a $(2,1)$-form (namely, ∂_a cancels off the $(3,0)$ part). Since Ω is harmonic, these are harmonic forms, and one can show that they are linearly independent, so that they span the whole $H^{1,2}(X)$.

It is worthwhile to note that (1.37a) can be recovered from F-theory, neglecting non-perturbative contributions[25] of order $\sim \mathrm{e}^{-\frac{\pi}{g_s}}$, as the orientifold weak coupling limit

[24] It should be noted that this is not the only form in which they appear in the literature. For instance, the -2 factor in K_K is sometimes replaced with a -3. The relation between these conventions amounts to a redefinition of the moduli fields, which can be found in [18].

[25] These are actually a feature of F-theory compactifications, which often ameliorate singularity issues of pure type IIB orientifold compactifications.

of the full Kähler potential (at tree-level)

$$K = K_K(T, \overline{T}) + K_c(z, \overline{z}); \tag{1.42a}$$

$$K_K(T, \overline{T}) = -2 \log \frac{1}{\ell_M^8} \operatorname{Vol}(Z); \tag{1.42b}$$

$$K_c(z, \overline{z}) = -\log \int_Z \Omega_4 \wedge \overline{\Omega}_4. \tag{1.42c}$$

where z are the complex structure moduli of the elliptically fibered four-fold Z, Ω_4 is the holomorphic 4-form of Z, and ℓ_M is the eleven-dimensional Planck length, which is related to the type IIB string length by

$$\ell_M^3 = \ell_s^2 \, L, \tag{1.43}$$

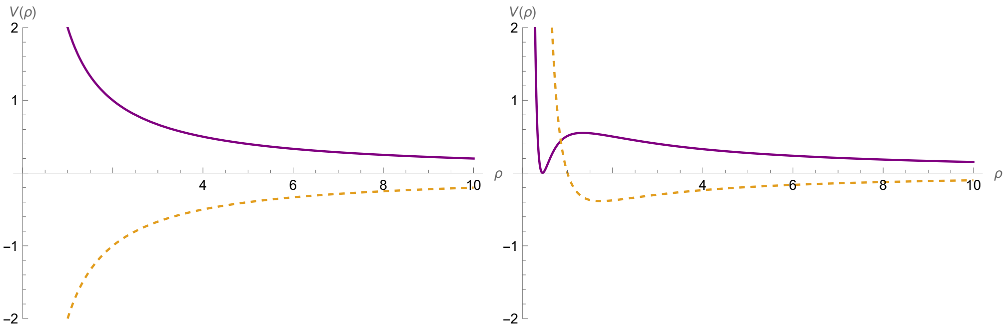
where L is the square root of the area v of the elliptic fiber[26]. In particular, this concretely shows that from the F-theory point of view the complex structure moduli of Z encompass the complex structure moduli of X together with the axio-dilaton and the D7-brane deformation moduli, and that their total number (from an F-theory compactification) is $2h^{1,3}(Z)$. Furthermore, they can be compactly labeled by $D_a\Omega_4$, $a = 1, \ldots, h^{1,3}(Z)$, as we showed above in the case of a CY three-fold.

This correspondence goes actually further, and in fact one can count all of the moduli of a type IIB string compactification using the Hodge numbers of Z alone, except for the D3-brane moduli, which come directly from M2-brane moduli. In particular, in the F-theory framework one need not worry about the orientifold-even or -odd character of the deformations, since this has become a piece of geometric information. Since one Kähler modulus gets lost in getting from M-theory to F-theory (the area of the elliptic fiber is shrunk to zero), there are $h^{1,1}(Z) - 1$ Kähler moduli. M-theory has only one form potential C_3, with field strength $G_4 = \mathrm{d}C_3$, and a dual potential C_6 defined by $G_7 = \star_{11} G_4$. C_6 axions correspond to C_4 axions, and there are $h^{1,1}(Z)$ of them (one deformation has to be removed since it is related to the shrinking elliptic fiber). C_3 axions generate B_2 and C_2 axions, and their number is given by $2h^{1,2}(Z)$. $\mathbb{R}^{1,2}$-filling M2-branes correspond exactly to spacetime-filling D3-branes, and they come in equal number, i.e. $6N_{D3}$.

1.2.2 Dine-Seiberg problem

One might object that there is no reason to worry about the moduli-space problem, since eventually SUSY is going to be spontaneously broken in the four-dimensional EFT, so that all of its non-renormalization theorems cease to apply, and all the moduli are virtually guaranteed to receive masses through quantum corrections. However, relying on quantum effects in order to stabilize moduli poses a serious threat to computational

[26]Since the internal manifold Z is Kähler, the area v does not depend on the point on the base of fibration. In presence of warping, the internal manifold is conformally Kähler and v does vary from point to point of the base. More precisely, for the choice of warp factors in (1.6), it has form $v(y) = v_0 \, \mathrm{e}^{\frac{4}{3} A(y)}$, where v_0 is a constant area.



(a) *Lowest order* ***(b)*** *Lowest order + higher orders*

Figure 1.2: *Qualitative forms of the quantum-generated scalar potential for ρ. On the right, the solid line is obtained from the addition of* two *higher order corrections, and it has a (possibly long-lived) metastable Minkowski vacuum, while the dashed line comes from the inclusion of just one correction, and it has a stable Anti-de-Sitter (AdS) vacuum. The solid line could also be slightly modified to give a metastable de Sitter (dS) vacuum.*

control over these corrections, as we are now going to argue. This is known as the *Dine-Seiberg problem* [5]. Since parametric control over the supergravity approximation and over the perturbative regime in string compactifications are serious necessities that one needs to achieve, and given that these are controlled by the VEVs of the Kähler moduli and of the dilaton respectively, this is an important lesson that has far-reaching consequences.

Consider a modulus ρ, such as the volume V_X or the inverse string coupling $\mathrm{e}^{-\phi}$, such that its limit $\rho \to +\infty$ corresponds to the weak coupling region, where the tree-level low-energy effective action is valid. As we stated, while at tree level $V_{\text{tree}}(\rho) = 0$, we expect quantum corrections to generate an effective potential $V(\rho)$ in the four-dimensional EFT, and by the above assumption it will need to obey the property

$$\lim_{\rho \to +\infty} V(\rho) = 0, \tag{1.44}$$

since we are considering a string compactification which does not stabilize any modulus. Assuming the leading order term in the potential to be a power law[27]

$$V_{\text{LO}}(\rho) \sim a\, \rho^n, \tag{1.45}$$

we see that there are only two qualitative possibilities, in figure 1.2 (a), corresponding to the choice $V > 0$ or $V < 0$ at infinity. In the former case, ρ has a runaway dynamics that brings it to infinity, while in the latter ρ is pushed towards zero, in the strongly coupled region of the theory. Neither of these two possibilities is viable, since they do not stabilize $\langle \rho \rangle$ at a finite value. Thus, we are forced to appeal to higher order (polynomial)

[27] This is reasonable if we think of this correction as a perturbative quantum effect. Nonetheless, the conclusion of this argument does not change even generalizing it to transcendental functional dependence.

corrections in order to have a local minimum appear[28], like in figure 1.2 (b). Notice that (1.44) guarantees that a runaway direction is present at all orders, which is often called *Dine-Seiberg vacuum.* This is a feature of potentials of quantum nature, whenever (1.44) holds. The issue, however, is that assuming ρ is stabilized at a finite value via higher order corrections to the quantum scalar potential implies that higher order corrections are comparable to leading order ones at the minimum point $\langle\rho\rangle$, namely that ρ is stabilized at the strong coupling region. One should then consider all other higher order corrections too, which we lack the computational tools to do in the absence of extended SUSY ($\mathcal{N} \geq 2$ in four dimensions), so that we unavoidably lose control.

Notice that at the core of this argument lies the interplay of $\langle\rho\rangle$ controlling the regime of the EFT and the EFT governing the dynamics of ρ itself. This tells us that, ideally, we should resort to some other way other than quantum corrections to stabilize the Kähler and complex structure moduli, and that whenever we cannot do without them, parameteric control should be addressed carefully.

1.2.3 Flux compactifications and de Sitter vacua

Considering string compactifications in the supergravity approximation, up to now we have implicitly assumed that the VEVs of all the supergravity p-form field strengths F_p were vanishing. The common jargon in string theory literature for the VEVs of F_p is *fluxes.* Naturally, if a string compactification admits non-vanishing VEVs for its p-form field strengths, classical electric and magnetic fluxes (1.24) and (1.25) are generated, hence the name. Dirac quantization requires the fluxes F_p to be integrally quantized, namely they need to satisfy

$$\int_{\Sigma_I} F_p = \ell_s^{p-1}\, N^I \in \ell_s^{p-1}\, \mathbb{Z} \tag{1.46}$$

for some suitably normalized[29] field strength F_p, and for any homology class p-cycle representative $\Sigma_I \in H_p(X)$, $I = 1, \ldots, b_p(X)$. Equivalently stated,

$$F_p = \ell_s^{p-1}\, N^I\, \Sigma_I, \tag{1.47}$$

where Poincaré duality is understood. Thus, fluxes are discrete degrees of freedom, when present. In general, setting the fluxes to zero is not a consistent assumption, since they too are subject to the equations of motion for the compactification, their respective *Bianchi identities*, a consistency condition associated with the brane content of the compactification called *tadpole cancellation condition*, and maximal symmetry. Since one is usually interested in SUSY low-energy EFTs, SUSY conditions are often solved instead of the EOMs[30]. Moreover, one can expect that turning on fluxes provides a *backreaction*

[28] For instance, like in the Lennard-Jones potential $V(r) \sim \alpha\left(\frac{\sigma}{r}\right)^{12} - \beta\left(\frac{\sigma}{r}\right)^{6}$, which in the present case would correspond to the dashed line of figure 1.2 (b).

[29] The ℓ_s factor is due to the fact that $[F_p] = \ell^{p-1}$.

[30] In the case of global supersymmetry, solutions to the supersymmetry constraints are automatically solutions to the equations of motion. In the local case, this is no longer true, unless the Bianchi identities are imposed as well [23].

on the external and internal geometry, i.e. the Einstein equations for the ten-dimensional metric are expected to be modified by a non-trivial flux-induced stress-energy tensor.

String compactifications involving non-trivial fluxes are called *flux compactifications*. Due to the fact that fluxes backreact on the geometry, in flux compactifications one needs to employ the most generic background compatible with maximal symmetry, namely warped ones of the form (1.6). More precisely, fluxes end up *sourcing* the warp factor. As it turns out, fluxes are indeed able to stabilize at least part of the compactification moduli at classical level. This is good news, since it allows us to evade the Dine-Seiberg problem. However, a very general no-go theorem due to Maldacena and Nuñez [13] shows that the simplest flux compactifications setups (namely using the supergravity approximation, smooth internal geometry, no branes included) are completely ruled out from reproducing *de Sitter* low-energy EFTs, which are phenomenologically favoured. This is bad news, since it tells us that we need to work with mildly pathological setups in order to even hope to find dS string vacua. This difficulty in producing dS vacua from string theory has recently given rise to doubts regarding even their existence, see for instance the proposal of [42].

No-go theorems

Let us first give a very direct derivation[31] of the Maldacena-Nuñez no-go theorem for the case of type IIB string theory, which is the more relevant one for this thesis, and for the KKLT scenario at large. Let us start from type IIB supergravity,

$$\begin{aligned} S_{IIB} =& \frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10}X \sqrt{-\hat{g}} \left[R - \frac{|\mathrm{d}\tau|^2}{2(\mathrm{Im}\,\tau)^2} - \frac{|G_3|^2}{2\mathrm{Im}\,\tau} - \frac{|F_5|^2}{4} \right] + \\ & - \frac{i}{8\kappa_{10}^2} \int \frac{1}{\mathrm{Im}\,\tau} C_4 \wedge G_3 \wedge \overline{G_3}. \end{aligned} \tag{1.48}$$

Here we use the conventions of §5.1.1, and the fermionic sector is omitted. As we mentioned at (1.13), this is the α'-leading order approximation of type IIB string theory, and in this limit one identifies

$$\frac{1}{\kappa_{10}^2} = \frac{4\pi}{\ell_s^8}. \tag{1.49}$$

Since we are dealing with fluxes, let us compactify this theory on a warped background of the form (1.6). By Poincaré invariance, G_3 is only allowed to have legs along the internal manifold, while F_5 necessarily takes the form

$$F_5 = (1 + \star_{10})\, \mathrm{d}\alpha \wedge \mathrm{dVol}_4, \tag{1.50}$$

where $\alpha(y)$ is a function of the internal coordinates only, and dVol_4 is the spacetime volume form. The ten-dimensional Einstein equations are

$$R_{MN} = \kappa_{10}^2 \left(T_{MN} - \frac{1}{8}\, \hat{g}_{MN}\, T \right), \tag{1.51}$$

[31] Found in [23].

where the ten-dimensional stress-energy tensor is computed by

$$T_{MN} = -\frac{2}{\sqrt{-\hat{g}}}\,\frac{\delta S_{IIB}}{\hat{g}^{MN}}, \tag{1.52}$$

and T is its trace. The external components of (1.51) can be computed from (1.48) to be

$$R_{\mu\nu} = -\frac{1}{4}\,\hat{g}_{\mu\nu}\left(\frac{1}{2\,\mathrm{Im}\,\tau}\,|G_3|^2 + \mathrm{e}^{-8A}\,|\mathrm{d}\alpha|^2\right). \tag{1.53}$$

On the other hand, the ten-dimensional Ricci tensor can be expressed in terms of the unwarped four-dimensional one as follows,

$$R_{\mu\nu} = R^{(4)}_{\mu\nu} - \mathrm{e}^{4A}\Delta A\, g_{\mu\nu}, \tag{1.54}$$

where $\Delta = g^{mn}\nabla_m\nabla_n$. Maximally symmetric spacetimes are Einstein, and in four dimensions it holds

$$R_{\mu\nu} = \Lambda\, g_{\mu\nu}, \tag{1.55}$$

where Λ is the cosmological constant, and $g_{\mu\nu} = \mathrm{e}^{-2A}\hat{g}_{\mu\nu}$ is the external unwarped metric. Plugging (1.54) into (1.53) yields the equation for the warp function

$$\Delta A = \frac{\mathrm{e}^{4A}}{8\,\mathrm{Im}\,\tau}\,|G_3|_0^2 + \frac{1}{4}\,\mathrm{e}^{-8A}\,|\mathrm{d}\alpha|_0^2 + \mathrm{e}^{-4A}\,\Lambda, \tag{1.56}$$

where $|\omega_p|_0^2$ is defined like $|\omega_p|^2$ but using the unwarped internal metric instead of the warped one. This can be recast as an equation of motion for the warp factor

$$\Delta \mathrm{e}^{4A} = \frac{\mathrm{e}^{8A}}{2\,\mathrm{Im}\,\tau}\,|G_3|_0^2 + \mathrm{e}^{-4A}\left(|\mathrm{d}\alpha|_0^2 + \left|\mathrm{d}\mathrm{e}^{4A}\right|_0^2\right) + 4\Lambda. \tag{1.57}$$

The no-go theorem is a simple consequence of this equation. Let us assume that externally we have Mink_4 or dS_4, that is $\Lambda \geq 0$. Integrating both sides over X, since the internal space is compact we get zero on the l.h.s., while on the r.h.s. we have a sum of positive definite terms, therefore (1.57) implies that $\Lambda = 0$ and

$$\int_X \mathrm{d}^6y\,\sqrt{g_6}\,|G_3|_0^2 = \int_X \mathrm{d}^6y\,\sqrt{g_6}\,|\mathrm{d}\alpha|_0^2 = 0. \tag{1.58}$$

Assuming that X is non-singular, (1.58) implies $G_3 = \mathrm{d}\alpha = 0$. We conclude that, on smooth geometries[32], type IIB supergravity has no Kaluza-Klein compactifications down to dS_4 spacetime, while it might admit KK compactifications down to Mink_4, but only with vanishing fluxes (except for F_1) and thus constant warping. On the contrary, (1.57) does not obstruct AdS flux compactifications ($\Lambda < 0$), and as it turns out these vacua are abundant[33].

[32] Assuming X has a smooth geometry also rules out the inclusion of branes in the compactification, since they introduce localized (i.e. delta-like) physical quantities on their worldvolume.

[33] Notable examples of these are Freund-Rubin vacua, like type IIB string theory on $\mathrm{AdS}_5 \times S^5$ with non-trivial F_5 flux.

As we already mentioned, the above no-go theorem for type IIB string compactifications actually has a larger scope. This is known as Maldacena-Nuñez no-go theorem, and it can be stated as follows. Let us consider any D-dimensional gravity theory, whose gravitational dynamics is given exactly by the Einstein-Hilbert action (i.e. without curvature corrections coming from string theory for instance), possibly coupled to arbitrary massless fields (e.g. scalars, p-forms, non-abelian gauge fields) with positive definite kinetic terms and with zero or negative potential (possibly depending on the scalars). Let us compactify this theory on a manifold X, with coordinates y^m, down to an a-dimensional ($a < D$) vacuum solution via a warped compactification of the form

$$\mathrm{d}s_D^2 = \Omega^2(y) \left(\mathrm{d}s_a^2(x) + \mathrm{d}s_X^2(y) \right) , \tag{1.59}$$

where $\mathrm{d}s_a^2(x)$ is maximally symmetric, namely AdS, Minkowski or dS. Let us also assume that X is compact, and that the warp factor is regular everywhere on X. Then, *there are no compactifications down to de Sitter, and no compactifications down to Minkowski except if only $F_1 \neq 0$ or $F_{D-1} \neq 0$*[34]*, in which case we get Minkowski with constant warp factor (which can be reabsorbed by coordinate redefinition).*

String theory immediately violates this theorem's hypothesis, with α' curvature corrections to the action and with the inclusion of open strings, but the core message of this result is rather that one should get one's hands dirty while looking for de Sitter vacua from string theories with the above ingredients. Including localized objects like D-branes and O-planes into the type IIB supergravity argument presented above clearly exemplifies how the no-go theorem can be evaded. D-branes and O-planes act as sources for the localized part of the stress-energy tensor

$$T_{MN}^{\mathrm{loc}} = -\frac{2}{\sqrt{-\hat{g}}} \frac{\delta S_{\mathrm{loc}}}{\delta \hat{g}^{MN}}, \tag{1.60}$$

where S_{loc} is the worldvolume action of the localized sources. In the case of D7-branes wrapped around a four-cycle Σ, assuming vanishing worldvolume fluxes, the relevant terms of the localized action at leading order in α' are [23]

$$S_{\mathrm{loc}}^{D7} = -T_{\mathrm{D}p} \int_{\mathbb{R}^{1,3}\times\Sigma} \mathrm{d}^8\xi \, \sqrt{-\tilde{g}} + \mu_7 \int_{\mathbb{R}^{1,3}\times\Sigma} C_8 - \mu_3 \int_{\mathbb{R}^{1,3}\times\Sigma} C_4 \wedge \frac{p_1(R)}{48}. \tag{1.61}$$

Here ξ are the worldvolume coordinates, $\tilde{g}$ is the pullback of the background metric to the brane worldvolume, C_8 is defined by $\mathrm{d}C_8 = \star \mathrm{d}C_0$, μ_7 is the electric charge of the brane defined in (1.24), induced by the electric coupling with C_8 (i.e. the magnetic coupling with F_1), and μ_3 is the induced D3-charge on the D7-brane by curvature corrections, which at lowest order are due to the first Pontryagin class of Σ:

$$p_1(R) = -\frac{1}{2} \frac{1}{(2\pi)^2} \operatorname{tr} R^{\wedge 2}, \tag{1.62}$$

[34]Recall that $F_{D-1} = \star F_1$.

where $R = (R^a{}_b)$ is the curvature two-form. In particular, μ_3 is actually higher order in α', precisely given by [38]

$$\mu_3 = (2\pi)^4 \alpha'^2 \mu_7. \tag{1.63}$$

Since the total (net) D3-charge of a compactification enters the Bianchi identity for F_5, as well as the warp factor EOMs and the tadpole cancellation condition, induced D3-charge on D7-branes by means of curvature corrections is a feature one should take into account, even though it is higher order in the α' expansion. It is straightforward to see that (1.60) contributes to (1.57) with a contribution proportional to the current

$$\mathcal{J}_{\text{loc}} = \frac{1}{4}\left(T^m_m - T^\mu_\mu\right), \tag{1.64}$$

where the indices are contracted using the *warped* metric (i.e. there are warp factors involved in the sum). More precisely, one finds that the full EOM for the warp factor including localized sources is given by

$$\Delta \mathrm{e}^{4A} = \frac{\mathrm{e}^{8A}}{2\,\mathrm{Im}\,\tau}\,|G_3|_0^2 + \mathrm{e}^{-4A}\left(|\mathrm{d}\alpha|_0^2 + \left|\mathrm{d}\mathrm{e}^{4A}\right|_0^2\right) + 4\Lambda + 2\kappa_{10}^2 \mathrm{e}^{2A}\mathcal{J}_{\text{loc}}. \tag{1.65}$$

For instance, in the case of D7-branes, one can see that this new term does bring a negative contribution due to the curvature corrections presented above, possibly balancing the rest of the positive terms and thus giving a way to evade the no-go theorem [23].

Let us remark that, in this context, from (1.65) and from the Bianchi identity for F_5 (1.69), one can show that at tree-level F_5 is completely determined by the warp factor, and more precisely [23]

$$\alpha = \mathrm{e}^{4A}, \tag{1.66}$$

where we refer to the notation used in (1.50).

Tadpole cancellation condition

Let us consider type IIB string theory compactifications. Localized sources like D3-branes, anti-D3-branes, O3-planes and D7-branes contribute to the net D3-charge of the compactification

$$Q_3^{\text{loc}} = \int_X \rho_3^{\text{loc}}, \tag{1.67}$$

where ρ_3 is the D3-charge density 6-form due to localized objects. It contains delta functions centered on the sources[35], for instance D3-branes, O3-planes, fluxes on D7-branes and D7-branes curvature corrections:

$$\rho_3^{\text{loc}} = \sum_{I=1}^{N_{D3}} \delta_I^{(6)} - \frac{1}{4}\sum_{J=1}^{N_{O3}} \delta_J^{(6)} + \dots. \tag{1.68}$$

Non-vanishing Q_3 modifies the Bianchi identity for the self-dual field strength F_5 to

$$\mathrm{d}F_5 = H_3 \wedge F_3 - 2\,\kappa_{10}^2 T_3 \rho_3^{\text{loc}}, \tag{1.69}$$

[35]In the case of D7-branes, the D3-charge is smeared over the brane.

where $T_3 = \frac{2\pi}{\ell_s^4}$ is the D3-brane tension, so that using (1.9)

$$2\kappa_{10}^2 T_3 = \ell_s^4. \tag{1.70}$$

This modification of the bulk Bianchi identity can be physically understood identifying $F_3 \wedge H_3$ with the D3-charge density induced by bulk fluxes, so that

$$\rho_3 = \frac{1}{\ell_s^4} F_3 \wedge H_3 + \rho_3^{\mathrm{loc}}, \tag{1.71}$$

and (1.69) becomes

$$\mathrm{d}F_5 + \ell_s^4\, \rho_3 = 0. \tag{1.72}$$

Integrating (1.72) over the internal space X, and using the fact that X is compact, one arrives at the tadpole cancellation condition $Q_3 = 0$, or equivalently

$$\frac{1}{\ell_s^4} \int_X F_3 \wedge H_3 + Q_3^{\mathrm{loc}} = 0. \tag{1.73}$$

This shows that the EOMs[36] require the total D3-brane charge of the compactification to vanish.

This condition also admits an F-theory interpretation. The EOM for the M-theory field strength G_4 including N_{M2} M2-branes and the first curvature correction to the action is given by [25]

$$\mathrm{d} \star_{11} G_4 = \frac{1}{2} G_4 \wedge G_4 - \ell_M^6 I_8(R) + \ell_M^6 \sum_{I=1}^{N_{M2}} \delta^{(8)}_{M2,\, I}, \tag{1.74}$$

where $I_8(R)$ is an 8-form built out of complete contractions of the curvature 2-form. Integrating over the compact internal four-fold Z one finds

$$\frac{1}{2\,\ell_M^6} \int_Z G_4 \wedge G_4 + N_{M2} = \frac{\chi(Z)}{24} =: Q_c, \tag{1.75}$$

where $\chi(Z)$ is the Euler characteristic of Z. Schematically decomposing[37]

$$G_4 = H_3 \wedge L\, \mathrm{d}x + F_3 \wedge L\, \mathrm{d}y, \tag{1.76}$$

where x, y are the real coordinates of the elliptic fiber, and recalling (1.43), one finds in the type IIB weak coupling limit the condition

$$\frac{1}{\ell_s^4} \int_X F_3 \wedge H_3 + N_{D3} - Q_c = 0, \tag{1.77}$$

[36] Since F_5 is self-dual, its Bianchi identity is equivalent to its EOM.

[37] This expression should not be taken too seriously. It does hold as is in the case of type IIB bulk fluxes, but it hides the $SL(2,\mathbb{Z})$ twisting that F-theory geometrically embeds.

where $N_{D3} = N_{M2}$. This is actually equivalent to (1.73), even though we need to state the dictionary between the two expressions. Indeed, the $F_3 \wedge H_3$ flux density in (1.77) already includes the brane-flux contributions from D7-branes present in the F-theory compactification, which enters (1.73) through Q_3, together with the naive bulk fluxes contribution; on the contrary, Q_3 in (1.73) also contains contributions from D7-branes curvature corrections, which enter (1.77) explicitly, corresponding to a D3-brane charge $-Q_c$. Notice how D3-brane charge from curvature corrections on D7-branes are due to the Euler characteristic of the elliptically fibered four-fold from the F-theory point of view.

This tadpole cancellation condition is a crucial contraint on fluxes in compactifications, and it is the ultimate reason as to why string theory is not infinitely finely tunable.

Complex structure moduli stabilization

Let us consider type IIB compactifications as obtained from F-theory. Turning on flux yields the tree-level scalar potential for the moduli in $\mathbb{R}^{1,2}$ [38]

$$V_{\text{tree}}(G) = \frac{2\pi}{\ell_M^9}\frac{1}{2}\int_Z G_4 \wedge \star_Z G_4. \tag{1.78}$$

The presence of the Hodge star makes it clear that this is a scalar potential for the metric moduli. However, rescaling the external metric in order to recover the canonical normalization $m_p^2/2$ of the Einstein-Hilbert term, one finds that (1.78) generates a runaway potential for the internal volume modulus of the form of the solid line in figure 1.2 (a), unless we turn it off imposing

$$\int_Z G_4 \wedge \star_Z G_4 = 0, \tag{1.79}$$

which means $G_4 = 0$ in the case of smooth Z. This is a consequence of the no-go theorem presented above, and it can be avoided introducing branes and higher order corrections.

Turning on the F-theory flux G_4 and including curvature corrections and M2-branes contributions generates the scalar potential for the moduli

$$V_{IIB} = \frac{2\pi}{\ell_s^4}\frac{1}{\ell_M^6}\int_Z G_{4,-} \wedge \star_Z G_{4,-}, \tag{1.80}$$

where $G_{4,-}$ is the anti-self-dual[39] (ASD) part of G_4. This can also be recast in terms of type IIB spectrum as

$$V_{IIB} = \frac{2\pi}{\ell_s^8}\int_X \frac{1}{\operatorname{Im}\tau}\, G_{3,-} \wedge \star_X \overline{G_{3,-}}, \tag{1.81}$$

[38] Recall F-theory is recovered compactifying M-theory over $\mathbb{R}^{1,2} \times Z$, and then applying T-duality fiberwise.

[39] On a D-dimensional space with euclidean metric, the square of the Hodge star acting on k-forms is given by $\star^2 = (-1)^{k(D-k)}$. Therefore, on Z and on 4-forms $\star^2 = 1$, which allows for real eigenvalues ± 1. These correspond to SD and ASD 4-forms. On the contrary, on X and on 3-forms $\star^2 = -1$, thus 3-forms on X (like G_3) can only be imaginary-self-dual (ISD) or imaginary-anti-self-dual (IASD).

where $G_{3,-}$ is the imaginary-anti-self-dual (IASD) part of G_3. Once again, after performing a rescaling of the external metric in order to bring the Einstein-Hilbert term to the canonical normalization

$$g_{\mu\nu} \mapsto \frac{m_p^2}{\mathrm{Vol}(X)}\, g_{\mu\nu}, \tag{1.82}$$

(1.81) becomes a runaway potential for the volume modulus of X, which would destabilize the compactification. In order to escape this problem, however, thanks to curvature corrections we do not need to set the fluxes to zero, but we need to impose that G_3 is ISD, that is

$$G_3 = i \star_X G_3, \tag{1.83}$$

or equivalently $G_4 = \star_Z G_4$. Notice that this implies that the flux should be harmonic, and since X is a CY three-fold with $\chi(X) \neq 0$, this also implies that G_3 is primitive by the Lefschetz decomposition. Moreover, on a n-dimensional Kähler manifold (n odd), a harmonic $(n-k,k)$-form with Lefschetz spin ℓ satisfies [25]

$$\star\, \omega_n = (-1)^{k+\ell}(-i)\,\omega_n, \tag{1.84}$$

therefore (1.83) constraints G_3 to be primitive ($\ell = 0$) and of the form $(2,1)+(0,3)$. The condition (1.83) can actually be interpreted as the classical supersymmetric EOM for the fluxes. Thanks to the presence of the Hodge star, this is the condition stabilizing the metric moduli of the compactification, while the fluxes are left as tunable discrete degrees of freedom.

The ISD condition (1.83) can actually be recast as F-flatness and D-flatness conditions for the compactification moduli. The F-flatness condition lives over the complex structure moduli space, and it takes the form[40]

$$D_a W(z) = 0, \tag{1.85}$$

where we introduced the so-called Gukov-Vafa-Witten (GVW) superpotential

$$W(z) = \frac{1}{\ell_M^3} \int_Z G_4 \wedge \Omega_4. \tag{1.86}$$

The D-flatness condition can be expressed analogously as

$$D_J \tilde{W} = 0, \tag{1.87}$$

where

$$D_J = \partial_J + \partial_J K_J; \tag{1.88a}$$

$$K_J = -\frac{1}{2}\log \int_Z \frac{J^4}{4!}; \tag{1.88b}$$

$$\tilde{W}(J) = \frac{1}{\ell_M^3} \int_Z G_4 \wedge J^2. \tag{1.88c}$$

[40] When dealing with complex structure deformations, the (p,q)-rank of a form is unfixed. Therefore, although only the $(0,4)$ and $(4,0)$ parts of G_4 contribute to W, (1.85) only constraints $G_4^{1,3}$ and $G_4^{3,1}$.

Despite its appearance, since J is real, $\tilde{W}(J)$ cannot be interpreted as a superpotential, so that (1.87) should really be regarded as a D-flatness condition.

In the case of smooth Z, one can show that (1.87) is always satisfied[41], so we will not consider it in the following. In presence of intersecting 7-branes one should assess whether or not it should be imposed separately. This leaves us with the GVW F-flatness condition (1.85), which only stabilizes the complex structure moduli of Z, and once this is satisfied the scalar potential for the remaining moduli (1.81) is flat and set to zero. In particular, this shows that this kind of flux compactifications leave all Kähler moduli unstabilized at tree level. Therefore, we will need to resort to quantum effects in order to stabilize the latter, bringing the Dine-Seiberg problem back into the picture.

Finally, as one would expect, (1.81) can be recast according to the standard four-dimensional supergravity scalar potential formula. Reinstating the κ_4 factors, one finds

$$\begin{aligned} V_{IIB} &= \frac{1}{4\pi\kappa_4^4}\,\mathrm{e}^{\kappa_4^2(K_c+K_K)}\,\left(h^{\alpha\overline{\beta}}D_\alpha W\overline{D_\beta W} - 3\kappa_4^2\,|W|^2\right) \\ &= \frac{1}{4\pi\kappa_4^4}\,\mathrm{e}^{\kappa_4^2(K_c+K_K)}\,h^{a\bar{b}}D_a W\overline{D_b W}, \end{aligned} \tag{1.89}$$

where a, b run over the complex structure moduli only, α, β run over Kähler and complex structure moduli, $h_{\alpha\overline{\beta}} = \kappa_4^2\partial_\alpha\partial_{\overline{\beta}}K$ is the metric moduli space metric, and we denoted by the same symbol as above the covariant derivative over the whole moduli space[42]

$$D_\alpha = \partial_\alpha + \kappa_4^2\,\partial_\alpha K. \tag{1.90}$$

Notice that in the rigid limit $\kappa_4^2 \to 0$, namely for non-compact internal space according to (1.16), (1.89) reduces to the standard rigid supersymmetry formula for the F-term scalar potential[43]. This shows that the GVW superpotential can be used to enforce SUSY by imposing $DW = 0$. In particular, (1.89) tells us that V_{IIB} has a *no-scale* structure, namely it does not stabilize the Kähler moduli, since their contribution to DW cancels with the $-3\,|W|^2$ term:

$$\sum_{\alpha,\overline{\beta}\in\text{Kähler moduli}} h^{\alpha\overline{\beta}}\partial_\alpha K\partial_{\overline{\beta}}K = 3. \tag{1.91}$$

We already noted that the no-scale structure forces us to resort to quantum corrections, which is indeed troublesome. On the other hand, a positive byproduct is that at tree level we can take the KK scale to be large, so that the supergravity approximation is justified.

As one would expect, in the weak coupling limit the GVW superpotential for the complex structure moduli of the four-fold Z splits into the sum of a bulk part, the

[41] In particular, one can prove that G_4 has no $\ell = 2$ part, which makes it primitive.

[42] This does not lead to any ambiguity since the derivative of the Kähler potential only selects its relevant part.

[43] The $1/\kappa_{(4)}^2$ factor in front of (1.89) diverges in the $\kappa_{(4)} \to 0$ limit. This is an artifact due to the rescaling (1.82), which is singular in this limit, and thus it should not be taken.

superpotential for the complex structure moduli of X, and a brane part, the superpotential for the D7-brane deformation moduli,

$$W(z) = W_b(\tau, \psi) + W_{D7}(\psi, \phi), \tag{1.92}$$

where

$$W_b(\tau, \psi) = \frac{1}{\ell_s^2} \int_X G_3 \wedge \Omega_3; \tag{1.93}$$

$$W_{D7}(\psi, \phi) \simeq \int_{\Gamma_5} (F_2 - B_2) \wedge \Omega_3, \tag{1.94}$$

where Γ_5 is a 5-chain in Z going from the D7-brane to the O7-plane (needed to cancel the tadpole), and F_2 is the brane flux. At leading order in g_s, (1.85) splits into the F-flatness conditions

$$\partial_\tau W + (\partial_\tau K_\tau) W = 0 \tag{1.95a}$$

$$\partial_\psi W + (\partial_\psi K_\psi) W = 0 \tag{1.95b}$$

$$\partial_\phi W_{D7} = 0. \tag{1.95c}$$

All of these weak coupling expansions hold up to non-perturbative corrections, which arise exclusively in the F-theory picture.

A final remark on supersymmetry. Assuming that (1.85) stabilizes *all* complex structure moduli of Z, one would end up with a Minkowski vacuum and broken SUSY since

$$D_{T_A} W = \kappa_4^2 \, (\partial_{T_A} K_K) W \neq 0, \tag{1.96}$$

unless $W = 0$ at its minimum. This can happen accidentally, or imposing[44] that $G_4^{0,4} = G_4^{4,0} = 0$, i.e. that

$$G_3^{0,3} = 0. \tag{1.97}$$

Another way to restore SUSY without constraining the flux is to break the no-scale structure of the scalar potential by introducing quantum corrections to the GVW superpotential depending on the Kähler moduli. This is the path chosen in KKLT scenarios.

1.2.4 Non-perturbative corrections

Quantum effects in string theory compactifications have a number of origins. They can arise both from the α' stringy expansion[45] and from the g_s expansion. Moreover, non-perturbative effects and instantonic solutions can also arise on branes, which are in turn non-perturbative solutions in g_s, and they can in principle affect the moduli space of the low-energy theory.

[44] Recall that G_4 is real, while G_3 is complex. The condition on G_4 is straightforward from $W = 0$, while the condition on G_3 comes from $W_b = 0$.

[45] Since α' controls the quantum string worldsheet perturbative expansion, these can still be regarded as quantum effects.

While α'-quantum corrections to the moduli Kähler potential are scarcely constrained, much more can be said about quantum corrections to the effective superpotential for compactification moduli. Thanks to classical non-renormalization theorems from supersymmetry, such quantum effects can only be of non-perturbative origin[46]. Two main classes of such effects, whose string theory embedding have been extensively studied, are instantons and gaugino condensation on branes, which we are now going to review.

Instantons

Supersymmetric instantons in string theory compactifications can give rise to corrections to the moduli superpotential. These are of the form [25]

$$W_{\text{inst}} = \Lambda^3 \, \mathrm{e}^{2\pi i \, n^A T_A}, \tag{1.98}$$

where Λ^3 is a holomorphic function of non-Kähler moduli (called *Pfaffian*), including complex structure moduli, T_A are a basis of complexified Kähler moduli, and $n^A \in \mathbb{Z}$. Notice that (1.98) is invariant under integer shifts of $\mathrm{Re}\, T_A$, namely it preserves the axion shift symmetry. In the type IIB picture, (1.98) arises from a euclidean D3-brane wrapped around an internal divisor

$$D_4 = n^A D_{4,A}. \tag{1.99}$$

This is often called an E3-brane instanton. In the F-theory picture, (1.98) arises from M5-branes wrapped around the entire elliptic fiber[47] and D_4 in the base, namely wrapping $D_6 = n^A D_{6,A}$.

The existence of a E3-brane instantons does not imply that they will contribute to the superpotential. A sufficient condition for an E3-brane instanton to give non-zero contribution is carrying the minimal amount of fermionic zero-modes, that is two. These are the Goldstino fields living on the D3-brane worldvolume, due to the breaking of half SUSY by the brane[48] (recall that stable branes are BPS). In particular, such E3-brane instantons necessarily wrap rigid divisors, since otherwise its deformation moduli would correspond to bosonic zero modes on the D3-brane worldvolume, and by the remaining SUSY they would introduce fermionic zero modes.

Gaugino condensation

Let us consider N coincident spacetime-filling D7-branes wrapping a rigid divisor $D_4 = n^A D_{4,A}$ in X. Barring subtle complications, their low-energy dynamics is described by a pure $SU(N)$ super-Yang-Mills (SYM) theory (i.e. without coupled chiral matter) in $D = 4$ dimensions. The rigidity of the divisor, namely the absence of holomorphic global

[46] This can be quickly understood from holomorphicity of the superpotential and from the fact that the axion shift symmetry of the complexified Kähler modulus T_A is exact, so that all perturbative corrections must preserve it.

[47] These are the only M5 instantons that retain finite action in the F-theory limit of vanishing fiber area.

[48] Maximally SUSY instantons in $D = 4$ dimensions have $\mathcal{N} = 4$ Weyl spinor supercharges.

sections of its normal bundle, is crucial for gaugino condensation to occur on D7-branes. Indeed, D7-brane deformation moduli would appear in the low-energy spectrum of the worldvolume gauge theory as a Higgs branch, spontaneously breaking the gauge group completely at low energies. The complexified gauge coupling of the SYM theory is given by the complexified Kähler modulus associated with D_4, namely

$$\tau(T) = n^A T_A = \frac{\theta_{YM}(T)}{2\pi} + \frac{4\pi i}{g_{YM}^2(T)}. \tag{1.100}$$

This can be seen from the low-energy expansion of the DBI action for the D7-brane stack, which describes its worldvolume dynamics. Notice that the normalization factor in (1.100) is fixed to one by requiring the axion shift symmetry of T_A to match the non-perturbative SYM symmetry $\tau \to \tau + n$.

Let us show how (1.100) comes about in the case of $N = 1$ D7-brane wrapping a rigid divisor D. In the static gauge and neglecting the couplings with bulk fluxes, the bosonic part of the DBI action in the string frame takes the form [23]

$$S_{DBI}^{D7} = -\frac{2\pi}{\ell_s^8} \int_{D7} \mathrm{d}^8\sigma\, \mathrm{e}^{-\phi} \sqrt{-\det\left(g^{(s)} + \frac{\ell_s^2}{2\pi} F\right)} \tag{1.101}$$

where the deformation scalars are absent due to the rigidity of D, $g^{(s)} = \mathrm{e}^{\phi/2} g^{(e)}$ is the induced metric in the string frame, and in the static gauge $\mathrm{d}^8\sigma = \mathrm{d}^4x\, \mathrm{d}^4\xi$ is the worldvolume measure splitting. Let us expand this actions in an α' series, using the identity

$$\det(1 + \epsilon A) = 1 + \epsilon \operatorname{tr} A + \epsilon^2 \frac{(\operatorname{tr} A)^2 - \operatorname{tr} A^2}{2} + \mathcal{O}(\epsilon^3), \tag{1.102}$$

we find

$$\begin{aligned} S_{DBI}^{D7} = &- \frac{2\pi}{\ell_s^8} \left(\int_D \mathrm{d}^4\xi \sqrt{\tilde{g}_4^{(e)}} \right) \int_{\mathbb{R}^{1,3}} \mathrm{d}^4x\, \mathrm{e}^{\phi} \sqrt{-g_4^{(e)}} \\ &- \left(\frac{1}{4\pi} \frac{1}{\ell_s^4} \int_D \mathrm{d}^4\xi \sqrt{\tilde{g}_4^{(e)}} \right) \frac{1}{2} \int_{\mathbb{R}^{1,3}} F_2 \wedge \star_4 F_2, \end{aligned} \tag{1.103}$$

where $\tilde{g}_4^{(e)}$ and $g_4^{(e)}$ are the determinant of the internal and external induced metrics respectively, in an unwarped background and in the Einstein frame. Notice that the first term in (1.103) yields the four-dimensional tension contribution, which is going to be canceled by the O7-plane tension in a realistic setup. In the non-abelian case of N coinciding D7-branes, the second term in (1.103) readily generalizes to

$$S_{stack} = -\left(\frac{1}{4\pi} \frac{1}{\ell_s^4} \int_D \mathrm{d}^4\xi \sqrt{\tilde{g}_4^{(e)}} \right) \frac{1}{2} \int_{\mathbb{R}^{1,3}} \operatorname{tr} F_2 \wedge \star_4 F_2 + \dots, \tag{1.104}$$

which allows us to identify, using (1.34),

$$\frac{1}{g_{YM}^2} = \frac{1}{\ell_s^4} \operatorname{Vol}(D) = \frac{1}{4\pi} \operatorname{Im} T, \tag{1.105}$$

which completes to (1.100) by supersymmetry. Notice that here we choose the trace normalization for the gauge group generators

$$\operatorname{tr} T^a T^b = \frac{\delta^{ab}}{2}. \tag{1.106}$$

In the rest of this work, we will actually use the equivalent choice for the complexified Kähler modulus

$$\rho = -iT, \tag{1.107}$$

so that (1.100) actually takes the form

$$\tau(\rho) = i\,\rho. \tag{1.108}$$

Notice that (1.108) in components reads

$$\operatorname{Re}\rho = \frac{4\pi}{g_{YM}^2}; \tag{1.109a}$$

$$\operatorname{Im}\rho = -\frac{\theta_{YM}}{2\pi}. \tag{1.109b}$$

In particular, $\operatorname{Im}\rho$ is periodic with period 2π. This is due to large gauge transformations of C_4, see §3.4 below (3.21) for an introduction to the matter.

At energies lower than the non-perturbative scale of the worldvolume gauge theory, gauginos λ (fermionic superparterns of the gauge fields in the $\mathcal{N} = 1$ chiral multiplet) condense in pairs [7]. This adds a new non-trivial gauge-invariant VEV, which makes its appearance in the EFT and sources highly non-trivial modifications of the string theory background. The generic form of the superpotential contribution from gaugino condensation taking plane on a D7-brane stack is

$$W_{np}(T) = N\,\mu_0^3\,\mathrm{e}^{\frac{2\pi i\,T}{N}}. \tag{1.110}$$

In §4.4.2 we work out the details on how to derive this. Here μ_0^3 is a UV scale associated with the running of $\tau(T)$, which depends on non-Kähler moduli, including complex-structure moduli.

Notice that instantons corrections (1.98) and gaugino condensation corrections (1.110) have a very similar form, in particular the exponential arguments suggest that gaugino condensation may be interpreted as a sort of *fractional* instantonic contribution. This intuition can actually be made explicit, and one can show that the four-dimensional gaugino condensate superpotential is related to a three-dimensional M5-brane instanton superpotential [25]. This provides a unification of the two non-perturbative effects, at least at the superpotential level.

1.2.5 The KKLT scenario

The Kachru-Kallosh-Linde-Trivedi (KKLT) scenario [15], proposed in 2003, brings together all the ingredients discussed above to achieve a dS vacuum in type IIB string theory compactifications while retaining reasonable control over quantum corrections. While the scenario delineates a general strategy to how one could in principle achieve dS vacua with exponentially small cosmological constant and spontaneously broken supersymmetry, this is by no means a guarantee of success, due to the plethora of details that the scenario ignores. Indeed, as of today no realistic explicit models of dS vacua from type IIB string theory have been constructed, and the task may very well be impossible. As we already mentioned, motivated doubts have recently arisen about the existence altogether of dS vacua in string theory (de Sitter conjecture [42]), which implies that even the KKLT scenario should not work. Indeed, it has been conjectured that this construction is not consistent [51].

The KKLT recipe splits into two movements. First, one stabilizes the complex structure moduli with fluxes (§1.2.3) and the Kähler moduli with non-perturbative corrections to the GVW superpotential (§1.2.4) in such a way that supersymmetry is preserved. This produces a SUSY AdS_4 vacuum. Then, one breaks all supersymmetry by introducing spacetime-filling $\overline{D}3$-branes[49], which also provide a quantum correction to the scalar potential able to *uplift* the minimum to a positive value. In order to perform the uplift and end up with an exponentially small value for the cosmological constant, the contribution from anti-D3-brane is redshifted by means of a highly warped geometry known as *Klebanov-Strassler solution* (or throat).

Moduli stabilization

Let us consider type IIB/F-theory flux compactification, and let us assume that one of the non-perturbative effects of §1.2.4 contribute to the effective superpotential. The flux is quantized according to

$$G_4 = \ell_M^3 N^I \Sigma_{4,I}, \tag{1.111}$$

where $\Sigma_{4,I}$ is a basis of 4-cycles wrapping a 1-cycle in the elliptic fiber, and Poincaré duality is understood. For simplicity, let us assume that there is only one Kähler modulus T. Moreover, let us also assume that g_s has been already stabilized at a small value with a proper tuning of fluxes[50]. The total superpotential of the compactification in F-theory language is then given by

$$W(z,T) = W_{\mathit{flux}}(z) + \mu_0^3(z)\, \mathrm{e}^{2\pi i a T}, \tag{1.112}$$

where

$$a = \begin{cases} 1 & \text{E3-brane instanton} \\ 1/N & \text{Gaugino condensation on N D7-branes} \end{cases} \tag{1.113}$$

and W_{flux} is the GVW superpotential (1.86). The UV scale μ_0^3 is naturally order 1 in string units[51]. Since (1.112) is dimensionless in our convention, and W_{flux} is in string

[49] Anti-D3-branes, or $\overline{D}3$-branes, break the SUSY generators along their worldvolume. Since we chose them transverse to the internal space, they break spacetime supersymmetry.

[50] This is possible at least in some cases.

[51] The string mass scale is defined by the effective scalar potential prefactor in (1.89), given by $m_s^4 \sim m_p^4\, \mathrm{e}^{\kappa_4^2 K_K}$.

units, we should take

$$\mu_0^3 \sim 1. \tag{1.114}$$

Crucially, let us assume that the quantum effect in (1.112) is small with respect to the classical flux contribution. Then, the complex structure moduli are stabilized by the classical EOMs up to small corrections:

$$D_a W_{flux}(z) = 0. \tag{1.115}$$

The flux superpotential takes the explicit form

$$W_{flux}(z) = N^I \Pi_I(z), \tag{1.116}$$

where we introduced the periods

$$\Pi_I(z) = \int_{\Sigma_{4,I}} \Omega_4\,(z). \tag{1.117}$$

For a generic G_4 flux, all the complex structure moduli are stabilized[52] by (1.115) at values $z^a = z_0^a$. From the scalar potential (1.89), we see their masses at weak coupling are of order

$$m_z \sim \frac{|G_4|\, m_p}{\mathrm{Vol}(X)}\, g_s \sim \frac{|G_4|\, m_s}{(\mathrm{Vol(X)})^{\frac{1}{2}}}\, g_s, \tag{1.118}$$

where $|G_4|$ is some measure of the flux in string units, and $\mathrm{Vol}(X)$ is in string units as well. The size of the flux is estimated from the tadpole cancellation condition (1.75), which reads here

$$\frac{1}{2}\, Q_{IJ} N^I N^J + N_{D3} = Q_c, \tag{1.119}$$

where $Q_{IJ} = \Sigma_{4,I} \cdot \Sigma_{4,J}$ is the intersection matrix. Introducing no D3-branes in the compactification, we estimate

$$|G_4| \sim \sqrt{Q_c}. \tag{1.120}$$

This makes the number of possible fluxes finite, although typically exponentially large. One can show that this still allows one to expect a large number of vacua with exponentially small $|W_0| := |W_{flux}(z_0)|$ to exist [17], once the complex structure moduli are stabilized. This is desirable in order to stabilize the Kähler modulus at a large value, which justifies the supergravity approximation and the tree-level approximation for the Kähler potential, so we are going to assume it.

After stabilization of the complex structure moduli, we are left with a superpotential for the Kähler modulus

$$W(T) = W_0 + \mu_0^3(z_0)\, \mathrm{e}^{2\pi i a T}, \tag{1.121}$$

where W_0 is exponentially small and μ_0^3 is order 1. Thanks to the quantum contribution in (1.121), we can now stabilize the Kähler modulus imposing supersymmetry,

$$D_T W(T) = 0. \tag{1.122}$$

[52] Actually, this is plausible but not obvious, given that fluxes are constrained by the tadpole cancellation condition.

Using a field redefinition, (1.37b) can be taken to the form

$$K_K(T,\overline{T}) = -3\log \operatorname{Im} T, \tag{1.123}$$

and using this, (1.122) becomes

$$2\pi i a T \sim \log\left(\frac{W_0}{\mu_0^3}\right)^{-1}. \tag{1.124}$$

Thanks to the fact that W_0 is exponentially small, (1.124) stabilizes T at order 10 or more, as stated above. The Kähler modulus receives a mass from (1.89) of order

$$m_T \sim \mathrm{e}^{2\pi i a T_0} m_s \sim |W_0|\, m_s, \tag{1.125}$$

which is exponentially small with respect to (1.118). This is consistent with our choice of integrating out the complex structure moduli before the Kähler modulus.

We managed to stabilize in principle all moduli in a SUSY vacuum. The effective scalar potential (1.89) then generates a cosmological constant

$$\Lambda = -3\,\frac{1}{4\pi\kappa_4^2}\,\mathrm{e}^K\,|W|^2 \sim -m_s^4\,\mathrm{e}^{2\pi a T_0} \sim -m_s^4\,|W_0|^2\,, \tag{1.126}$$

which is necessarily negative. Thus, we found a AdS_4 vacuum with exponentially small cosmological constant. Notice that W_0 is determined by the $(0,3)$ part of G_3, which is allowed to be non-vanishing even requiring SUSY, thanks to the presence of non-perturbative corrections to W.

Actually, the scalar potential for the Kähler modulus generated by (1.121) can be explicitly computed to be

$$V_T = \frac{\alpha}{\sigma^2}\left[\frac{4}{3}\,(2\pi a)^2\,\mu_0^6\sigma^2\mathrm{e}^{-4\pi a\sigma} + 4A^2(2\pi a)\sigma\,\mathrm{e}^{-4\pi a\sigma} - W_0^2\right], \tag{1.127}$$

where $T = i\,\sigma$, α is a real constant, and W_0 is assumed real. This potential is of the form of the dashed line in figure 1.2 (b), which is expected being the result of a tree level contribution plus a quantum effect.

Uplift to dS

As we already mentioned, adding $\overline{D3}$-branes to the compactification breaks SUSY completely. This also adds a negative tadpole, which needs to be canceled by flux, and a positive energy density

$$\delta V = 2\,\frac{a_0^4 T_3}{g_s}\,\frac{1}{\sigma^3}, \tag{1.128}$$

where a_0 is the warp factor value at the location of the brane. In order to uplift the AdS minimum from an exponentially small *negative* value to an exponentially small *positive* value, it is possible to employ the Klebanov-Strassler solution. Indeed, it has been shown

[12] that there do exist warped type IIB compactifications with regions of substantial warping. More precisely, these models have warped throats, with warp factor reaching the minimal value

$$\mathrm{e}^{A_{min}} \sim \exp -\frac{2\pi K}{3g_s M}, \tag{1.129}$$

which is parametrized by properly tuned flux integers M and K. Furthermore, it has been shown that the presence of ISD flux in the Klebanov-Strassler solution drives $\overline{D3}$-branes towards the bottom of the throat, where they experience the maximal redshift (1.129), exponentially suppressing their contribution to the scalar potential. Putting everything together[53], one ends up with a potential looking like the solid line of figure 1.2 (b), which admits a metastable dS vacuum with exponentially suppressed cosmological constant.

The need for an exponentially small W_0, and the need to hit exactly the value (1.1) for the cosmological constant may sound like a problem for naturalness, seemingly invoking fine tuning. However, a feature of KKLT scenarios is to admit exponentially large numbers of vacua with approximately the same W_0 and Λ, both close to zero, forming a so-called *discretuum*. If there are at least 10^{120} of such dS vacua, there should be at least order 1 vacua satisfying (1.1). In this sense, naturalness of KKLT is not an issue thanks to the landscape of string theory, as long as the anthropic principle does not bother us.

1.3 Gaugino condensation from ten dimensions

Gaugino condensation plays a central role in the KKLT scenario. Together with E3-brane instantons, it provides a suitable quantum effect which enable us to stabilize the Kähler moduli. One major point of criticism, however, is that the whole argument for Kähler moduli stabilization in KKLT is based on the low-energy four-dimensional EFT, and on its effective superpotential. In light of the objections moved against it, a ten-dimensional understanding of the non-perturbative phenomena would put KKLT on a much firmer footing. While achieving such a description would certainly be of major academic interest, at first glance it does not seem reasonable to even expect that a formulation of such low-energy phenomena making use of the high-energy formalism of warped flux compactifications even exists. However, this is a faulty line of thought. Consider a type IIB flux compactification with bulk KK scale Λ_{KK}^{bulk}, exhibiting a confining sector on a D7-brane stack endowed with a confining scale Λ_{np}. The D7-brane stack needs to wrap an internal rigid divisor D, therefore the UV cutoff scale for the effective SYM theory supported on its worldvolume is given by the KK scale Λ_{KK}^{D7} associated to D, and confinement is naturally assumed to be a low-energy phenomenon, so that it always holds

$$\Lambda_{KK}^{D7} \gg \Lambda_{np}. \tag{1.130}$$

On the other hand, it is reasonable to expect a ten-dimensional description of gaugino condensation to exist whenever it holds

$$\Lambda_{np} \gtrapprox \Lambda_{KK}. \tag{1.131}$$

[53] Together with the new term (1.128), the scalar potential gets slightly modified, see [15].

Therefore, a well-defined regime to describe ten-dimensional effects of gaugino condensation exists if and only if the compactification satisfies the scale hierarchy

$$\Lambda_{KK}^{D7} \gg \Lambda_{KK}^{bulk}. \tag{1.132}$$

In such case, one can in principle devise a setup such that the complex structure moduli conspire in order to achieve the complete scale hierarchy

$$\Lambda_{KK}^{D7} \gg \Lambda_{np} \gtrapprox \Lambda_{KK}. \tag{1.133}$$

Whenever this happens, it is reasonable to expect a ten-dimensional description of gaugino condensation to exist, which should match the four-dimensional supergravity EFT employed by KKLT when one integrates out all massive Kaluza-Klein modes. Clearly, if the complex structure moduli allow Λ_{np} to drop below Λ_{KK}^{bulk}, the only viable description of gaugino condensation is a four-dimensional one. Notice that (1.132) and (1.133) are guaranteed to hold in local models since Λ_{KK}^{bulk} is sent to zero, which justifies the ten-dimensional analysis of §5.

1.3.1 State of the art

In type II compactifications to four dimensions, requiring $\mathcal{N} = 2$ supersymmetry in the low-energy EFT in the absence of fluxes leads to the requirement that the internal manifold X be CY. This can be seen imposing the Killing spinor equation found by setting the SUSY variation of the gravitinos in the supergravity approximation to zero. Calabi-Yau manifolds are Kähler manifolds with exactly $SU(3)$ holonomy[54]. Reduction[55] of the holonomy group can be effectively understood in terms of existence of a globally defined $SU(3)$-covariantly constant spinor, which makes the $U(1)$ factor of $U(3)$ trivially realized [6]. Therefore, $\mathcal{N} = 2$ $D = 4$ SUSY in unfluxed compactifications can be recast as the union of an algebraic condition, the existence of a globally defined spinor, and a differential one, it being covariantly constant. In presence of fluxes, both conditions are modified imposing to find $\mathcal{N} = 2$ or $\mathcal{N} = 1$ SUSY in four dimensions [21]. The algebraic condition becomes that the Whitney sum bundle $TX \oplus T^{\star}X$ should have $SU(3) \times SU(3)$ structure. This implies the existence of two globally defined pure spinors. These can be represented as polyforms, i.e. sums of differential forms of different rank (odd in type IIB, even in type IIA). The differential condition becomes a set of differential equations for the pure spinors involving the fluxes, the warp factor and the dilaton. These are usually referred to as *supersymmetry conditions*. These equations are most easily formulated employing the formalism of *generalized complex geometry* [30, 19]. Therefore, we surmise that this should be the mathematical framework one needs to employ in order to achieve a proper ten-dimensional description of flux vacua.

A ten-dimensional embedding of gaugino condensation in the supergravity approximation is expected to manifest itself as a modification of the fluxes EOMs as well as a

[54] The holonomy group of a space is defined as the structure group of its associated frame bundle.
[55] The most general holonomy group of a Kähler three fold is $U(3)$.

deformation of the background geometry. This is due to the fact that a non-vanishing value for the gaugino bilinear VEV $\langle\lambda\lambda\rangle$, which has a four-dimensional origin and it is artificially injected into the ten-dimensional framework, would turn on Chern-Simons corrections coming from quadratic self-interactions terms, or from interaction with bulk fluxes in the DBI action

$$\delta S \sim \int_{D7} \mathrm{d}^8\sigma \sqrt{-g}\, G_3\lambda\lambda, \tag{1.134}$$

as well as localized currents entering the supersymmetry conditions. Indeed, it has been shown in [28] that non-perturbative effects on D7-branes in type IIB compactifications source IASD fluxes at leading order in $\langle\lambda\lambda\rangle$. This is a ten-dimensional quantum correction to the tree-level ISD G_3 flux. In general, IASD primitive contributions to G_3 can be of the form $(1,2)+(3,0)$, but [28] finds only a $(1,2)$ term due to gaugino condensation at leading order. The appearance of a IASD correction to the flux is exactly due to the modification of the EOM for G_3 induced by $\langle\lambda\lambda\rangle$. On the other hand, it has been shown in [24] that E3-brane instantons and gaugino condensation on D7-branes (condensing D7-branes) in type II compactifications destabilize the ordinary CY complex structure of the internal manifold, making it a genuine generalized complex one. This happens independently of the presence of background fluxes. Therefore, a ten-dimensional embedding of gaugino condensation, also referred to as a *geometrization* of the non-perturbative contribution it brings to the effective four-dimensional superpotential, seems to manifest itself at leading order in a twofold manner: it sources an IASD contribution to the flux G_3, and it deforms the internal space into a proper generalized complex geometry.

A more explicit analysis of the fluxes sourced by gaugino condensation on D-branes in type II compactifications has been performed in [31]. This work studies condensing D5-, D6- and D7-branes and solves the supersymmetry conditions, updated with the non-perturbative contributions, in the framework of generalized complex geometry and in a perturbative approach, stopping at leading order in the gaugino condensate $\langle\lambda\lambda\rangle$. Although the case of D7-branes turns out to be a pathological one, for reasons we will discuss in §5, they find that condensing D7-brane do source IASD $(1,2)+(3,0)$ components to G_3, and they provide an explicit (although local) SUSY solution for it in absence of tree-level fluxes. This validates the findings of [28], since the $(3,0)$ part of G_3 found in [31] actually turns out to vanish in the setup of the former (constant dilaton over the internal space).

A parallel line of research [49, 45, 46, 31] studied the form of the on-shell gaugino-condensate action, as coming from the low-energy expansion of the DBI action. This is of interest, given that [28] shows how it could source IASD fluxes. In particular, due to the localized nature of the currents gaugino condensation introduces in the generalized complex geometry equations, the issue resides in the fact that the bulk + D7-branes action appears to contain delta functions (from bulk and branes) and square delta functions (from bulk fluxes only) centered on the D7-brane stack. This is both physically and mathematically problematic, since square delta functions are not well defined, and on-shell actions cannot really be divergent[56]. While the form of the $G_3\lambda\lambda$ coupling has

[56]Integrating out one delta function would leave the second one, making the action divergent. This is

been explicitly computed multiple times in the literature (see e.g. [49, 31]), the higher order quartic gaugino coupling from D7-branes is far harder to come by, with no real consensus about its precise form. Attempts at computing the quartic coupling can be found in [49], which performs an involved dimensional reduction, and in [46], which proposes a so-called *perfect square* form of the gaugino action, in order to conveniently cancel divergences.

The very recent work [50] brings together many of the lessons learned from the above literature. It argues that properly employing a *smearing* procedure of the D7-brane non-perturbative effect already proposed in [24, 26]

$$\delta_D^2 \mapsto \gamma\, \mathrm{e}^{2A-\phi} J, \tag{1.135}$$

where D is the 4-cycle wrapped by the D7-branes, A is the warp function, ϕ is the dilaton, γ is a constant and J is the internal Kähler form, one is actually able to reproduce the predictions of the effective four-dimensional superpotential of [15] from the modified fluxes and geometry. This justifies *a posteriori* the smearing procedure (1.135). Furthermore, it addresses the issues of possible divergences in the on-shell action, exhibiting its form and concluding that no perfect square form arises and that the complete on-shell action is indeed divergent, which in turn implies the need for local counterterms.

1.3.2 An explicit supergravity solution?

As we already mentioned, all of the above papers proceed assuming a non-vanishing VEV for the gaugino bilinear $\langle\lambda\lambda\rangle$ at low energies, and adopting a perturbative approach for the corrections it introduces. The origin of this VEV is purely four-dimensional, which is the very picture one is trying to reproduce in a ten-dimensional framework. Thus, the logic employed in these analysis is somewhat faulty, although very practical and reasonable. In an effort to get rid of this somewhat subtle inconsistency, [34] studies a specific type IIB *local* compactification where gaugino condensation does occur. More precisely, this paper finds an *explicit* family of supergravity solutions for type IIB string theory compactified on the simplest del Pezzo cone, i.e. the $\mathbb{P}^2$ cone $\mathcal{O}_{\mathbb{P}^2}(-3)$, including four D7-branes and one O7-plane[57] wrapping the $\mathbb{P}^2$ base of the cone, in a so-called *near-stack* limit, namely zooming close to the $\mathbb{P}^2$ base of the cone, where the D7-brane stack lies[58]. This solution is found solving the supersymmetry conditions for AdS_4 compactifications, which in the decompactification limit yield a Mink_4 vacuum, using the mathematical framework of generalized complex geometry. The central point, as the authors argue, is that this explicit solution should already encode the ten-dimensional effect of non-perturbative dynamics on 7-branes, without including any gauge theory four-dimensional input. Indeed, the proposed solution does carry a *dynamic $SU(2)$ structure*, as opposed to ordinary $SU(3)$ structures, which has been shown to be required

the reasoning, at least.

[57] Due to the presence of the O7-plane, one could also talk of eight half-D7-branes in the upstairs geometry brought down to four in the downstairs geometry.

[58] In this sense, the supergravity solution exhibited therein is doubly local.

by gaugino condensation on D7-branes (see e.g. [43]), and which is strictly related to generalized complex geometry.

This thesis studies the same model as the one considered in [34], both from a four-dimensional perspective (in chapter 3 and chapter 4) and from a ten-dimensional one (in chapter 5), but without taking any near-stack limit. Moreover, we do inject the four-dimensional information $\langle\lambda\lambda\rangle \neq 0$, which allows us to directly build upon the existing literature. Employing the results of [40, 27] for warped compactifications, we find the explicit form of the low-energy EFT for the chiral field associated to the $\mathbb{P}^2$ warped volume[59], and we show that gaugino condensation sources a runaway potential which pushes it to infinity. This implies that *type IIB compactifications on* $\mathcal{O}_{\mathbb{P}^2}(-3)$ *with four D7-branes and one O7-plane wrapped around* $\mathbb{P}^2$ *are unstable, and they cannot be regarded as a suitable playground to probe a ten-dimensional description of gaugino condensation on D7-branes.* In this sense, the family of supergravity solutions found in [34] cannot be interpreted as a near-stack limit of a putative stable type IIB supergravity solution on the whole $\mathbb{P}^2$ cone. We also move to ten-dimensions, and making use of the results of [31] we investigate possible leading order perturbations to the background metric. Performing an *ad hoc* trivialization procedure, we find the explicit global equations of motion for the leading order metric perturbation, and we show that they admit both stationary and time-dependent solutions. We argue that the former solution is related to the unstable vacuum found in the four-dimensional analysis, while a subclass of the latter can be associated with the $\mathbb{P}^2$ expansion phenomenon found in §4.

[59]In collaboration with prof. Luca Martucci.

Chapter 2

Geometry of the $\mathbb{P}^2$ cone

The aim of this chapter is to provide the necessary facts for the analysis of the following chapters. The topics are organized in order of increasing technicality. In order to derive the low-energy four-dimensional EFT for the $\mathbb{P}^2$ cone, we will follow the constructions of [40, 39, 27]. This requires us to lay some mathematical foundations for the geometry of the $\mathbb{P}^2$ cone, which make up the majority of the chapter. Moreover, the ten-dimensional analysis we will tackle next requires us to build upon some results of [31] specialized to this particular geometric playground. This prompts us to state some more results about delta functions and delta 2-forms on the $\mathbb{P}^2$ cone, which are found in §2.9.3.

In section 2.1 we provide some motivation as to why the complex cone over $\mathbb{P}^2$ appears to be a promising internal space candidate in order to study gaugino condensation in type IIB compactifications, and we provide a direct construction of it as a toric variety (i.e. via homogeneous coordinates). This serves as a basis for setting up an atlas. In section 2.2 we give a telegraphic presentation of the divisor group of the space, of the linear equivalences between its class representatives and of their intersection products. This hopefully provides an intuitive understanding of the space, as well as introducing intersection products as an effective tool to compute integrals, which we will use in the rest of the chapter. In section 2.3 we present the atlas we are going to use to describe the $\mathbb{P}^2$ cone. This is a crucial tool both for the four-dimensional analysis of chapters 3 and 4 and for the ten-dimensional analysis of chapter 5. In section 2.4 we present the isometry group of the space. This will be of prime relevance in the ten-dimensional analysis of chapter 5. Section 2.5 delves into the cohomology and homology groups both of the $\mathbb{P}^2$ cone and of the generic Kähler-Einstein del Pezzo cone. This is needed in order to work out the form of the Lagrangian for local del Pezzo models, as we do throughout §3. Section 2.6 reviews the asymptotic behavior of the Green's function associated to the Hodge-de Rham Laplacian, which completes the discussion of §3.2. In section 2.7 we introduce the Eguchi-Hanson metric for the $\mathbb{P}^2$ cone, and its associated Kähler form. This is a closed-form Ricci-flat and maximally symmetric metric for the space. It plays a minor role in the four-dimensional analysis of §3, while it covers a major one in the constructions and results of §5. Section 2.8 defines the Kähler modulus v of the space, and relates it explicitly to the $\mathbb{P}^2$ volume in the Eguchi-Hanson metric. This is a central quantity in §3, since it enters local del Pezzo models as a function of the chiral field ρ and of the rest of moduli. Finally, section 2.9 reviews three possible representatives for the single linearly independent cohomology class of 2-forms of the space, namely the primitive form dual to the $\mathbb{P}^2$, the Kähler form and the $\mathbb{P}^2$ delta 2-form. More precisely, we state their explicit form, their potential and some notable properties. The four-dimensional analysis in §3 makes use of §2.9.2 and partly of §2.9.1, while the ten-dimensional one in §5 makes full use of §2.9.3 and §2.9.1.

We suggest the reader to first go through the less technical sections from §2.1 to §2.4 (included), in order to get acquainted with the internal space structure, and to skip the rest of the chapter, jumping to §3. The rest of the mathematical notions and objects will be accompanied by references to this chapter, so that the reader can go back at its own pace. We have tried to keep this chapter short, leaving to appendix A some details and proofs, as well as some mathematical lore about the considered topics.

2.1 Motivation and construction

We are strictly interested in studying the effects of gaugino condensation occurring on a stack of D7-branes wrapped around an internal 4-cycle, from both a four-dimensional and a ten-dimensional point of view. A convenient class of internal manifolds fit for this purpose are complex cones over Kähler-Einstein del Pezzo surfaces M[1]. We will call these *local del Pezzo models*. One could generally think of these as some limit of a putative proper compactification, where one zooms in on the del Pezzo surface, which plays the role of the 4-cycle where we will wrap the stack of D7-branes. The infinite volume of the internal manifolds decouples four-dimensional gravity[2], which simplifies the ten-dimensional dynamics of the system. Moreover, gaugino condensation only occurs on coinciding D7-branes if the low-energy EFT supported on the stack is pure $\mathcal{N} = 1$ SYM, namely if the is no massless charged matter in the spectrum of the theory. Notably, the D7-brane deformation moduli can potentially break this condition if they stay massless at low energy, since they would enter the supersymmetric gauge theory as a Higgs branch, preventing gaugino condensation from taking place [15]. Complex cones over Kähler-Einstein del Pezzo surfaces avoid this problem, since the del Pezzo base is a rigid divisor of the geometry, namely it does not admit any holomorphic deformation. They also come with the very welcome feature of admitting no complex structure deformations, which would otherwise be promoted to a modulus of a string theory compactified on it, so that their complex structure is entirely fixed by the one of the underlying unresolved complex cone. This leads to a pure $\mathcal{N} = 1$ SYM theory supported on the D7-branes at low energies[3]. Finally, complex cones over Kähler-Einstein del Pezzo surfaces are (non-compact) Calabi-Yau manifolds, which is required in order to retain $\mathcal{N} = 1$ SUSY in the four-dimensional low-energy effective theory, presented in §3.

[1]More precisely, by complex cone over a Kähler-Einstein del Pezzo surface M we mean the canonical line bundle over M, denoted $\mathcal{K}_M$.

[2]See around (1.16).

[3]We discuss the gauge group of this theory in §4.1.

Up to biholomorphisms, *there are only eight Kähler-Einstein del Pezzo surfaces*, which are $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and the blowup of $\mathbb{P}^2$ at $3 \leq k \leq 8$ generic points, denoted dP_k [36]. In our notation, $\mathbb{P}^n$ denotes the $2n$-dimensional complex projective space. In this work we will study the local model compactified on the complex cone over $dP_0 = \mathbb{P}^2$.

The complex cone over $\mathbb{P}^2$ is defined as the canonical line bundle over $\mathbb{P}^2$, therefore our internal manifold of choice is[4]

$$X_0 := \mathcal{O}_{\mathbb{P}^2}(-3). \tag{2.1}$$

This can be obtained as the crepant[5] resolution of the orbifold $\mathbb{C}^3/\mathbb{Z}_3$, namely from blowing up the orbifold singularity at the origin[6]. Thus, X_0 is a line bundle over $\mathbb{P}^2$, endowed with a Calabi-Yau structure, and it can be effectively described as a toric variety[7] as follows. Consider the four complex coordinates $(Z^1, Z^2, Z^3, Z^4) \in \mathbb{C}^4$ charged under a single $U(1)$ with charge vector

$$Q = (1, 1, 1, -3) \tag{2.2}$$

and with FI parameter $\xi > 0$. The D-flatness condition arising from the corresponding gauged linear sigma model is

$$\left|Z^1\right|^2 + \left|Z^2\right|^2 + \left|Z^3\right|^2 - 3\left|Z^4\right|^2 = \xi, \qquad \xi > 0 \tag{2.3}$$

and the equivalence relation holding on the moduli space of the theory is

$$(Z^1, Z^2, Z^3, Z^4) \sim (\lambda Z^1, \lambda Z^2, \lambda Z^3, \lambda^{-3} Z^4), \qquad \lambda \in \mathbb{C}^\star. \tag{2.4}$$

The points of $\mathbb{C}^4$ that cannot be gauge-transformed with (2.4) into a solution of the D-flatness condition (2.3) make up the SR-ideal, which in this case is

$$\Xi_\xi = \left\{(Z^1, Z^2, Z^3, Z^4) \in \mathbb{C}^4 : Z^1 = Z^2 = Z^3 = 0\right\}. \tag{2.5}$$

Thus,

$$\mathcal{O}_{\mathbb{P}^2}(-3) = \frac{\mathbb{C}^4 \setminus \Xi_\xi}{\sim}. \tag{2.6}$$

Notice that choosing $\lambda = \mathrm{e}^{\frac{2\pi i}{3}}$ in (2.4), one gets

$$(Z^1, Z^2, Z^3, Z^4) \sim \left(\mathrm{e}^{\frac{2\pi i}{3}} Z^1, \mathrm{e}^{\frac{2\pi i}{3}} Z^2, \mathrm{e}^{\frac{2\pi i}{3}} Z^3, Z^4\right), \tag{2.7}$$

which shows that the action of $\mathbb{Z}_3$ in $\mathbb{C}^3/\mathbb{Z}_3$ is embedded in the action of the gauged $U(1)$, which is what we expect from resolving the orbifold. The local coordinates description will show how one retrieves $\mathbb{C}^3/\mathbb{Z}_3$ away from the resolved origin.

Finally, let us note that, since the complex cone over $\mathbb{P}^2$ is a negative line bundle, it admits no global holomorphic sections.

[4] Recall that $\mathcal{O}_{\mathbb{P}^2}(-1)$ is the tautological line bundle over $\mathbb{P}^2$ and that $\mathcal{O}_{\mathbb{P}^2}(-n)$ is its n-th tensor power.

[5] A crepant resolution preserves the canonical class of the singular space. A crepant resolution of a Calabi-Yau cone is still Calabi-Yau.

[6] The origin is a singular point for $\mathbb{C}^3/\mathbb{Z}_3$, since the deficit angle $\frac{4\pi}{3}$ from the $\mathbb{Z}_3$ quotient induces a delta-like curvature concentrated in the origin. Recall that the blowup operation consists in excising the orbifold singularity and replacing it with a finite-sized $\mathbb{P}^2$.

[7] Recall that toric varieties can be readily understood as the moduli space of supersymmetric gauged linear sigma models.

2.2 Divisors

A divisor $D = \sum_I n_I D^I$ of X is a formal sum of codimension one holomorphic submanifolds D^I with integer coefficients n_I [25]. The description of $\mathcal{O}_{\mathbb{P}^2}(-3)$ as a toric variety provides us with a natural complete set of divisors[8]:

$$D_\alpha = \left\{ (Z^1 : Z^2 : Z^3 : Z^4) \in \mathcal{O}_{\mathbb{P}^2}(-3) : Z^\alpha = 0 \right\}, \tag{2.8}$$

where $\alpha = 1, 2, 3, 4$. Let us denote $E := D_4$, and from (2.4) we easily see that E is biholomorphic to $\mathbb{P}^2$, namely this is the resolved divisor. Notice also from (2.4) that D_i, $i = 1, 2, 3$, are non-compact divisors, while E is compact.

Since $\mathcal{O}_{\mathbb{P}^2}(-3)$ is realized as a toric variety with just one $U(1)$, we expect only one linearly independent divisor class. Indeed, two possible gauge invariant and rational combinations of the homogeneous coordinates of $\mathcal{O}_{\mathbb{P}^2}(-3)$ are

$$\frac{Z^i}{Z^j}, \qquad Z^4 \left(Z^i\right)^3, \tag{2.9}$$

where $i, j = 1, 2, 3$. These are globally defined, hence they provide the following linear equivalences:

$$D_1 = D_2 = D_3, \qquad E = -3D_1. \tag{2.10}$$

This shows that there is indeed only one independent divisor class of $\mathcal{O}_{\mathbb{P}^2}(-3)$, which we can represent with the compact 4-cycle E.

2.2.1 Intersection products

From (2.5) one immediately sees that $D_1 \cap D_2 \cap D_3 = \emptyset$, namely

$$D_1 \cdot D_2 \cdot D_3 = 0. \tag{2.11}$$

From (2.10) it follows that $D_i^2 \cdot D_j = 0$ and $D_i^3 = 0$, $i, j = 1, 2, 3$. On the other hand, $D_1 \cap D_2 \cap E = \left\{ (Z^1 : Z^2 : Z^3 : Z^4) \in \mathbb{C}^4 : Z^1 = Z^2 = Z^4 = 0, \left|Z^3\right|^2 = \xi \right\}$, thus the D-flatness condition admits a single point solution up to $U(1)$ equivalences, namely

$$D_i \cdot D_j \cdot E = 1. \tag{2.12}$$

Using (2.10) one also finds

$$D_i \cdot E \cdot E = -3, \qquad E \cdot E \cdot E = 9. \tag{2.13}$$

One can also construct compact holomorphic curves like $C_1 = D_1 \cdot E$, and since $D_1 \cap E = \left\{ (Z^1 : Z^2 : Z^3 : Z^4) \in \mathbb{C}^4 : Z^1 = Z^4 = 0, \left|Z^2\right|^2 + \left|Z^3\right|^2 = \xi \right\} \simeq \mathbb{P}^1$, we see that $C_1 \simeq \mathbb{P}^1$ in $\mathbb{P}^2$ [9]. It follows that

$$C_i \cdot E = -3. \tag{2.14}$$

[8] The colon denotes homogeneous coordinates.

[9] For instance, take $(Z^2, Z^3) = \lambda(Z_0^2, Z_0^3)$, with $(Z_0^2, Z_0^3) \neq (0, 0)$ fixed and $\lambda \in \mathbb{C}^\star$, so we can interpret the D-flatness condition as a gauge fixing for $|\lambda|$, leaving $U(1)$ to be modded out. This is indeed $\mathbb{P}^1$.

Similarly, one can construct non-compact holomorphic curves like $\tilde{C}_1 = D_2 \cdot D_3 = \left\{(Z^1 : Z^2 : Z^3 : Z^4) \in \mathbb{C}^4 : Z^2 = Z^3 = 0, \left|Z^1\right|^2 - 3\left|Z^4\right|^2 = \xi\right\}$, with intersection product

$$\tilde{C}_i \cdot E = 1. \tag{2.15}$$

2.3 Local coordinates

From the SR-ideal (2.5) it follows that an open covering of $\mathcal{O}_{\mathbb{P}^2}(-3)$ is given by

$$\mathcal{U}_{(i)} = \left\{(Z^1 : Z^2 : Z^3 : Z^4) \in \mathcal{O}_{\mathbb{P}^2}(-3) : Z^i \neq 0\right\} \tag{2.16}$$

for $i = 1, 2, 3$. One can also define the open set

$$\mathcal{U}_{(4)} = \left\{(Z^1 : Z^2 : Z^3 : Z^4) \in \mathcal{O}_{\mathbb{P}^2}(-3) : Z^4 \neq 0\right\}, \tag{2.17}$$

and one immediately sees that $\mathcal{U}_{(4)} \subset \bigcup_{i=1}^{3} \mathcal{U}_{(i)}$.

On $\mathcal{U}_{(i)}$ we define the following local coordinates:

$$\begin{aligned} u^a_{(i)} &= \frac{Z^a}{Z^i} \qquad a = 1, 2, 3, \quad a \neq i \\ \xi_{(i)} &= Z^4 \left(Z^i\right)^3 \end{aligned} \tag{2.18}$$

These coordinates are completely invariant under (2.4). On $\mathcal{U}_{(4)}$ we define the following local coordinates:

$$z^i = Z^i \left(Z^4\right)^{\frac{1}{3}}. \tag{2.19}$$

Notice that a residual $\mathbb{Z}_3$ invariance is left from (2.4), acting on these coordinates as

$$\mathbb{Z}_3 : z^i \mapsto \mathrm{e}^{\frac{2\pi i}{3}} z^i. \tag{2.20}$$

A $\mathbb{Z}_3$-invariant combination of these coordinates is

$$r^2 = z^i \overline{z}_i. \tag{2.21}$$

Let us stress that these coordinates are defined for $r^2 \neq 0$. This shows that $\mathcal{U}_{(4)} \simeq \mathbb{C}^{\star\,3}/\mathbb{Z}_3$, as we expect from this resolved orbifold. The change of coordinates on $\mathcal{U}_{(i)} \cap \mathcal{U}_{(j)}$ is

$$\begin{aligned} u^a_{(j)} &= \frac{u^a_{(i)}}{u^j_{(i)}} \qquad a \neq i, j \\ u^i_{(j)} &= \frac{1}{u^j_{(i)}} \\ \xi_{(j)} &= \left(u^j_{(i)}\right)^3 \xi_{(i)} \end{aligned} \tag{2.22}$$

The change of coordinates on $\mathcal{U}_{(4)} \cap \mathcal{U}_{(i)}$ is

$$\begin{aligned} z^j &= u^j_{(i)} \xi^{\frac{1}{3}}_{(i)} \qquad j \neq i \\ z^i &= \xi^{\frac{1}{3}}_{(i)} \\ r^2 &= \left|\xi_{(i)}\right|^{\frac{2}{3}} (1 + \rho^2_{(i)}), \qquad \rho^2_{(i)} = u^a_{(i)} \overline{u}_{(i)a} \end{aligned} \tag{2.23}$$

with inverse

$$\begin{aligned} u^a_{(i)} &= \frac{z^a}{z^i} \\ \xi_{(i)} &= \left(z^i\right)^3 \end{aligned} \tag{2.24}$$

Note that from (2.22) one sees that on $\mathcal{U}_{(i)} \cap \mathcal{U}_{(j)}$ it holds

$$1 + \rho^2_{(j)} = \frac{1 + \rho^2_{(i)}}{\left|u^j_{(i)}\right|^2}. \tag{2.25}$$

This shows that the combination $r^2 = \left|\xi_{(i)}\right|^{\frac{2}{3}} (1 + \rho^2_{(i)})$ is indeed globally defined.

Notice that the resolved divisor $E \simeq \mathbb{P}^2$ can be covered with the open sets $\mathcal{U}_{(i)} \cap E$, $i = 1, 2, 3$, and with local coordinates $\left(u^a_{(i)}\right)$ over each one of these. Therefore, on $\mathcal{U}_{(i)}$ the $\left(u^a_{(i)}\right)$ describe the point on the base of the cone, while $\xi_{(i)}$ describes the point on the fiber above it. Moreover, since $\mathbb{P}^2 \simeq E = \left\{Z^4 = 0\right\}$, $\mathcal{U}_{(4)}$ does not contain the base of the cone, which would correspond to $r^2 = 0$. Indeed, extending the r^2 coordinate to $\mathcal{U}_{(i)}$ as

$$r^2 = \left|\xi_{(i)}\right|^{\frac{2}{3}} (1 + \rho^2_{(i)}), \tag{2.26}$$

it holds $\mathbb{P}^2 \cap \mathcal{U}_{(i)} \simeq \left\{r^2 = 0\right\}$, i.e. $\mathbb{P}^2$ can be locally described as the zero of the holomorphic section

$$h_{(i)}\left(\xi_{(i)}\right) = \frac{1}{\ell_s^3} \xi_{(i)}, \tag{2.27}$$

where the string length scale has been added to make this dimensionless. As already stated, since $\mathcal{O}_{\mathbb{P}^2}(-3)$ is a negative line bundle, it admits no global holomorphic sections, which makes the task of extending (2.27) hopeless.

2.4 Maximal symmetry

Recall that $\mathcal{O}_{\mathbb{P}^2}(-3)$ is obtained as crepant resolution of the orbifold $\mathbb{C}^3/\mathbb{Z}_3$, effectively blowing up the orbifold singularity at $z^i = 0$ to a finite-sized $\mathbb{P}^2$ (which is therefore the exceptional divisor of the resolution). The maximal symmetry of X_0 is inherited from that of the singular orbifold, as we now explain. This is a continuous symmetry, which is not in contrast with X_0 being CY, thanks to the fact that it is non-compact. When endowed with the euclidean metric, the (holomorphic) isometry group of $\mathbb{C}^3$ is $U(3)$, acting in the defining representation on its coordinates (z^i). Upon $\mathbb{Z}_3$ identification, the isometry group of $\mathbb{C}^3/\mathbb{Z}_3$ becomes $U(3)/\mathbb{Z}_3$. Once the resolution of the singular point is performed, X_0 still admits a local chart with the $\mathbb{C}^3/\mathbb{Z}_3$ coordinates z^i, defined for $r^2 \neq 0$. Therefore, in a natural way X_0 admits as maximal symmetry the group $U(3)/\mathbb{Z}_3$, whose action is defined by its defining representation when acting on the (z^i) coordinates. Since the actions of $U(3)$ and of $U(3)/\mathbb{Z}_3$ on X_0 are the same, it is equivalent to state that the maximal symmetry of X_0 is $U(3)$.

In order to better understand how $U(3)$ acts on the internal geometry, it is useful to consider the well known decomposition

$$U(3) \simeq \frac{SU(3) \times U(1)}{\mathbb{Z}_3} \simeq PU(3) \times U(1), \tag{2.28}$$

where $PU(3) \simeq U(3)/U(1) \simeq SU(3)/\mathbb{Z}_3$ is the projective unitary group. The actions of $SU(3)$ and $PU(3)$ on $\mathbb{P}^2$ are equivalent. Therefore, the holomorphic isometry group of X_0 can be decomposed as[10]

$$U(3)/\mathbb{Z}_3 \simeq PU(3) \times U(1)/\mathbb{Z}_3. \tag{2.29}$$

Since $PU(3)$ is the holomorphic isometry group of $\mathbb{P}^2$, this elegantly suggests that the action of $U(3)/\mathbb{Z}_3$ on $X_0 = \mathcal{O}_{\mathbb{P}^2}(-3)$ corresponds to $SU(3)$ acting in the defining representation on the homogeneous coordinates of $\mathbb{P}^2$, and to the action of $U(1)/\mathbb{Z}_3$ on the (z^i) given by $z^i \mapsto \mathrm{e}^{\frac{i\alpha}{3}} z^i$, with $\alpha \sim \alpha + 2\pi$. Thus, $U(3)$ consists of the isometries of the base of the complex cone together with the rotations of the fiber above it, as one would expect.

The action of $U(3)$ can even be defined on the homogeneous coordinates of $\mathcal{O}_{\mathbb{P}^2}(-3)$ directly. For $U(3)$ to act in the defining representation on the (z^i), one easily sees from (2.19) that it must act of the (Z^i) in the defining representation, while Z^4 needs to be a singlet:

$$U(3) : \begin{cases} Z^i \longmapsto U^i{}_j Z^j \\ Z^4 \longmapsto Z^4 \end{cases} \tag{2.30}$$

where $U \in U(3)$. This shows that the exceptional divisor $\mathbb{P}^2$ is $U(3)$-invariant. Similarly, the $U(1)$ factor in (2.28) acts as

$$U(1) : \begin{cases} Z^i \longmapsto \mathrm{e}^{i\alpha} Z^i \\ Z^4 \longmapsto Z^4 \end{cases} , \qquad \alpha \in \mathbb{R}. \tag{2.31}$$

2.5 Topology

In this section, we review the absolute and relative cohomologies of the $\mathbb{P}^2$ complex cone, and their relation, which we make use of in §3.8. We then extend these results to the

[10]Distribution of the quotient on the direct product factors is legitimate since $\mathbb{Z}_3$ can be see as a normal subgroup of $SU(3)$ or $U(1)$.

generic Kähler-Einstein del Pezzo cones, which play a crucial role in the construction of local del Pezzo models, found in §3. An introduction to short exact sequences and relative cohomology is found in the appendices A.1 and A.2.

2.5.1 $\mathbb{P}^2$ cone

Recall the Betti numbers $b_p(X_0)$ are defined as the dimension of $H^p(X_0;\mathbb{R})$. The Hodge diamond of $\mathcal{O}_{\mathbb{P}^2}(-3)$ is given by

$$\begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 0 & & 0 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{matrix} \tag{2.32}$$

Being non-compact, $\mathcal{O}_{\mathbb{P}^2}(-3)$ has an exact volume form, which is why $b_6(X_0) = 0$[11]. Since $\mathcal{O}_{\mathbb{P}^2}(-3)$ is connected, $b_0(X_0) = 1$. The boundary of $\mathcal{O}_{\mathbb{P}^2}(-3)$ is clearly the same as the boundary of the unresolved orbifold $\mathbb{C}^3/\mathbb{Z}_3$, namely

$$Y_0 := \partial X_0 = S^5/\mathbb{Z}_3. \tag{2.33}$$

This is a regular five-dimensional Sasaki-Einstein manifold, namely it can be described as a $U(1)$ fibration over a Kähler-Einstein del Pezzo surface[12]. Thus, $\mathcal{O}_{\mathbb{P}^2}(-3)$ can be described both as a complex cone over $\mathbb{P}^2$ or as the resolution of a real Calabi-Yau cone over Y_0, denoted $C(Y_0) = \mathbb{R}_{>0} \times Y_0$, with metric

$$\mathrm{d}s^2_{C(Y_0)} = \mathrm{d}r^2 + r^2 \mathrm{d}s^2_{Y_0}. \tag{2.34}$$

Thus, the orbifold $\mathbb{C}^3/\mathbb{Z}_3$ can also be described as a Calabi-Yau cone over Y_0

$$\mathbb{C}^3/\mathbb{Z}_3 \simeq C(S^5/\mathbb{Z}_3). \tag{2.35}$$

The resolved real Calabi-Yau cone structure implies that [20]

$$b_1(X_0) = b_5(X_0) = 0. \tag{2.36}$$

The Calabi-Yau holomorphic 3-form $\Omega \in H^{3,0}(X_0;\mathbb{R})$ is exact[13]. The only non-trivial cohomology groups[14] are the $H^{1,1}(X_0;\mathbb{R})$, which is generated by the $(1,1)$-harmonic form

[11] This can also be justified using Poincaré duality: $H^6(X_0;\mathbb{R}) \simeq H_0(X_0, Y_0;\mathbb{R}) = H_0^c(X_0;\mathbb{R}) = 0$, where the last equality is due to the fact that the only connected component of X_0 is non-compact.

[12] Indeed, notice that in our case Y_0 is then a $U(1)$ fibration over $\mathbb{P}^2$ up to a $\mathbb{Z}_3$ identification, which is isomorphic to $S^5/\mathbb{Z}_3$ by the Hopf fibration.

[13] On a non-compact Kähler manifold, harmonic forms can be exact and non-vanishing. This is the case for Ω.

[14] For an introduction to relative cohomology, see appendix A.

ω dual to the only compact divisor $E \simeq \mathbb{P}^2$, and $H^{2,2}(X_0;\mathbb{R})$, the Poincaré-dual group to $H_2(X_0,Y_0;\mathbb{R})$.

As for the topology of the boundary, it has the same homology of a 5-sphere up to torsion[15]. Therefore,

$$b_i(Y_0) = 0 \qquad i = 1,\ldots,4 \tag{2.38a}$$

$$b_0(Y_0) = b_5(Y_0) = 1 \tag{2.38b}$$

and[16]

$$H_1(Y_0;\mathbb{Z}) = H_3(Y_0;\mathbb{Z}) = \mathbb{Z}_3. \tag{2.39}$$

The volume form of S^5 generates the top-dimensional cohomology group of Y_0.

Applying (A.8) to X_0 provides us with an important relation between the non-trivial homology and cohomology groups of the space. Thanks to $b_1(Y_0) = b_3(Y_0) = 0$, (A.8) implies

$$0 \longrightarrow H^2(X_0,Y_0;\mathbb{R}) \longrightarrow H^2(X_0;\mathbb{R}) \longrightarrow 0, \tag{2.40}$$

which is equivalent to

$$H^2(X_0,Y_0;\mathbb{R}) \simeq H^2(X_0;\mathbb{R}), \tag{2.41}$$

which implies in turn

$$b_2(X_0) = b_4(X_0). \tag{2.42}$$

Dualizing (2.40) yields the homology analogous

$$H_2(X_0;\mathbb{R}) \simeq H_2(X_0,Y_0;\mathbb{R}). \tag{2.43}$$

On the other hand, applying (non-compact) Poincaré duality to (2.40) yields

$$H_4(X_0;\mathbb{R}) \simeq H_4(X_0,Y_0;\mathbb{R}), \tag{2.44}$$

with cohomology analogous

$$H^4(X_0,Y_0;\mathbb{R}) \simeq H^4(X_0;\mathbb{R}). \tag{2.45}$$

The isomorphisms (2.40, 2.43, 2.44, 2.45) admit a natural generalization into short exact sequences in the generic del Pezzo cone case. The sequence (2.40) and its dualizations

[15] Recall that n-th homology group of a topological space X, with coefficients in $\mathbb{Z}$, by the universal coefficient theorem has the general form

$$H_n(X;\mathbb{Z}) = \left(\bigoplus_{i=1}^{b_n(X)} \mathbb{Z}\right) \oplus \left(\bigoplus_{j=1}^{m} G_j\right), \tag{2.37}$$

where $m \in \mathbb{Z}_{\geq 0}$ and G_j are (finite) cyclic groups. The discrete part of $H_n(X;\mathbb{Z})$ makes up its torsion.

[16] Thanks to the $\mathbb{Z}_3$ quotient, closed curves on $S^5/\mathbb{Z}_3$ come in three flavors: those that would also be closed on the 5-sphere, and those whose extrema are identified under the $\mathbb{Z}_3$ action. Take a point $p \in S^5$, let p_k, $k = 1,2,3$, the three images of p under $\mathbb{Z}_3$, and consider a curve with one extremum in p, then choosing the other extremum to be one of the three p_k yields three inequivalent cycles in $S^5/\mathbb{Z}_3$. Therefore $H_1(Y_0;\mathbb{Z}) = \mathbb{Z}_3$.

do not hold in the case of (co)homologies with integral coefficients, due to presence of torsion terms arising from (2.39). Therefore, we expect cyclic groups $\mathbb{Z}_3$ to show up in place of the 0's. In particular, we expect $H^2(X_0;\mathbb{Z})$ and $H^4(X_0;\mathbb{Z})$ to be isomorphic up to cyclic factors, namely

$$H^2_{free}(X_0;\mathbb{Z}) \simeq H^4_{free}(X_0;\mathbb{Z}), \tag{2.46}$$

where

$$H^p_{free}(X_0;\mathbb{Z}) = H^p(X_0;\mathbb{Z})/\mathrm{Tor}\, H^p(X_0;\mathbb{Z}). \tag{2.47}$$

Except for the Kähler form $J \in H^{1,1}(X_0;\mathbb{R})$ and the holomorphic 3-form $\Omega \in H^{3,0}(X_0;\mathbb{R})$, due to Dirac's quantization condition, we will work with integrally quantized forms and cycles, namely elements of $H^p(\bullet;\mathbb{Z})$ and $H_p(\bullet;\mathbb{Z})$. See §3.4 for more details on this choice.

The non-compact divisors D_i define the same relative homology class $[D] := [D_i] \in H_4(X_0, Y_0;\mathbb{Z})$[17]. Any one of their boundaries $\Sigma_i := \partial D_i$ is non-zero, and they represent the same homology class $[\Sigma] := [\Sigma_i]$ generating $H_3(Y_0;\mathbb{Z}) = \mathbb{Z}_3$. This implies that

$$3[\Sigma] = 0, \tag{2.48}$$

which is consistent with the previously derived linear equivalence (2.10).

One can also show that the resolved divisor E is not Spin, see appendix A.

2.5.2 General del Pezzo cone

Notice that (2.44) means that the $\mathbb{P}^2$ cone does not admit non-compact divisor classes[18], essentially due to the fact that $b_3(Y_0) = 0$, that is due to the particularly simple horizon topology (2.38). This fact can equivalently be seen as a consequence of $b_2(\mathbb{P}^2) = 1$[19]. Indeed, $H_2(\mathbb{P}^2;\mathbb{Z})$ is generated by the single class of curves $[C] = [C_i]$ we introduced in §2.2. The restriction of X_0 over C_i is the divisor D_i, which is linearly equivalent to E, thus showing that there are no non-compact divisor classes.

This does not hold for the other complex cones over Kähler-Einstein del Pezzo surfaces $M = \mathbb{P}^1 \times \mathbb{P}^1, dP_k$, where $k = 0, 3, \ldots, 8$. Let us denote these, with their respective horizon, by

$$X = \mathcal{K}_M; \qquad Y = \partial\mathcal{K}_M. \tag{2.50}$$

This is motivated by the fact that X can be constructed as the canonical bundle of M. The generic Kähler-Einstein del Pezzo cone has non-compact divisor classes. The precise

[17] Recall that, by non-compact Poincaré duality, $H_4(X_0, Y_0;\mathbb{Z}) \simeq H^2(X_0;\mathbb{Z})$.

[18] By non-compact divisor classes we mean divisor classes without compact representatives.

[19] In general, it holds

$$H_p(\mathbb{P}^p;\mathbb{Z}) = \begin{cases} \mathbb{Z} & p \text{ even} \\ 0 & \text{otherwise} \end{cases}. \tag{2.49}$$

result for the general case is

$$b_4(\mathcal{K}_M) = 1 \tag{2.51a}$$

$$b_2(\mathcal{K}_M) = \begin{cases} 1+1 & \text{if } M = \mathbb{P}^1 \times \mathbb{P}^1 \\ 1+k & \text{if } M = dP_k,\ k = 0, 3, \dots, 8 \end{cases} \tag{2.51b}$$

$$b_3(\partial\mathcal{K}_M) = \begin{cases} 1 & \text{if } M = \mathbb{P}^1 \times \mathbb{P}^1 \\ k & \text{if } M = dP_k,\ k = 0, 3, \dots, 8 \end{cases} \tag{2.51c}$$

$$b_i(\mathcal{K}_M) = 0 \qquad i = 1, 3, 5, 6. \tag{2.51d}$$

Here (2.51a) shows that del Pezzo cones always admit only one compact divisor class, while (2.51c) shows that different bases of the cone lead to different horizon topologies. This result can be understood from the Betti numbers of the base:

$$b_2(M) = \begin{cases} 1+1 & \text{if } M = \mathbb{P}^1 \times \mathbb{P}^1 \\ 1+k & \text{if } M = dP_k,\ k = 0, 3, \dots, 8 \end{cases} \tag{2.52}$$

Indeed, $\mathbb{P}^1 \times \mathbb{P}^1$ admits two independent 2-cycles $\mathcal{C}_1$ and $\mathcal{C}_2$, the two $\mathbb{P}^1$'s, while dP_k admits $1+k$ 2-cycles $\mathcal{C}, \mathcal{E}_\sigma$, $\sigma = 1, \dots, k$, where $\mathcal{C}$ is the pullback of the generator of $H_2(\mathbb{P}^2; \mathbb{Z})$ under the blowdown map $\pi : dP_k \to \mathbb{P}^2$, and $\mathcal{E}_\sigma$ are the exceptional curves associated to the k blowups in $\mathbb{P}^2$. One can then introduce a basis of independent divisors for $H_4(\mathcal{K}_M, \partial\mathcal{K}_M; \mathbb{Z})$ by restricting $\mathcal{K}_M$ over each of these curves.

Let us emphasize that a remarkable and characteristic feature of del Pezzo cones is (2.51d). Thanks to $b_3(X) = 0$, string theories compactified on a del Pezzo cone *do not admit complex structure moduli*, which greatly simplifies their low-energy dynamics[20]. Moreover, it makes it possible to write down short exact sequences which generalize the isomorphisms (2.40, 2.43, 2.44, 2.45) for the generic del Pezzo cone:

$$0 \longrightarrow H^2(X, Y; \mathbb{R}) \longrightarrow H^2(X; \mathbb{R}) \longrightarrow H^2(Y; \mathbb{R}) \longrightarrow 0; \tag{2.53a}$$

$$0 \longrightarrow H_2(Y; \mathbb{R}) \longrightarrow H_2(X; \mathbb{R}) \longrightarrow H_2(X, Y; \mathbb{R}) \longrightarrow 0; \tag{2.53b}$$

$$0 \longrightarrow H_4(X; \mathbb{R}) \longrightarrow H_4(X, Y; \mathbb{R}) \longrightarrow H_3(Y; \mathbb{R}) \longrightarrow 0; \tag{2.53c}$$

$$0 \longrightarrow H^3(Y; \mathbb{R}) \longrightarrow H^4(X, Y; \mathbb{R}) \longrightarrow H^4(X; \mathbb{R}) \longrightarrow 0. \tag{2.53d}$$

Using (A.4), these imply

$$b_2(X) = b_4(X) + b_3(Y), \tag{2.54}$$

which generalizes (2.42). Notice that (2.51c) follows from (2.51b) and (2.51a) thanks to (2.54). In this sense, $b_3(Y_0) = 0$ is the reason why the topology of $\mathcal{K}_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-3)$ is the simplest.

[20] Thanks to $b_3(X) = 0$, Kähler-Einstein del Pezzo cones fall into the classes of compactification backgrounds analyzed in [27].

2.6 Hodge-de Rham Green's function

Let us consider the Green's function $G(y; y_0)$ for the Hodge-de Rham operator associated with the Ricci-flat metric on the generic del Pezzo cone $\mathcal{K}_M$. These are of interest when one studies solutions for the warp factor equation of motion (3.8). They are defined by the equation

$$\Delta G(y; y_0) = \ell_s^4\, \delta(y - y_0), \tag{2.55}$$

where $y_0 \in \mathcal{K}_M$ is fixed, where we introduced a string length factor in order to make the Green's function dimensionless, and where we introduced the Hodge-de Rham Laplacian for the Eguchi-Hanson metric $\Delta = -\nabla^m \nabla_m$ (its action on general p-forms is given by (A.58)). They are symmetric, i.e. they obey [40]

$$G(y; y_I) = G(y_I; y). \tag{2.56}$$

They do not admit a closed form expression, but their asymptotic behavior far away from the $\mathbb{P}^2$ base is determined by the Green's functions on the unresolved cone. Thanks to $\mathcal{K}_M$ being the resolution of a real Calabi-Yau cone $C(Y)$ over a Sasaki-Einstein 5-fold Y, the metric of $\mathcal{K}_M$ asymptotically has form (2.34). In components, this is

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 g_{Y\,ab} \end{pmatrix} \tag{2.57}$$

with inverse

$$g^{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} g_Y{}^{ab} \end{pmatrix} \tag{2.58}$$

Using (2.57) and the generic action of the Hodge-de Rham operator (A.58) we find that its action on $C(Y)$ is given by

$$\Delta_{C(Y)} f(y) = \left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r^2} \Delta_Y \right) f(y). \tag{2.59}$$

Therefore, one easily sees that the Green's function on the real cone $C(Y)$ is given by[21]

$$G_c(y; y_0) = \frac{1}{4\mathrm{Vol}(Y)} \frac{\ell_s^4}{r^4}, \tag{2.61}$$

where $r^2 = (y - y_0)^m (y - y_0)^n \delta_{mn}$. The Green's function $G(y; y_0)$ of $\mathcal{K}_M$ then is asymptotically described by the Green's function on $C(Y)$:

$$G(y; y_0) \sim_\infty \frac{1}{4\mathrm{Vol}(Y)} \frac{\ell_s^4}{r^4}, \tag{2.62}$$

where $r^2 = y^m y^n \delta_{mn}$ in the $r^2 \sim \infty$ limit.

[21] Indeed, given $\varphi \in \mathcal{S}(\mathbb{R}^6)$,

$$\begin{aligned} \langle \Delta \left(\frac{1}{4\mathrm{Vol}(Y)} \frac{1}{r^4} \right) | \varphi \rangle &= \frac{1}{4\mathrm{Vol}(Y)}\, 4 \int r^5 \mathrm{d}r\, \mathrm{dVol}(Y) \left(\frac{\mathrm{d}}{\mathrm{d}r} \frac{1}{r^5} \right) \varphi \\ := -\frac{1}{\mathrm{Vol}(Y)} \int r^5 \mathrm{d}r\, \mathrm{dVol}(Y) \frac{1}{r^5} \frac{\mathrm{d}}{\mathrm{d}r} \varphi &= \varphi(0). \end{aligned} \tag{2.60}$$

2.7 Metric and Kähler form

Finding an explicit Kähler and Ricci-flat metric for a compact Calabi-Yau space is a notoriously difficult problem. In fact, currently none is known [37]. This is mainly due to the fact that compact CY manifolds do not admit continuous isometries, allowing at most discrete symmetries. On the other hand, non-compact CY manifolds do allow for continuous isometries, which can greatly simplify the form of the Khaler potential. Thanks to $\mathcal{O}_{\mathbb{P}^2}(-3)$ being non-compact, it is fairly easy to write down an explicit Ricci-flat metric for it assuming maximal symmetry, see §A.6. The maximally symmetric Ricci-flat metric for the resolved $\mathbb{C}^3/\mathbb{Z}_3$ is the so-called Eguchi-Hanson geometry [6]. In the $\mathcal{U}_{(4)}$ patch, it takes the form[22]

$$\mathrm{d}s^2_{X_0} = \left(1+\frac{c^6}{r^6}\right)^{\frac{1}{3}} \left(\delta_{i\bar{\jmath}} - \frac{c^6}{r^6}\left(1+\frac{c^6}{r^6}\right)^{-1} \frac{\bar{z}_i z_{\bar{\jmath}}}{r^2}\right) \mathrm{d}z^i \mathrm{d}\bar{z}^{\bar{\jmath}} \tag{2.63}$$

where $c \in \mathbb{R}$ is a constant controlling the volume of the resolved $\mathbb{P}^2$, and we use the notation

$$z_{\bar{\imath}} = \delta_{\bar{\imath}j} z^j. \tag{2.64}$$

Setting $c = 0$ one recovers the singular orbifold metric. For general $c > 0$, this is a one-parameter family of metrics, and any physical application thereof requires c to be eventually fixed to a certain positive value. Notice that this metric is flat in the limit $r^2 \to \infty$, and together with the residual $\mathbb{Z}_3$ action acting on the (z^i) this shows that $\partial X_0 \simeq S^5/\mathbb{Z}_3$. The metric components in (z^i) coordinates are

$$g_{i\bar{\jmath}} = \left(1+\frac{c^6}{r^6}\right)^{\frac{1}{3}} \left(\delta_{i\bar{\jmath}} - \frac{c^6}{r^6}\left(1+\frac{c^6}{r^6}\right)^{-1} \frac{\bar{z}_i z_{\bar{\jmath}}}{r^2}\right). \tag{2.65}$$

The singularity in $r^2 = 0$ is just a coordinate singularity, and the metric is regular also on the resolved $\mathbb{P}^2$ passing to the coordinates $\left(\xi_{(i)}, u^a_{(i)}\right)$. By construction, this satisfies

$$\det(g_{i\bar{\jmath}}) = 1, \tag{2.66}$$

or equivalently $\sqrt{g_6} = 8$, where $g_6 = \det(g_{mn})$. Its inverse is easily found to be

$$g^{i\bar{\jmath}} = \left(1+\frac{c^6}{r^6}\right)^{-\frac{1}{3}} \left(\delta^{i\bar{\jmath}} + \frac{c^6}{r^6}\frac{z^i \bar{z}^{\bar{\jmath}}}{r^2}\right). \tag{2.67}$$

The Kähler form on $\mathcal{U}_{(4)}$ is then given by

$$\begin{aligned} J &= i g_{i\bar{\jmath}} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{\jmath}} \\ &= i\left(1+\frac{c^6}{r^6}\right)^{\frac{1}{3}} \left(\delta_{i\bar{\jmath}} - \frac{c^6}{r^6}\left(1+\frac{c^6}{r^6}\right)^{-1} \frac{\bar{z}_i z_{\bar{\jmath}}}{r^2}\right) \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{\jmath}} \end{aligned} \tag{2.68}$$

[22] Notice this is manifestly $U(3)$-symmetric, thanks to the fact that $\mathcal{U}_{(4)} = \{r^2 \neq 0\}$ is.

Thanks to the Kähler structure, one can also write this as

$$J = i\left(C(r^2)\delta_{i\bar{\jmath}} + C'(r^2)\bar{z}_i z_{\bar{\jmath}}\right) \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{\jmath}} \tag{2.69}$$

where

$$C(r^2) = \left(1 + \frac{c^6}{r^6}\right)^{\frac{1}{3}}. \tag{2.70}$$

This can be recast in terms of the Kähler potential on $\mathcal{U}_{(4)}$ as

$$J = \frac{1}{2}\mathrm{d}\mathrm{d}^c \mathcal{C}_{(4)}(r^2) \tag{2.71}$$

where[23]

$$\mathrm{d}^c = i\left(\bar{\partial} - \partial\right), \tag{2.72}$$

and the Kähler potential on $\mathcal{U}_{(4)}$ is given by.

$$\mathcal{C}_{(4)}(r^2) = \int_\star^{r^2} C(y)\mathrm{d}y. \tag{2.73}$$

Its explicit expression is

$$\begin{aligned}\mathcal{C}_{(4)}(r^2) = c^2\left(1 + \frac{r^6}{c^6}\right)^{\frac{1}{3}} &+ \\ -\frac{c^2}{6}\left[2\sqrt{3}\,\mathrm{arctg}\left(\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}}\left(1 + \frac{r^6}{c^6}\right)^{\frac{1}{3}}\right) + 3\log\frac{\frac{r^2}{c^2}}{\left(1 + \frac{r^6}{c^6}\right)^{\frac{1}{3}} - 1}\right] &+ const.\end{aligned} \tag{2.74}$$

Notice that this potential exhibits a singularity in $r^2 = 0$, with asymptotic behavior

$$\mathcal{C}_{(4)}(r^2) \sim_0 c^2 \log(r^2). \tag{2.75}$$

On the other hand, J is regular for $r^2 = 0$ just like the metric is, therefore the patching rules of the potential need to get rid of this singularity. More precisely, we can choose the patching rules for the Kähler potential as follows:

$$\mathcal{C}_{(i)} = \mathcal{C}_{(4)} - \frac{c^2}{3}\log\left|\xi_{(i)}\right|^2 \qquad on \quad \mathcal{U}_{(i)} \cap \mathcal{U}_{(4)} \tag{2.76a}$$

$$\mathcal{C}_{(i)} = \mathcal{C}_{(j)} - c^2\log\left|u^i_{(j)}\right|^2 \qquad on \quad \mathcal{U}_{(i)} \cap \mathcal{U}_{(j)} \tag{2.76b}$$

Indeed, using the global definition of r^2 (2.26) and setting $r^2 = 0$ in (2.76b) using (2.74), we find

$$\mathcal{C}_{(i)}|_{\mathbb{P}^2} = c^2 \log\left(1 + \rho^2_{(i)}\right), \tag{2.77}$$

[23]Notice that $\frac{1}{2}\mathrm{d}\mathrm{d}^c = i\partial\bar{\partial}$.

which is the Kähler potential for the Fubini-Study metric[24]. This shows that on $\mathcal{U}_{(i)} \cap \mathcal{U}_{(4)}$,

$$J|_{\mathbb{P}^2} = c^2 J_{FS} = c^2 \, \frac{i}{1+\rho^2_{(i)}} \left(\delta_{a\bar{b}} - \frac{\overline{u}_{(i)a} u_{(i)\bar{b}}}{1+\rho^2_{(i)}} \right) \mathrm{d}u^a_{(i)} \wedge \mathrm{d}\overline{u}^{\bar{b}}_{(i)}, \tag{2.78}$$

and that (2.63) induces the Fubini-Study metric on $\mathbb{P}^2$:

$$\mathrm{d}s^2_{\mathbb{P}^2} = c^2 \mathrm{d}s^2_{FS} = \frac{c^2}{1+\rho^2_{(i)}} \left(\delta_{a\bar{b}} - \frac{\overline{u}_{(i)a} u_{(i)\bar{b}}}{1+\rho^2_{(i)}} \right) \mathrm{d}u^a_{(i)} \mathrm{d}\overline{u}^{\bar{b}}_{(i)}. \tag{2.79}$$

From a direct computation, one can find the explicit form of the metric and the Kähler form on $\mathcal{U}_{(i)}$, in the (ξ, u^a) coordinates (dropping the pedix ${}_{(i)}$), starting from (2.63) and (2.68) and applying the change of coordinates. The Kähler form becomes

$$J = \left(c^6 + r^6\right)^{\frac{1}{3}} J_{FS} + \frac{i}{3} \frac{\left(1+\rho^2\right)^2}{\left(c^6+r^6\right)^{\frac{2}{3}}} \left[\overline{u}_a \, \xi \, \mathrm{d}u^a \wedge \mathrm{d}\overline{\xi} - u_{\overline{a}} \, \overline{\xi} \, \mathrm{d}\overline{u}^{\overline{a}} \wedge \mathrm{d}\xi + \frac{1}{3}(1+\rho^2) \mathrm{d}\xi \wedge \mathrm{d}\overline{\xi} \right], \tag{2.80}$$

where we used (2.26) and the change of coordinates identities

$$\mathrm{d}z^i \wedge \mathrm{d}\overline{z}_i = |\xi|^{\frac{2}{3}} \left[\mathrm{d}u^a \wedge \mathrm{d}\overline{u}_a + \frac{1}{3} \overline{u}_a \, \mathrm{d}u^a \wedge \frac{\mathrm{d}\overline{\xi}}{\overline{\xi}} - \frac{1}{3} u_{\overline{a}} \, \mathrm{d}\overline{u}^{\overline{a}} \wedge \frac{\mathrm{d}\xi}{\xi} + \frac{1}{9}(1+\rho^2) \frac{\mathrm{d}\xi}{\xi} \wedge \frac{\mathrm{d}\overline{\xi}}{\overline{\xi}} \right] \tag{2.81a}$$

$$\overline{z}_i \mathrm{d}z^i = |\xi|^{\frac{2}{3}} \left[\overline{u}_a \mathrm{d}u^a + \frac{1}{3}(1+\rho^2) \frac{\mathrm{d}\xi}{\xi} \right] \tag{2.81b}$$

Notice that (2.80) is indeed regular in $r^2 = 0$, and it implies (2.78) once this is pulled-back onto $\mathbb{P}^2$. Then, the metric in these coordinates immediately follows:

$$\mathrm{d}s^2_{X_0} = \left(c^6 + r^6\right)^{\frac{1}{3}} \mathrm{d}s^2_{FS} + \frac{1}{3} \frac{\left(1+\rho^2\right)^2}{\left(c^6+r^6\right)^{\frac{2}{3}}} \left[\overline{u}_a \, \xi \, \mathrm{d}u^a \, \mathrm{d}\overline{\xi} + u_{\overline{a}} \, \overline{\xi} \, \mathrm{d}\overline{u}^{\overline{a}} \, \mathrm{d}\xi + \frac{1}{3}(1+\rho^2) \, \mathrm{d}\xi \, \mathrm{d}\overline{\xi} \right]. \tag{2.82}$$

In passing, notice that the Kähler form and the metric evaluated on $\xi = 0$, but without taking the pullback to $\mathbb{P}^2$, are of the form

$$J|_{\xi=0} = c^2 J_{FS} + \frac{i}{9\,c^4}(1+\rho^2)^3 \mathrm{d}\xi \wedge \mathrm{d}\overline{\xi} \tag{2.83a}$$

$$\mathrm{d}s^2_{X_0}|_{\xi=0} = c^2 \mathrm{d}s^2_{FS} + \frac{1}{9\,c^4}(1+\rho^2)^3 \mathrm{d}\xi \, \mathrm{d}\overline{\xi} \tag{2.83b}$$

which shows that they are of block-diagonal form: the fiber legs do not mix with the base legs on the resolved divisor, making ∂_ξ and ∂_{u^a} orthogonal on the $\mathbb{P}^2$. Explicitly, the metric components at $\xi = 0$ are

$$g_{\alpha\overline{\beta}}|_{\xi=0} = \begin{pmatrix} g_{a\overline{b}}|_{\xi=0} & 0 \\ 0 & g_{\xi\overline{\xi}}|_{\xi=0} \end{pmatrix} \tag{2.84}$$

[24]Eduard Study (/ˈʃtuːdi/ SHTOO-dee), more properly Christian Hugo Eduard Study (23 March 1862 - 6 January 1930), was a German mathematician.

where $\alpha, \beta \in \{a, \xi\}$, and

$$g_{a\bar{b}}|_{\xi=0} = \frac{c^2}{1+\rho^2}\left(\delta_{a\bar{b}} - \frac{\overline{u}_a u_{\overline{b}}}{1+\rho^2}\right) \tag{2.85a}$$

$$g_{\xi\bar{\xi}}|_{\xi=0} = \frac{1}{9c^4}(1+\rho^2)^3 \tag{2.85b}$$

The block-diagonal form makes it easier to compute the inverse metric at $\xi = 0$:

$$g^{a\bar{b}}|_{\xi=0} = \frac{1+\rho^2}{c^2}\left(\delta^{a\bar{b}} + u^a \overline{u}^{\bar{b}}\right) \tag{2.86a}$$

$$g^{\xi\bar{\xi}}|_{\xi=0} = \frac{9c^4}{(1+\rho^2)^3} \tag{2.86b}$$

Finally, notice that in the limit $r^2 \to \infty$ (and taking the pullback to ∂X_0) we get

$$J|_{\partial X_0} = i\,\mathrm{d}z^i \wedge \mathrm{d}\overline{z}_i \neq 0 \tag{2.87}$$

which shows that this is not a compactly supported 2-form, so that $J \in H^2(X_0;\mathbb{R})$.

2.8 Kähler modulus

Let us introduce a parametrization for the Kähler cone. Since $b_2(X_0) = 1$, there is only one linearly independent $(1,1)$-cohomology class of $H^2(X_0, Y_0;\mathbb{Z})$, which can be represented by its Poincaré-dual homology class $[E] \in H_4(X_0;\mathbb{Z})$[25]. Therefore, we can choose to decompose J is this basis, i.e.

$$[J] = v[E] \tag{2.88}$$

where $v \in \mathbb{R}$ is our choice of parametrization for the only (uncomplexified) Kähler modulus of the model. Admittedly, by an abuse of notation, here we denote a cohomology class by its Poincaré-dual. The Kähler cone condition states that for any 2-homology class with holomorphic curve representative C it holds

$$\int_C J > 0. \tag{2.89}$$

Therefore, choosing $C = C_i = D_i \cdot E$ we can compute explicitly (2.89)[26]:

$$\begin{aligned}\int_{C_i} J &= ic^2 \int_{\mathbb{C}} \left(1 - \frac{|y|^2}{1+|y|^2}\right) \frac{\mathrm{d}y \wedge \mathrm{d}\overline{y}}{1+|y|^2} \\ &= 4\pi c^2 \int_0^\infty \frac{r\mathrm{d}r}{(1+r^2)^2} = 2\pi c^2\end{aligned} \tag{2.90}$$

[25] Recall that Poincaré duality in this non-compact context involves relative (co)homology and take the form $H^n_{free}(X_0;\mathbb{Z}) \simeq H^{free}_{6-n}(X_0, Y_0;\mathbb{Z})$, $H^n_{free}(X_0, Y_0;\mathbb{Z}) \simeq H^{free}_{6-n}(X_0;\mathbb{Z})$, where *free* singles out the non-torsion component of the (co)homology group. See [14] and [27] for more details. Since $b_3(Y_0) = 0$, it holds $H^2(X_0;\mathbb{Z}) \simeq H^2(X_0, Y_0;\mathbb{Z})$ up to cyclic factors.

[26] Recall $\mathrm{d}y \wedge \mathrm{d}\overline{y} = -2i\,\mathrm{d}\mathrm{Re}(y) \wedge \mathrm{d}\mathrm{Im}(y)$.

which justifies the choice $c \in \mathbb{R}$ as a consequence of the Kähler cone condition. On the other hand, using (2.14)

$$\int_{C_i} J = v(C_i \cdot E) = -3v, \tag{2.91}$$

which shows that

$$v = -\frac{2\pi c^2}{3}. \tag{2.92}$$

Thus, the Kähler cone condition forces our Kähler modulus to be negative,

$$v < 0. \tag{2.93}$$

Let us give an intuitive understanding to this quantity. The volume of the resolved divisor, being a holomorphic codimension one submanifold, can be computed by[27]

$$\mathrm{Vol}\left(\mathbb{P}^2\right) = \int_E \frac{1}{2} J \wedge J = \frac{v^2}{2}(E \cdot E \cdot E) = \frac{9v^2}{2} = 2\pi^2 c^4, \tag{2.97}$$

where we used (2.88). This shows how the Kähler modulus, or alternatively c^2, controls the volume of the resolved divisor.

2.9 Harmonic forms and other representatives

The Calabi-Yau structure of $\mathcal{O}_{\mathbb{P}^2}(-3)$ is equivalent to the existence of a holomorphic $(3,0)$-form Ω, which is harmonic and covariantly constant. Being the resolution of the orbifold $\mathbb{C}^3/\mathbb{Z}_3$, X_0 inherits the same CY structure as $\mathbb{C}^3$ on $\mathcal{U}_{(4)}$, namely

$$\Omega = \mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3. \tag{2.98}$$

Due to the non-compactness of X_0, this is exact and non-vanishing despite being harmonic. Moreover, the pullback of Ω to Y_0 does not vanish, thus $\Omega \in H^{3,0}(X_0;\mathbb{R})$.

From $b_2(\mathcal{O}_{\mathbb{P}^2}(-3)) = 1$ we know that there is only one linearly independent harmonic form in $H^2(X_0, Y_0; \mathbb{Z})$. There are, however, a few notable representatives of these linearly dependent cohomology classes, which we are now going to review.

[27] Referring to §2.9, notice that

$$\mathrm{Vol}\left(\mathbb{P}^2\right) = \int_E \frac{1}{2} J \wedge J \neq \int_{X_0} \frac{1}{2} J \wedge J \wedge \omega = 0, \tag{2.94}$$

by primitivity of ω, contrary to the naive expectation. Indeed, care has to be applied when dealing with Poincaré duality and integrals of non-compactly supported forms on compact divisors. In this case, this is due to (2.124), so that the correct relation is

$$\int_E \frac{1}{2} J \wedge J = \int_{X_0} \frac{1}{2} (J \wedge J)|_c \wedge \omega = \frac{9v^2}{2} = \mathrm{Vol}(\mathbb{P}^2), \tag{2.95}$$

where $(J \wedge J)_c = v^2\, \omega \wedge \omega$ is the compactly supported part of $J \wedge J$. An alternate definition is

$$\int_{X_0} (J \wedge J)_c \wedge \omega = \int_{X_0} J \wedge J \wedge \delta^2_{\mathbb{P}^2}. \tag{2.96}$$

2.9.1 Primitive 2-form

There is a unique primitive representative for the cohomology class dual[28] to the resolved divisor $\mathbb{P}^2$. This is given by[29]

$$\omega = \frac{1}{2}\mathrm{d}\left(A(r^2)\mathrm{d}^c r^2\right) = \frac{1}{2}\mathrm{d}\mathrm{d}^c\kappa_{(4)}(r^2), \tag{2.99}$$

where

$$A(r^2) = -\frac{3}{2\pi}\frac{1}{r^2\left(1+\frac{r^6}{c^6}\right)^{\frac{2}{3}}}, \tag{2.100}$$

and where we introduced the local potential $\kappa_{(4)}$, which on $\mathcal{U}_{(4)}$ is given by

$$\kappa_{(4)}(r^2) = -\int_{r^2}^{\infty} A(y)\mathrm{d}y = \frac{3}{2\pi}\int_{\frac{r^2}{c^2}}^{\infty}\frac{1}{u\left(1+u^3\right)^{\frac{2}{3}}}\mathrm{d}u. \tag{2.101}$$

This can be computed explicitly:

$$\kappa_{(4)} = \frac{1}{4\pi}\left[2\sqrt{3}\,\mathrm{arctg}\left(\frac{1}{\sqrt{3}}+\frac{2}{\sqrt{3}}\left(1+\frac{r^6}{c^6}\right)^{\frac{1}{3}}\right)+3\log\frac{r^2}{(c^6+r^6)^{\frac{1}{3}}-c^6}\right]-\frac{\sqrt{3}}{4}. \tag{2.102}$$

The expression of the form in local coordinates (z^i) on $\mathcal{U}_{(4)}$ is

$$\begin{aligned}\omega &= i(A(r^2)\delta_{i\bar{\jmath}} + A'(r^2)\bar{z}_i z_{\bar{\jmath}})\mathrm{d}z^i\wedge \mathrm{d}\bar{z}^{\bar{\jmath}}\\ &= i\left(A(r^2)\partial\bar{\partial}r^2 + A'(r^2)\partial r^2\wedge\bar{\partial}r^2\right),\end{aligned} \tag{2.103}$$

where $A(r^2)$ is given by (2.100). By primitivity, it also holds

$$J\wedge J\wedge\omega = 0. \tag{2.104}$$

Notice that in the limit $r^2\to\infty$, $\omega|_{Y_0} = 0$ [30], which shows that this is a compactly supported form, $\omega\in H^2(X_0,Y_0;\mathbb{Z})$. This is clearly singular in $r^2=0$, as a consequence of the singularity of these coordinates on $\mathbb{P}^2$, with asymptotic behavior

$$\kappa_{(4)} \sim_0 -\frac{3}{2\pi}\log r^2. \tag{2.105}$$

However, ω is regular on the resolved $\mathbb{P}^2$, since we can cancel this singular behavior by means of patching rules similar to (2.76a, 2.76b):

$$\kappa_{(i)} = \kappa_{(4)} + \frac{1}{2\pi}\log\left|\xi_{(i)}\right|^2 \qquad on \quad \mathcal{U}_{(i)}\cap\mathcal{U}_{(4)}; \tag{2.106a}$$

$$\kappa_{(i)} = \kappa_{(j)} + \frac{3}{2\pi}\log\left|u^i_{(j)}\right|^2 \qquad on \quad \mathcal{U}_{(i)}\cap\mathcal{U}_{(j)}. \tag{2.106b}$$

[28]The Poincaré-dual of a 4-cycle on X_0 is necessarily a 2-form. It needs to be a $(1,1)$-form because $\mathbb{P}^2$ is a divisor.
[29]See §A.5.1 for a derivation.
[30]This is true regardless of the pullback to Y_0 being taken or not.

Notice that, from these patching rules and (2.100) we read

$$\kappa_{(i)}|_{\mathbb{P}^2} = -\frac{3}{2\pi}\log\left(1+\rho^2_{(i)}\right), \tag{2.107}$$

which is proportional to the Fubini-Study potential. In other terms, we found that

$$\omega|_{\mathbb{P}^2} = -\frac{3}{2\pi c^2} J|_{\mathbb{P}^2}. \tag{2.108}$$

This should not surprise us. Being both harmonic $(1,1)$-forms in a 1-dimensional cohomology group, they should be proportional, up to exact terms due to the non-compactness of X_0[31]. Either way, this is enough to conclude that

$$[\omega] = -\frac{3}{2\pi c^2}[J]. \tag{2.109}$$

This is consistent with (2.92). We will come back to the precise relation between the two forms.

One can compute the explicit form of ω on $\mathcal{U}_{(i)}$ in the coordinates (ξ, u^a) starting from (2.103) and using the change of coordinates (2.23). One finds

$$\begin{aligned}\omega = &- i\,\frac{3}{2\pi}\frac{1}{\left(1+\frac{r^6}{c^6}\right)^{\frac{2}{3}}}\Bigg\{J_{FS} - \frac{2}{c^6}\,|\xi|^2\,\frac{1+\rho^2}{1+\frac{r^6}{c^6}}\,\overline{u}_a\,u_{\overline{b}}\,\mathrm{d}u^a\wedge\mathrm{d}\overline{u}^{\overline{b}} + \\ &- \frac{2}{3\,c^6}\frac{(1+\rho^2)^2}{1+\frac{r^6}{c^6}}\left[\overline{\xi}\,\mathrm{d}\xi\wedge u^a\mathrm{d}\overline{u}_a + \xi\,\overline{u}^{\overline{a}}\,\mathrm{d}u_{\overline{a}}\wedge\mathrm{d}\overline{\xi} + \frac{1}{3}(1+\rho^2)\,\mathrm{d}\xi\wedge\mathrm{d}\overline{\xi}\right]\Bigg\},\end{aligned} \tag{2.110}$$

where we used the change of coordinates identities (2.81), recalling that $\partial\overline{\partial}r^2 = \mathrm{d}z^i\wedge\mathrm{d}\overline{z}_i$ and $\partial r^2 = \overline{z}_i\mathrm{d}z^i$. In principle, the local potential $\kappa_{(i)}$ could be directly derived integrating (2.110), but the patching rules (2.106a, 2.106b) are a faster option.

Let us stress that, by definition of Poincaré-dual, the 2-form ω satisfies

$$\int_E \alpha_4 = \int_{X_0}\alpha_4\wedge\omega \tag{2.111}$$

for every every compactly supported $\alpha_4 \in H^4(X_0, Y_0;\mathbb{R})$. Thanks to (2.122), this also implies that

$$\int_{X_0}\omega\wedge\omega\wedge\omega = \int_E \omega\wedge\omega = 9; \tag{2.112a}$$

$$\int_{X_0} J\wedge\omega\wedge\omega = \int_E J\wedge\omega = \int_{E\cdot E} J = -3\int_C J = -6\pi c^2. \tag{2.112b}$$

[31]Co-exact terms are technically allowed, as but will not show up. Notice that in order to enter a closed form, a co-exact term $\mathrm{d}^\dagger\lambda_{p+1}$ is constrained to satisfy $\mathrm{d}\mathrm{d}^\dagger\lambda_{p+1} = 0$.

2.9.2 Kähler form

The Kähler form of a Kähler manifold M is always harmonic, since it is covariantly constant and

$$\mathrm{d}^\dagger \alpha_p = -\frac{1}{(p-1)!}\nabla^k \alpha_{km_1\ldots m_{p-1}} \mathrm{d}x^{m_1} \wedge \ldots \wedge \mathrm{d}^{m_{p-1}}, \tag{2.113}$$

in the convention $\mathrm{d}^\dagger = (-1)^{D(p+1)+1} \star \mathrm{d}\star$. Being $H^{1,1}(X_0, Y_0; \mathbb{R})$ one-dimensional, this implies that it must be proportional to the other harmonic form we found ω, up to exact terms, namely

$$J = J_0 + v\,\omega, \tag{2.114}$$

where J_0 is exact and $v \in \mathbb{R}$. The presence of an exact term is actually to be expected, since $\frac{1}{v}J \in H^2(X_0;\mathbb{Z})$, while $\omega \in H^2(X_0, Y_0; \mathbb{Z})$, and since the absolute and the relative cohomologies of $\mathcal{O}_{\mathbb{P}^2}(-3)$ are isomorphic only up to $\mathbb{Z}_3$ identifications. This is consistent with (2.88) and (2.109), with $v = -\frac{2\pi c^2}{3}$. Let us check this relation by direct computation. Using (2.71) and (2.101), we find

$$J_0 = \frac{1}{2}\mathrm{d}\mathrm{d}^c \left(\mathcal{C}_{(4)} - v\kappa_{(4)}\right) =: \frac{1}{2}\mathrm{d}\mathrm{d}^c k_0, \tag{2.115}$$

so that the local potential for J_0 is given by

$$\begin{aligned} k_0(r^2) &= \mathcal{C}_{(4)}(r^2) - v\kappa_{(4)}(r^2) = \int_\star^{r^2} \left(1 + \frac{c^6}{y^3}\right)^{-\frac{2}{3}} \mathrm{d}y \\ &= c^2\left(1+\frac{r^6}{c^6}\right)^{\frac{1}{3}}, \end{aligned} \tag{2.116}$$

where we choose $\star$ such that the constant additive term is set to zero. This is for later convenience. In particular, it is consistent with the gauge fixing choice (3.50a) with respect to the c^2 dependence of the potential, see §3.8 for the details. From the patching rules (2.76a, 2.76b, 2.106a, 2.106b) we see that this is actually globally defined, which makes J_0 exact, as expected. Notice also that this is regular in $r^2 = 0$, like J and ω. Therefore, J_0 on $\mathcal{U}(4)$ can be recast as

$$J_0 = i\left(D(r^2)\delta_{i\bar{\jmath}} + D'(r^2)\bar{z}_i z_{\bar{\jmath}}\right) \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{\jmath}} \tag{2.117}$$

where

$$D(r^2) = \left(1 + \frac{c^6}{r^6}\right)^{-\frac{2}{3}}. \tag{2.118}$$

Thus, (2.114) implies that the Eguchi-Hanson Kähler potential can be recast as

$$\mathcal{C} = k_0 + v\kappa. \tag{2.119}$$

Moreover, by direct computation[32]

$$J_0 \wedge \omega \wedge \omega = 8\,\frac{\mathrm{d}}{\mathrm{d}r^2}\left[r^6(A^2(r^2)D(r^2))\right] \mathrm{d}r^2 \wedge \mathrm{d}\Omega_5 \tag{2.120}$$

[32] Recall we use the real orientation $\mathrm{d}^6 y = \bigwedge_{i=1}^3 \mathrm{d}u^i \wedge \mathrm{d}v^i$, given complex coordinates y^i, $i = 1, 2, 3$, and their real representation $y^j = u^j + iv^j$.

thus, using (2.118) and (2.100), we find[33]

$$\int_{X_0} J_0 \wedge \omega \wedge \omega = 0. \tag{2.122}$$

This also shows that, using (2.104),

$$\int_{X_0} J_0 \wedge J_0 \wedge \omega = -v^2 \int_{X_0} \omega^3 = -9v^2. \tag{2.123}$$

Notice, in particular, that

$$\int_{X_0} J_0 \wedge J_0 \wedge \omega \neq \int_E J_0 \wedge J_0 = 0, \tag{2.124}$$

where we used

$$J_0|_E = 0, \tag{2.125}$$

which is implied by (2.108).

2.9.3 Delta 2-form for $\mathbb{P}^2$

The delta 2-form associated with the base $\mathbb{P}^2$ of the complex cone satisfies the characterizing condition

$$\int_{X_0} \omega_4 \wedge \delta^{2}_{\mathbb{P}^2} = \int_{\mathbb{P}^2} \omega_4 \tag{2.126}$$

for *any* 4-form ω_4 globally defined on X_0.

In the patch $\mathcal{U}_{(i)} = \{Z^i \neq 0\}$, $i = 1, 2, 3$, with coordinates $(\xi_{(i)}, u^a_{(i)})$, $a = 1, 2$, $\mathbb{P}^2$ is locally described by $\operatorname{Re}\xi_{(i)} = \operatorname{Im}\xi_{(i)} = 0$. Using equation (3.23) from [25], we find

$$\delta^{2}_{\mathbb{P}^2} = \frac{i}{2}\delta(\xi_{(i)})\mathrm{d}\xi_{(i)} \wedge \mathrm{d}\bar{\xi}_{(i)}, \tag{2.127}$$

where

$$\delta(\xi) = \delta(\operatorname{Re}\xi)\delta(\operatorname{Im}\xi). \tag{2.128}$$

Globally, this form can be written as

$$\delta^{2}_{\mathbb{P}^2} = \frac{1}{2}\mathrm{d}\mathrm{d}^c\mathcal{P} \tag{2.129}$$

[33]Explicitly,

$$\begin{aligned}\int_{X_0} J_0 \wedge \omega \wedge \omega &= 8\int_{X_0} \frac{\mathrm{d}}{\mathrm{d}r^2}\left[r^6(A^2(r^2)D(r^2))\right]\mathrm{d}r^2 \wedge \mathrm{d}\Omega_5 \\ &= \frac{18}{\pi^2}\operatorname{Vol}(S^5/\mathbb{Z}_3)\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}r^2}\left(\frac{r^6}{c^4}\left(1+\frac{r^6}{c^6}\right)^{-2}\right)\mathrm{d}r^2 \\ &= \frac{18}{\pi^2}\operatorname{Vol}(S^5/\mathbb{Z}_3)\left(\frac{r^6}{c^6}\left(1+\frac{r^6}{c^4}\right)^{-2}\right)\Bigg|_0^\infty = 0.\end{aligned} \tag{2.121}$$

where d^c is defined in (2.72), and $\mathcal{P}$ is the local potential defined by

$$\begin{aligned} \mathcal{P}_{(i)} &= \frac{1}{2\pi}\log\left|\xi_{(i)}\right|^2 \qquad on \quad \mathcal{U}_{(i)} \\ \mathcal{P}_{(4)} &= const \in \mathbb{R} \qquad on \quad \mathcal{U}_{(4)} \end{aligned} \tag{2.130}$$

This can be checked using the well-known identities[34]

$$\partial_\xi \frac{1}{\bar{\xi}} = \partial_{\bar{\xi}} \frac{1}{\xi} = \pi\delta(\xi). \tag{2.132}$$

The patching rules for this potential are

$$\mathcal{P}_{(i)} = \mathcal{P}_{(j)} + \frac{3}{2\pi}\log\left|u^i_{(j)}\right|^2 \qquad on \quad \mathcal{U}_{(i)} \cap \mathcal{U}_{(j)} \tag{2.133a}$$

$$\mathcal{P}_{(i)} = \mathcal{P}_{(4)} + \frac{1}{2\pi}\log\left|\xi_{(i)}\right|^2 \qquad on \quad \mathcal{U}_{(i)} \cap \mathcal{U}_{(4)} \tag{2.133b}$$

In particular, we see that $\delta^2_{\mathbb{P}^2} = 0$ on $\mathcal{U}_{(4)}$, which is expected since $\mathbb{P}^2 \cap \mathcal{U}_{(4)} = \varnothing$. The patching rules follow from the changes of coordinates (2.23, 2.22).

By definition, $\delta^2_{\mathbb{P}^2}$ is a non-harmonic[35] $(1,1)$-form belonging to the cohomology class dual to the resolved $\mathbb{P}^2$, whose primitive representative is ω. Moreover, since $\delta^2_{\mathbb{P}^2}$ is compactly supported (its support is $\mathbb{P}^2$ itself), $\delta^2_{\mathbb{P}^2} \in H^2(X_0, Y_0; \mathbb{Z})$. Therefore, there must exist a globally defined (non-closed) 1-form Λ_1 such that

$$\omega = \delta^2_{\mathbb{P}^2} + \mathrm{d}\Lambda_1. \tag{2.134}$$

This is indeed the case. Just by looking at the patching rules (2.106a, 2.106b, 2.133a, 2.133b) one sees that the combination of potentials

$$\kappa - \mathcal{P} = \kappa_{(4)}(r^2) \tag{2.135}$$

is globally defined, with a singularity in $r^2 = 0$, and it yields the globally defined 1-form

$$\Lambda_1 = \frac{1}{2}\mathrm{d}^c\kappa_{(4)}, \tag{2.136}$$

[34] This is straightforward using Cauchy's formula and Stoke's theorem. Let γ be a closed curve centered at the origin of $\mathbb{C}$, D^1 the surface enclosed by γ, then

$$\begin{aligned} \int_{D^1} \delta(z)\,\mathrm{d}x \wedge \mathrm{d}y = 1 &= \int_{\gamma=\partial D^1} \frac{\mathrm{d}z}{2\pi i}\frac{1}{z} = \int_{D^1} \mathrm{d}\left(\frac{1}{z}\right) \wedge \frac{\mathrm{d}z}{2\pi i} \\ &= \int_{D^1} \bar{\partial}\left(\frac{1}{z}\right) \wedge \frac{\mathrm{d}z}{2\pi i} = \int_{D^1} \frac{1}{2\pi i}\left(\frac{\partial}{\partial \bar{z}}\frac{1}{z}\right)\mathrm{d}\bar{z} \wedge \mathrm{d}z \\ &= \frac{1}{\pi}\int_{D^1} \frac{\partial}{\partial \bar{z}}\left(\frac{1}{z}\right)\mathrm{d}x \wedge \mathrm{d}y, \end{aligned} \tag{2.131}$$

where $z = x + iy$. This is consistent with equation (5.33) of [8].

[35] If it were harmonic, then it would have to be proportional to the primitive form ω, i.e. $\delta^2_{\mathbb{P}^2} = a\,\omega$. Since $\delta^2_{\mathbb{P}^2}$ is compactly supported, there can be no further exact terms. This implies that $\delta^2_{\mathbb{P}^2}$ is primitive, which is a contradiction since $\iota_J\delta^2_{\mathbb{P}^2} \propto \delta(r^2) \neq 0$, where r^2 is understood as globally defined.

which is singular on $\mathbb{P}^2$, and does satisfy (2.134). Notice that this solution is not unique, but it is defined up to closed 1-forms. Since $b_1(X_0) = 0$, all closed 1-forms are exact, therefore the general solution to (2.134) is

$$\tilde{\Lambda}_1 = \Lambda_1 + \mathrm{d}f\left(z^i, \overline{z}^{\overline{\imath}}\right), \tag{2.137}$$

where $f(z^i, \overline{z}^{\overline{\imath}})$ is regular in $r^2 = 0$.

Scalar delta function for $\mathbb{P}^2$

Let us define a scalar delta-function localized on $\mathbb{P}^2$ starting from the delta 2-form:

$$\delta^{(0)}_{\mathbb{P}^2} = J \lrcorner \delta^2_{\mathbb{P}^2}. \tag{2.138}$$

Recal the definition of interior product for 2-forms $\alpha_2 \lrcorner \beta_2 = \frac{1}{2}\alpha^{mn}\beta_{mn}$. In §A.5.3 we show that

$$\delta^{(0)}_{\mathbb{P}^2} = \frac{9}{2}\frac{c^4}{\left(1+\rho^2\right)^3}\,\delta(\xi) \qquad on \quad \mathcal{U}_{(i)}. \tag{2.139}$$

Despite its appearance, this is globally defined. One can make this manifest with the following equivalent representation:

$$\delta^{(0)}_{\mathbb{P}^2} = \frac{1}{2}\Delta\kappa_{(4)} = \frac{9}{2\pi}\,c^4\,\delta(r^6), \tag{2.140}$$

where Δ is the Hodge-de Rham Laplacian for the Eguchi-Hanson metric (2.63), whose action of $U(3)$-symmetric functions is given by

$$\Delta f(r^2) = -\frac{2}{r^4}\left(c^6+r^6\right)^{-\frac{1}{3}}\left[r^2(c^6+r^6)\frac{\mathrm{d}^2}{\mathrm{d}(r^2)^2} + (c^6+3r^6)\frac{\mathrm{d}}{\mathrm{d}r^2}\right] f(r^2). \tag{2.141}$$

Chapter 3

Local del Pezzo models

In this chapter we present the four-dimensional low-energy $\mathcal{N} = 1$ SUSY EFT for the moduli fields associated to the local warped compactifications of weakly coupled type IIB string theory on the general complex cone $X = \mathcal{K}_M$ over a Kähler-Einstein del Pezzo surface M, with horizon $Y = \partial X$. We presented the topology of $\mathcal{K}_M$ in §2.5.2. We do so by adapting the construction of [40], which is crucially based in turn on the results of [39], to Kähler-Einstein del Pezzo cones, and keeping the universal modulus a non-vanishing, in order to retrieve Mink_4 externally. We later specialize to the $M = \mathbb{P}^2$ cone case. We will neglect non-perturbative corrections to the EFT, and we will adopt the supergravity approximation. The entirety of this chapter has been obtained in collaboration with prof. Luca Martucci, and has been based on unpublished notes.

In section 3.1 we state the ten-dimensional supergravity background of local del Pezzo models, possibly including N D3-branes. Section 3.2 reviews how the universal modulus enters warped type IIB compactifications, and identifies it in the present model. In §3.3 reviews some relevant mathematical notions about Kähler-Einstein del Pezzo cones, which are crucial in order to define the moduli of the low-energy four-dimensional EFT. In particular, we define flat form potentials, and we introduce a basis of harmonic 2-forms which allows us to describe the relevant flat deformations of the background. In §3.4 we use results from [40, 27] to define the moduli fields of the background, and its non-dynamical marginal parameters. In §3.5 we define the local potentials associated to some relevant 2-forms of the constructions, which enter the low-energy EFT directly. Section 3.6 we use some results from [39, 40] to introduce a chiral parametrization of the Kähler modulus of the background, which we introduced in §3.4. In §3.7 we exhibit the effective Lagrangian of local del Pezzo models, and we find its Kähler potential. Finally, §3.8 specializes this construction to the special case of the $\mathbb{P}^2$ cone, the simplest del Pezzo cone.

3.1 Supergravity background

Let us consider warped type IIB backgrounds in the Einstein frame of the form

$$\mathrm{d}s_{10}^2 = \mathrm{e}^{2A}\mathrm{d}s_{\mathbb{R}^{1,3}}^2 + \mathrm{e}^{-2A}\mathrm{d}s_X^2 \tag{3.1}$$

where $\mathrm{d}s_{\mathbb{R}^{1,3}}^2$ is the four-dimensional Minkowski metric, $X = \mathcal{K}_M$ is the complex cone over a Kähler-Einstein del Pezzo surface M (see §2.5.2), and $\mathrm{d}s_X^2$ is a Ricci-flat Kähler metric for X. Due to $b_3(X) = 0$ (see §2.5.2), the complex structure of the del Pezzo cone is completely fixed by the one of the underlying singular Calabi-Yau cone $C(Y)$.

We include no D7-branes in the compactification, so that the axio-dilaton $\tau = C_0 + i\mathrm{e}^{-\phi}$ is constant on the internal manifold, and we assume

$$\frac{1}{g_s} = \operatorname{Im}\tau \gg 1, \tag{3.2}$$

in order to attain a weak coupling regime for the string theory. We will allow, however, for the presence of N spacetime-filling D3-branes.

The brane content implies that the fluxes $F_1 = \mathrm{d}C_0$ and $G_3 = F_3 - \tau H_3$ are classically vanishing,

$$F_1 = G_3 = 0 \qquad \text{classically.} \tag{3.3}$$

On the other hand, F_5 is non-vanishing due to a non-trivial warp factor. From (1.66), we have

$$F_5 = (1 + \star_{10})\, \mathrm{d}\mathrm{e}^{4A(y)} \wedge \mathrm{d}^4 x. \tag{3.4}$$

Notice that (3.4) satisfies the correct quantization condition[1]:

$$\int_Y F_5 = -\ell_s^4 N. \tag{3.6}$$

Let us point out that the tadpole cancellation condition in this non-compact context does not provide any further constraint as long as (3.3) holds. Indeed, integrating both sides of (1.72) over X, due to the fact that X *does* have a boundary Y, we get by Stokes' theorem

$$\int_Y F_5 + \ell_s^4\, N = 0, \tag{3.7}$$

where we used $\int_X \rho_3^{\text{loc}} = N$. Then, the quantization condition (3.6) exactly ensures that this holds.

[1] Indeed, from $\star_{10}\left(\mathrm{d}\mathrm{e}^{4A(y)} \wedge \mathrm{d}^4 x\right) = \star_X \mathrm{d}\mathrm{e}^{-4A(y)}$ we see $i_Y^\star F_5 = \star_X \mathrm{d}\mathrm{e}^{-4A(y)}$. Then, using (3.13) and the fact $\star_X \mathrm{d}r = r^5 \mathrm{dVol}(Y)$, we find

$$\int_Y F_5 = \int_Y \star_X \mathrm{d}\mathrm{e}^{-4A(y)} = -4\mathrm{Vol}(Y) R^4 = -\ell_s^4 N. \tag{3.5}$$

In this setting, the only source to warping are D3-branes, according to the equation of motion for the warp factor [40]

$$\Delta \mathrm{e}^{-4A} = \ell_s^4 \star_X \sum_{I=1}^{N} \delta_I^6, \tag{3.8}$$

where $\star_X$ is the Hodge star with respect to the unwarped internal metric, δ_I^6 are delta 6-forms localized at the positions $Z_I \in X$ of the D3-branes[2], and for dimensional reasons we introduced the string length, which we define as

$$\ell_s = 2\pi\sqrt{\alpha'}, \tag{3.9}$$

where α' is the Regge slope parameter, which controls the stringy perturbative expansion.

3.2 Universal modulus

In unwarped compactifications without fluxes, the universal modulus parametrizes rescalings of the Kähler form of the internal space

$$J \mapsto aJ. \tag{3.10}$$

Due to scale invariance of the vacuum Einstein equations, this is a Kähler modulus. While in a proper compactification it is a dynamical field, in a local model it becomes non-dynamical, because its ten-dimensional kinetic term diverges in the decompactification limit, as one can see applying the rescaling (3.10) to the Kähler potential for the Kähler moduli (1.37b).

In warped compactifications, its identification becomes more subtle, as it is explained in [39]. The equation of motion for the warp factor (3.8) admits the general solution[3]

$$\mathrm{e}^{-4A(y)} = a + \sum_{I=1}^{N} G(y; y_I), \tag{3.12}$$

where $G(y; y_I)$ are the Green's functions of the Hodge-de Rham operator Δ for the metric $\mathrm{d}s_X^2$ in (3.1), y^m are the real coordinates on X, and $a \in \mathbb{R}$ is a constant. Recalling the asymptotic behavior of the Green's functions (2.62), we find

$$\mathrm{e}^{-4A(y)} \sim_\infty a + \frac{R^4}{r^4}, \tag{3.13}$$

[2]We denote respectively by y_I^m and Z_I^i the real and complex coordinates for the position of the D3-branes on X. We advise the reader not to confuse the notation Z_I^i for the generic holomorphic local chart of the del Pezzo cone with the homogeneous coordinates (Z^i) introduced in §2.1 to define the $\mathbb{P}^2$ cone.

[3]It is a simple check:

$$\int_X \left(\star_X \sum_I \delta_I^6 \right) (\tilde{y})\, G(y;\tilde{y})\, \mathrm{d}^6\tilde{y} = \left(\sum_I \delta_I^6, \star_X G(y;\tilde{y}) \right) = \sum_I G(y; y_I). \tag{3.11}$$

where

$$R^4 = \frac{N\,\ell_s^4}{4\mathrm{Vol}(Y)}. \tag{3.14}$$

The universal modulus can be identified with variations of the constant a. In our local background, the universal modulus is non-dynamical, therefore a can be regarded as a *marginal parameter* of the low-energy four-dimensional EFT.

Since we are interested in the recovering the asymptotic geometry $\mathrm{Mink}_4 \times C(Y)$ far away from the exceptional divisor of X, i.e. for $r^2 \sim \infty$, where $r^2 = y^m y^n \delta_{mn}$, we will assume

$$a > 0. \tag{3.15}$$

This is in contrast with [40], which takes the so-called *near-horizon limit* boundary condition

$$a = 0, \tag{3.16}$$

in order to recover the asymptotic geometry $\mathrm{AdS}_5 \times Y$. This chapter is devoted to reviewing the construction found in [40] accounting for this modification of the asymptotic geometry, in order to apply it to del Pezzo cones.

3.3 Flat form potentials and harmonic forms

The moduli fields associated with the background described in §3.1 parametrize its flat deformations, namely those that do not lead to changes in its energy. Due to $b_3(X) = 0$, there are no flat deformations of the complex structure of X. Therefore, the only background deformations we need to consider are those of the (internal) Kähler form J (Kähler moduli), and of the supergravity potentials C_2, C_4 and B_2 (axionic moduli). These are essentially given by closed 2- and 4-forms. Therefore, in order to define these moduli precisely we need to identify a basis in the cohomology of X for 2- and 4-forms. However, we also need to take into account gauge invariance associated with the axionic moduli (essentially due to their natural shift symmetry), and the complication due to the fact that the internal space is non-compact. For this reason, in §3.3.1 we introduce flat forms and their gauge invariance, while in §3.3.2 we use the topological results of §2.5.2 in order to define a base in cohomology for 2-forms, and for their dual divisors.

3.3.1 Gauge invariance of flat form potentials

As we already mentioned in §1.2.1, axion moduli are roughly due to changes in the cohomology class of supergravity potentials, which preserve field strengths. Let us be more precise. The paper [27] precisely studies the low-energy four-dimensional EFTs for type IIB string theory compactified on (crepant resolutions of) Calabi-Yau cones $C(Y)$, where Y is a Sasaki-Einstein 5-fold, with N spacetime-filling D3-branes, which are allowed to move away from the tip of the cone[4]. However, it assumes the near-horizon

[4]Both resolving the Calabi-Yau cone and allowing the D3-branes to move away from the tip of the cone are deformations of the exact solution of type IIB string theory $\mathrm{AdS}_5 \times Y$, which plays a central

limit (3.16). Following its presentation of form field moduli, let us consider a generic p-form field strength G, with $(p-1)$-form local potential C, so that locally $G = \mathrm{d}C$. The point is that, even in a given coordinate patch, C is not uniquely determined. Given a $(p-1)$-form $C^\natural$ such that $G = \mathrm{d}C^\natural$, then all other potentials giving rise to the same field strength are

$$C = C^\natural + C^\flat, \tag{3.17}$$

where $C^\flat$ is a *flat* $(p-1)$-form, namely it is closed. While $C^\natural$ is possibly locally defined, e.g. in the case of C_4, we assume $C^\flat$ is globally defined in order to treat it with cohomology[5].

Naively, the gauge transformations of C are the ones that preserve its field strength G, namely adding elements of $H^{p-1}(\mathbb{R}^{1,3} \times X; \mathbb{R}) \simeq H^{p-1}(X; \mathbb{R})$, like in (3.17). However, on a quantum level potential fields are physical by themselves, and indeed $(p-2)$-branes will couple electrically with C by means of the standard Chern-Simons term

$$S_{CS} = \mu \int_{D(p-2)} C, \tag{3.18}$$

where μ is the electric coupling. Then, invariance of the path integral partition functional implies in particular that $\exp[iS_{CS}]$ be gauge invariant, which allows us to identify the generic gauge transformation of C

$$C \mapsto C + \frac{2\pi}{\mu}\,\alpha, \tag{3.19}$$

where $\alpha \in H^{p-1}(X; \mathbb{Z})$, namely α is a closed $(p-1)$-form such that its integral over a basis of $H_{p-1}(X)$ is an integer. In the special case of an exact α, (3.19) is called a *small* gauge transformation, otherwise it is called a *large* gauge transformation. By the universal coefficient theorem, restricting the coefficients from $\mathbb{R}$ to $\mathbb{Z}$ in general gives rise to a torsion component of the cohomology group, defined as

$$\mathrm{Tor}\, H^{p-1}(X; \mathbb{Z}) = \ker\left(H^{p-1}(X; \mathbb{Z}) \to H^{p-1}(X; \mathbb{R})\right), \tag{3.20}$$

see also footnote 15 in §2.5.1. As a consequence, the inclusion of $H^{p-1}(X; \mathbb{Z})$ into $H^{p-1}(X; \mathbb{R})$ is identified as the *free* component of $H^{p-1}(X; \mathbb{Z})$, namely $H^{p-1}_{\mathit{free}}(X, \mathbb{Z}) \simeq H^{p-1}(X; \mathbb{Z})/\mathrm{Tor}\, H^{p-1}(X; \mathbb{Z})$. This is the non-cyclic part of $H^{p-1}(X; \mathbb{Z})$. Therefore, it would appear that physical flat deformations of $(p-1)$-form potentials are classified by

$$H^{p-1}(X; \mathbb{R})/H^{p-1}_{\mathit{free}}(X; \mathbb{Z}). \tag{3.21}$$

This is still naive, but it will suffice for our purposes. In fact, it does not hold for RR form potentials, since in this case large gauge transformations are actually linked to $SL(2; \mathbb{Z})$

role in exemplifying the AdS/CFT conjecture. The former deformations are called *baryonic*, while the latter are called *mesonic*.

[5]This is restrictive, since the most general flat form field is not globally defined. Indeed, the field strength G typically obeys a quantization condition which constrains it to belong to $H^p(X; \mathbb{Z})$, rather than $H^p(X; \mathbb{R})$. On the other hand, $C^\flat \in H^{p-1}(X; \mathbb{R})$, so that imposing it to be flat by $G^\flat = \mathrm{d}C^\flat = 0$ shows that flat field strengths are classified by $\ker(H^p(X; \mathbb{Z}) \to H^p(X; \mathbb{R})) \simeq \mathrm{Tor}\, H^{p-1}(X; \mathbb{Z})$. Clearly, non-trivial elements of $\mathrm{Tor}\, H^{p-1}(X; \mathbb{Z})$ cannot be written as $\mathrm{d}C^\flat$ for some globally defined $C^\flat$.

gauge symmetry of type IIB string theory, whose ultimate effect is to twist (3.21) by a B contribution (see [27] for the details). In any case, (3.21) (or any generalization thereof) implies that flat deformations of C are periodic (or belong to a twisted torus). This means that *all axionic moduli we will defined are periodic.*

3.3.2 Cohomology and homology bases

Let us turn to the task of defining a basis in cohomology for 2-forms. Motivated by the Hodge theorem, which holds in the compact case, it is natural to seek harmonic representatives for each cohomology class in $H^2(X;\mathbb{R})$. On the other hand, due to the non-compact nature of X we should also consider the relative cohomology $H^2(X,Y;\mathbb{R})$. This is the group of compactly supported and closed 2-forms on X. Its relation with the absolute cohomology of X is described in §2.5.2. The salient feature for this analysis is that, thanks to $b_3(X)=0$, we can write the short exact sequence (2.53a), which implies the isomorphism[6]

$$H^2(Y;\mathbb{R}) \simeq H^2(X;\mathbb{R})/H^2(X,Y;\mathbb{R}), \tag{3.22}$$

and the dimensional splitting $b_2(X) = b_4(X) + b_3(Y)$, found in (2.54). Intuitively, (3.22) ensured that all non-trivial 2-forms on the horizon Y are obtained as a pullback of non-compactly supported 2-forms on X, modulo adding compactly supported terms (which are killed by the pullback). On the other hand, it holds $b_4(X)=1$ (see (2.51a)), namely there is only one linearly independent compactly supported 2-form on X. Therefore, a basis of $H^2(X;\mathbb{R})$ is given by the pair of harmonic 2-forms (ω,χ_σ), where $\omega \in H^2(X;\mathbb{Z})$ generates $H^2(X;\mathbb{R})$, while $\chi_\sigma \in H^2(X;\mathbb{Z})$, $\sigma = 1,\dots,b_3(Y)$, generate $H^2(Y;\mathbb{R})$ via pullback onto Y. Note that we choose ω and χ_σ to be integrally quantized. The quantization conditions for this basis can be explicitly stated as follows: given a basis of 2-cycles $C^a \in H_2(X)$, we require

$$N^a = \int_{C^a} \omega \in \mathbb{Z}; \qquad M^a{}_\sigma = \int_{C^a} \chi_\sigma \in \mathbb{Z}. \tag{3.23}$$

We assume (ω,χ_σ) to be integrally quantized so that there exists a dual divisor basis, which makes the computation of integrals far more convenient, at the cost of introducing torsion components $\operatorname{Tor} H^p(X;\mathbb{Z})$ in the cohomology groups. Notice that due to the quotient in (3.22) the χ_σ can always be shifted by some multiple of the compactly supported generator ω, namely by

$$\chi_\sigma \mapsto \chi_\sigma + n_\sigma\,\omega, \qquad n_\sigma \in \mathbb{Z}. \tag{3.24}$$

Thus, the choice of the non-compactly-supported harmonic 2-forms χ_σ is non-canonical[7]. This non-canonical division into compactly supported and non-compactly-supported harmonic 2-forms (ω,χ_σ) allows us to define the moduli fields and the marginal parameters of the EFT, see §3.4.

[6] This is an application of (A.3). An introduction to relative cohomology is found in §A.2.

[7] This is due to the fact that (2.53a) does *not* split.

The asymptotic behaviors of these harmonic forms are given by [27]

$$\|\omega\|^2 \sim_\infty \frac{1}{r^{8+\mu}}; \tag{3.25a}$$

$$\|\chi_\sigma\|^2 \sim_\infty \frac{1}{r^4}, \tag{3.25b}$$

where $\mu > 0$. This shows that only the harmonic generators of $H^2(X,Y;\mathbb{Z})$ are L_2-normalizable, which we checked in the case of $X = \mathcal{K}_{\mathbb{P}^2}$, see (A.52), finding $\mu = 4$. As it is shown in [40], the asymptotics (3.25a, 3.25b) also imply that these harmonic representatives are *primitive*.

Thanks to the asymptotics (3.25a, 3.25b) we can define the warped and unwarped norms of ω:[8]

$$\mathcal{M} = \int_X \omega \wedge \star_X \omega \tag{3.26a}$$

$$\mathcal{G} = \int_X \mathrm{e}^{-4A} \omega \wedge \star_X \omega \tag{3.26b}$$

These norms enter the four-dimensional EFT. Let us stress that (3.26b) is only defined for ω, and not for χ_σ. Indeed, assuming $a \neq 0$ implies that the space of L_2^w-normalizable forms coincides with the space of L_2-normalizable forms, where the L_2^w product is the warped product which defines the norm (3.26b). This is a major qualitative distinction between this analysis and the one found in [40], which will affect the low-energy EFT spectrum. Using the solution for the warp factor (3.12), we find the relation

$$\mathcal{G} = a\,\mathcal{M} + \sum_{I=1}^{N} \int_{X;\,\tilde{y}} G(\tilde{y}; y_I)\,(J \wedge \omega \wedge \omega)\,(\tilde{y}). \tag{3.27}$$

Now, let us look for a basis of 4-cycles in $H_4(X;\mathbb{R})$. Thanks to the fact that $H^2(X;\mathbb{Z})$ is isomorphic to the Picard group[9] of X, ω and χ_σ which we defined above admit a divisor basis as their Poincaré dual. Let us denote with $E \in H_4(X;\mathbb{Z})$ and $D_\sigma \in H_4(X,Y;\mathbb{Z})$ the divisors Poincaré-dual to the forms ω and χ_σ respectively. Recall that $H_4(X,Y;\mathbb{R})$ is the group of 4-chains C of X such that $\partial C \in H_3(Y;\mathbb{R})$, namely such that they are closed up to a 3-cycle of the horizon. Clearly, E is a closed 4-cycle of X. On the other hand, due to the short exact sequence (2.53c), D_σ are non-compact 4-chains such that $\partial D_\sigma \in H_3(Y;\mathbb{Z})$ define non-trivial 3-cycles in Y.

[8] In the more general case $b_4(X) > 1$ (which cannot be achieved with del Pezzo cones), these would be positive-definite metrics on the space of L_2-normalizable harmonic forms ω_α, given by $\mathcal{M}_{\alpha\beta} = \int_X \omega_\alpha \wedge \star_X \omega_\beta$ and $\mathcal{G}_{\alpha\beta} = \int_X \mathrm{e}^{-4A} \omega_\alpha \wedge \star_X \omega_\beta$. Their dimensions are found using the formula $[\star_X] = \ell^{D-2p}$, where D is the real dimension of the ambient space X and p is the order of the form, so that $[\mathcal{M}] = [\mathcal{G}] = \ell^2$.

[9] Here, it can be described as the group of divisors of X up to linear equivalences. See [40] and references therein.

Let us define the intersection products

$$\mathcal{I} = E \cdot E \cdot E = \int_X \omega \wedge \omega \wedge \omega; \tag{3.28a}$$

$$\mathcal{I}_\sigma = E \cdot E \cdot D_\sigma = \int_X \omega \wedge \omega \wedge \chi_\sigma; \tag{3.28b}$$

$$\mathcal{I}_{\sigma\rho} = E \cdot D_\sigma \cdot D_\rho = \int_X \omega \wedge \chi_\sigma \wedge \chi_\rho. \tag{3.28c}$$

These will enter the four-dimensional EFT. In the case of the $\mathbb{P}^2$ cone, (3.28a) has been computed in (2.13). Once again, notice that the non-compact divisors D_σ are defined only up to a linear shift by the compact divisor E, namely

$$D_\sigma \mapsto D_\sigma + n_\sigma E, \qquad n_\sigma \in \mathbb{Z}, \tag{3.29}$$

which is the Poincaré-dual version of (3.24).

3.4 Moduli and marginal parameters

The moduli fields of this class of compactifications are the following:

- Open string moduli: the N positions of the D3-branes $Z_I \in X$, $I = 1, \ldots, N$.
- Closed string moduli: $\begin{cases} \text{Kähler moduli} \\ \text{Axion moduli of } B_2, C_2, \text{and } C_4 \end{cases}$.

In particular, let us stress the fact that there are no complex structure moduli, thanks to $h^{1,2}(X) = 0$.

Linear flat deformations of supergravity potentials and of the Kähler form, depending on $\mathbb{R}^{1,3}$ coordinates, give rise to *moduli fields* and *marginal parameters* of the low-energy four-dimensional EFT of the compactification. While the former make up the spectrum of the EFT, the latter are entirely non-dynamical. Establishing the number and the nature of the moduli, as opposed to the marginal parameters, is a crucial part of this analysis. Due to (3.15), the number of moduli in local del Pezzo models is different than the ones found in [27], and reviewed in [40]. The guiding principle for establishing whether a ten-dimensional deformation gives rise to a dynamical field in the low-energy four-dimensional EFT is checking that *its kinetic term coming from the ten-dimensional action is finite*. In this section we will present the results from [40] relevant for us, whenever they still apply to out setup, or their proper modification holding in this case.

3.4.1 C_2 and B_2 moduli

Let us consider C_2 and B_2 moduli first. By supersymmetry, C_2 and B_2 moduli come in the same number. In [27] it is shown that it is consistent to set to zero the fluctuations of the axio-dilaton, while deforming the supergravity potentials and the Kähler form. Flat linear deformations of the two 2-form potentials in general are non-compactly supported, i.e. they are elements of $H^2(X;\mathbb{R})$. Moreover, since G_3 vanishes in our background, in our notation we can take $C_2^\natural = B_2^\natural = 0$. Following [40], we expand a flat linear deformation of the G_3 potential in the cohomology basis defined above:

$$\delta C_2(x) - \tau\, \delta B_2(x) = \ell_s^2\, \left(\beta(x)\, \omega + \lambda^\sigma(x)\, \chi_\sigma\right), \tag{3.30}$$

where $\beta(x)$ and $\lambda^\sigma(x)$ are complex-valued[10] fields depending on $\mathbb{R}^{1,3}$ coordinates x. Thanks to the fact that in the analysis of [27] the warp factor drops out of the computation for the case of C_2 and B_2 moduli, the results presented in [40] hold equally well in our case. On the one hand, the deformation β defines the axion modulus associated to C_2 and B_2, since it controls a compactly supported cohomology class shift. On the other hand, the non-compactly supported deformations λ^σ correspond to marginal parameters of the EFT, and in particular they do not depend on x. Thus, there is only one axion modulus and $b_3(Y)$ marginal parameters from C_2 and B_2.

Note that, as one can see from the definition (3.30), the non-perturbative shift symmetry $\tau \mapsto \tau + n$, $n \in \mathbb{Z}$, implies that *at perturbative level only* $\operatorname{Im}\beta$ *and* $\operatorname{Im}\lambda^\sigma$ *enter the Kähler potential of the four-dimensional EFT.*

3.4.2 C_4 moduli

Let us pass to C_4 moduli. Due to (3.6), F_5 is not exact and $C_4^\natural$ is not globally defined. Its deformations belong to $H^4(X;\mathbb{R})$, but due to the short exact sequence (2.53d), it holds

$$H^4(X;\mathbb{R}) \simeq H^4(X, Y;\mathbb{R})/H^3(Y;\mathbb{R}). \tag{3.31}$$

This tells us that, somewhat counterintuitively, one can always regard a generic deformation of C_4 as compactly-supported, up to compactly-supported 3-forms $\mathrm{d}\Lambda_3$, where the pullback onto Y of Λ_3 defines a non-trivial cohomology class in $H^3(Y;\mathbb{R})$. This makes $\mathrm{d}\Lambda_3$ non-exact in $H^4(X,Y;\mathbb{R})$. Therefore, compactly supported deformations of C_4 fall into two classes: those of the form $\mathrm{d}\Lambda_3$, and all the others. In analogy to the case of B_2 and C_2 moduli treated above, this suggests that *all* C_4 deformations give rise to C_4 moduli, and no marginal parameters. This is indeed the case if one assumes the near-horizon boundary condition (3.16), as it is pointed out in [40], thanks to a general result from [27].[11] However, in our case of interest $a \neq 0$ this no longer holds. Indeed, [27] proves that the C_4 moduli are in one-to-one correspondence with L_2^w-normalizable harmonic forms, namely with respect to the warped metric

$$g_X^w = \mathrm{e}^{-4A}\, g_X. \tag{3.32}$$

As we already pointed out below (3.26b), (3.15) implies that (in the notation of [27])

$$\mathcal{H}^2_{L_2}(X, \mathrm{e}^{-4A}\, g_X) \simeq \mathcal{H}^2_{L_2}(X, g_X), \tag{3.33}$$

[10] This is necessary since $\overline{\omega} = \omega$ and $\overline{\chi_\sigma} = \chi_\sigma$.

[11] Indeed, it is shown in [27] that in the near-horizon limit of the compactification, the internal space develops an isolated conical singularity at $r = \infty$, which in turn implies the above result.

contrary to the near-horizon case. Therefore, we conclude that in local del Pezzo models only compactly-supported deformations of C_4, which belong to $H^4(X,Y;\mathbb{R})$, give rise to C_4 axion moduli, while its non-compactly supported ones generate non-dynamical parameters. Thus, there is only $b_4(X)=1$ axion modulus associated to C_4, which is given roughly by[12]

$$\int_E \delta C_4 = \int_X C_4 \wedge \delta_E^2, \tag{3.34}$$

while there are $b_3(Y)$ (see §2.5.2) marginal parameters coming from C_4 deformations, given by

$$\int_{D_\sigma} \delta C_4 = \int_X C_4 \wedge \chi_\sigma. \tag{3.35}$$

However, these would-be C_4 marginal parameters are actually unphysical, so that they do not show up in the EFT. This is due to the fact that they are associated to deformations of C_4 of the form

$$\delta C_4 = \gamma\, \mathrm{d}\Lambda_3. \tag{3.36}$$

As we explained below (3.31), this is a cohomologically non-trivial deformation in $H^4(X,Y;\mathbb{R})$, but it is an exact deformation in $H^4(X;\mathbb{R})$, namely it is a small gauge transformation (see §3.3.1).

3.4.3 Kähler moduli

Linear deformations of the Kähler form are constrained by the Kähler condition

$$\mathrm{d}J = 0, \tag{3.37}$$

therefore they are necessarily flat. Following [40], let us expand the Kähler form into the harmonic 2-form basis we defined above:[13]

$$J = J_0 + v\,\omega + u^\sigma\,\chi_\sigma =: \hat{J}_0 + v\,\omega, \tag{3.38}$$

where v and u^σ are real constants, and J_0 is an exact $(1,1)$-form. The exact component J_0 appears even in the $M=\mathbb{P}^2$ case, as shown in §2.9.2, and it is due to the fact that J itself is not compactly supported. Then, a linear flat Kähler deformation takes the form

$$\delta J(x) = \delta J_0(x) + \delta v(x)\,\omega + \delta u^\sigma(x)\,\chi_\sigma, \tag{3.39}$$

where $v(x)$ and $u^\sigma(x)$ are real-valued functions on external spacetime, and $\delta J_0(x)$ is an exact $(1,1)$-form field over $\mathbb{R}^{1,3}$ as well. Once again, [27] states that the ten-dimensional kinetic terms induce a *warped* norm for metric perturbations, i.e. a scalar product with respect to the warped metric (3.32). Due to (3.33), in the absence of the near-horizon limit only compactly supported Kähler deformations are normalizable, which makes them

[12] Allowing for a non-trivial F_5 flux, due to D3-branes, complicates the definition of the C_4 axions [40].

[13] Notice that, since $[J]=\ell^2$ and $[\omega]=[\chi_\sigma]=1$, the Kähler moduli and marginal parameters are dimensionful: $[v]=[u^\sigma]=\ell^2$.

dynamical[14]. Therefore, analogously to the case of B_2 and C_2 moduli, δv defines the sole Kähler modulus of the compactification, while δu^σ are the $b_3(Y)$ marginal parameters associated to the Kähler deformations, and they are constant. The generic deformed Kähler form then is given by

$$J + \delta J = J_0(x) + v(x)\,\omega + u^\sigma\,\chi_\sigma. \tag{3.40}$$

We take (3.40) as the definition of the Kähler modulus v and marginal parameters u^σ. Notice that, contrary to the case of C_4, the non-compactly supported flat linear deformations of the Kähler form are physical, since they are not exact. We also define the non-compactly supported part of J,

$$\hat{J}_0 = J_0 + u^\sigma \chi_\sigma. \tag{3.41}$$

Notice that the Kähler moduli and marginal parameters are constrained by the identity

$$\mathcal{I}v + \mathcal{I}_\sigma u^\sigma = \int_X J \wedge \omega \wedge \omega = -\mathcal{M}, \tag{3.42}$$

where we used the primitivity of ω, namely (A.45), and the identity[15] $\int J_0 \wedge \omega \wedge \omega = 0$. This could also be taken as an alternative definition for $\mathcal{M}$.

3.4.4 Summary

In conclusion, the local del Pezzo models spectrum and non-dynamical parameters set comprise:

- $3N$ D3-brane chiral moduli $Z_I \in X$;
- one Kähler modulus v;
- one B_2 and C_2 axion β;
- one C_4 axion;
- $b_3(Y)$ B_2 and C_2 marginal parameters λ^σ;
- $b_3(Y)$ Kähler marginal parameters u^σ;
- the constant axio-dilaton τ;
- the non-dynamical universal modulus a.

Here, $b_3(Y)$ for the generic del Pezzo cone is given by (2.51c). In particular, notice that in the case of the $\mathbb{P}^2$ cone, the associated local del Pezzo model has no marginal parameters. This is a subset of the spectrum of the EFTs exhibited in [40].

[14] This is expected from supersymmetry, since Kähler and C_4 deformations are required to pair into complexified Kähler moduli in the EFT.

[15] We check this in the $\mathbb{P}^2$ cone case in (2.122).

3.5 Local potentials

In order to write down the four-dimensional EFT for the moduli fields we defined above, we will need to use the (local) potentials for some of the 2-forms we introduced. More precisely, we are interested into the harmonic forms ω and χ_σ, introduced in §3.3.2, and the exact component of the Kähler form J_0, which we defined in (3.40). Thanks to the $\partial\overline{\partial}$ lemma, local potentials always exist for closed 2-forms on Kähler manifolds. Therefore, we can always write[16]

$$\omega = i\partial\overline{\partial}\kappa(z,\overline{z};v) \tag{3.43a}$$

$$\chi_\sigma = i\partial\overline{\partial}\xi_\sigma(z,\overline{z};v) \tag{3.43b}$$

$$J_0 = i\partial\overline{\partial}k_0(z,\overline{z};v) \tag{3.43c}$$

where z^i are complex local coordinates of X, κ and ξ_σ are local potentials, while k_0 is a *globally* defined potential, since J_0 is exact by definition. Recall the definition of the Dolbeault operators $\partial\omega_p = \frac{\partial\omega_{m_1\ldots m_p}(z,\overline{z})}{\partial z^i}\mathrm{d}z^i \wedge \mathrm{d}y^{m_1} \wedge \ldots \wedge \mathrm{d}y^{m_p}$. Notice that they all depend on the Kähler modulus v, since ω and χ_σ are harmonic, while J_0 is part of the harmonic decomposition of J, and harmonicity is a metric-dependent notion. We can also define the local potential of $\hat{J}_0$, the non-compactly supported part of J:

$$\hat{k}_0 = k_0 + u^\sigma\xi_\sigma. \tag{3.44}$$

Thanks to these definitions, from the decomposition of the Kähler form (3.40) we see that the Kähler potential of the background internal metric is given by

$$\mathcal{K} = k_0 + v\,\kappa + u^\sigma\xi_\sigma = \hat{k}_0 + v\,\kappa, \tag{3.45}$$

so that $J = i\partial\overline{\partial}\mathcal{K}$.

Local potentials transform under change of coordinates like a Kähler potential [40], namely we have generic patching rules of the form

$$\kappa(z,\overline{z}) \mapsto \kappa(z,\overline{z}) + \chi(z) + \overline{\chi}(\overline{z}), \tag{3.46}$$

where $\chi(z)$ is a holomorphic function of the complex coordinates z^i of X. Here, $\chi(z)$ is necessarily (v, u^σ)-independent, since it is associated with the invariance of the complex structure of X under holomorphic deformations of its local coordinates. Therefore, the derivative of the local potentials with respect to v or u^σ are globally defined functions. On the other hand, (3.43a)-(3.43c) define the local potentials only up to a (v, u^σ)-dependent, but point-independent function. Since the EFT for the moduli fields will directly depend on these local potentials, a fixing of their moduli dependence is in order.

[16]We explicitly include the Kähler modulus dependence, however remember that Kähler marginal parameters dependence is also allowed. The dimensions are given by $[\kappa] = [\xi_\sigma] = 1$, $[k_0] = \ell^2$.

As it is explained in [40], a convenient and consistent choice to fix such degeneration is choosing the derivative of the local potentials with respect to v as follows:

$$\frac{\partial\kappa}{\partial v}(y) = \frac{2}{\ell_s^4}\int_{X;\,\tilde{y}} G(y;\tilde{y})\,(J\wedge\omega\wedge\omega)\,(\tilde{y}) \tag{3.47a}$$

$$\frac{\partial\xi_\sigma}{\partial v}(y) = \frac{2}{\ell_s^4}\int_{X;\,\tilde{y}} G(y;\tilde{y})(J\wedge\omega\wedge\chi_\sigma)(\tilde{y}) \tag{3.47b}$$

where $G(y;y_0)$ are the Hodge-de Rham Green's functions on X. It should be stressed that these conditions do not change turning on $a \neq 0$, but they could be modified adding a constant term on the r.h.s. of (3.47a) and (3.47b). We choose not to do so, which affects the form of the Kähler potential in §3.7.

Notice that this also fixes the dependence on v of k_0 and of $\hat{k}_0$. Indeed, (2.88) implies that

$$\frac{\partial[J]}{\partial v} = [\omega]. \tag{3.48}$$

In particular, this holds for the harmonic representatives of their respective classes, so that

$$\frac{\partial J}{\partial v} = \omega. \tag{3.49}$$

Therefore, the expansion (3.38) together with (3.49) impose

$$\frac{\partial k_0}{\partial v} = -v\,\frac{\partial\kappa}{\partial v} - u^\sigma\,\frac{\partial\xi_\sigma}{\partial v}; \tag{3.50a}$$

$$\frac{\partial\hat{k}_0}{\partial v} = -v\,\frac{\partial\kappa}{\partial v}. \tag{3.50b}$$

This only leaves the usual freedom to add constant terms to the potentials, and it will be our fixing of choice to write down the moduli EFT. It should be noted that, while cohomology classes do not depend on (v, u^σ), harmonic representatives J and ω do, so that (3.49) does not follow solely from (3.38). Moreover, (3.50a) shows that J_0 depends on v too (other than u^σ).

3.6 The ρ chiral field

Because of four-dimensional supersymmetry, the physical quantity entering the EFT of unwarped compactifications are the *complexified* Kähler moduli, which are defined as [25]

$$T_A = \frac{1}{\ell_s^4}\int_{D_{4,A}} \left(C_4 + i\,\mathrm{dVol}(D_{4,A})\right), \tag{3.51}$$

where $\{D_{4,A}\}_A$ is a basis of divisors of the internal space. According to this prescription, the Kähler moduli come together with the C_4 axion moduli via complexification, and the resulting complexified Kähler modulus T_A inherits shift symmetry only by *real* constants:

$$T_A \mapsto T_A + \alpha_A, \qquad \alpha_A \in \mathbb{R}, \tag{3.52}$$

as a consequence of the axion shift symmetry[17] of $\int_{D_{4,A}} C_4$. In the following, we will largely refer to the complexified Kähler modulus as simply Kähler modulus.

In our model, since the only compact divisor is $E \simeq \mathbb{P}^2$, the cone base, we expect the Kähler modulus to roughly take the form, interchanging real and imaginary part of (3.51) and neglecting warping effects,

$$\rho \sim \frac{1}{\ell_s^4} \int_{\mathbb{P}^2} \left(\frac{1}{2} J \wedge J + i C_4 \right) = \frac{1}{\ell_s^4} \left(\mathrm{Vol}(\mathbb{P}^2) + i \int_{\mathbb{P}^2} C_4 \right), \tag{3.53}$$

with axion symmetry

$$\rho \mapsto \rho + i\alpha, \qquad \alpha \in \mathbb{R}. \tag{3.54}$$

Classical shift symmetry of ρ, together with four-dimensional supersymmetry, implies that $\mathrm{Im}\,\rho$ *does not enter the low-energy four-dimensional Kähler potential at perturbative level.*

In unwarped compactifications, (3.51) provides the so-called *chiral parametrization* of the background Kähler modulus, namely the scalar fields T_A enter the four-dimensional low-energy EFT as the bottom component of a $\mathcal{N} = 1$ superfield. This parametrization of Kähler moduli is desiderable, since four-dimensional supersymmetry prescribes holomorphy of a number of objects with respect to them, like superpotentials or D3-brane instanton actions. However, due to the inclusion of mobile D3-branes in our model, a neat identification of ρ is complicated by the presence of a non-trivial warp factor. This issue has been successfully addressed in [39], identifying the precise modification of (3.53) by means of local four-dimensional superconformal symmetry and holomorphy of the action of probe D3-brane instantons. As it turns out, the Kähler modulus v is not a chiral parametrization of the background fields, thus it needs to be exchanged for a chiral field entering the EFT as the bottom component of a superfield. Let us call this the ρ chiral field, whose real part will play the role of a *chiral* Kähler modulus.

The precise parametrization of the real part of the ρ chiral field in term of the background moduli defined in §3.4 can be found restoring the contribution from B_2 and C_2 axions in (A.13) of [40], which can be found in equation (3.2) therein. In particular, note that the background D3-brane charge Q_6^{bg} defined there is vanishing in our background, since the only source of D3-brane charge at tree level included here are the D3-branes themselves, and in our background G_3 is vanishing. Therefore, the real part of the ρ chiral fields in terms of the background moduli is given by[18]

$$\begin{aligned} \mathrm{Re}\,\rho = \frac{a}{\ell_s^4} \left(\frac{1}{2} \mathcal{I} v^2 + \mathcal{I}_\sigma v u^\sigma + \frac{1}{2} \mathcal{I}_{\sigma\rho} u^\sigma u^\rho \right) + \frac{1}{2} \sum_{I=1}^{N} \kappa(Z_I, \overline{Z}_I; v) + \\ - \frac{1}{2\,\mathrm{Im}\,\tau} \mathcal{I} \left(\mathrm{Im}\,\beta\right)^2 - \frac{1}{\mathrm{Im}\,\tau} \mathcal{I}_\sigma \mathrm{Im}\,\beta\,\mathrm{Im}\,\lambda^\sigma, \end{aligned} \tag{3.55}$$

where $\mathcal{I}, \mathcal{I}_\sigma, \mathcal{I}_{\sigma\rho}$ are the intersection products of the divisor classes of X, which were defined in (3.28a, 3.28b, 3.28c), and $\kappa(Z_I, \overline{Z}_I; v)$ is the local potential for the compactly

[17] At non-perturbative level, this breaks down to a discrete shift symmetry, due to large gauge transformations described in §3.3.1.

[18] Notice that by this definition, ρ is dimensionless, like in (3.51).

supported form ω Poincaré dual to the (compact) exceptional divisor E, which we defined in (3.43a). It should be noted that, while [40] defines a $\mathrm{Re}\,\rho$ field for *all* $b_2(X)$ Kähler deformations, due to non-vanishing universal modulus of our background $a \neq 0$, only compactly supported ones give rise to dynamical moduli, so that one is left with (3.55) only.

The parametrization (3.55) should be understood as an implicit definition of the real Kähler modulus v in terms of the real part of of ρ, and in terms of the imaginary part of β and of the D3-brane moduli Z_I^i. Indeed, one can compute, using (3.42), (3.26b), (A.45) and (3.47a),

$$\begin{aligned}\frac{\partial \mathrm{Re}\,\rho}{\partial v} &= \frac{1}{\ell_s^4}\int_{X;\,\tilde{y}}\left(a+\sum_{I=1}^{N} G(\tilde{y};y_I)\right)(J\wedge\omega\wedge\omega)\,(\tilde{y}) \\ &= \frac{1}{\ell_s^4}\int_X \mathrm{e}^{-4A} J\wedge\omega\wedge\omega = -\frac{1}{\ell_s^4}\mathcal{G},\end{aligned} \tag{3.56}$$

where $\mathcal{G}$ is the warped norm of ω, defined in (3.26b). Therefore, $\frac{\partial \mathrm{Re}\,\rho}{\partial v} \neq 0$, and (3.55) can be locally inverted, yielding v as a function of $\mathrm{Re}\,\rho$, $\mathrm{Im}\,\beta$ and Z_I. As it is explained in [39], β and Z_I^i are chiral moduli, and the universal modulus, although non-vanishing, is non-dynamical, therefore this completes our search for chiral moduli parametrizations.

Notice that the parametrization (3.55) implies that in presence of D3-branes, $\mathrm{Re}\,\rho$ is locally defined because the potential κ is. More precisely, under change of coordinates the local potential κ transforms according to the patching rule (3.46), which implies that under the very same change of coordinates $\mathrm{Re}\,\rho$ transforms as

$$\mathrm{Re}\,\rho \mapsto \mathrm{Re}\,\rho + \frac{1}{2}\chi(z) + \frac{1}{2}\overline{\chi}(\overline{z}). \tag{3.57}$$

This is a manifestation of the fact that $\mathrm{Re}\,\rho$ is actually a section of the moduli space [40]. More precisely, the total moduli space M has a fiber bundle structure, where the ρ moduli space is fibered over the D3 moduli space and the β moduli space

$$M_\rho \hookrightarrow M \overset{\pi}{\to} M_{D3} \times M_\beta. \tag{3.58}$$

This will also be apparent from the form of the EFT, where a covariant exterior derivative is induced from the fibration structure. Notice that, in the case of only one D3-brane, the D3 moduli space is a copy of X.

Note that (3.55) defines v as a globally defined function over the total moduli space M, thanks to the fact that both sides of the equation transform accordingly under change of coordinates. Its derivatives with respect to $\mathrm{Re}\,\rho$, $\mathrm{Im}\,\beta$, and Z_I are extracted from

(3.55). They are given by

$$\frac{\partial v}{\partial \mathrm{Re}\,\rho} = -\ell_s^4\,\mathcal{G}^{-1}; \tag{3.59a}$$

$$\frac{\partial v}{\partial \mathrm{Im}\,\beta} = -\frac{\frac{\partial \mathrm{Re}\,\rho}{\partial \mathrm{Im}\,\beta}}{\frac{\partial \mathrm{Re}\,\rho}{\partial v}} = -\frac{\ell_s^4}{\mathrm{Im}\,\tau}\mathcal{G}^{-1}\left(\mathcal{I}\,\mathrm{Im}\,\beta + \mathcal{I}_\sigma \mathrm{Im}\,\lambda^\sigma\right); \tag{3.59b}$$

$$\frac{\partial v}{\partial Z_I^i} = -\frac{\frac{\partial \mathrm{Re}\,\rho}{\partial Z_I^i}}{\frac{\partial \mathrm{Re}\,\rho}{\partial v}} = \frac{\ell_s^4}{2}\,\mathcal{G}^{-1}\mathcal{A}_i^I, \tag{3.59c}$$

where

$$\mathcal{A}_i^I := \frac{\partial \kappa(Z_I, \overline{Z}_I; v)}{\partial Z_I^i}. \tag{3.60}$$

Let us point out that here $\frac{\partial}{\partial Z_I^i}$ denotes the *partial* derivative with respect to Z_I^i, and not the *total* one, namely one should only derive the explicit dependence with respect to Z_I^i, keeping v fixed. Due to the patching rule for the local potential (3.46), (3.60) is locally defined, with transformation law under change of coordinates given by

$$\mathcal{A}_i^I \mapsto \mathcal{A}_i^I + \frac{\partial \chi(Z_I)}{\partial Z_I^i}. \tag{3.61}$$

This might be puzzling, since v is globally defined over M. However, $\frac{\partial v}{\partial Z_I^i}$ is necessarily locally defined, since the derivative is taken only along the base of the fibration M_{D3}, while v also depends on the fiber $\mathrm{Re}\,\rho$.

Let us conclude with a couple of comments. The derivative of $\frac{\partial \mathrm{Re}\,\rho}{\partial v}$ (3.56) can also be integrated to give an immediate interpretation to (3.55). Using (3.49) and recalling that ω is Poincaré-dual to the exceptional divisor E, we can recast the above expression as

$$\frac{\partial \mathrm{Re}\,\rho}{\partial v} = \frac{1}{\ell_s^4}\frac{\partial}{\partial v}\int_E \mathrm{e}^{-4A}\left(\frac{1}{2}\,J\wedge J\right), \tag{3.62}$$

which implies

$$\mathrm{Re}\,\rho = \frac{1}{\ell_s^4}\int_E \mathrm{e}^{-4A}\mathrm{dVol}(E) + \ldots, \tag{3.63}$$

where the dots are some non-holomorphic function of β, Z_I and of the marginal parameters u^σ and λ^σ. Thus, the real part of ρ is given by the *warped* volume of the compact divisor E, corrected by G_3 moduli and D3-brane moduli. This is the correct generalization of (3.51) to a warped background, and in the case of constant warping the Kähler modulus v becomes an equivalent chiral parametrization of the background modulus to $\mathrm{Re}\,\rho$.

Moreover, (3.59a) provides us with a more direct way to compute the warped norm of ω defined in (3.26b):

$$\mathcal{G} = -\ell_s^4\frac{\partial \mathrm{Re}\,\rho}{\partial v} = -a\left(\mathcal{I}v + \mathcal{I}_\sigma u^\sigma\right) - \frac{\ell_s^4}{2}\sum_{I=1}^N \frac{\partial \kappa(Z_I, \overline{Z}_I; v)}{\partial v}. \tag{3.64}$$

Let us stress that (3.64) in particular shows that $\mathcal{G}$ is globally defined, despite the fact that $\operatorname{Re}\rho$ is not, since the holomorphic function $\chi(z)$ in the patching rule (3.57) is v-independent. This is consistent with the definition (3.26b) for $\mathcal{G}$.

3.7 Effective action and Kähler potential

The low-energy four-dimensional EFT of weakly coupled type IIB string theory compactified on a Kähler-Einstein del Pezzo cone, including only D3-branes, is a rigid $\mathcal{N}=1$ supersymmetric field theory for the moduli $\operatorname{Re}\rho$, $\operatorname{Im}\beta$ and Z_I, and for their fermionic superpartners. By supersymmetry, at perturbative level it has vanishing superpotential, since the moduli have no classical scalar potential.

The effective Lagrangian of this model *at perturbative level* can be obtained from that of the holographic EFTs studied in [40], under the assumptions that the D3-branes are not coincident[19] and that the two-derivative approximation of supergravity is sensible. In this work we will neglect non-perturbative string corrections to the Kähler potential of the EFT. In particular, due to the modifications to the spectrum of the EFT once the near-horizon limit is lifted, which we explored in §3.4, one only needs to get rid of the degrees of freedom that have become non-dynamical. These are given by the ρ chiral fields associated to non-compactly supported Kähler deformations.

Not all of the moduli listed in §3.4 contribute to the Kähler potential of the EFT at perturbative level. As we noted below (3.30), only $\operatorname{Im}\beta$ enters the theory at perturbative level. Analogously, below (3.54) we explained that only $\operatorname{Re}\rho$ enters the Kähler potential at perturbative level. Thus, *the set of independent chiral fields entering the Kähler potential of the four-dimensional EFT at perturbative level consists of* $\operatorname{Re}\rho$, $\operatorname{Im}\beta$, *and* Z_I.

The theory comprises N decoupled $U(1)$ $\mathcal{N}=1$ super-Yang-Mills sectors supported on the mobile D3-branes, describing the evolution of their worldvolume field strengths F^A ($A=1,\dots,N$), and an $\mathcal{N}=1$ supersymmetric interacting Lagrangian of the chiral moduli

$$\mathcal{L} = \mathcal{L}_{SYM} + \mathcal{L}_{chiral}, \tag{3.65}$$

where the decoupled N $U(1)$ SYM sectors are given by [40]

$$\mathcal{L}_{SYM} = -\frac{1}{4\pi}\sum_{A=1}^{N}\left(\operatorname{Im}\tau\, F^A\wedge\star_{D3}F^A + \operatorname{Re}\tau\, F^A\wedge F^A\right) + \text{fermions}, \tag{3.66}$$

and the chiral sector is given by

$$\begin{aligned}\mathcal{L}_{chiral} = {} & -\pi\,\mathcal{G}^{-1}\nabla\rho\wedge\star_4\nabla\overline{\rho} - \frac{2\pi}{\ell_s^4}\sum_{I=1}^{N} g_{i\overline{\jmath}}(Z_I,\overline{Z}_I)\,\mathrm{d}Z_I^i\wedge\star_4\mathrm{d}\overline{Z}_I^{\overline{\jmath}} \\ & -\frac{\pi}{\ell_s^4\operatorname{Im}\tau}\,\mathcal{M}\,\mathrm{d}\beta\wedge\star_4\mathrm{d}\overline{\beta} + \text{fermions},\end{aligned} \tag{3.67}$$

[19] In the language of [27], assuming that mesonic deformations of the background are indeed present.

where we define the covariant exterior derivative

$$\nabla\rho := \mathrm{d}\rho - \mathcal{A}_i^I\,\mathrm{d}Z_I^i - \frac{i}{\operatorname{Im}\tau}\left(\mathcal{I}\operatorname{Im}\beta + \mathcal{I}_\sigma\operatorname{Im}\lambda^\sigma\right)\mathrm{d}\beta, \tag{3.68}$$

and the D3-brane metric $g_{i\bar{\jmath}}$ is the Calabi-Yau metric on X, which we introduced in the background (3.1). Recall that $\mathcal{G}$ and $\mathcal{M}$ are respectively the warped and unwarped norms of ω, defined[20] in (3.26b) and (3.26a), $\mathcal{I}$ and $\mathcal{I}_\sigma$ are the intersection products defined in (3.28a, 3.28b), and $\mathcal{A}_i^I$ are the derivatives of the local potential κ for ω, defined in (3.60). Since we will only be interested in the bosonic dynamics of the EFT, we omitted the details of the bosonic sector. This is one of the main original results of this work, and it has been obtained in collaboration with prof. Luca Martucci; it is based on unpublished notes.

Notice that, since $M_{D3} \simeq X$, the moduli space metric on M_{D3} is naturally given by the Calabi-Yau metric on X; note that it also depends on the chiral fields $\operatorname{Re}\rho$ and $\operatorname{Im}\beta$ through the Kähler modulus v. Moreover, the covariant exterior derivative (3.68) signals that M_ρ is fibered over $M_{D3} \times M_\beta$. Note that it is invariant under change of coordinates on M_{D3}, namely under the simultaneous action of the transformations (3.57) and (3.61).

Let us prove that (3.67) is supersymmetric, by exhibiting a Kähler potential that generates it. This is given by a proper modification of the one exhibited by [40] keeping the universal modulus a finite. The final result is

$$\begin{aligned}\ell_s^4\,K =& -\frac{4\pi}{\ell_s^4}\,a\left(\frac{1}{3}\mathcal{I}\,v^3 + \frac{1}{2}\mathcal{I}_\sigma v^2 u^\sigma\right) + 2\pi\sum_{I=1}^{N}\hat{k}_0(Z_I,\overline{Z}_I;v)\\ &- \frac{4\pi}{\operatorname{Im}\tau}\,u^\sigma\left(\frac{1}{2}\,\mathcal{I}_\sigma\left(\operatorname{Im}\beta\right)^2 + \mathcal{I}_{\sigma\rho}\operatorname{Im}\beta\operatorname{Im}\lambda^\rho\right).\end{aligned} \tag{3.69}$$

The corresponding Lagrangian is obtained as[21]

$$\mathcal{L}_{chiral} = \left(\int \mathrm{d}^2\theta\,\mathrm{d}^2\overline{\theta}\,K(\Phi,\overline{\Phi})\right)\mathrm{d}^4x = -K_{A\overline{B}}\,\mathrm{d}\phi^A\wedge\star_4\mathrm{d}\overline{\phi}^{\overline{B}} + \text{fermions} + \ldots, \tag{3.70}$$

where $\left(\Phi^A\right) = \left(\Phi^\rho,\Phi^\beta,\Phi^I\right)$ are the chiral superfields with bottom components $(\phi^A) = (\rho,\beta,Z_I)$, $K_{A\overline{B}} = \frac{\partial^2 K}{\partial\Phi^A\partial\overline{\Phi}^{\overline{B}}}$, and the dots are irrelevant higher order kinetic contributions.

One can explicitly check that (3.67) follows from (3.69) computing the second derivatives of the Kähler potential. Using (3.50b), (3.42), (2.56), (3.45) and (3.59a - 3.59c),

[20] From a practical standpoint, $\mathcal{G}$ can be computed from (3.64), and $\mathcal{M}$ from (3.42).

[21] Using the four-dimensional signature convention $(-,+,+,+)$, and the Grassmannian integration convention $\int \mathrm{d}^2\theta\,\theta^2 = 1$.

one finds

$$\frac{\mathrm{d}K}{\mathrm{d}v} = \frac{4\pi}{\ell_s^8}\, v\,\mathcal{G}; \tag{3.71a}$$

$$\frac{\mathrm{d}K}{\mathrm{d}\rho} = \frac{1}{2}\frac{\mathrm{d}K}{\mathrm{dRe}\,\rho} = \frac{1}{2}\frac{\partial v}{\partial\mathrm{Re}\,\rho}\frac{\partial K}{\partial v} = -\frac{2\pi}{\ell_s^4}\, v; \tag{3.71b}$$

$$\frac{\mathrm{d}K}{\mathrm{d}\beta} = \frac{1}{2i}\frac{\partial K}{\partial\mathrm{Im}\,\beta} + \frac{\partial v}{\partial\beta}\frac{\partial K}{\partial v} = \frac{2\pi i}{\ell_s^4\,\mathrm{Im}\,\tau}\left[v\left(\mathcal{I}\mathrm{Im}\,\beta + \mathcal{I}_\sigma\mathrm{Im}\,\lambda^\sigma\right) + u^\sigma\left(\mathcal{I}_\sigma\mathrm{Im}\,\beta + \mathcal{I}_{\sigma\rho}\mathrm{Im}\,\lambda^\rho\right)\right]; \tag{3.71c}$$

$$\frac{\mathrm{d}K}{\mathrm{d}Z_I^i} = \frac{2\pi}{\ell_s^4}\frac{\partial\hat{k}_0}{\partial Z_I^i} + \frac{\partial v}{\partial Z_I^i}\frac{\partial K}{\partial v} = \frac{2\pi}{\ell_s^4}\left(\frac{\partial\hat{k}_0}{\partial Z_I^i} + \mathcal{A}_i^I v\right); \tag{3.71d}$$

and finally the second derivatives in agreement with (3.67):

$$\frac{\mathrm{d}^2K}{\mathrm{d}\rho\mathrm{d}\overline{\rho}} = \pi\,\mathcal{G}^{-1}; \tag{3.72a}$$

$$\frac{\mathrm{d}^2K}{\mathrm{d}\beta\mathrm{d}\overline{\rho}} = -\frac{i\pi}{\mathrm{Im}\,\tau}\,\mathcal{G}^{-1}(\mathcal{I}\,\mathrm{Im}\,\beta + \mathcal{I}_\sigma\mathrm{Im}\,\lambda^\sigma); \tag{3.72b}$$

$$\frac{\mathrm{d}^2K}{\mathrm{d}Z_I^i\mathrm{d}\overline{\rho}} = -\pi\,\mathcal{G}^{-1}\mathcal{A}_i^I; \tag{3.72c}$$

$$\frac{\mathrm{d}^2K}{\mathrm{d}\beta\mathrm{d}\overline{\beta}} = \frac{\pi}{\ell_s^4\,\mathrm{Im}\,\tau}\,\mathcal{M} + \frac{\pi}{(\mathrm{Im}\,\tau)^2}\,\mathcal{G}^{-1}\left(\mathcal{I}\,\mathrm{Im}\,\beta + \mathcal{I}_\sigma\mathrm{Im}\,\lambda^\sigma\right)^2; \tag{3.72d}$$

$$\frac{\mathrm{d}^2K}{\mathrm{d}Z_I^i\mathrm{d}\overline{\beta}} = \frac{i\pi}{\mathrm{Im}\,\tau}\,\mathcal{G}^{-1}\mathcal{A}_i^I(\mathcal{I}\,\mathrm{Im}\,\beta + \mathcal{I}_\sigma\mathrm{Im}\,\lambda^\sigma); \tag{3.72e}$$

$$\frac{\mathrm{d}^2K}{\mathrm{d}Z_I^i\mathrm{d}\overline{Z}_I^{\overline{\jmath}}} = 2\pi g_{i\overline{\jmath}} + \pi\ell_s^4\mathcal{G}^{-1}\mathcal{A}_i\overline{\mathcal{A}}_{\overline{\jmath}}. \tag{3.72f}$$

Here we use the total derivative notation in order to make it clear when one should also take into account the implicit dependence of v on all the other chiral fields. Notice how the gauge fixings (3.47a, 3.47b) directly affect the form of the local del Pezzo Kähler potential (3.69).

3.8 EFT of the local $\mathbb{P}^2$ model (no D7-branes)

Let us specialize the above construction to the $\mathbb{P}^2$ cone

$$X_0 := \mathcal{K}_{\mathbb{P}^2}, \tag{3.73}$$

that is we choose as a base for the del Pezzo cone the simplest del Pezzo surface $M = \mathbb{P}^2$.

3.8.1 Background specifics and chiral moduli

We choose the internal background metric $\mathrm{d}s_X^2$ introduced in (3.1) to be the Eguchi-Hanson metric. See §2.7 for its explicit form. Recall that it depends on the Eguchi-Hanson parameter $c \in \mathbb{R}$, which controls the size of the resolution, and which is related to the Kähler modulus v by (2.92).

As we explained in §2.3, the $\mathbb{P}^2$ cone can be completely covered by three local patches $\mathcal{U}_{(i)}$, $i = 1, 2, 3$, where X_0 looks like a $\mathbb{P}^2$ with local coordinates u^a, $a = 1, 2$, and with a complex fiber above described by the complex coordinate ξ. Another convenient local patch is $\mathcal{U}_{(4)}$, which includes the $\mathbb{P}^2$ cone outside the $\mathbb{P}^2$ base, and over which we can place three complex coordinates z^i, $i = 1, 2, 3$, with the identification $z^i \sim \mathrm{e}^{\frac{2\pi i}{3}}$, so that X_0 looks locally like $\mathbb{C}^3/\mathbb{Z}_3$.

The horizon of the $\mathbb{P}^2$ cone is given by $Y_0 = \partial X_0 = S^5/\mathbb{Z}_3$, see §2.5.1. From the topology of the horizon (2.38), it follows in particular that

$$b_3(Y_0) = 0. \tag{3.74}$$

From the summary of §3.4 we conclude that *there are no marginal parameters in the local $\mathbb{P}^2$ model.*

The unique primitive and compactly supported 2-form ω which is Poincaré-dual to the exceptional divisor $\mathbb{P}^2$ is explicitly described in §2.9.1, together with its local potential κ. On $\mathcal{U}_{(4)}$ one should use $\kappa_{(4)}$, whose explicit form is found in (2.101), while on $\mathcal{U}_{(i)}$ one should use $\kappa_{(i)}$, which is defined by the patching rules (2.106a, 2.106b).

The only surviving intersection number is $\mathcal{I}$, defined in (3.28a), which we computed in (2.13), finding

$$\mathcal{I} = 9. \tag{3.75}$$

The axionic modulus from C_2 and B_2 is defined by the flat deformation

$$\delta C_2(x) - \tau\, \delta B_2(x) = \ell_s^2 \beta(x)\, \omega, \tag{3.76}$$

and the Kähler modulus v by

$$J + \delta J = J_0(x) + v(x)\, \omega. \tag{3.77}$$

By the Kähler cone condition (2.89), we showed that v is forced to be negative in this background. The explicit expression of the exact 2-form J_0 is found in §2.9.2, together with its global potential k_0. Due to the fact that there are no marginal parameters in the EFT, the potential for the non-compactly supported part of J, defined in (3.44), is given by

$$\hat{k}_0(z, \overline{z}; v) = k_0(z, \overline{z}; v) = -\frac{3v}{2\pi}\left(1 - \frac{8\pi^3}{27v^3}\, r^6\right)^{\frac{1}{3}}, \tag{3.78}$$

where in the second equality we used (2.116) and (2.92).

Including N D3-branes in the background, the parametrization of $\mathrm{Re}\,\rho$ in terms of the background moduli (3.55) becomes

$$\mathrm{Re}\,\rho = \frac{9a}{2\,\ell_s^4}\, v^2 + \frac{1}{2}\sum_{I=1}^{N} \kappa(z_I, \overline{z}_I; v) - \frac{9}{2\,\mathrm{Im}\,\tau}\, (\mathrm{Im}\,\beta)^2\,, \tag{3.79}$$

where both $\operatorname{Re}\rho$ and the local potential κ for ω should be evaluated in the chosen local patch. Recall that one should invert (3.79) in order to retrieve v as a function of the chiral moduli, as we explained in §3.6.

Due to the parametrization (3.79), $\operatorname{Re}\rho$ is local, namely it is defined up to a holomorphic or anti-holomorphic function, since the potential κ is, according to the patching rule (3.46). When the D3-branes are away from the $\mathbb{P}^2$ base of the cone, one can use the local potential $\kappa_{(4)}$ found in (2.101), which defines $\operatorname{Re}\rho_{(4)}$, while if one needs to describe a D3-brane moving also on the $\mathbb{P}^2$ base, the local expression $\kappa_{(i)}$ defined in (2.106a) should be employed, which defines $\operatorname{Re}\rho_{(i)}$. The ρ chiral field patching rules are easily derived from the ones of κ in (2.106a, 2.106b):

$$\operatorname{Re}\rho_{(i)} = \operatorname{Re}\rho_{(4)} + \frac{1}{4\pi}\log\left|\prod_{I=1}^{N}\left(\xi_{(i)}\right)_I\right|^2 \qquad \textit{on} \quad \mathcal{U}_{(i)}\cap\mathcal{U}_{(4)}; \tag{3.80a}$$

$$\operatorname{Re}\rho_{(i)} = \operatorname{Re}\rho_{(j)} + \frac{3}{4\pi}\log\left|\prod_{I=1}^{N}\left(u^i_{(j)}\right)_I\right|^2 \qquad \textit{on} \quad \mathcal{U}_{(i)}\cap\mathcal{U}_{(j)}. \tag{3.80b}$$

Due to the complicated dependence of κ from v (see its explicit expression (2.101) keeping in mind the relation between c^2 and v (2.92)) a global inversion of (3.79) is impossible as long as D3-branes are included in the model, but local inversions are within computational reach.

The unwarped norm (3.26a) is computed using (3.42) and (3.75), finding

$$\mathcal{M} = -9v. \tag{3.81}$$

The warped norm (3.26a) is computed using (3.64), finding

$$\begin{aligned}\mathcal{G} &= -9av - \frac{\ell_s^4}{2}\sum_{I=1}^{N}\frac{\partial\kappa(z_I,\bar{z}_I;v)}{\partial v}\\ &= -9av - \frac{3\,\ell_s^4}{4\pi v}\sum_{I=1}^{N}\left(1-\frac{8\pi^3}{27v^3}\,r_I^6\right)^{-\frac{2}{3}},\end{aligned} \tag{3.82}$$

where in the second equality we used (2.101) and[22]

$$\frac{\partial\kappa(z,\bar{z};v)}{\partial v} = \frac{3}{2\pi v}\left(1-\frac{8\pi^3}{27v^3}\,r^6\right)^{-\frac{2}{3}}. \tag{3.83}$$

Notice that for $r^2\sim\infty$ (3.83) displays the asymptotic behavior

$$\frac{\partial\kappa(z,\bar{z};v)}{\partial v} \sim_\infty \frac{27v}{8\pi^3 r^4}, \tag{3.84}$$

[22]Since the patching rule functions do not depend on v, the derivative of κ with respect to it is globally defined and it can be computed starting from the expression of the potential on any local patch, like $\kappa_{(4)}$ in (2.101).

which is indeed consistent with the prescription (3.47a), thanks to the asymptotic behavior of conical Green's functions $\sim \frac{1}{r^4}$ in (2.62) with $\mathrm{Vol}(Y_0) = \frac{\pi^3}{3}$, thus the local potential κ as defined in (2.101) has the correct v dependence. Moreover, notice that k_0 specified by (2.116) does satisfy the consistency condition (3.50a), using (3.83), which justifies the choice of vanishing additive term discussed below (2.116).

3.8.2 Effective action and Kähler potential

The chiral Lagrangian of the EFT is found from (3.67) with the explicit expressions (3.82) and (3.81), which yields

$$\begin{aligned}\mathcal{L}_{\mathbb{P}^2} = & \frac{\pi}{9av}\left[1 + \frac{\ell_s^4}{12\pi a v^2}\sum_{I=1}^{N}\left(1 - \frac{8\pi^3}{27v^3}\, r_I^6\right)^{-\frac{2}{3}}\right]^{-1} \nabla\rho \wedge \star_4 \nabla\overline{\rho} + \\ & - \frac{2\pi}{\ell_s^4}\sum_{I=1}^{N} g_{i\overline{\jmath}}(z_I, \overline{z}_I)\, \mathrm{d}z_I^i \wedge \star_4 \mathrm{d}\overline{z}_I^{\overline{\jmath}} + \frac{9\pi v}{\ell_s^4\, \mathrm{Im}\,\tau}\, \mathrm{d}\beta \wedge \star_4 \mathrm{d}\overline{\beta} + \text{fermions},\end{aligned} \tag{3.85}$$

with covariant exterior derivative

$$\nabla\rho = \mathrm{d}\rho - \mathcal{A}_i^I\, \mathrm{d}z_I^i - \frac{9i}{\mathrm{Im}\,\tau}\, \mathrm{Im}\,\beta\, \mathrm{d}\beta. \tag{3.86}$$

The Kähler potential (3.69) simplifies to

$$\begin{aligned}\ell_s^4\, K_{\mathbb{P}^2} &= -\frac{4\pi a \mathcal{I}}{3\,\ell_s^4}\, v^3 + 2\pi \sum_{I=1}^{N} k_0(z_I, \overline{z}_I; v) \\ &= -\frac{12\pi a}{\ell_s^4}\, v^3 - 3v \sum_{I=1}^{N}\left(1 - \frac{8\pi^3}{27v^3}\, r_I^6\right)^{\frac{1}{3}}.\end{aligned} \tag{3.87}$$

All of these expressions are not completely explicit in the sense that they still require a local inversion of (3.79) in order to obtain an expression for $v = v(\mathrm{Re}\,\rho, \mathrm{Im}\,\beta; z_I, \overline{z}_I)$.

It should be noted that, thanks to the internal space being non-compact, its Ricci-flat metric is explicitly known, given by (2.63), as well as the local potential $\kappa_{(4)}$ for the Poincaré-dual form to its exceptional divisor (2.101) and the Kähler potential (2.74). Thus, the Lagrangian (3.85) provides an example of explicit D3-brane kinetic term.

In the following, we consider in detail the case of $N = 0$ and $N = 1$ D3-branes in the model.

3.8.3 No D3-branes

Let us assume there are $N = 0$ D3-branes in the local $\mathbb{P}^2$ model. The ρ chiral field parametrization (3.79) becomes

$$\mathrm{Re}\,\rho = \frac{9a}{2\,\ell_s^4}\, v^2 - \frac{9}{2\,\mathrm{Im}\,\tau}\, (\mathrm{Im}\,\beta)^2\,. \tag{3.88}$$

Due to the absence of D3-branes, this can be globally inverted, yielding

$$v = -\frac{\ell_s^2}{3}\sqrt{\frac{2}{a}\left(\operatorname{Re}\rho + \frac{9}{2\operatorname{Im}\tau}\,(\operatorname{Im}\beta)^2\right)}, \tag{3.89}$$

where we used (2.93). The Kähler potential (3.87) shrinks further to

$$K_{\mathbb{P}^2} = -\frac{12\pi a}{\ell_s^8}\,v^3, \tag{3.90}$$

and plugging (3.89) back into (3.90), we find a global and explicit expression:

$$K_{\mathbb{P}^2} = -\frac{8\pi}{9\,\ell_s^2}\sqrt{\frac{2}{a}}\left(\operatorname{Re}\rho + \frac{9}{2\operatorname{Im}\tau}\,(\operatorname{Im}\beta)^2\right)^{\frac{3}{2}}. \tag{3.91}$$

The warped norm (3.82) before the substitution of v becomes

$$\mathcal{G} = -9av. \tag{3.92}$$

Likewise, the Lagrangian (3.85) reduces to

$$\begin{aligned}\mathcal{L}_{\mathbb{P}^2} = &-\frac{\pi}{3\,\ell_s^2}\left[2a\left(\operatorname{Re}\rho + \frac{9}{2\operatorname{Im}\tau}\,(\operatorname{Im}\beta)^2\right)\right]^{-\frac{1}{2}}\nabla\rho\wedge\star_4\nabla\overline{\rho} + \\ &-\frac{3\pi}{\ell_s^2\operatorname{Im}\tau}\sqrt{\frac{2}{a}\left(\operatorname{Re}\rho + \frac{9}{2\operatorname{Im}\tau}\,(\operatorname{Im}\beta)^2\right)}\,\mathrm{d}\beta\wedge\star_4\mathrm{d}\overline{\beta} + \text{fermions},\end{aligned} \tag{3.93}$$

with covariant exterior derivative

$$\nabla\rho = \mathrm{d}\rho - \frac{9i}{\operatorname{Im}\tau}\operatorname{Im}\beta\,\mathrm{d}\beta. \tag{3.94}$$

3.8.4 One D3-brane

Let us include only $N = 1$ D3-brane in the model. The ρ chiral field parametrization (3.79) is given by

$$\operatorname{Re}\rho = \frac{9a}{2\,\ell_s^4}\,v^2 + \frac{1}{2}\,\kappa(z,\overline{z};v) - \frac{9}{2\operatorname{Im}\tau}\,(\operatorname{Im}\beta)^2. \tag{3.95}$$

A global inversion of this expression is an impossible task. We will perform a local inversion in §**??**, once the orientifold projection got us rid of the β degree of freedom. The Kähler potential (3.87) becomes

$$\ell_s^4\,K_{\mathbb{P}^2} = -\frac{12\pi a}{\ell_s^4}\,v^3 - 3v\left(1 - \frac{8\pi^3}{27v^3}\,r^6\right)^{\frac{1}{3}}, \tag{3.96}$$

and the warped norm (3.82) is given by

$$\mathcal{G} = -9av - \frac{3\,\ell_s^4}{4\pi v}\left(1 - \frac{8\pi^3}{27v^3}\,r^6\right)^{-\frac{2}{3}}. \tag{3.97}$$

Finally, the EFT Lagrangian (3.85) is given by

$$\begin{aligned}\mathcal{L}_{\mathbb{P}^2} = \frac{\pi}{9av}&\left[1 + \frac{\ell_s^4}{12\pi av^2}\left(1 - \frac{8\pi^3}{27v^3}\,r^6\right)^{-\frac{2}{3}}\right]^{-1}\nabla\rho \wedge \star_4 \nabla\overline{\rho} + \\ &- \frac{2\pi}{\ell_s^4}\,g_{i\overline{\jmath}}(z,\overline{z})\,\mathrm{d}z^i \wedge \star_4 \mathrm{d}\overline{z}^{\overline{\jmath}} + \frac{9\pi v}{\ell_s^4\,\mathrm{Im}\,\tau}\,\mathrm{d}\beta \wedge \star_4 \mathrm{d}\overline{\beta} + \text{fermions},\end{aligned} \tag{3.98}$$

with covariant exterior derivative

$$\nabla\rho = \mathrm{d}\rho - \mathcal{A}_i\,\mathrm{d}z^i - \frac{9i}{\mathrm{Im}\,\tau}\,\mathrm{Im}\,\beta\,\mathrm{d}\beta. \tag{3.99}$$

Chapter 4

D7-branes and gaugino condensation

In the previous chapter we derived the form of the low-energy four-dimensional EFT for type IIB string theory compactified on the $\mathbb{P}^2$ cone, including N D3-branes and *no D7-branes*, and neglecting non-perturbative string corrections to the Kähler potential. We are now ready to include four D7-branes and one O7-plane wrapped around the $\mathbb{P}^2$ base[1]. This completes the background to the one considered in [34], when no D3-branes are included in the model. Gaugino condensation occurring on the D7-brane stack generates an effective scalar potential of non-perturbative nature for the chiral field $\mathrm{Re}\,\rho$, which injects non-trivial dynamics into the low-energy EFT described in §3.8. Our goal is to compute the effective scalar potential entering the EFT at leading order in the gaugino condensate, and to find its vacuum structure. We will show that the effective scalar potential is runaway, which implies that $\mathrm{Re}\,\rho$ is pushed to infinity, rendering the compactification unstable.

In §4.1 we review some details of worldvolume gauge theories in string compactifications relevant for our background, in order to include the stack of four D7-branes and one O7-plane. Section 4.2 describes how the inclusion of the orientifold plane wrapped around the $\mathbb{P}^2$ projects out of the EFT spectrum one of its chiral fields, more specifically the axionic chiral field associated to B_2 and C_2. In §4.3 we exhibit the Kähler potential of the local $\mathbb{P}^2$ model once the D7- and O7-stack is included, neglecting possible non-perturbative corrections. This is done by dropping from the Kähler potential of the local $\mathbb{P}^2$ model found in §3.8 the chiral fields which have projected out by orientifold projection. In §4.4 we derive in two ways the effective superpotential, and we compute the effective scalar potential of the EFT coming from gaugino condensation considering a generic SYM theory with gauge coupling dependent on a chiral background sector. In particular, we show that the non-perturbative effect generates a runaway scalar potential for the real part of the ρ chiral field, which makes the whole compactification unstable. In §4.5 we study the classical motion of $\mathrm{Re}\,\rho$ and we compute its precise runaway behavior.

[1]See §1.1.2 for an introduction to orientifold planes, and §1.1.3 for an introduction to D-branes in type II string theories.

4.1 Worldvolume gauge theories and tadpole cancellation

The dynamics of an open string starting and ending on a D-brane can be described by a theory supported on the worldvolume of the D-brane. At energies lower than the string scale, only the string zero modes have relevant dynamics, and they constitute its spectrum. In a string compactification, the dynamics of a single D7-brane wrapped around a divisor D of the internal space is described at energies lower than the KK scale associated to D by an effective $U(1)$ gauge theory of massless fields (scalars, spinors and vectors) localized on the brane[2]. On general grounds, N coincident D7-branes wrapping a supersymmetric[3] 4-cycle in the internal space support on their worldvolume a supersymmetric $U(N)$ gauge theory, whose low-energy spectrum is determined also by the chosen 4-cycle[4]. The $\mathbb{P}^2$ base in $\mathcal{O}_{\mathbb{P}^2}(-3)$ is a *vanishing* 4-cycle. This makes it so that the gauge theory supported on a stack of D7-branes wrapped around the $\mathbb{P}^2$ is going to present a gauge anomaly[5]. Cancellation of this gauge anomaly requires adding non-compact D7-branes (*flavor* branes) and/or O7-planes. The resulting gauge theories have a rather intricate moduli space and non-perturbative dynamics which are not fully understood yet. The O7-planes cancelling the charges of the D7-branes modify the gauge group from $U(N)$ to $SO(2N)$ [25]. While flavor branes can cancel anomalies of an arbitrary number of D7-branes, in order to do so with the O7-planes the number of D7-branes needs to be a multiple of a fixed integer, given by the ratio between the orientifold charge and the D7-brane charge. In our setup of gauge anomaly cancellation via (internally) compact O7-planes in $\mathcal{O}_{\mathbb{P}^2}(-3)$, the RR charge of the O7-plane is constrained by anomaly cancellation to be

$$Q_{O7} = -4Q_{D7}, \tag{4.1}$$

and the O7-plane is also required to be wrapped around the same 4-cycle as the D7-branes, namely the $\mathbb{P}^2$ base. Therefore, wrapping a stack of four D7-branes and one O7-plane wrapping the $\mathbb{P}^2$ base of $\mathcal{O}_{\mathbb{P}^2}(-3)$ yields a gauge anomaly-free $SO(8)$ gauge theory[6]. Moreover, tadpole cancellation via (4.1) means that the net D7-brane charge of our setup is vanishing. Since there are no lower-dimensional branes or orientifold planes, the only leftover brane charge we expect to find in the low-energy EFT is the lower-dimensional

[2] Supersymmetric branes support vector supermultiplets.

[3] A supersymmetric 4-cycle is a divisor of the internal space. Supersymmetry of the 4-cycle is required in order to make the D7-branes BPS states, which ensures their stability.

[4] Up to cyclic group quotients, which affects only global properties, $U(N) \simeq SU(N) \times U(1)$. The $U(1)$ gauge group factor is actually associated to a decoupled sector of the gauge theory, therefore we could equivalently just consider the $SU(N)$ gauge theory supported on the D7-brane stack worldvolume, leaving the decoupled $U(1)$ sector as understood.

[5] D7-branes wrapping vanishing 4-cycles are also called *color* 7-branes. See [32] for more details on this topic.

[6] Furthermore, it has been shown in [10] that since $\mathbb{P}^2$ is not spin (see §A.3), the gauge theory supported on the D7-branes worldvolume also presents a global anomaly, whose cancellation can in principle break the gauge group down to at most $U(4)$ (see also [34]). It is not understood how the presence of the O7-plane affects the anomaly cancellation requirement, but in principle it could prevent gaugino condensation from occurring by breaking $SO(8)$ down to a non-asymptotically free gauge group. Thus, in the entirety of this work we will assume that gaugino condensation does take place in the geometric setup we study.

D3-brane charge induced by curvature corrections (higher order in α') on the D7-brane stack, which is going to be proportional to the Euler characteristic of the wrapped 4-cycle $\chi(\mathbb{P}^2) = 3$, and they will source a warp factor of the form in (3.1). We described this kind of induced brane charge below (1.61). However, these corrections are not central in our analysis, therefore we will neglect them in this work.

To make contact with the F-theory picture, recall that O7-planes in the weak coupling limit of F-theory are recovered as a pair of D7-branes with non-perturbatively small separation scale at small coupling [25]. Thus, in this picture, the $\mathbb{P}^2$ base is wrapped by *six* 7-branes. Indeed, notice that $U(6)$ and $SO(8)$ carry the same dual Coxeter number [11, 23]:

$$N_{\mathfrak{c}} = 6. \tag{4.2}$$

Since $N_{\mathfrak{c}}$ defines the number of vacua of a SYM theory, which is a physical quantity, this is necessary in order to embed this type IIB orientifold vacuum into F-theory.

4.2 Orientifold projection

We studied the local $\mathbb{P}^2$ model *including no D3-branes* in §3.8.3. Let us consider how its spectrum is modified once a O7-plane is wrapped around the $\mathbb{P}^2$ base.

Wrapping four D7-branes and one O7-plane around the $\mathbb{P}^2$ base yields an orientifold compactification, with associated orientifold involution leaving the $\mathbb{P}^2$ invariant and changing the sign of the holomorphic 3-form Ω, see §1.1.2. Since Ω on $\mathcal{U}_{(4)}$ (the local patch excluding the $\mathbb{P}^2$ base defined in (2.17)) is given by (2.98), we immediately see that the orientifold involution in our model is defined on $\mathcal{U}_{(4)}$ by

$$\sigma : z^i \mapsto -z^i. \tag{4.3}$$

Its action on the homogeneous coordinates can be taken to be

$$\sigma : Z^i \mapsto -Z^i, \tag{4.4}$$

so that on $\mathcal{U}_{(i)}$, where X_0 looks locally like $\mathbb{P}^2$ with coordinates (u^a), $a = 1, 2$, and with a complex fiber $\xi \in \mathbb{C}$ above it, see (2.16), this action reads

$$\sigma : \begin{cases} \xi \mapsto -\xi \\ u^a \mapsto u^a \end{cases} . \tag{4.5}$$

This shows that the orientifold involution acts by inverting the complex fiber over each point of the $\mathbb{P}^2$ base. From (4.4) we also see that σ belongs to the $U(1)$ factor of the $U(3)$ isometry group in (2.28).

Although the D7-brane tadpole has been cancelled locally by introducing the O7-plane coinciding with the D7-brane stack, the orientifold plane still binds us to work either with the double cover of $\mathcal{O}_{\mathbb{P}^2}(-3)$, over which the $SL(2,\mathbb{Z})$ doublet (B_2, C_2) is single-valued, or with $\mathcal{O}_{\mathbb{P}^2}(-3)$ directly, but at the cost of including the non-trivial monodromy $-1 \in SL(2,\mathbb{Z})$ circling around the D7-brane stack, namely $\xi = 0$ in $\mathcal{U}_{(i)}$. We will opt for the latter option. D7-brane charge cancellation also ensures that at perturbative level the EFT (3.67) still holds for the spectrum of the theory that survives the orientifold projection.

In order to determine the spectrum of the four-dimensional EFT associated to the considered orientifold compactification, let us examine the local $\mathbb{P}^2$ model spectrum before orientifold projection. Thanks to the trivial cohomology of the internal horizon (2.38), the decomposition of C_2 and B_2 flat deformations (3.30) takes the simple form

$$\delta C_2(x) - \tau \delta B_2(x) = \ell_s^2\, \beta(x)\, \omega, \tag{4.6}$$

where x are the external spacetime coordinates. From (2.103) we see that

$$\sigma^\star \omega = \omega, \tag{4.7}$$

which shows that the geometry of the background does not allow B_2 and C_2 to have any orientifold-odd deformations.

Including now the D7-branes and the O7-plane, the prescription (1.17) shows that only orientifold-odd components of B_2 and C_2 survive the orientifold projection. Thus, the axion β is projected out of the EFT by the inclusion of the O7-plane. On the other hand, from (3.40) we find that a generically deformed Kähler form in the $\mathbb{P}^2$ model is given by

$$J = J_0 + v(x)\, \omega, \tag{4.8}$$

thus from (4.7) and (1.17) we also see that the ρ chiral field is not projected out of the theory. This holds even after complexification, since $H^4(X_0;\mathbb{R}) \ni C_4^\flat$ is generated by J^2 and $\omega \wedge J$, which are both orientifold-even. Therefore, *the spectrum of the low-energy EFT from this orientifold compactification consists only of the chiral field* ρ. Notice that, due to the fact that β has disappeared from the spectrum, the moduli space (3.58) is no longer fibered, and $\mathrm{Re}\,\rho$ is globally defined over M_ρ.

4.3 Kähler potential of the local $\mathbb{P}^2$ model

Let us compute the Kähler potential of the local $\mathbb{P}^2$ model after inclusion of the D7-brane stack and the O7-plane. This is done modifying the Kähler potential of local del Pezzo models (3.69), by accounting for the action of the orientifold projection on the spectrum of the EFT.

The parametrization of the ρ chiral field in our model is obtained from (3.88) dropping the β terms, and it reads

$$\mathrm{Re}\rho = \frac{9v^2}{2\,\ell_s^4}\, a = \frac{a}{\ell_s^4}\, \mathrm{Vol}(\mathbb{P}^2). \tag{4.9}$$

where we used (2.97). Let us point out that the non-perturbative definition of $\mathrm{Re}\,\rho$ in presence of the D7-branes and O7-plane stack is obtained from (3.63), which simplifies to

$$\mathrm{Re}\,\rho = \frac{1}{\ell_s^4} \int_{\mathbb{P}^2} \mathrm{e}^{-4A} \mathrm{dVol}(\mathbb{P}^2), \tag{4.10}$$

assuming no spacetime-filling D3-branes are present. In (4.10) the warp factor is sourced only by quantum effects, and it provides a ten-dimensional definition for $\operatorname{Re}\rho$, which will come in handy when comparing the ten-dimensional results to the four-dimensional ones in §5.5.3. Classically[7], (4.10) reduces to (4.9).

Inverting (4.9) is straightforward, and recalling the negativity condition for v (2.93) it yields

$$v = -\frac{\ell_s^2}{3}\sqrt{\frac{2}{a}\operatorname{Re}\rho}. \tag{4.11}$$

The warped norm (3.92), substituting (4.11), becomes

$$\mathcal{G} = 3\ell_s^2\sqrt{2a\operatorname{Re}\rho}. \tag{4.12}$$

The Kähler potential receives no explicit modifications from the O7-plane projection, so that (3.90) still holds in form. However, in principle we expect non-perturbative corrections due to the gaugino condensation to contribute to the Kähler potential:

$$K = K_{cl} + K_{np}, \tag{4.13}$$

where[8] $K_{cl} \propto \mathrm{e}^{-\frac{\pi}{3}\rho}$. In this work we neglect such non-perturbative corrections.

Thus, we find a global explicit expression for the (perturbative) Kähler potential for the local $\mathbb{P}^2$ model with the D7- and O7-stack, plugging (4.11) in (3.90):

$$K = \frac{1}{\ell_s^2}\frac{8\pi}{9}\sqrt{\frac{2}{a}}\,(\operatorname{Re}\rho)^{\frac{3}{2}}, \tag{4.14}$$

with second derivative and its inverse

$$K_{\rho\overline{\rho}} = \frac{1}{\ell_s^2}\,\frac{\pi}{3}\,\frac{1}{\sqrt{2a\operatorname{Re}\rho}}; \tag{4.15a}$$

$$K^{\rho\overline{\rho}} = \ell_s^2\,\frac{3}{\pi}\sqrt{2a\operatorname{Re}\rho}. \tag{4.15b}$$

Therefore, directly from (4.15a), or dropping the β terms in the Lagrangian (3.93) and in the exterior derivative (3.94), the kinetic part of the Lagrangian takes the precise form[9]

$$\mathcal{L}_{kin,\,\rho} = -\frac{1}{\ell_s^2}\,\frac{\pi}{3}\,\frac{1}{\sqrt{2a\operatorname{Re}\rho}}\,\partial_\mu\rho\,\partial^\mu\overline{\rho}\,\mathrm{d}^4x. \tag{4.16}$$

Notice that the kinetic Lagrangian (4.16) is singular in $\operatorname{Re}\rho = 0$. This corresponds to a zero-sized $\mathbb{P}^2$, namely a compactification on the singular orbifold $\mathbb{C}^3/\mathbb{Z}_3$. All of the construction of chapter 2 falls apart in this limit, so this comes as no surprise. As for the universal modulus a, defined in (3.12), due to the induced D3-brane charge, even in the absence of D3-branes the warp factor is not going to be constant. In particular, a is going to receive α' corrections. However, we already explained in §4.1 that we will neglect these corrections. Thus, we will carry out all the computations keeping a generic, and at the end one could fix $a = 1$, corresponding to Minkowski externally.

[7] Recall that we are neglecting warping due to D7-brane curvature corrections.

[8] This is due to $\langle S\rangle = \mu_0^3\mathrm{e}^{-\frac{\pi}{3}\rho}$, where $\langle S\rangle$ is the gaugino condensate, see (4.53).

[9] We include the kinetic term for $\operatorname{Im}\rho$ as well, since it enters for instance the effective superpotential (4.47).

4.4 Effective scalar potential from gaugino condensation

The Kähler potential (4.14) is not a complete description of the EFT for the local $\mathbb{P}^2$ including the D7- and O7-stack. Indeed, the theory also displays a superpotential. The stack of four D7-branes and one O7-plane wrapped around the $\mathbb{P}^2$ supports, due to gauge group modification by the orientifold plane, a rigid $\mathcal{N}=1$ $SO(8)$ pure glue SYM theory. This theory is well known to exhibit gaugino condensation in the IR [7]. This means that the bilinear formed by the Weyl fermions $\lambda_\alpha = \lambda^a_\alpha T^a$, which are the superpartners of the spin one massless and non-abelian gauge fields $v_\mu = v^a_\mu T^a$, picks up a VEV[10]:

$$\langle \mathrm{tr}\,\lambda\lambda \rangle = \frac{1}{2}\langle \lambda^{\alpha a}\lambda^a_\alpha \rangle \neq 0, \tag{4.17}$$

where T^a are the $SO(8)$ generators, with the normalization convention

$$\mathrm{tr}\,T^a T^b = \frac{\delta^{ab}}{2}. \tag{4.18}$$

This section is devoted to the computation of the effective scalar potential due to gaugino condensation, assuming that no D3-branes are included in the background. The first step to do that is computing the effective superpotential from gaugino condensation, which is done in §4.4.2. This can be achieved both from the Veneziano-Yankielowicz superpotential [3], or from direct inspection of the F-flatness condition of the low-energy EFT, consisting of an EFT of moduli together with a Super-Yang-Mills theory supported on the D7-brane stack. As it is explained in §4.1 and below (1.100), the latter is given by a gauge-singlet chiral moduli sector with kinetic Lagrangian (4.16) plus an $SO(8)$ SYM theory with dynamical gauge coupling, i.e. depending on the background moduli. For this reason and in order to fix the notation, in §4.4.1 we review the generic setting of SYM with dynamical gauge coupling.

4.4.1 Field-dependent SYM gauge coupling

Let us start with a review of rigid non-abelian $\mathcal{N}=1$ SYM theories. We will adopt the conventions $(-,+,+,+)$ signature and $\epsilon_{0123}=+1$, and we will work in the $\mathcal{N}=1$ superspace formalism[11]. We employ the normalization convention (4.18) for the generators of the gauge group. We denote by σ^i the Pauli matrices, $\sigma^\mu = (1,\sigma^i)$, $\overline{\sigma}^\mu = (1,-\sigma^i)$, with index structure $(\sigma^\mu)_{\alpha\dot\beta}$ and $(\overline{\sigma}^\mu)^{\dot\alpha\beta}$. Let us use the convention $\epsilon^{12}=\epsilon^{\dot1\dot2}=+1$ and $\epsilon_{12}=\epsilon_{\dot1\dot2}=-1$, so that spinor indices are raised and lowered as follows,

$$\lambda^\alpha = \epsilon^{\alpha\beta}\lambda_\beta, \qquad \lambda_\alpha = \epsilon_{\alpha\beta}\lambda^\beta, \tag{4.19}$$

and the same holds for dotted indices. Undotted indices are contracted northwest to southeast, while dotted indices are contracted southwest to northeast; in particular, λ_α and $\overline{\chi}^{\dot\alpha}$ are column spinors, while λ^α and $\overline{\chi}_{\dot\alpha}$ are row spinors. Denoting with x^μ the

[10] Notice that $\mathrm{tr}\,\lambda\lambda$ is the lowest dimensional gauge-invariant combination involving the gaugino fields.

[11] For an introduction to the subject, see e.g. [48].

four-dimensional spacetime coordinates, and with θ, $\overline{\theta}$ the Grassmannian coordinates, we define the covariant supersymmetric derivatives

$$D_\alpha = \partial_\alpha + i(\sigma^\mu\overline{\theta})_\alpha\partial_\mu; \tag{4.20a}$$
$$\overline{D}_{\dot\alpha} = \overline{\partial}_{\dot\alpha} + i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu. \tag{4.20b}$$

Defining the new complex coordinate $y^\mu = x^\mu + i(\theta\sigma^\mu\overline{\theta})$ and performing the change of coordinates in superspace $(x^\mu, \theta, \overline{\theta})$, they take the form

$$D_\alpha = \partial_\alpha + 2i(\sigma^\mu\overline{\theta})_\alpha\partial_\mu; \tag{4.21a}$$
$$\overline{D}_{\dot\alpha} = \overline{\partial}_{\dot\alpha}. \tag{4.21b}$$

The vector superfield in the Wess-Zumino gauge takes the form

$$V = (\theta\sigma^\mu\overline{\theta})v_\mu(x) + i\theta^2\overline{\theta}\overline{\lambda}(x) - i\overline{\theta}^2\theta\lambda(x) + \frac{1}{2}\theta^2\overline{\theta}^2 D(x), \tag{4.22}$$

where $v_\mu = v^a_\mu T^a$. We define the covariant derivative and the field strength:

$$\nabla_\mu = \partial_\mu - i[v_\mu, \cdot]; \tag{4.23a}$$
$$F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu - i[v_\mu, v_\nu]. \tag{4.23b}$$

The supersymmetric non-abelian field strength is defined as

$$\begin{aligned} W_\alpha &= -\frac{1}{4}\overline{D}^2\left(\mathrm{e}^{-V}D_\alpha\mathrm{e}^{V}\right) \\ &= -i\lambda_\alpha(y) + i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} + \theta_\alpha D(y) + \theta^2\left(\sigma^\mu\nabla_\mu\overline{\lambda}(y)\right)_\alpha, \end{aligned} \tag{4.24}$$

where

$$\sigma^{\mu\nu} = \frac{1}{4}\left(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu\right), \tag{4.25}$$

with index structure $(\sigma^{\mu\nu})_\alpha{}^\beta$. Let us define the gauge-invariant composite chiral superfield[12]

$$S := -\frac{1}{16\pi^2}\mathrm{tr}\, W^\alpha W_\alpha =: s(y) + \sqrt{2}\theta\chi_S(y) + \theta^2 f_S(y), \tag{4.26}$$

where

$$s(y) = \frac{1}{16\pi^2}\,\mathrm{tr}\,\lambda\lambda; \tag{4.27a}$$
$${\chi_S}^\alpha(y) = -\frac{\sqrt{2}}{16\pi^2}\,\mathrm{tr}\,\left[(\lambda\sigma^{\mu\nu})^\alpha F_{\mu\nu} - i\lambda^\alpha D\right]; \tag{4.27b}$$
$$f_S(y) = -\frac{1}{16\pi^2}\,\mathrm{tr}\,\left[D^2 - 2i\lambda\sigma^\mu\nabla_\mu\overline{\lambda} - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - \frac{i}{2}F_{\mu\nu}\tilde{F}^{\mu\nu}\right], \tag{4.27c}$$

[12]Notice that the prefactor choice $S = \frac{1}{(4\pi i)^2}\,\mathrm{tr}\,W^\alpha W_\alpha$ is such that $\mathrm{Re}\,(4\pi i\tau(\phi)S) = \frac{(4\pi i)^2}{g^2}\mathrm{Re}\,S - 2\theta\mathrm{Im}\,S = \frac{1}{g^2}\mathrm{Re}\,\mathrm{tr}\,W^\alpha W_\alpha + \frac{\theta}{8\pi^2}\mathrm{Im}\,\mathrm{tr}\,W^\alpha W_\alpha$.

where $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. This will be the supermultiplet which the IR EFT of pure SYM is built on.

The worldvolume $SO(8)$ gauge theory describes the dynamics of the superfield S, and it is coupled to the background chiral superfield Φ^ρ, whose bottom component is the scalar field ρ, through the holomorphic gauge coupling $\tau(\Phi^\rho)$, which is itself a chiral superfield, and whose bottom component is given by

$$\tau(\rho) = \frac{\theta_{YM}}{2\pi} + \frac{4\pi i}{g^2_{YM}}. \tag{4.28}$$

The precise definition of the holomorphic gauge coupling is given by

$$\tau(\rho) = i\,\rho, \tag{4.29}$$

where the i factor is required in order to match the perturbative and non-perturbative parts of the two quantities. We discussed the need for this normalization below (1.100), as well as a justification for the dependence of the gauge coupling on the background chiral field associated to the 4-cycle.

For the sake of being slightly more general, let us consider N background chiral fields, with components

$$\Phi^I = \phi^I(y) + \sqrt{2}\theta\chi^I(y) + \theta^2 f^I(y), \tag{4.30}$$

and at the end we will go back to the $N=1$ case we are interested in. These are $SO(8)$ singlets, i.e. they live in its trivial representation. The pure gauge SYM action with holomorphic coupling depending on N background singlet chiral fields Φ^I is given by

$$\begin{aligned} S_{\text{gauge}} &= \frac{1}{8\pi i}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\tau(\Phi)\,\mathrm{tr}\,W^\alpha W_\alpha + \text{c.c.} = \mathrm{Re}\,\frac{1}{4\pi i}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\tau(\Phi)\,\mathrm{tr}\,W^\alpha W_\alpha \\ &= 2\pi i\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\tau(\Phi)\,S + \text{c.c.} = \mathrm{Re}\,4\pi i\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\tau(\Phi)\,S \\ &= \mathrm{Re}\,4\pi i\int \mathrm{d}^4x\left[f_S\,\tau(\phi) + \left(sf^I - \chi^I\chi_S\right)\tau_I(\phi) - \frac{s}{2}\chi^I\chi^J\tau_{IJ}(\phi)\right] \\ &= -4\pi\int \mathrm{d}^4x\left[\mathrm{Im}\,\left(f_S\,\tau(\phi)\right) + \mathrm{Im}\,\left(sf^I - \chi^I\chi_S\right)\tau_I(\phi) - \mathrm{Im}\,\frac{s}{2}\,\chi^I\chi^J\tau_{IJ}(\phi)\right], \end{aligned} \tag{4.31}$$

where $\tau_I(\phi) = \frac{\partial\tau(\phi)}{\partial\phi^I}$ and $\tau_{IJ}(\phi) = \frac{\partial^2\tau(\phi)}{\partial\phi^I\partial\phi^J}$. Using

$$\chi^I\chi_S = -\frac{\sqrt{2}}{16\pi^2}\,\mathrm{tr}\,\left[(\lambda\sigma^{\mu\nu}\chi^I)F_{\mu\nu} - i\lambda\chi^I D\right], \tag{4.32}$$

we can decompose the action as follows:

$$\begin{aligned} S_{\text{gauge}} &= -4\pi\int \mathrm{d}^4x\left[\mathrm{Re}\,f_S\,\mathrm{Im}\,\tau(\phi) + \mathrm{Im}\,f_S\,\mathrm{Re}\,\tau(\phi)\right] + \\ &- 4\pi\,\mathrm{Im}\int \mathrm{d}^4x\left[s\,f^I\tau_I(\phi) + \frac{\sqrt{2}}{16\pi^2}\,\mathrm{tr}\,\left((\lambda\sigma^{\mu\nu}\lambda^I)F_{\mu\nu} - i\lambda\chi^I\,D\right)\tau_I(\phi) - \frac{s}{2}\,\chi^I\chi^J\tau_{IJ}(\phi)\right] \\ &=: S_{\text{SYM}} + S_{\Phi-\text{SYM}}. \end{aligned} \tag{4.33}$$

Using the explicit expressions

$$\mathrm{Re}\, f_S(y) = -\frac{1}{16\pi^2}\,\mathrm{tr}\,\left[D^2 - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - 2i\,\lambda\sigma^\mu\nabla_\mu\overline{\lambda} + i\nabla_\mu(\lambda\sigma^\mu\overline{\lambda})\right]; \tag{4.34a}$$

$$\mathrm{Im}\, f_S(y) = -\frac{1}{16\pi^2}\,\mathrm{tr}\,\left[\frac{1}{2}F_{\mu\nu}\tilde{F}^{\mu\nu} - \nabla_\mu(\lambda\sigma^\mu\overline{\lambda})\right]; \tag{4.34b}$$

and the identity valid for any chiral superfield Φ

$$\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\Phi(y) = \int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\Phi(x), \tag{4.35}$$

we arrive at the quasi-explicit gauge action

$$\begin{aligned} S_{\mathrm{gauge}} &= \mathrm{Re}\,4\pi i \int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\tau(\Phi)\,S \\ &= S_{\mathrm{SYM}} + S_{\Phi-\mathrm{SYM}}, \end{aligned} \tag{4.36}$$

where

$$\begin{aligned} S_{\mathrm{SYM}} = \int \mathrm{d}^4x \Bigg[&\frac{1}{g^2_{\mathrm{YM}}}\left(-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - i\lambda^a\sigma^\mu\nabla_\mu\overline{\lambda}^a + \frac{i}{2}\nabla_\mu(\lambda^a\sigma^\mu\overline{\lambda}^a) + \frac{1}{2}D^aD^a\right) + \\ &+ \frac{\theta_{\mathrm{YM}}}{32\pi^2}\left(F^a_{\mu\nu}\tilde{F}^{a\mu\nu} - 2\nabla_\mu(\lambda^a\sigma^\mu\overline{\lambda}^a)\right)\Bigg]; \end{aligned} \tag{4.37a}$$

$$\begin{aligned} S_{\Phi-\mathrm{SYM}} = -4\pi\,\mathrm{Im}\int \mathrm{d}^4x \Bigg[&s\,f^I\tau_I(\phi) + \frac{\sqrt{2}}{16\pi^2}\,\mathrm{tr}\,\left((\lambda\sigma^{\mu\nu}\chi^I)F_{\mu\nu} - i\lambda\chi^I\,D\right)\tau_I(\phi) + \\ &- \frac{s}{2}\,\chi^I\chi^J\tau_{IJ}(\phi)\Bigg]. \end{aligned} \tag{4.37b}$$

Here we kept for completeness the total derivative terms, which do not contribute for spacetimes without boundary. Notice that, in the case of non-dynamical gauge coupling $\tau = \mathrm{const}$, we retrieve from (4.36) the usual pure gauge $\mathcal{N} = 1$ SYM action in the holomorphic scheme:

$$S_{\mathrm{gauge}} = \int \mathrm{d}^4x \left[\frac{1}{g^2_{\mathrm{YM}}}\left(-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - i\lambda^a\sigma^\mu\nabla_\mu\overline{\lambda}^a + \frac{1}{2}D^aD^a\right) + \frac{\theta_{\mathrm{YM}}}{32\pi^2}F^a_{\mu\nu}\tilde{F}^{a\mu\nu}\right]. \tag{4.38}$$

4.4.2 Effective superpotential

The (bottom component of the) composite superfield S becomes massive at low energies, being the pseudo-Goldstone boson associated to the spontaneous breaking at low energies of the anomalous chiral symmetry of pure SYM by means of its VEV [7, 3]:

$$\langle S\rangle = \langle s\rangle = \frac{1}{16\pi^2}\,\langle \mathrm{tr}\,\lambda\lambda\rangle \neq 0. \tag{4.39}$$

Thus, the full theory describing the SYM sector and the decoupled background chiral fields sector at low energies generates an EFT for the chiral fields only. This theory has a non-trivial superpotential for the chiral superfield $\tau(\Phi^\rho)$ generated by the SYM sector (4.36) by integrating out the heavy superfield S.

In this section we derive the explicit form of the effective superpotential from gaugino condensation in two ways. First we derive it using the Veneziano-Yankielowicz superpotential, which describes the low-energy dynamics of a pure SYM theory. Then, we comment on the vacuum choice associated with gaugino condensation. Finally, we show that the effective scalar superpotential can also be obtained from inspection of the F-flatness condition of the full theory including the SYM sector and the background ρ modulus sector, just assuming a non-vanishing VEV (4.39). The latter derivation provides a more direct intuition for the form of the effective superpotential, which is the reason why we include it, whereas the former derivation ties the form of the effective superpotential to the effective low-energy dynamics of the SYM sector.

From the Veneziano-Yankielowicz superpotential

The effective superpotential describing the dynamics of the chiral superfield S at low energies in pure $\mathcal{N}=1$ SYM is notoriously given by the Veneziano-Yankielowicz superpotential [3]

$$W_{\rm VY} = N_{\mathfrak{c}}\, S \left(1 - \log \frac{S}{\mu_0^3}\right), \tag{4.40}$$

where $N_{\mathfrak{c}}$ is the dual Coxeter number of the gauge group of the SYM theory (in our setup, this is given by (4.2)), μ_0 is an energy scale, and the normalization will be justified shortly. In the case of pure $\mathcal{N}=1$ SYM with non-dynamical gauge coupling τ, (4.40) singles out the unique SUSY vacuum

$$\langle s \rangle = \mu_0^3, \tag{4.41}$$

showing that μ_0^3 should be seen as the non-perturbative scale of the theory, entering explicitly the low-energy scalar potential. However, we are interested in a theory including a singlet chiral field Φ^ρ and a dynamical gauge coupling $\tau(\rho)$, and in this case we will show μ_0 is actually a UV scale above the non-perturbative scale $|\Lambda|$:

$$\mu_0 > |\Lambda|\,. \tag{4.42}$$

The $\mathcal{N}=1$ SYM sector of the IR EFT, neglecting the S contribution to the Kähler potential[13], is given by

$$\mathcal{L}_{\rm SYM,IR} = 2\pi i \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \tau(\Phi^\rho)\, S + \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, W_{\rm VY} + \text{c.c.}, \tag{4.43}$$

[13]As explained in [3], the Kähler potential for S would be of the form $(\overline{S}S)^{\frac{1}{3}}$. Clearly, in order to integrate S out of the action at tree-level one has to neglect its kinetic contribution.

so that at low energies the SYM sector generates an effective superpotential for $\tau(\Phi^\rho)$ given by

$$\tilde{W}_{\rm eff}(S,\Phi^\rho) = 2\pi i\, \tau(\Phi^\rho)\, S + N_{\mathfrak{c}}\, S\left(1 - \log\frac{S}{\mu_0^3}\right). \tag{4.44}$$

Assuming ρ is stabilized with a mass lower than that of S, one can consistently integrate out S as in the standard top-down EFT approach, namely solving the classical EOMs coming from (4.44):

$$\frac{\partial \tilde{W}_{\rm eff}(s,\rho)}{\partial s} = 0, \tag{4.45}$$

which yields $s = \mu_0^3\, {\rm e}^{\frac{2\pi i\tau}{N_{\mathfrak{c}}}}$. This is completed to the superfield relation

$$S = \mu_0^3\, {\rm e}^{\frac{2\pi i\tau(\Phi^\rho)}{N_{\mathfrak{c}}}}, \tag{4.46}$$

which provides the expression of the heavy superfield S in terms of the light superfield Φ^ρ. Therefore, we find the effective superpotential for the IR EFT of the ρ chiral field integrating out S, namely plugging (4.46) into (4.44):

$$W_{\rm eff}(\Phi^\rho) = N_{\mathfrak{c}}\, S = N_{\mathfrak{c}}\, \mu_0^3\, {\rm e}^{\frac{2\pi i\tau(\Phi^\rho)}{N_{\mathfrak{c}}}}. \tag{4.47}$$

The assignment (4.46) can be easily understood as follows. For a SYM theory with dual Coxeter number of the gauge group $N_{\mathfrak{c}}$, the complexified non-perturbative scale defined at the UV scale μ_0 takes the well known form[14]

$$\Lambda = \mu_0\, {\rm e}^{\frac{2\pi i\tau}{3N_{\mathfrak{c}}}}, \tag{4.48}$$

where $\tau := \tau(\langle\rho\rangle)$ is the classic complexified gauge coupling at the high-energy scale μ_0. Recall that this is obtained from dimensional transmutation of the one-loop exact running for $\tau_{\rm cl} = \frac{4\pi i}{g^2_{\rm YM}}$, and from its subsequent completion with the non-peturbative theta angle contribution. This defines the chiral superfield

$$\Lambda^3(\Phi^\rho) = \mu_0^3\, {\rm e}^{\frac{2\pi i\tau(\Phi^\rho)}{N_{\mathfrak{c}}}}. \tag{4.49}$$

On the other hand, the IR superfield S is naturally set by the non-perturbative scale, up to a phase factor that we fix to one[15]:

$$S = \Lambda^3(\Phi^\rho), \tag{4.50}$$

and this is exactly (4.46). This allows us to correctly identify μ_0 as a UV scale far above the gaugino condensate scale $|\Lambda|$, and it also justifies the normalization choice for (4.40).

[14]Recall that this scale is indeed independent of μ_0, thanks to the renormalization group equation satisfied by $\tau(\mu)$.

[15]This shows that the value of the gaugino condensate (and of the non-perturbative scale of the worldvolume gauge theory) is fixed by the volume of the $\mathbb{P}^2$.

On the SYM vacuum choice

The assignment (4.46) reads in components

$$s(y) = \Lambda^3(\rho); \tag{4.51a}$$

$$\chi_{S\,\alpha}(y) = \frac{2\pi i}{N_c}\,\Lambda^3(\rho)\,\chi^\rho_\alpha \tau_\rho(\rho); \tag{4.51b}$$

$$f_S(y) = \frac{2\pi i}{N_c}\,\Lambda^3(\rho)\left[f^\rho \tau_\rho(\rho) - \frac{1}{2}\left(\tau_{\rho\rho}(\rho) + \frac{2\pi i}{N_c}\tau_\rho(\rho)\tau_\rho(\rho)\right)\chi^\rho\chi^\rho\right]. \tag{4.51c}$$

Recall that the notation is defined in (4.30). This implies that the IR superfield S takes the VEV[16]

$$\langle S\rangle = \langle\Lambda^3(\Phi^\rho)\rangle = \Lambda^3(\langle\rho\rangle)\left(1 + \theta^2\,\frac{2\pi i}{N_c}\,\langle f^\rho\rangle \tau_\rho(\langle\rho\rangle)\right), \tag{4.52}$$

and on SUSY vacua $\langle f^\rho\rangle = 0$, so that

$$\langle S\rangle = \Lambda^3 = \mu_0^3\,e^{\frac{2\pi i\tau}{N_c}} = \langle s\rangle, \tag{4.53}$$

which shows why (4.39) holds. In particular, the assignment (4.46) provides S with a non-vanishing VEV. Notice that (4.53) explicitly shows also that a pure SYM theory admits N_c inequivalent vacua

$$\langle S\rangle^{(k)} = e^{\frac{2\pi i k}{N_c}}\,\langle S\rangle \qquad k = 0,\dots,N_c - 1, \tag{4.54}$$

and that one can jump from one to the next by the natural shift $\tau \mapsto \tau + 1$. Indeed, pure $\mathcal{N} = 1$ SYM theory admits an anomalous $U(1)$ R-symmetry $\lambda \mapsto e^{i\alpha}\lambda$, explicitly broken at quantum level to $\mathbb{Z}_{2N_c}$ due to the fact that it corresponds to a shift in the YM angle $\theta \to \theta - n\alpha$ (i.e. $\tau \to \tau - \frac{n}{2\pi}\,\alpha$) where n is the number of zero modes of λ in a $k = 1$ instanton solution, and further spontaneously broken to $\mathbb{Z}_2$ by (4.53). The N_c inequivalent vacua $e^{\frac{2\pi i k}{N_c}}\,\langle S\rangle$ yield different superpotentials, namely

$$W^{(k)}_{\text{eff}} = N_c\,S^{(k)}\left(1 - \frac{2\pi i k}{N_c}\right) = e^{\frac{2\pi i k}{N_c}}\,W_{\text{eff}}\left(1 - \frac{2\pi i k}{N_c}\right), \tag{4.55}$$

where $S^{(k)} = e^{\frac{2\pi i k}{N_c}}\,S$, and S is given by (4.46). Our vacuum choice is $k = 0$, corresponding to $\theta_{YM} \in [0, 2\pi)$, which restricts the span of $\text{Im}\rho$ down to

$$\text{Im}\,\rho \in (-1, 0], \tag{4.56}$$

thanks to (4.29), or more explicitly due to (1.109b).

[16]Recall that by Lorentz invariance of the vacuum $\langle\chi_S\rangle = 0$.

Alternate derivation via VEV insertion

The effective superpotential (4.47) can also be derived simply assuming that S acquires a VEV given by (4.53) and upon inspection of the F-flatness condition of the full theory.

Let us consider the setup introduced in §4.4.1, with N background chiral fields Φ^I. The full IR action for the SYM theory and the chiral background superfields, before integrating out the heavy degrees of freedom, is given by

$$S_{\text{full}} = S_{\text{SYM}} + S_{\Phi-\text{SYM}} + \int \mathrm{d}^4x\, \mathcal{L}_{\text{matter}}, \tag{4.57}$$

where

$$\begin{aligned}\mathcal{L}_{\text{matter}} &= \int \mathrm{d}^2\theta\, \mathrm{d}^2\overline{\theta}\, K(\Phi, \overline{\Phi}) + \left(\int \mathrm{d}^2\theta\, W(\Phi) + \text{c.c.}\right) \\ &= K_{IJ}(\phi, \overline{\phi}) \left(\partial_\mu \overline{\phi}^I \partial^\mu \phi^J - i\chi^I \sigma^\mu \partial_\mu \chi^J + \overline{f}^I f^J\right) + \\ &+ \left(\partial_I W(\phi) f^I - \frac{1}{2}\partial^2_{IJ} W(\phi) \chi^I \chi^J + \text{c.c.}\right) + \mathcal{O}(K_{IJK})\end{aligned} \tag{4.58}$$

We have included a generic effective superpotential $W(\phi)$, possibly allowing for a classical contribution from the UV theory to the Φ^I sector. However, we are going to set this to zero at the end of the computation, since the chiral fields are moduli. Moreover, we do not allow for Fayet-Iliopoulos terms, since they do not play a role in this analysis. The F-term associated to the Φ^I is defined as[17]

$$\mathcal{F}_{\phi^I} := -\frac{\delta}{\delta f^I} S_{\text{kin}}\bigg|_{\text{ren}}, \tag{4.59}$$

where we keep only the renormalizable terms[18]. This yields the F-term

$$\mathcal{F}_{\phi^I} = -K_{IJ}(\phi, \overline{\phi})\, \overline{f}^J. \tag{4.60}$$

Imposing the EOM for the auxiliary field f^I

$$\frac{\delta}{\delta f^I} S = 0 \tag{4.61}$$

gives its on-shell value

$$\mathcal{F}_{\phi^I} = \partial_I W - \frac{1}{8\pi i}\, \tau_I(\phi)\, \mathrm{tr}\, \lambda\lambda + \mathcal{O}(K_{IJK}). \tag{4.62}$$

Similarly we define the D-terms associated to the vector superfield V^a:

$$\mathcal{D}_a := \frac{\delta}{\delta D^a} S_{\text{SYM}}, \tag{4.63}$$

[17] The sign choice is a matter of convention.

[18] The higher derivative terms are going be be negligible in the low energy limit.

which yields

$$\mathcal{D}_a = \frac{1}{g^2_{\rm YM}(\phi)} D_a. \tag{4.64}$$

Imposing the EOM for D^a, we find its on-shell expression:

$$\mathcal{D}_a = \operatorname{Im}\left(\frac{\sqrt{2}}{8\pi i}\,\tau_I(\phi)\lambda_a\chi^I\right). \tag{4.65}$$

Now let us assume that gaugino condensation does occur, namely that

$$\langle s\rangle \neq 0. \tag{4.66}$$

By Lorentz invariance, we still require

$$\langle v^a_\mu\rangle = \langle\lambda^a\rangle = 0, \tag{4.67}$$

while $\langle\phi^I\rangle \neq 0$. In fact, we are interested in the dynamics of the only scalar field of our EFT. The F- and D-terms of our theory (4.62, 4.65), once we take their VEVs, become[19]

$$\mathcal{F}_{\phi^I} = \partial_I W(\langle\phi^I\rangle) - \frac{1}{8\pi i}\tau_I(\langle\phi^I\rangle)\,\langle\operatorname{tr}\lambda\lambda\rangle \tag{4.68a}$$

$$\mathcal{D}_a = 0 \tag{4.68b}$$

These contribute to the classical Lagrangian, describing the dynamics of the VEVs, which we are now going to determine.

Denoting $\langle\phi^I\rangle$ with ϕ^I for easiness of notation, the classical action is given by

$$\begin{aligned}\langle S_{\rm full}\rangle &= \langle S_{\rm SYM}\rangle + \langle S_{\Phi-{\rm SYM}}\rangle + \langle S_{\rm matter}\rangle\\ &= -4\pi \operatorname{Im}\int {\rm d}^4x\,\langle s\rangle\langle f^I\rangle\tau_I(\phi) +\\ &+\int {\rm d}^4x\left[K_{IJ}(\phi,\overline{\phi})\left(\partial_\mu\overline{\phi}^I\partial^\mu\phi^J + \langle\overline{f}^I\rangle\langle f^J\rangle\right) + \left(\partial_I W(\phi) f^I + \text{c.c.}\right)\right].\end{aligned} \tag{4.69}$$

Thus, using (4.60) we find the classical Lagrangian

$$\begin{aligned}\mathcal{L}_{\rm cl} =& K_{IJ}(\phi,\overline{\phi})\,\partial_\mu\overline{\phi}^I\partial^\mu\phi^J +\\ &+ K^{IJ}(\phi,\overline{\phi})\left[\mathcal{F}_{\phi^I}\overline{\mathcal{F}}_{\phi^J} - \left(\partial_I W(\phi)\overline{\mathcal{F}}_{\phi^I} + 2\pi i\,\langle s\rangle\tau_I(\phi)\overline{\mathcal{F}}_{\phi^J} + \text{c.c.}\right)\right],\end{aligned} \tag{4.70}$$

and by the on-shell expression of the F-term (4.68a) we find the scalar potential

$$\begin{aligned}V(\phi,\overline{\phi}) &= K^{IJ}(\phi,\overline{\phi})\mathcal{F}_{\phi^I}\overline{\mathcal{F}}_{\phi^J}|_{\text{on-shell}}\\ &= K^{IJ}(\phi,\overline{\phi})\Big(\partial_I W(\phi) + 2\pi i\,\tau_I(\phi)\langle s\rangle\Big)\left(\overline{\partial}_I\overline{W}(\phi) - 2\pi i\,\overline{\tau}_I(\phi)\overline{\langle s\rangle}\right).\end{aligned} \tag{4.71}$$

This is but the standard SUSY formula for the F-term scalar potential.

[19] By an abuse of notation we denote them in the same way.

Therefore, since $K_{IJ}(\phi,\overline{\phi})$ is positive definite, SUSY vacua are found imposing the F-flatness condition (the D-flatness condition (4.68b) is already satisfied here)

$$\mathcal{F}_{\phi^I} = \partial_I W(\phi) - \frac{1}{8\pi i}\,\tau_I(\phi)\,\langle \mathrm{tr}\,\lambda\lambda\rangle = 0. \tag{4.72}$$

From the components of S (4.27b, 4.27c) and from the VEVs of the SYM vector multiplet (4.67), we see that it holds

$$\langle f_S\rangle = \langle \chi_S\rangle = 0 \tag{4.73}$$

which is consistent with the assignment of S in terms of the background modulus (4.46) (albeit we are not assuming it here), and it shows that the value for the VEV of S (4.39) holds. Thus, the F-flatness condition (4.72) can be recast as[20]

$$\partial_I W(\phi) + 2\pi i\,\tau_I(\phi)\langle S\rangle = 0. \tag{4.74}$$

Notice that assuming the definition of $\langle S\rangle$ in terms of the background moduli

$$\langle S\rangle = \mu_0^3\,\mathrm{e}^{\frac{2\pi i\tau(\Phi)}{N_\mathfrak{c}}}, \tag{4.75}$$

which generalizes (4.53), then (4.74) can also be written as

$$\partial_I W_{\mathrm{eff}}(\phi) := \partial_I\left(W(\phi) + N_\mathfrak{c}\langle S\rangle\right) = 0 \tag{4.76}$$

where we used the relation[21]

$$\langle S\rangle = \frac{1}{2\pi i}\frac{\partial}{\partial\tau}W_{\mathrm{eff}}(\tau,\phi), \tag{4.77}$$

which is implied by (4.75).

Let us specialize to the local $\mathbb{P}^2$ model including the D7- and O7-stack. We consider only one background chiral field Φ^ρ with bottom component ρ, and we set the perturbative superpotential to zero $W(\rho) = 0$. Then, we find

$$W_{\mathrm{eff}}(\rho) = N_\mathfrak{c}\,\Lambda^3 = N_\mathfrak{c}\,\mu_0^3\,\mathrm{e}^{\frac{2\pi i\tau(\rho)}{N_\mathfrak{c}}}, \tag{4.78}$$

from which we recover the effective superpotential (4.47) once this is extended to the full chiral superfield.

[20] This is equation (2.16) of [31].

[21] This is equation (30) of [7]. However, there this relation between gaugino condensate and effective superpotential is derived from a more general argument. Indeed, the pure SYM sector once the heavy superfield S has been integrated out should yield an effective superpotential $W_{\mathrm{eff}}(\tau,\phi)$, depending on $\tau(\phi)$ exclusively through a coupling proportional to $\langle S\rangle$, like in the UV Lagrangian (4.31).

4.4.3 Effective scalar potential

Let us compute the scalar potential from gaugino condensation in the local $\mathbb{P}^2$ model, including the D7- and O7-stack. Let us explicitly set the dual Coxeter number of the gauge group $SO(8)$:

$$N_{\mathfrak{c}} = 6. \tag{4.79}$$

Using (4.29), the effective superpotential is given by

$$W_{\text{eff}}(\rho) = 6\,\mu_0^3\,\mathrm{e}^{-\frac{\pi}{3}\rho}. \tag{4.80}$$

The F-flatness condition for (4.80) is

$$\mathcal{F}_\rho = \frac{\partial W_{\text{eff}(\rho)}}{\partial\rho} = -2\pi\,\mu_0^3\,\mathrm{e}^{-\frac{\pi}{3}\rho} = 0. \tag{4.81}$$

Thus, SUSY is preserved only with ρ at infinity:

$$\rho = +\infty. \tag{4.82}$$

This is not a vacuum since $\operatorname{Re}\rho = +\infty$ is at infinite distance in the moduli space. Indeed, the ρ moduli space metric is given by the second derivative of the Kähler potential (4.15a), namely

$$g_{\rho\bar{\rho}} = K_{\rho\bar{\rho}} = \frac{\pi}{3\ell_s^2}\frac{1}{\sqrt{2a\operatorname{Re}\rho}}. \tag{4.83}$$

Therefore, the M_ρ distance from a point $\star$ and $\operatorname{Re}\rho = +\infty$ is given by

$$d_\rho(\star,\infty) = \int_{[\star,\infty]\times(-1,0]} \sqrt{g}\,\mathrm{dRe}\,\rho\wedge\mathrm{dIm}\,\rho, \tag{4.84}$$

and using $\sqrt{g} = 2\det g_{i\bar{\jmath}}$, we find

$$d_\rho(\star,\infty) = \frac{\pi}{3\ell_s^2}\sqrt{\frac{2}{a}}\int_\star^\infty \frac{1}{\sqrt{\operatorname{Re}\rho}}\mathrm{dRe}\,\rho = +\infty. \tag{4.85}$$

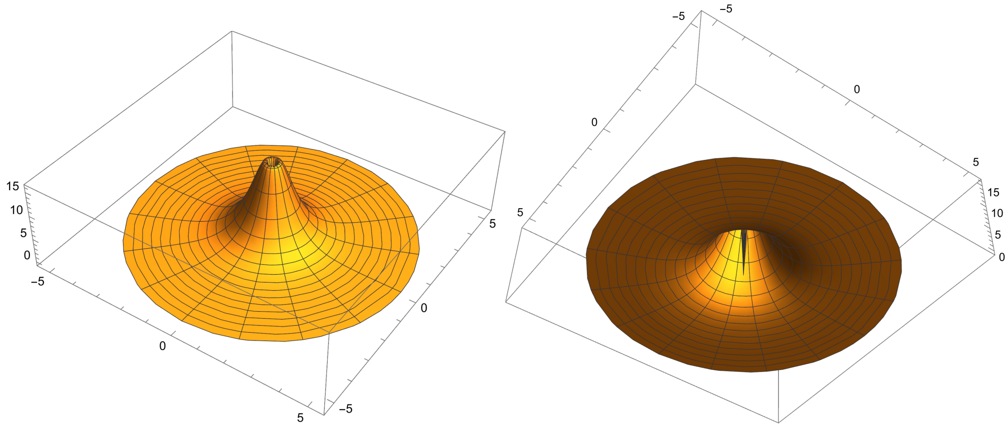
This is the first indication that the compactification on the $\mathbb{P}^2$ cone we chose is unstable, since at $\operatorname{Re}\rho = +\infty$ the $\mathbb{P}^2$ base is in the decompactification limit. In passing, notice that $\operatorname{Re}\rho = 0$ is at finite distance in the moduli space, since

$$d_\rho(0,\star) = \frac{\pi}{3\ell_s^2}\sqrt{\frac{2}{a}}\int_0^\star \frac{1}{\sqrt{\operatorname{Re}\rho}}\mathrm{dRe}\,\rho < \infty. \tag{4.86}$$

From (4.81), (4.71) and (4.15b) we find the effective scalar potential of the EFT for the ρ chiral field:

$$V(\rho) = 12\pi\,\mu_0^6\,\ell_s^2\,\sqrt{2a\operatorname{Re}\rho}\,\mathrm{e}^{-\frac{2\pi}{3}\operatorname{Re}\rho}. \tag{4.87}$$

Therefore, combining the results (4.16) and (4.87), the full effective Lagrangian for the



(a) Top *(b) Bottom*

Figure 4.1: *Effective scalar potential for the ρ chiral field (4.87), plotted in units of $\mu_0^6 \ell_s^2 \sqrt{a}$. The radial coordinate is* Reρ*, while the polar coordinate is* Imρ*, spanning $(-1, 0]$. See around (4.56) for details.*

local $\mathbb{P}^2$ model including four D7-branes and one O7-plane wrapped around the $\mathbb{P}^2$ is given by

$$\mathcal{L}_{\mathbb{P}^2} = -\frac{1}{\ell_s^2}\,\frac{\pi}{3}\,\frac{1}{\sqrt{2a\,\mathrm{Re}\,\rho}}\,\partial_\mu\rho\,\partial^\mu\overline{\rho} - 12\pi\,\mu_0^6\,\ell_s^2\,\sqrt{2a\,\mathrm{Re}\,\rho}\,\mathrm{e}^{-\frac{2\pi}{3}\,\mathrm{Re}\,\rho}, \tag{4.88}$$

where we use the $(-,+,+,+)$ signature convention. Recall that this result does not include non-perturbative corrections to the Kähler potential. This is one of the main results of this work. Figure 4.1 shows a plot of this scalar potential from the top and from the bottom. Its rotation symmetry is clearly due to its independence of $\mathrm{Im}\,\rho$. This kind of dependence could only be sourced by non-perturbative string effects different from gaugino condensation, which we are nonetheless not concerned about, since the dynamics of $\mathrm{Im}\,\rho$ has no relevance with respect to the stability of the compactification.

It should be noted that, since the Lagrangian (4.16) is singular in $\mathrm{Re}\rho = 0$, only nonvanishing field configurations can be regarded as valid vacua. This potential then selects $\mathrm{Re}\rho = +\infty$ as the only value of ρ which preserves SUSY. This is not a vacuum, as we showed above. Its asymptotic behaviors are

$$V(\rho, r^2) \sim^*_{\mathrm{Re}\rho\sim+\infty} \mathrm{e}^{-\frac{2\pi}{3}\mathrm{Re}\rho} \tag{4.89a}$$

$$V(\rho, r^2) \sim^*_{\mathrm{Re}\rho\sim 0^+} \sqrt{\mathrm{Re}\rho}. \tag{4.89b}$$

We can write (4.87) completely in terms of the gaugino condensate, making its non-perturbative nature manifest. Indeed, from (4.53) we have

$$|\langle S\rangle|^2 = \mu_0^6\,\mathrm{e}^{-\frac{2\pi}{3}\,\mathrm{Re}\,\rho}, \tag{4.90}$$

which can be inverted as long as $\langle S\rangle \neq 0$:[22]

$$\operatorname{Re}\rho = \frac{3}{\pi}\,\log\frac{\mu_0^3}{|\langle S\rangle|}. \tag{4.91}$$

Therefore, (4.87) can be recast as

$$V(\rho) = 12\,\sqrt{6\pi a}\,\ell_s^2\,|\langle S\rangle|^2\,\left(\log\frac{\mu_0^3}{|\langle S\rangle|}\right)^{\frac{1}{2}}. \tag{4.92}$$

This expression, and (4.91) allow us to revisit the interpretation of the SUSY vacua of (4.92). Imposing $V(\rho) = 0$, there are two solutions:

- $|\langle S\rangle|^2 = 0$, i.e. $\operatorname{Re}\rho = +\infty$, which is not a vacuum because of (4.85). This is the so-called *Dine-Seiberg vacuum* introduced in §1.2.2;
- $|\langle S\rangle|^2 = \mu_0^6$, i.e. $\operatorname{Re}\rho = 0$, which is a non-legitimate vacuum, since at energies $E \sim \mu_0$ there is no gaugino condensate. Thus, setting $|\langle S\rangle|^2 = \mu_0^6$ is inconsistent. This is the physical interpretation as to why $\operatorname{Re}\rho = 0$ is not an allowed value for the ρ chiral field.

Therefore, the local $\mathbb{P}^2$ model without D3-branes and with four D7-branes and one O7-plane wrapped around the $\mathbb{P}^2$ base does not admit any SUSY vacuum. We will show in the next section that this implies that the only physical cosmological-like solutions for the evolution of $\operatorname{Re}\rho$ in time are runaway.

4.5 Cosmological evolution

Assuming no D3-branes in the local $\mathbb{P}^2$ model, we have shown that the scalar potential for the modulus ρ generated by gaugino condensation occurring on an stack of four D7-branes and one O7-plane wrapped around the $\mathbb{P}^2$ base is given by

$$V(\rho) = 12\pi\,\mu_0^6\,\ell_s^2\,\sqrt{2a\operatorname{Re}\rho}\,\mathrm{e}^{-\frac{2\pi}{3}\operatorname{Re}\rho}. \tag{4.93}$$

Let us study some simple dynamics associated with this potential. In particular, in a cosmological spirit, let us study *radial* solutions depending only on time, namely of the form $(\operatorname{Re}\rho_t, \operatorname{Im}\rho_0)$, where $\operatorname{Im}\rho_0$ is fixed and $\operatorname{Re}\rho_t$ is solely a function of time t. In this whole section, we will set

$$\ell_s = 1. \tag{4.94}$$

4.5.1 Qualitative analysis

In order to assess qualitatively the kind of cosmological-like solutions associated to the scalar potential (4.93), let us establish a simple fact about one-dimensional Lagrangian dynamical systems in presence of a non-trivial kinetic function.

[22]Notice that thanks to (4.42), this yields $\operatorname{Re}\rho > 0$.

Non-trivial kinetic terms and $d = 1$ dynamics

Consider the following general one-dimensional Lagrangian

$$\mathcal{L} = f(\rho)\dot{\rho}^2 - V(\rho), \tag{4.95}$$

where ρ is a real classical degree of freedom, depending only on time, $f(\rho) > 0$ for all $\rho > 0$ by requiring the kinetic matrix to be positive definite, and the dot denotes the time derivative.

The associated equation of motion is

$$\ddot{\rho} + \frac{1}{2}\frac{f'(\rho)}{f(\rho)}\dot{\rho}^2 = -\frac{1}{2}\frac{1}{f(\rho)}V'(\rho). \tag{4.96}$$

This can be recast into the form

$$\ddot{\mathcal{F}}(\rho) = -\frac{1}{2}\frac{V'(\rho)}{f(\rho)^{\frac{1}{2}}}, \tag{4.97}$$

where

$$\mathcal{F}(\rho) = \int^{\rho} f(u)^{\frac{1}{2}}\mathrm{d}u. \tag{4.98}$$

Then, one can view this equation as an ODE for $\mathcal{F}(\rho)$ up to an overall rescaling. Then, let us consider the change of coordinate[23]

$$x = \alpha\mathcal{F}(\rho) \qquad \alpha > 0, \tag{4.99}$$

and we end up with the ODE

$$\ddot{x} = -\frac{1}{2}\alpha^2\frac{\mathrm{d}}{\mathrm{d}x}V\left(\mathcal{F}^{-1}\left(\frac{x}{\alpha}\right)\right). \tag{4.100}$$

A prime integral of this dynamical system is the usual mechanical energy

$$H(x,\dot{x}) = \frac{\dot{x}^2}{2} + \nu(x), \tag{4.101}$$

where the effective potential $\nu(x)$ is defined by

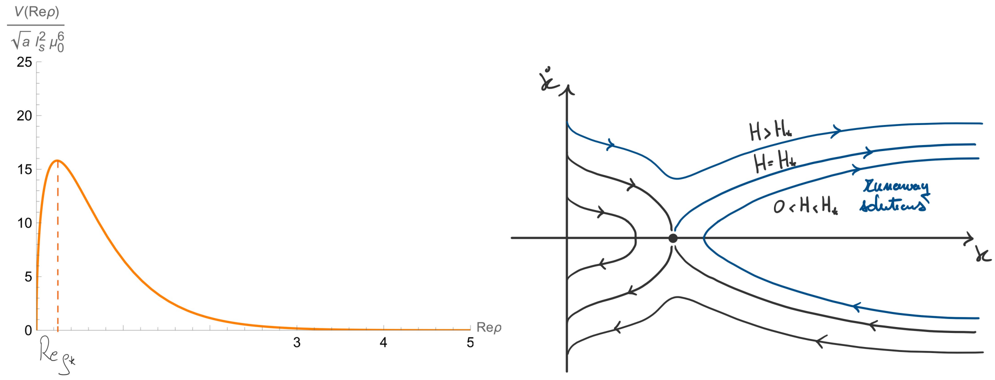
$$\ddot{x} = -\frac{\mathrm{d}}{\mathrm{d}x}\nu(x). \tag{4.102}$$

Thus, from (4.100) one immediately reads the form of the effective potential

$$\nu(x) = \frac{\alpha^2}{2}V\left(\mathcal{F}^{-1}\left(\frac{x}{\alpha}\right)\right), \tag{4.103}$$

up to an additive constant which we set to zero.

[23]Indeed, thanks to the assumption $f(\rho) > 0$, $\mathcal{F}(\rho)$ is locally invertible on $\mathbb{R}_{>0}$ and orientation-preserving. Since we are dealing with functions $\mathbb{R}_{>0} \to \mathbb{R}_{>0}$, local invertibility implies global invertibility.



(a) **(b)**

Figure 4.2: *(a) One-dimensional plot of the scalar potential for* $\mathrm{Re}\,\rho$ *in (4.93) in units* $\mu_0^6\ell_s^2\sqrt{a}$*; (b) Qualitative phase portrait for* x_t *associated with the prime integral* $H(x,\dot{x}) = \frac{1}{2}\dot{x}^2 + V(x)$*, where* x *is the alternate coordinate to* $\mathrm{Re}\,\rho$ *defined in (4.99).*

We conclude that $V(x)$ and $\nu(x)$ are just related by an orientation-preserving one-dimensional change of coordinate (and an overall rescaling). In particular, this implies that $V(x)$ and $\nu(x)$ share general features like the limits for $x \to 0$ and $x \to \infty$, and their stationary points[24]. As a consequence, the phase portraits associated with the prime integrals $H_{\mathit{fictitious}} = \frac{1}{2}\dot{x}^2 + V(x)$ and $H = \frac{1}{2}\dot{x}^2 + \nu(x)$ will also share the same general features, like runaway orbits and stable solutions.

General features

As we showed above, before trying and solving the precise dynamics associated to (4.87) as embedded in the correct EFT, the classical time-dependent playground we are considering allows us to proceed with an analysis of the phase portrait associated with the fictitious prime integral $H(x,\dot{x}) = \frac{1}{2}\dot{x}^2 + V(x)$, in order to find out the main general features of the solutions. Figure 4.2 shows the profile of the scalar potential (4.93) with respect to $\mathrm{Re}\,\rho$, and it qualitatively displays the phase portrait of the scalar potential (4.87). From the explicit expression of the potential (4.93) one readily finds that it has an unstable de Sitter vacuum in

$$\mathrm{Re}\rho_\star = \frac{3}{4\pi}, \tag{4.104}$$

which corresponds to a constant orbit of fictitious energy $H_\star = V(x_\star)$, where $x = x_\star$ is the value for the *ad hoc* coordinate x corresponding to $\mathrm{Re}\,\rho_\star$. This looks rather unusual, but it does seem to be corroborated by the existence of a stationary ten-dimensional perturbative solution, presented in §5.5.

The feature we are interested in is the existence of runaway orbits. For the *ad hoc* variable x, these have been denoted in blue in figure 4.2. They are of three kinds:

[24]Namely, if $x_\star$ is stationary for $\nu(x)$, then $\mathcal{F}^{-1}\left(\frac{x_\star}{\alpha}\right)$ is stationary for $V(x)$.

Low-energy runaway orbits They have energy $0 < H < H_\star$, and they can all be obtained with initial data $(x_0, 0)$, with $x_0 > x_\star$.

Separatrix runaway orbit This single runaway orbit has energy $H = H_\star$, and it has initial data $(x_0, \dot{x}_0)$, with $x_0 > x_\star$ and $\dot{x}_0 > 0$ such that $H_0 = H_\star$.

High-energy runaway orbits They have energy $H > H_\star$, and they can be obtained with initial data $(x_0, \dot{x}_0)$ with $x_0 > 0$ and $\dot{x}_0 > 0$ such that $H_0 > H_\star$. In particular, they are the only kind of runaway solution such that one can choose $x_0 > x_\star$.

Except for the unstable constant solution at (4.104), all other orbits either reach $x = x_\star$ in an infinite amount of time, or they reach $x = 0$ in a finite amount of time.

In the next subsection we will show that the *ad hoc* coordinate (4.99) is related to $\mathrm{Re}\,\rho$ in such a way that the three runaway orbits described above are also present in the $\mathrm{Re}\,\rho$ phase portrait (figure 4.3 (b)). The main difference between the phase portraits of $\mathrm{Re}\,\rho$ and of x (figure 4.2 (b)) is that all orbits approaching $\mathrm{Re}\,\rho = 0$ reach it in an *infinite* amount of time. This is due the fact that in $\mathrm{Re}\,\rho = 0$ the Lagrangian of the system (4.105) is singular.

4.5.2 The dynamical system

The full theory for $\mathrm{Re}\rho$ in our cosmological setup, namely for radial motions with $\mathrm{Im}\,\rho = 0$ and without any D3-brane, is given by (4.88) setting spacial derivatives to zero:

$$\mathcal{L}_\rho = \frac{\pi}{3}\frac{1}{\sqrt{2a\mathrm{Re}\rho}}\dot{\mathrm{Re}\rho}^2 - 12\pi\,\mu_0^6\,\sqrt{2a\,\mathrm{Re}\,\rho}\,\mathrm{e}^{-\frac{2\pi}{3}\mathrm{Re}\,\rho}. \tag{4.105}$$

This yields the equation of motion

$$\ddot{\mathrm{Re}\rho} - \frac{1}{4}\frac{1}{\mathrm{Re}\rho}\dot{\mathrm{Re}\rho}^2 = 24\pi a\,\mu_0^6\left(\mathrm{Re}\rho - \frac{3}{4\pi}\right)\mathrm{e}^{-\frac{2\pi}{3}\mathrm{Re}\rho}. \tag{4.106}$$

Following the same construction outlined in §4.5.1, one can recast this ODE in the simpler form

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\mathrm{Re}\rho\right)^{\frac{3}{4}} = 18\pi a\,\mu_0^6\left(\left(\mathrm{Re}\rho\right)^{\frac{3}{4}} - \frac{3}{4\pi}\left(\mathrm{Re}\rho\right)^{-\frac{1}{4}}\right)\mathrm{e}^{-\frac{2\pi}{3}\mathrm{Re}\rho}. \tag{4.107}$$

Therefore, let us solve this ODE for the new *ad hoc* variable

$$x = \left(\mathrm{Re}\rho\right)^{\frac{3}{4}}. \tag{4.108}$$

The ODE for x is given by

$$\ddot{x} = 18\pi a\,\mu_0^6\left(x - \frac{3}{4\pi}x^{-\frac{1}{3}}\right)\mathrm{e}^{-\frac{2\pi}{3}x^{\frac{4}{3}}}. \tag{4.109}$$

The standard prime integral for this dynamical system is given by the mechanical energy

$$H(x,\dot{x}) = \frac{1}{2}\dot{x}^2 + \nu(x), \tag{4.110}$$

where[25]

$$\begin{aligned}\nu(x) &= -18\pi a\,\mu_0^6 \int \left(x - \frac{3}{4\pi}x^{-\frac{1}{3}}\right) \mathrm{e}^{-\frac{2\pi}{3}x^{\frac{4}{3}}}\,\mathrm{d}x \\ &= \frac{81a}{4}\,\mu_0^6\,x^{\frac{2}{3}}\mathrm{e}^{-\frac{2\pi}{3}x^{\frac{4}{3}}}.\end{aligned} \tag{4.111}$$

As expected, the phase portrait associated to the prime integral (4.110) is qualitatively the same as the one shown in figure 4.2 (b). The only constant solution is found imposing $\nu'(x) = 0$, which yields

$$x_\star = \left(\frac{3}{4\pi}\right)^{\frac{3}{4}}, \tag{4.112}$$

accordingly with the location of the unstable vacuum (4.104) and with the definition of the variable (4.108). This constant solution has energy[26]

$$H_\star = \frac{81a}{4}\,\mu_0^6\sqrt{\frac{3}{4\pi}}\,\mathrm{e}^{-\frac{1}{2}}. \tag{4.113}$$

The existence of the energy (4.110) allows us to lower the degree of the ODE (4.109) of one unit, meaning we just need to solve

$$\frac{1}{2}\dot{x}^2 + \frac{81a}{4}\,\mu_0^6\,x^{\frac{2}{3}}\mathrm{e}^{-\frac{2\pi}{3}x^{\frac{4}{3}}} = h, \qquad h > 0. \tag{4.114}$$

Using the relation (4.108) and the form of the prime integral (4.110) for x, we conclude that the prime integral for the dynamical system (4.106) is given by

$$F\left(\operatorname{Re}\rho, \dot{\operatorname{Re}}\,\rho\right) = \frac{\left(\dot{\operatorname{Re}}\,\rho\right)^2}{2\sqrt{\operatorname{Re}\rho}} + 36a\,\mu_0^6\,\sqrt{\operatorname{Re}\rho}\,\mathrm{e}^{-\frac{2\pi}{3}\operatorname{Re}\rho}. \tag{4.115}$$

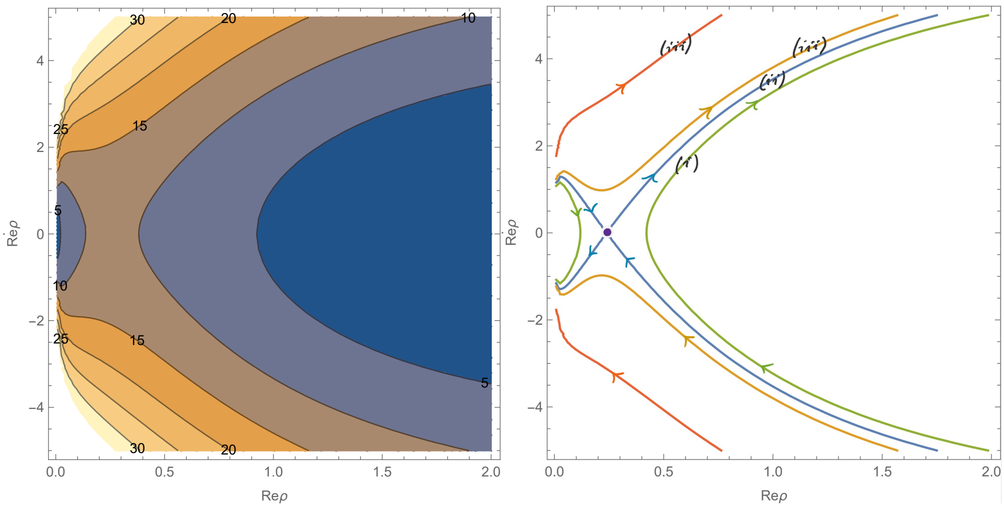
Therefore, all orbits solving (4.106) belong to level sets of (4.115), which are displayed in figure 4.3 (a). The resulting phase portrait for $\operatorname{Re}\rho$ is found in figure 4.3 (b). Upon inspection of the $\operatorname{Re}\rho$ phase portrait, we conclude that the general features of the runaway solutions found in the x phase portrait in §4.5.1 translate exactly to $\operatorname{Re}\rho$ orbits. More precisely, we conclude that the Lagrangian (4.105) induces three kinds of runaway solutions:

Low-energy runaway orbits They have energy $0 < H < H_\star$, and they can all be obtained with initial data $(\operatorname{Re}\rho_0, 0)$, with $\operatorname{Re}\rho_0 > \frac{3}{4\pi}$.

Separatrix runaway orbit This single runaway orbit has energy $H = H_\star$, and it has initial data $(\operatorname{Re}\rho_0, \dot{\operatorname{Re}}\,\rho_0)$, with $\operatorname{Re}\rho_0 > \frac{3}{4\pi}$ and $\dot{\operatorname{Re}}\,\rho_0 > 0$ such that $H_0 = H_\star$.

[25] One could alternatively use formula (4.103) in §4.5.1 using $\alpha = \frac{3}{4}\sqrt{\frac{3}{\pi}}(2a)^{\frac{1}{4}}$ and $\mathcal{F}(\rho) = \frac{4}{3}\sqrt{\frac{\pi}{3}}\frac{1}{(2a)^{\frac{1}{4}}}\rho^{\frac{3}{4}}$.

[26] Here and onward we will loosely refer to the prime integral H as *energy*, albeit H does not have the physical dimensions of one.



(a) (b)

Figure 4.3: *(a) Level sets of the prime integral (4.115) in the plane* $(\mathrm{Re}\,\rho, \mathrm{Re}\,\dot{\rho})$*. They represent orbits of the Lagrangian (4.105). (b) Phase portrait for* $\mathrm{Re}\,\rho$ *in the* $(\mathrm{Re}\,\rho, \mathrm{Re}\,\dot{\rho})$ *plane. Runaway orbits come in three classes: (i) low-energy runaway orbits; (ii) separatrix runaway orbits; (iii) high-energy runaway orbits.*

High-energy runaway orbits They have energy $H > H_\star$, and they can be obtained with initial data $(\mathrm{Re}\,\rho_0, \mathrm{Re}\,\dot{\rho}_0)$ with $\mathrm{Re}\,\rho_0 > 0$ and $\mathrm{Re}\,\dot{\rho}_0 > 0$ such that $H_0 > H_\star$. In particular, they are the only kind of runaway solution such that one can choose $\mathrm{Re}\,\rho_0 > \frac{3}{4\pi}$.

As for the other orbits, the phase portrait in figure 4.3 (b) shows (modulo computational limitations) they either reach $\mathrm{Re}\,\rho = \frac{3}{4\pi}$ or $\mathrm{Re}\,\rho = 0$ in an infinite amount of time[27], see figure 4.4 (b) for a closeup of the orbits' behavior around $\mathrm{Re}\,\rho = 0$. Since $\mathrm{Re}\,\rho = 0$ is not a legitimate vacuum of the theory, as we discussed in §4.4.3, the latter do not seem to be physically acceptable solutions. On the other hand, since $\mathrm{Re}\,\rho = \frac{3}{4\pi}$ is an unstable vacuum, the former orbits do appear to be physical, but upon small perturbations they either become runaway orbits or non-physical orbits. For this reason, we continue this analysis only considering runaway orbits exclusively, and specifically we study low-energy ones.

4.5.3 Explicit low-energy runaway solutions

As it is discussed in §4.5.1, runaway solutions of the ODE (4.114) are of three kinds: low-energy, separatrix and high-energy. In the following we will focus on low-energy

[27]The reason for this is that the full EOM (4.106), when both sides are multiplied by $\mathrm{Re}\,\rho$, admits $\mathrm{Re}\,\rho = 0$ as a (unstable) constant solution, together with the already known unstable vacuum $\mathrm{Re}\,\rho = \frac{3}{4\pi}$. Therefore, due to existence and uniqueness theorems, no orbit of (4.106) can reach $\mathrm{Re}\,\rho = 0$ in a finite amount of time, even though $\mathrm{Re}\,\rho = 0$ is not a solution of the ODE.

runaway solutions, given their simplicity. Intuitively, along these solutions the 4-cycle $\mathbb{P}^2$ starts with zero initial velocity, and if it is large enough it starts expanding.

In order to have a cosmological-like low-energy runaway solution, the initial conditions with respect to $\mathrm{Re}\,\rho$ read

$$\mathrm{Re}\,\rho_0 > \frac{3}{4\pi}; \qquad \dot{\mathrm{Re}}\,\rho_0 = 0. \tag{4.116}$$

These are found imposing low energy, namely $0 < h < H_\star$, and proper initial conditions, that is $(x, \dot{x}) = (x_0, 0)$ with $x_0 > x_\star$. With this input data, it immediately follows from (4.114) (imposing $\dot{x}_t > 0$) that low-energy runaway solutions solve the equation

$$\dot{x} = \sqrt{2h - \frac{81a}{2}\,\mu_0^6\,x^{\frac{2}{3}}\,\mathrm{e}^{-\frac{2\pi}{3}x^{\frac{4}{3}}}}, \tag{4.117}$$

so that they are implicitly defined by

$$\int_{x_0}^{x_t} \mathrm{d}x \,\frac{1}{\sqrt{2h - \frac{81a}{2}\,\mu_0^6\,x^{\frac{2}{3}}\,\mathrm{e}^{-\frac{2\pi}{3}x^{\frac{4}{3}}}}} = t, \tag{4.118}$$

where we set $t_0 = 0$, and the energy is fixed by the initial datum as

$$h = \frac{81a}{4}\,\mu_0^6\,x_0^{\frac{2}{3}}\,\mathrm{e}^{-\frac{2\pi}{3}x_0^{\frac{4}{3}}}. \tag{4.119}$$

This is hard to make explicit, so perturbative approximations for some regimes are in order if we want an explicit description of how the inflation of the internal 4-cycle occurs. The graph of a full low-energy runaway solution for $\mathrm{Re}\,\rho = \frac{3}{4\pi} + 1$ is found in figure 4.4 (a). Now, let us work out the precise runaway behavior of these solutions for early and late times with respect to t.

Early times

In order to find an explicit expression for the low-energy runaway solution at times $t \sim 0$, let us expand the ODE (4.117) for $x \sim x_0$, and let us stop at leading order. We find

$$\dot{x} = \sqrt{2h\left(\frac{2}{3}\,\frac{\frac{4\pi}{3}x_0^{\frac{4}{3}} - 1}{x_0}\right)}\,\sqrt{x - x_0} + o_{x_0}\left(\sqrt{x - x_0}\right). \tag{4.120}$$

This readily integrates to

$$x_t = x_0 + 9\pi a\,\mu_0^6\,\mathrm{e}^{-\frac{2\pi}{3}x_0^{\frac{4}{3}}}\,\frac{x_0^{\frac{4}{3}} - \frac{3}{4\pi}}{x_0^{\frac{1}{3}}}\,t^2 + o_0(t^2). \tag{4.121}$$

Recall now that $x = (\mathrm{Re}\rho)^{\frac{3}{4}}$, so one finds the solutions at early times

$$\mathrm{Re}\rho_t = \mathrm{Re}\rho_0 + 12\pi a\,\mu_0^6\,\mathrm{e}^{-\frac{2\pi}{3}\mathrm{Re}\rho_0}\left(\mathrm{Re}\rho_0 - \frac{3}{4\pi}\right)t^2 + o_0(t^2). \tag{4.122}$$

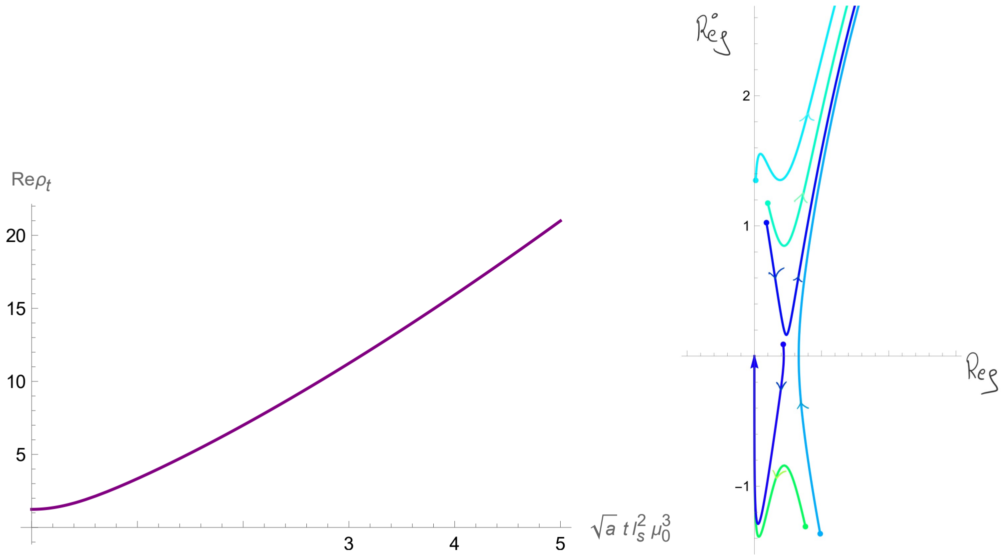


Figure 4.4: *(a) Graph of a low-energy runaway solution* $\mathrm{Re}\,\rho_t$ *for* $\mathrm{Re}\,\rho_0 = \frac{3}{4\pi} + 1$. *The time axis is in units of* $\frac{1}{\ell_s^2 \mu_0^3 \sqrt{a}}$. *(b) Detail of the* $\mathrm{Re}\,\rho$ *phase portrait around* $\mathrm{Re}\,\rho = 0$ *for some illustrative (non-maximal) orbits. Non-runaway orbits reach* $\mathrm{Re}\,\rho = 0$ *in an infinite amount of time.*

Therefore we find that the 4-cycle starts to inflate scaling with t^2, and from the phase portrait in figure 4.2 we know that this rate will only increase, as we will now show explicitly.

Notice that the solution (4.122) and the relation (4.90) allow us to study how the gaugino condensate evolves with time at the beginning of the expansion. Plugging (4.122) into (4.90) and neglecting $o_0(t^2)$ corrections one finds

$$|\langle S\rangle|^2 = \mu_0^6\, \mathrm{e}^{-\frac{2\pi}{3}\mathrm{Re}\,\rho_0} - 8\pi^2 a \left(\mathrm{Re}\,\rho_0 - \frac{3}{4\pi}\right) |\langle S\rangle|^4\, t^2 + o_0(t^2). \tag{4.123}$$

Let us adopt a perturbative approach in $|\langle S\rangle|^2$, in agreement with the analysis of §5. In this framework, the perturbative approach corresponds to assuming

$$\mathrm{Re}\,\rho_0 \gg \frac{3}{4\pi}. \tag{4.124}$$

Therefore, (4.123) at lowest order becomes

$$|\langle S\rangle|^2 = \mu_0^6\, \mathrm{e}^{-\frac{2\pi}{3}\mathrm{Re}\,\rho_0} + o_0(t^2). \tag{4.125}$$

We conclude that, at lowest order, the gaugino condensate is constant at early times. This allows us to rewrite (4.122) as (reinstating the ℓ_s factors)

$$\mathrm{Re}\rho_t = \mathrm{Re}\rho_0 + 12\pi a\, \ell_s^4 |\langle S\rangle|^2 \left(\mathrm{Re}\rho_0 - \frac{3}{4\pi}\right) t^2 + o_0(t^2), \tag{4.126}$$

which holds at lowest order in $|\langle S\rangle|^2$. This also shows that the characteristic time of early expansion is

$$\tau_{early} \simeq \frac{1}{\sqrt{12\pi a \left(\mathrm{Re}\,\rho_0 - \frac{3}{4\pi}\right)}} \frac{1}{\ell_s^2\,|\langle S\rangle|}, \tag{4.127}$$

and we assume $t \lessapprox \tau_{early}$.

Let us go one order further. Computing the next to leading order correction to (4.122) and using (4.90) to recast the result in terms of powers of the gaugino condensate, we find

$$\begin{aligned}\mathrm{Re}\rho_t =& \mathrm{Re}\rho_0 + 12\pi a\,\ell_s^4|\langle S\rangle|^2 \left(\mathrm{Re}\rho_0 - \frac{3}{4\pi}\right) t^2 + a^2\pi\,\ell_s^8\,|\langle S\rangle|^4 \left\{18\pi\left(\mathrm{Re}\,\rho_0 - \frac{3}{4\pi}\right) + \right.\\ &\left. -\frac{1}{2}\left[32\pi^2\left(\mathrm{Re}\,\rho_0 - \frac{15}{8\pi}\right)\mathrm{Re}\,\rho_0 - 9\right] + 96\pi^2\right\} \frac{\mathrm{Re}\,\rho_0 - \frac{3}{4\pi}}{\mathrm{Re}\,\rho_0}\, t^4 + o_0(t^4).\end{aligned} \tag{4.128}$$

Notice that this is also higher order in $\langle S\rangle$. Thus, at leading order in $|\langle S\rangle|^2$ there is no t^4 correction to the expansion of the $\mathbb{P}^2$.

Late times

At later times $t \sim +\infty$ we already know that $x_t \sim +\infty$ thanks to §4.5.1, so one immediately reads from equation (4.117) the leading order solution

$$x_t^{(0)} = \sqrt{2h}\,t, \tag{4.129}$$

where the exponential suppression of $\nu(x)$ allowed us to neglect it entirely.

Let us find the next to leading order correction to the solution. In order to do that, we start by expanding (4.117) keeping only the first order correction in the $x \sim +\infty$ limit. We get

$$\dot{x} = \sqrt{2h}\left(1 - \frac{1}{2}\left(\frac{x}{x_0}\right)^{\frac{2}{3}} \mathrm{e}^{-\frac{2\pi}{3}\left(x^{\frac{4}{3}} - x_0{}^{\frac{4}{3}}\right)} + o_\infty\left(x^{\frac{2}{3}}\mathrm{e}^{-x^{\frac{4}{3}}}\right)\right). \tag{4.130}$$

Let us parametrize the first order correction as

$$x_t^{(1)} = x_t^{(0)} + \delta_t^{(1)}, \tag{4.131}$$

where $x_t^{(0)}$ is given by (4.129), so that plugging this in (4.130) we find the ODE for the correction $\delta_t^{(1)}$. This yields, using (4.114),

$$\dot{\delta}_t^{(1)} = -\frac{81a}{4(2h)^{\frac{1}{6}}}\,\mu_0^6\,t^{\frac{2}{3}}\mathrm{e}^{-\frac{2\pi}{3}(2h)^{\frac{2}{3}}t^{\frac{4}{3}}}, \tag{4.132}$$

which integrates to

$$\delta_t^{(1)} = -\frac{243a}{8\pi h}\,\mu_0^6\left(\frac{3}{2\pi}\right)^{\frac{1}{4}} \int_0^{\left(\frac{2\pi}{3}\right)^{\frac{3}{4}}\sqrt{2h}\,t} \tau^{\frac{2}{3}}\mathrm{e}^{-\tau^{\frac{4}{3}}}\,\mathrm{d}\tau. \tag{4.133}$$

This is still hard to compute explicitly.

In order to find an explicit expression for this correction, we could proceed as follows. Since we are working in the regime $t \sim +\infty$, $\delta_t^{(1)}$ is weakly dependent on time, given the polynomial-exponential suppression of the integrand. Therefore, at leading order in this approximation, $\delta_t^{(1)}$ is a constant; more precisely,

$$\delta_t^{(1)} = -\frac{243a}{8\pi h}\,\mu_0^6 \left(\frac{3}{2\pi}\right)^{\frac{1}{4}} \int_0^\infty \tau^{\frac{2}{3}} \mathrm{e}^{-\tau^{\frac{4}{3}}}\,\mathrm{d}\tau + o_\infty\left(t^{\frac{2}{3}}\,\mathrm{e}^{-\frac{2\pi}{3}(2h)^{\frac{2}{3}} t^{\frac{4}{3}}}\right). \tag{4.134}$$

Now, one could directly evaluate this numerically. Alternatively, we could try and use an analytic approximation, which is what we choose to do. One finds[28]

$$\delta_t^{(1)} \simeq -\frac{9}{2\sqrt{2}}\left(\frac{3}{\mathrm{e}^2\pi^3}\right)^{\frac{1}{4}} x_0^{-\frac{2}{3}}\mathrm{e}^{\frac{2\pi}{3}x_0^{\frac{4}{3}}} + o_\infty\left(t^{\frac{2}{3}}\,\mathrm{e}^{-\frac{2\pi}{3}(2h)^{\frac{2}{3}} t^{\frac{4}{3}}}\right). \tag{4.140}$$

Therefore, plugging back (4.119) into (4.129) and adding the correction (4.140), one finds the solution at later times including the first order correction

$$x_t = 9\sqrt{\frac{a}{2}}\,\mu_0^3\, x_0^{\frac{1}{3}}\mathrm{e}^{-\frac{\pi}{3}x_0^{\frac{4}{3}}}\, t - \frac{9}{2\sqrt{2}}\left(\frac{3}{\mathrm{e}^2\pi^3}\right)^{\frac{1}{4}} x_0^{-\frac{2}{3}}\mathrm{e}^{\frac{2\pi}{3}x_0^{\frac{4}{3}}} + o_\infty\left(t^{\frac{2}{3}}\,\mathrm{e}^{-\frac{2\pi}{3}(2h)^{\frac{2}{3}} t^{\frac{4}{3}}}\right). \tag{4.141}$$

Going back to $\mathrm{Re}\rho_t = x_t^{\frac{4}{3}}$, we finally find the late time evolution of the 4-cycle volume

[28] We want to approximate analytically the integral

$$I = \int_0^\infty \tau^{\frac{2}{3}}\mathrm{e}^{-\tau^{\frac{4}{3}}}\,\mathrm{d}\tau. \tag{4.135}$$

Thanks to the polynomial-exponential suppression of the integrand, the leading contribution to the integral in (4.135) comes from the region around the maximum point of the integrand

$$\tau_\star = 2^{-\frac{3}{4}}. \tag{4.136}$$

For this reason, the approximation we will use is a variation of the steepest descent method. Let us define

$$v(\tau) = \frac{2}{3}\log\tau - \tau^{\frac{4}{3}}, \tag{4.137}$$

then we adopt the following gaussian approximation

$$\begin{aligned} I &= \int_0^\infty \mathrm{e}^{v(\tau)}\mathrm{d}\tau \\ &\simeq \mathrm{e}^{v(\tau_\star)}\int_{\mathbb{R}} \mathrm{e}^{-\frac{1}{2}|v''(\tau_\star)|(\tau-\tau_\star)^2}\mathrm{d}\tau \\ &= \mathrm{e}^{v(\tau_\star)}\sqrt{\frac{2\pi}{|v''(\tau_\star)|}}, \end{aligned} \tag{4.138}$$

which leads to

$$I \simeq \frac{3}{4}\,2^{-\frac{1}{4}}\sqrt{\frac{\pi}{\mathrm{e}}}. \tag{4.139}$$

One can check this approximation carries a 0.3% relative error.

along the low-energy runaway orbits:

$$\begin{aligned}\mathrm{Re}\rho_t =& 9^{\frac{4}{3}} \left(\frac{a}{2}\right)^{\frac{2}{3}} \mu_0^4 \,\mathrm{Re}\,\rho_0^{\frac{1}{3}} \,\mathrm{e}^{-\frac{4\pi}{9}\mathrm{Re}\,\rho_0} \,t^{\frac{4}{3}} - 27\,(18)^{\frac{1}{3}} \left(\frac{3}{\mathrm{e}^2\pi^3}\right)^{\frac{1}{4}} a^{\frac{1}{6}} \,\mathrm{Re}\,\rho_0^{-\frac{5}{12}} \mathrm{e}^{\frac{8\pi}{9}\mathrm{Re}\rho_0} \,t^{\frac{1}{3}} + \\ &+ o_\infty\left(t\,\mathrm{e}^{-\frac{2\pi}{3}(2h)^{\frac{2}{3}} t^{\frac{4}{3}}}\right).\end{aligned} \tag{4.142}$$

The scalar field Reρ grows like $t^{\frac{4}{3}}$, which is faster than the rate found for early times in equation (4.122), as expected.

Chapter 5

Metric perturbations from gaugino condensation

In §4 we found from the four-dimensional EFT that, precisely due to the condensation taking place on the D7-branes stack wrapped around the $\mathbb{P}^2$ base, the considered type IIB flux compactification on the $\mathbb{P}^2$ is necessarily unstable. The compact four-cycle inflates with time as a power law.

Following the recent efforts in the literature devoted to understanding gaugino condensation from a high-energy ten-dimensional perspective (see §1.3), in this chapter we are going to try and substantiate the instability claim of §4 from a ten-dimensional analysis. Contrary to [34], which purely solves the AdS supersymmetry conditions of the compactification in order to find the exhibited supergravity solution (in the infinite $\mathbb{P}^2$ volume limit) without explicitly assuming any non-vanishing gaugino condensate $\langle S \rangle \neq 0$, we will inject the assumption $\langle S \rangle \neq 0$ in our ten-dimensional analysis. Our approach will be based on the results of [31]. Since from our four-dimensional analysis in §4 it emerges that type IIB string theory on $\mathcal{O}_{\mathbb{P}^2}(-3)$ is unstable, we try to reproduce the inflation of the $\mathbb{P}^2$ base from a ten-dimensional point of view. In order to do that, we will try to compute the ten-dimensional equations of motion for *time-dependent* metric perturbations at leading order in the gaugino condensate.

In §5.1 we review type IIB supergravity, fixing our notation and conventions in the process, and we state the ten-dimensional setup which we are going to perturb. We also review the form of the stress-energy tensor from gaugino condensation as it is found in the literature today, including its putative quartic gaugino coupling, which we already introduced in §1.3.1. In §5.2 we tackle the problem of ten-dimensional metric perturbation with the standard first step of conveniently fixing the gauge. We make use of the ten-dimensional de Donder gauge together with imposing maximal symmetry, and we argue that this completely fixes the gauge and it removes a number of degrees of freedom. In §5.3 we present the precise form of G_3 sourced by gaugino condensation as it is found in [31]. We show that it is not globally defined, and we propose a *trivialization* of the characterizing equation for the 1-form inducing the generalized complex structure deformation, which admits global solutions. We compute the leading order contribution to the ten-dimensional stress-energy tensor coming from gaugino condensation using the global solution found above. In §5.4 we present the global equations of motion for the metric perturbations obtained by this procedure, leaving the contribution from the gaugino condensate stress-energy tensor generic. Indeed, this is only going to fix some boundary conditions of the solution, given its localized nature. In §5.5 we show that the equations of motion that we found do not admit trivial solutions, but they do admit both stationary and time-dependent ones. We exhibit the most general stationary solution, and we argue that it might be identified with the unstable vacuum found during the four-dimensional analysis in §4. We also display a class of time-dependent solutions, and we fix some of its free real parameters imposing that it should reproduce the $\mathbb{P}^2$ inflation phenomenon at early times found from the four-dimensional EFT analysis. From this viewpoint, this class of time-dependent solutions should represent a ten-dimensional description candidate at the metric level of gaugino condensation, for small time intervals.

5.1 Preliminaries

5.1.1 Type IIB supergravity

Type IIB supergravity has $\mathcal{N}=2$ SUSY in $D=10$ spacetime dimensions, corresponding to 32 (real) supersymmetry generators. Its bosonic particle content is divided into an NS-NS sector and a R-R sector. The NS-NS sector consists of the ten-dimensional metric $\hat{g}_{AB}$, the dilaton ϕ and the two-form B_2, while the R-R sector consists of the $p-$forms C_p, with $p=0,2,4$. The fermionic sector of the spectrum consists of two left-handed Majorana-Weyl gravitinos[1], and two right-handed Majorana-Weyl dilatinos. In the present work, however, we are exclusively interested in finding vacuum solutions of the equations of motion, therefore we will not need to worry about the fermionic sector of the theory. It is convenient to group the bosonic degrees of freedom in the following gauge-invariant[2] field strengths:

$$H_3 = \mathrm{d}B_2; \tag{5.1a}$$
$$F_1 = \mathrm{d}C_0; \tag{5.1b}$$
$$F_3 = \mathrm{d}C_2 - C_0 H; \tag{5.1c}$$
$$F_5 = \mathrm{d}C_4 - \frac{1}{2}C_2 \wedge H + \frac{1}{2}B \wedge \mathrm{d}C_2. \tag{5.1d}$$

[1]Being the gravitinos the gauge fields associated to local $\mathcal{N}=2$ SUSY, their presence is expected. Majorana-Weyl fermions exist in $D=2$ mod 8 (Lorentzian) spacetimes [4], and in ten dimensions they have 16 real independent components (possibly brought down to eight if they obey the Dirac equation). Thus, two Majorana-Weyl spinors in ten dimensions do match the 32 supersymmetry generators of $\mathcal{N}=2$ SUSY in $D=10$. Their chirality is peculiar to type IIB supergravity.

[2]We mean invariance with respect to the natural gauge transformations $\delta B_2 = \mathrm{d}\Lambda_1^{NS}$, $\delta C_p = \mathrm{d}\Lambda_{p-1}^{R}$.

The Bianchi identities for these field strengths are

$$\mathrm{d}H_3 = 0; \tag{5.2a}$$
$$\mathrm{d}F_1 = 0; \tag{5.2b}$$
$$\mathrm{d}F_3 = H_3 \wedge F_1; \tag{5.2c}$$
$$\mathrm{d}F_5 = H_3 \wedge F_3. \tag{5.2d}$$

The proper interpretation of (5.1a)-(5.1d) should be as local solutions of their respective Bianchi identities (5.2a)-(5.2d).

The bosonic part of the action for type IIB supergravity in the Einstein frame is [23]

$$S_{IIB} = S_{NS} + S_R + S_{CS}, \tag{5.3}$$

where

$$S_{NS} = \frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}X\sqrt{-\hat{g}}\left[R - \frac{1}{2}\left((\nabla\phi)^2 + \mathrm{e}^{-\phi}\,|H_3|^2\right)\right]; \tag{5.4a}$$
$$S_R = -\frac{1}{4\kappa_{10}^2}\int \mathrm{d}^{10}X\sqrt{-\hat{g}}\left(\mathrm{e}^{2\phi}\,|F_1|^2 + \mathrm{e}^{\phi}\,|F_3|^2 + \frac{1}{2}\,|F_5|^2\right); \tag{5.4b}$$
$$S_{CS} = -\frac{1}{4\kappa_{10}^2}\int C_4 \wedge H_3 \wedge F_3. \tag{5.4c}$$

Here $\hat{g}$ is the determinant of the background ten-dimensional metric, X^A are the ten-dimensional coordinates, κ_{10}^2 is the ten-dimensional Newton's constant given by (1.13), R is the Ricci scalar for the ten-dimensional Levi-Civita connection. Moreover, given a p-form F_p, we use the notation

$$|F_p|^2 = \frac{1}{p!}(F)_{M_1M_2\ldots M_p}(\overline{F})_{N_1N_2\ldots N_p}\hat{g}^{M_1N_1}\hat{g}^{M_2N_2}\cdots\hat{g}^{M_pN_p}. \tag{5.5}$$

More precisely, the global manifestly covariant action (5.3) yields the correct equations of motion for type IIB supergravity once they are supplemented with the self-duality constraint[3]

$$\star_{10} F_5 = F_5. \tag{5.6}$$

In our analysis, we will also make prominent use of a more compact formulation of type IIB supergravity, in terms of complexified scalars and complexified forms, which is better suited in order to achieve manifest covariance under its $SL(2,\mathbb{R})$ classical global symmetry [34] [4]. One can group the dilaton ϕ and the R-R axion C_0 into the so-called axio-dilaton

$$\tau = C_0 + i\mathrm{e}^{-\phi}. \tag{5.7}$$

[3]It is important to stress that the self-duality constraint is to be imposed at the level of the equations of motion, and not at the level of the action. In fact, this very same constraint is the whole reason why one cannot write down a globally defined covariant action for type IIB supergravity.

[4]Note that the $SL(2,\mathbb{R})$ classical global symmetry of type IIB supergravity is broken both by quantum and stringy effects down to the infinite discrete subgroup $SL(2,\mathbb{Z})$ [23].

Additionally, let us define the complex 3-form

$$G_3 = \mathrm{d}C_2 - \tau H_3 \tag{5.8}$$
$$= F_3 - i\mathrm{e}^{-\phi} H_3. \tag{5.9}$$

With these definitions, the action (5.3) can be recast in the form

$$\begin{aligned} S_{IIB} =& \frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}X\sqrt{-\hat{g}}\left[R - \frac{1}{2}\left(\mathrm{e}^{2\phi}\left|\mathrm{d}\tau\right|^2 + \mathrm{e}^{\phi}\left|G_3\right|^2 + \frac{1}{2}\left|F_5\right|^2\right)\right] + \\ &- \frac{i}{8\kappa_{10}^2}\int \mathrm{e}^{\phi} C_4 \wedge G_3 \wedge \overline{G_3}. \end{aligned} \tag{5.10}$$

The equivalence between (5.3) and (5.10) is made apparent by the identities

$$G_3 \wedge \overline{G_3} = 2i\mathrm{e}^{-\phi} F_3 \wedge H_3; \tag{5.11a}$$
$$\left|G_3\right|^2 = \left|F_3\right|^2 + \mathrm{e}^{-2\phi}\left|H_3\right|^2; \tag{5.11b}$$

The equations of motion for type IIB supergravity split in the Einstein equation for $\hat{g}_{AB}$ and in the equations of motion for the NS-NS and R-R forms. The Einstein equations for the ten-dimensional metric in the notation of (5.10) are [29]

$$R_{AB} - \frac{1}{2}\hat{g}_{AB}R = \kappa_{10}^2 T_{AB}^{(IIB)}, \tag{5.12}$$

where

$$\begin{aligned} T_{AB}^{(IIB)} =& \frac{1}{2\kappa_{10}^2}\left(\mathrm{e}^{2\phi}\nabla_{(A}\tau\nabla_{B)}\tau^\star + \mathrm{e}^{\phi}\left|G_3\right|_{AB}^2 + \frac{1}{2}\left|F_5\right|_{AB}^2\right) + \\ &- \frac{1}{4\kappa_{10}^2}\hat{g}_{AB}\left(\mathrm{e}^{2\phi}\left|\mathrm{d}\tau\right|^2 + \mathrm{e}^{\phi}\left|G_3\right|^2\right). \end{aligned} \tag{5.13}$$

Here we use the notation for a given p-form F_p

$$\left|F_p\right|_{AB}^2 = \frac{1}{(p-1)!}(F)_{(A|M_2M_3\dots M_p}(\overline{F})_{|B)N_2N_3\dots N_p}\hat{g}^{M_2N_2}\hat{g}^{M_3N_3}\cdots\hat{g}^{M_pN_p}, \tag{5.14}$$

and ∇_A is the ten-dimensional Levi-Civita (i.e. torsionless) connection. Moreover, we use the symmetrization notation

$$(A|\dots|B) = \frac{A\dots B + B\dots A}{2}, \tag{5.15}$$

with straightforward generalization to p indices. The rest of the equations of motion, in the notation of (5.3), are [29]

$$\mathrm{d}\star\mathrm{d}\phi = -\frac{1}{2}\mathrm{e}^{-\phi}H_3\wedge\star H_3 + \mathrm{e}^{2\phi}F_1\wedge\star F_1 + \frac{1}{2}\mathrm{e}^{\phi}F_3\wedge\star F_3; \tag{5.16a}$$

$$\mathrm{d}\left(\mathrm{e}^{2\phi}\star F_1\right) = -\mathrm{e}^{\phi}H_3\wedge\star F_3; \tag{5.16b}$$

$$\mathrm{d}\left(\mathrm{e}^{-\phi}\star H_3\right) = \mathrm{e}^{\phi}F_1\wedge\star F_3 + F_3\wedge\star F_5; \tag{5.16c}$$

$$\mathrm{d}\left(\mathrm{e}^{\phi}\star F_3\right) = -H_3\wedge\star F_5; \tag{5.16d}$$

$$\mathrm{d}\star F_5 = H_3\wedge F_3; \tag{5.16e}$$

$$\star F_5 = F_5. \tag{5.16f}$$

Notice that the self-duality constraint for F_5 implies the on-shell identity[5]

$$|F_5|^2 = 0. \tag{5.17}$$

5.1.2 Unperturbed background

We will compute at leading order in the gaugino condensate the equations of motion for the perturbations around a specific ten-dimensional background metric in the type IIB supergravity approximation. From this point on, the expressions *zero-order*, *leading order*, *lowest order* and *non-perturbative* will refer exclusively to the perturbative expansion in the gaugino condensate $\langle S\rangle$, unless we specify otherwise.

We assume that the zero-order stress-energy tensor for our background vanishes,

$$T^{(0)}_{AB} = 0, \tag{5.18}$$

where the apex $^{(0)}$ refers to the order in $\langle S\rangle$. This corresponds to setting to zero all fluxes and taking the axio-dilaton to be constant at zero-order:

$$\mathrm{d}\tau^{(0)} = G^{(0)}_3 = F^{(0)}_5 = H^{(0)}_3 = 0. \tag{5.19}$$

Since this is no longer a flux-compactification background, the warping function itself is vanishing at zero-order,

$$A^{(0)} = 0. \tag{5.20}$$

It should be noted that (5.20) technically neglects the background warping sourced by curvature corrections on the D7-brane stack, which we commented below (1.63). Since this effect is not central to our analysis, we will not take it into account. This is consistent with the four-dimensional analysis in §4, as in §4.1 we explain that we do not consider D7-brane backreaction on the warp factor. Moreover, notice that (5.20) corresponds to fixing the universal modulus introduced in §3.2 to

$$a = 1. \tag{5.21}$$

[5] This is easily seen using the identity $\star_{10}F_5\wedge F_5 = |F_5|^2\sqrt{-\hat{g}}\,\mathrm{d}^{10}X$.

We choose the Eguchi-Hanson Ricci-flat metric (see §2.7) for the internal manifold X_0, and Minkowski externally. This trivially solves the ten-dimensional Einstein equations in the vacuum. We denote the background metrics as follows,

$$\mathrm{d}s_{10}^{2\,(0)} = \hat{g}^{(0)}_{AB}\mathrm{d}X^A\mathrm{d}X^B \tag{5.22}$$

$$= \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + g^{(0)}_{i\bar{\jmath}}\mathrm{d}z^i\mathrm{d}\bar{z}^{\bar{\jmath}}. \tag{5.23}$$

Here X^A denotes ten-dimensional coordinates, x^μ denotes the four-dimensional coordinates and z^i denotes the complex six-dimensional coordinates. Recall that

$$\sqrt{g_6^{(0)}} = 8. \tag{5.24}$$

In this whole chapter, we will denote with a hat the ten-dimensional metric $\hat{g}_{AB}$. Its relation with the internal and external metrics in (5.22) is given by

$$\hat{g}_{\mu\nu} = \eta_{\mu\nu}; \tag{5.25a}$$

$$\hat{g}_{i\bar{\jmath}} = \frac{1}{2}\,g_{i\bar{\jmath}}; \tag{5.25b}$$

$$\hat{g}^{i\bar{\jmath}} = 2\,g^{i\bar{\jmath}}. \tag{5.25c}$$

As it is clear from (5.22), in this chapter we will use the following index notation:

- $A, B, \ldots$ denote ten-dimensional indices ($A = 1, \ldots, 10$).
- $m, n, \ldots$ denote real six-dimensional indices ($m = 1, \ldots, 6$).
- $i, j, \ldots$ denote complex six-dimensional indices ($i = 1, 2, 3$).
- $\mu, \nu, \ldots$ denote real four-dimensional indices ($\mu = 0, \ldots, 3$).

5.1.3 Gaugino condensate stress-energy tensor

The complete stress-energy tensor of our model at the non-perturbative level takes the form

$$T_{AB} = T^{(IIB)}_{AB} + T^{\langle S\rangle}_{AB}. \tag{5.26}$$

Here $T^{\langle S\rangle}_{AB}$ is the stress-energy tensor coming from the gaugino condensate coupling contributions to the Dirac-Born-Infeld action of the D7-branes in the ten-dimensional background,

$$T^{\langle S\rangle}_{AB} = -\frac{2}{\sqrt{-\hat{g}}}\frac{\delta S^{\langle S\rangle}_{D7}}{\delta\hat{g}^{AB}}, \tag{5.27}$$

while $T^{(IIB)}_{AB}$ is the type IIB supergravity stress-energy tensor sourced by the background fluxes induced by gaugino condensation via generalization of the complex geometry of the internal space. Therefore, at non-perturbative level the Einstein equations for the exact metric $\hat{g}_{AB}$ take the form

$$R_{AB} - \frac{1}{2}\hat{g}_{AB}R - \kappa_{10}^2 T_{AB}^{(IIB)} = \kappa_{10}^2 T_{AB}^{\langle S\rangle}. \tag{5.28}$$

The precise form of $S_{D7}^{\langle S\rangle}$ (and in turn of $T_{AB}^{\langle S\rangle}$) is not entirely known, and numerous efforts have been put in place in order to shed some light on it. More precisely, it is known that the action of a D7-brane stack contains a coupling between the G_3 flux and the gaugino bilinear $\lambda\lambda$. In a type IIB warped background

$$\mathrm{d}s_{10}^2 = \mathrm{e}^{2A}\mathrm{d}s_{\mathbb{R}^{1,3}}^2 + \mathrm{e}^{-2A}\mathrm{d}s_{X_0}^2, \tag{5.29}$$

this has been computed in the appendix of [31], and it has been re-derived in the appendix of [49], finding

$$S_{D7}^{G\lambda\lambda} = \mathrm{Re}\,\frac{4\pi i}{\ell_s^4}\int \mathrm{e}^{\frac{\phi}{2}}\,(G_3\cdot\Omega)\,\overline{s}\,\delta_D^{(0)}\sqrt{-g_4 g_6}\,\mathrm{d}^{10}X, \tag{5.30}$$

where $s = \frac{1}{16\pi^2}\mathrm{Tr}\,\lambda\lambda$ is the bottom component of the chiral superfield $S = -\frac{1}{16\pi^2}\mathrm{Tr}\,W^\alpha W_\alpha$, $\delta_D^{(0)}$ is the scalar delta-function centered on the internal divisor D wrapped by the D7-branes, Ω is the internal holomorphic 3-form normalized such that

$$\frac{i}{8}\,\Omega\wedge\overline{\Omega} = \sqrt{g_6}\,\mathrm{d}^6y, \tag{5.31}$$

and

$$G_3\cdot\Omega = \frac{1}{3!}G_{m_1m_2m_3}\Omega_{n_1n_2n_3}g^{m_1n_1}g^{m_2n_2}g^{m_3n_3}. \tag{5.32}$$

Here, y^m denote some real internal coordinates, g_4 the unwarped determinant of the external metric, and g_6 the unwarped determinant of the internal metric. On the other hand, it has been argued [44, 47, 49, 45] that the D7-brane action should also contain a quartic gaugino coupling, however as of today there is no general consensus about its specific form. All of these proposals agree nonetheless about the generic structure of the four-gaugino coupling. A precise proposal is found in [49], and in our notation it is given by

$$S_{D7}^{\lambda\lambda\lambda\lambda} = -\frac{\pi}{6}\,\nu\int\sqrt{-g_4g_6}\,\mathrm{e}^{-4A}\,(\Omega\cdot\overline{\Omega})\,|s|^2\,\delta_D^{(0)}\,\mathrm{d}^{10}X, \tag{5.33}$$

where we defined the inverse of the volume transverse to the divisor D, in the internal space X,

$$\nu := \frac{\int_D \sqrt{g_6}\,\mathrm{e}^{-4A}}{\int_X \sqrt{g_6}\,\mathrm{e}^{-4A}}. \tag{5.34}$$

All in all, the relevant part of the D7-brane action as far as gaugino condensation is concerned is

$$S_{D7}^{\langle S\rangle} = S_{D7}^{G\lambda\lambda} + S_{D7}^{\lambda\lambda\lambda\lambda}. \tag{5.35}$$

The stress-energy tensor $T_{AB}^{\langle S\rangle}$ computed from (5.35) inherits the very same decomposition into linear and quadratic dependence on the gaugino bilinear. Moreover, the fact that both the actions (5.30) and (5.33) are localized on the divisor D via the delta-function $\delta^{(0)}_D$ makes it clear that $T^{\langle S\rangle}_{AB}$ is localized as well. Therefore, once we turn the gaugino condensate on and we specialize to our case $D = \mathbb{P}^2$, the stress-energy tensor from D7-brane couplings takes the form[6]

$$T^{\langle S\rangle}_{AB} = T^{\lambda\lambda}_{AB} + T^{\lambda\lambda\lambda\lambda}_{AB}, \qquad T^{\lambda\lambda}, T^{\lambda\lambda\lambda\lambda} \propto \delta^{(0)}_{\mathbb{P}^2}. \tag{5.36}$$

Using (5.30) and assuming (5.33) is correct, one can explicitly compute the gaugino condensate stress-energy tensor. The result is[7]

$$T^{\lambda\lambda}_{\mu\nu} = \mathrm{Re}\left(\frac{4\pi i}{\ell_s^4}\,\mathrm{e}^{4A+\frac{\phi}{2}}(G_3\cdot\Omega)\,\overline{\langle S\rangle}\,\delta^{(0)}_{\mathbb{P}^2}\,\eta_{\mu\nu}\right); \tag{5.37a}$$

$$T^{\lambda\lambda}_{ij} = -\mathrm{Re}\left(\frac{4\pi i}{\ell_s^4}\,\mathrm{e}^{\frac{\phi}{2}}\,\overline{\langle S\rangle}\,\delta^{(0)}_{\mathbb{P}^2}\,(G_3\cdot\Omega)_{ij}\right); \tag{5.37b}$$

$$T^{\lambda\lambda\lambda\lambda}_{\mu\nu} = -\frac{\pi}{6}\,\nu\,\mathrm{e}^{-4A}\,(\Omega\cdot\overline{\Omega})\,|\langle S\rangle|^2\,\delta^{(0)}_{\mathbb{P}^2}\,\eta_{\mu\nu}, \tag{5.37c}$$

where

$$(G_3\cdot\Omega)_{mn} = \frac{1}{2}\,G_{(m|k_1k_2}\Omega_{|n)l_1l_2}\,g^{k_1l_1}g^{k_2l_2}. \tag{5.38}$$

Here we replaced the gaugino bilinear s with its vacuum expectation value $\langle S\rangle$. This assumes we are working at low enough energies for s to become non-dynamical, but high enough so that the ten-dimensional description does not break down[8].

Once the EOMs are have been solved *outside* of $\mathbb{P}^2$, (5.36) provides some boundary conditions. Let us point out that, due to (5.30), if G_3 has a singular component sourced by gaugino condensation $G_3^{sing} \propto \delta^{(0)}_{\mathbb{P}^2}$, then we would end up with

$$T^{\lambda\lambda}_{sing} \propto \left(\delta^{(0)}_{\mathbb{P}^2}\right)^2, \tag{5.39}$$

barring accidental cancellations. Here, the square of the scalar delta-function $\left(\delta^{(0)}_{\mathbb{P}^2}\right)^2$ makes its appearance. As we already mentioned in §1.3.1, this is actually ill-defined, but it could be interpreted as a divergence $\delta(0)$. For this reason, its cancellation in the Einstein equation is a necessity. Moreover, it should be stressed that, due to the non-compactness of X_0, in our setup it holds $\nu = 0$, so that

$$T^{\lambda\lambda\lambda\lambda}_{AB} = 0 \qquad \text{over all } X_0. \tag{5.40}$$

More generally, assuming (5.33) holds, this shows that local models are not fit to probe quartic gaugino couplings on D7-branes. In deriving the Einstein equations for the perturbation, we will remain agnostic about the explicit form of $T^{\langle S\rangle}$, keeping it implicit. Starting from §5.4.2, however, we will adopt as Ansatz for $T^{\langle S\rangle}$ a slight generalization of the form proposed in [49], in order to study some explicit solutions.

[6] Recall that, in the coordinates (z^i), $\mathbb{P}^2$ is defined by $r^2 = 0$. Therefore, $\delta^{(0)}_{\mathbb{P}^2} \propto \delta(r^2)$, which also has the correct dimension $[\delta^{(0)}_{\mathbb{P}^2}] = [\ell_s]^{-2}$ needed to make the previous actions dimensionless. Its precise expression is (2.140), see §2.9.3 and §A.5.3 for more details.

[7] Here we used the fact that $(F_3\cdot\Omega)g_{i\bar{\jmath}} = (F_3\cdot\Omega)_{i\bar{\jmath}}$ for any 3-form F_3.

[8] Clearly, this requires the scale hierarchy (1.133) to hold.

5.2 Fixing the gauge

The trace-reversed Einstein equations take the form

$$R_{AB} = \kappa_{10}^2 \overline{T}_{AB}; \tag{5.41}$$

$$\overline{T}_{AB} = T_{AB} - \frac{1}{8}\hat{g}_{AB}T, \tag{5.42}$$

where $T = T_{AB}\,\hat{g}^{AB}$, and the total stress energy tensor is given by (5.26). Let us introduce a generic perturbation to the background metric

$$\hat{g}_{AB} = \hat{g}^{(0)}_{AB} + h_{AB}, \tag{5.43}$$

where $\hat{g}^{(0)}_{AB}$ is given by (5.22). The linear perturbation to the ten-dimensional Ricci tensor is given by[9]

$$\delta R_{AB} = -\frac{1}{2}\left(\Box^{(0)}h_{AB} + \nabla^{(0)}_A\nabla^{(0)}_B h - \nabla^{(0)}_C\nabla^{(0)}_A h^C{}_B - \nabla^{(0)}_C\nabla^{(0)}_B h^C{}_A\right), \tag{5.44}$$

where indices are raised with the background metric, $h = h_{AB}\,\hat{g}^{(0)AB}$, the apex $\nabla^{(0)}_A$ denotes the Levi-Civita connection associated to the background metric $\hat{g}^{(0)}_{AB}$, and the background Laplace-Beltrami operator is defined as $\Box^{(0)} = \hat{g}^{(0)AB}\nabla^{(0)}_A\nabla^{(0)}_B$.

Naturally, in order to find physical solutions for the perturbation h_{AB} we should fix the gauge, analogously to how one proceeds in order to derive the form of gravitational waves in general relativity. In appendix B we schematically review how the story goes in this simpler case.

5.2.1 Ten-dimensional de Donder gauge

We choose the de Donder gauge

$$\Box\tilde{X}^M = 0, \tag{5.45}$$

where the tilde stresses the fact that we are performing a gauge transformation $\tilde{X}^A = X^A + \xi^A$ such that

$$\Box^{(0)}\xi^M = -\Box X^M. \tag{5.46}$$

We are going to assume that this equation admits at least one solution. In general, we do not expect this solution to be unique. The linearized de Donder gauge around a generic curved background takes the form

$$\nabla^{(0)}_A h^{AM} = \frac{1}{2}\nabla^{(0)M}h + h^{AB}\Gamma^{(0)M}_{AB}, \tag{5.47}$$

where $\Gamma^{(0)M}_{AB}$ denotes the Christoffel symbols of the background metric. Before plugging (5.47) in (5.44), we prefer to fix the gauge completely by specifying an explicit Ansatz for the perturbation. As we will see, this will simplify things considerably.

[9] Computed with the Mathematica package xAct.

5.2.2 Ansatz for the perturbation

Motivated by the time-dependent solution that we found for the ρ chiral field in the low-energy EFT, we will look for perturbations exhibiting maximal symmetry on the internal space $X_0 = \mathcal{O}_{\mathbb{P}^2}(-3)$ and obeying the cosmological principle externally, namely invariance under spacial rotations and translations. The external cosmological principle amounts to imposing $\mathbb{R}^3 \rtimes SO(3)$ symmetry. From §2.4 we known that the maximal internal symmetry is $U(3)$, thus we require our Ansatz to possess a $\mathbb{R}^3 \rtimes SO(3) \times U(3)$ symmetry.

The only $\mathbb{R}^3 \rtimes SO(3) \times U(3)$-invariant coordinates are the external time t and the internal distance from the cone base $r^2 = z^i \overline{z}_i$, therefore we can already assume to be in some coordinate system where the perturbation depends only on (t, r^2). At the non-perturbative level and in these coordinates, the most general ten-dimensional metric compatible with $\mathbb{R}^3 \rtimes SO(3) \times U(3)$ symmetry is

$$\hat{g}_{AB}\,\mathrm{d}X^A \mathrm{d}X^B = -2D_h \mathrm{d}t^2 + \mathrm{e}^{2A_h}\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + B_h \mathrm{d}r^2 \mathrm{d}t + C_h \mathrm{d}^c r^2 \mathrm{d}t + \mathrm{e}^{-2A_h} g_{i\overline{\jmath}} \mathrm{d}z^i \mathrm{d}\overline{z}^{\overline{\jmath}}, \tag{5.48}$$

where

$$\begin{aligned} g_{i\overline{\jmath}}\,\mathrm{d}z^i \mathrm{d}\overline{z}^{\overline{\jmath}} =& \mathrm{e}^{2F_h}\left(1+\frac{c^6}{r^6}\right)^{\frac{1}{3}} \mathrm{d}z^i \mathrm{d}\overline{z}_i - \mathrm{e}^{2G_h}\frac{c^6}{r^6}\left(1+\frac{c^6}{r^6}\right)^{-\frac{2}{3}} \frac{\partial r^2 \overline{\partial} r^2}{r^2} + \\ &+ H_h \left(\partial r^2\right)^2 + \overline{H}_h \left(\overline{\partial} r^2\right)^2 , \end{aligned} \tag{5.49}$$

where the A_h, B_h, C_h, D_h, F_h, G_h, H_h are all functions only of (t, r^2), with A_h, B_h, C_h, D_h, F_h, G_h real and H_h complex. Recall that $\partial r^2 = \overline{z}_i \mathrm{d}z^i$ is the holomorphic differential, or Dolbeault operator. By comparison with the background metric (5.22), the perturbation functions are required to vanish at zero-order. More precisely, we will show in §5.3.2 that the Einstein equations (5.41) imply that at leading order

$$A_h, B_h, C_h, D_h, F_h, G_h, H_h \sim |\langle S\rangle|^2 . \tag{5.50}$$

However, in a perturbative framework, the Ansatz (5.48, 5.49) is redundant when working at leading order in $\langle S\rangle$. In fact, in §C we show that it is always possible to set $D = 0$ and $H = 0$ by means of a change of coordinates[10], up to corrections of order $|\langle S\rangle|^4$, which is beyond leading order. Therefore, the minimal $\mathbb{R}^3 \rtimes SO(3) \times U(3)$-symmetric leading order metric Ansatz only involves five independent functions, and it is given by

$$\hat{g}_{AB}\mathrm{d}X^A \mathrm{d}X^B = \mathrm{e}^{2A_h}\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + B_h \mathrm{d}r^2 \mathrm{d}t + C_h \mathrm{d}^c r^2 \mathrm{d}t + \mathrm{e}^{-2A_h} g_{i\overline{\jmath}} \mathrm{d}z^i \mathrm{d}\overline{z}^{\overline{\jmath}}; \tag{5.51a}$$

$$g_{i\overline{\jmath}}\mathrm{d}z^i \mathrm{d}\overline{z}^{\overline{\jmath}} = \left(\mathrm{e}^{2F_h}\left(1+\frac{c^6}{r^6}\right)^{\frac{1}{3}} \delta_{i\overline{\jmath}} - \mathrm{e}^{2G_h}\frac{c^6}{r^6}\left(1+\frac{c^6}{r^6}\right)^{-\frac{2}{3}} \frac{\overline{z}_i z_{\overline{\jmath}}}{r^2}\right) \mathrm{d}z^i \mathrm{d}\overline{z}^{\overline{\jmath}}, \tag{5.51b}$$

[10]Notice that the time dependence of the perturbation functions make it so that the possibility of setting $H = 0$ by coordinate redefinition, that is assuming that gaugino perturbations preserve hermiticity of the internal metric, is *a priori* non-trivial.

where the A_h, B_h, C_h, F_h, G_h are all real functions of (t, r^2), and it is understood that all $\mathcal{O}(|\langle S\rangle|^4)$ terms on the r.h.s. should be discarded. Thus, the minimal $\mathbb{R}^3 \rtimes SO(3) \times U(3)$-symmetric Ansatz for the leading order ten-dimensional metric perturbation is

$$\begin{aligned} h_{AB}\mathrm{d}X^A\mathrm{d}X^B = {}& 2A_h\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + B_h\mathrm{d}r^2\mathrm{d}t + C_h\mathrm{d}^c r^2\mathrm{d}t+ \\ & + \left[2(F_h - A_h)\left(1+\frac{c^6}{r^6}\right)^{\frac{1}{3}}\delta_{i\bar{\jmath}} - 2(G_h - A_h)\frac{c^6}{r^6}\left(1+\frac{c^6}{r^6}\right)^{-\frac{2}{3}}\frac{\bar{z}_i z_{\bar{\jmath}}}{r^2}\right]\mathrm{d}z^i\mathrm{d}\bar{z}^{\bar{\jmath}}. \end{aligned} \tag{5.52}$$

5.2.3 Gauge-fixed Einstein equations

We will assume the Ansatz (5.52), together with the de Donder gauge (5.47), to fix the gauge completely. On general grounds, the de Donder gauge equation (5.46) around a flat background requires ten degrees of freedom to be fixed in order to admit a single solution. However, a solution to the equation may not even exist for complicated enough background metrics, even without fixing any degree of freedom. All in all, we surmise that the number of degrees of freedom we need to fix for (5.46) to admit a single solution lies between none and ten. On the other hand, a quick counting shows that the Ansatz (5.52) fixes at least 39 degrees of freedom[11], so that we expect it to fix the gauge completely and to impose at least 29 physical constraints on the solution. By a similar token, assuming the Einstein equations to provide at most 55 degrees of freedom to the perturbation (which is true for a flat background), imposing the Ansatz (5.52) together with the ten conditions from the de Donder gauge (5.47) would seem to leave us with at most six free degrees of freedom, before imposing further boundary conditions. In §5.5 we show that this does allow us to find a unique solution.

Notice that, thanks to the fact that we assumed the perturbation to preserve internal hermiticity, the de Donder gauge condition (5.47) simplifies to

$$\nabla^{(0)}_A h^{AM} = \frac{1}{2}\nabla^{(0)M} h, \tag{5.53}$$

where we used the fact that the Christoffel symbols for a Kähler manifold are *pure* in their indices, namely $\Gamma^{(0)k}_{i\bar{\jmath}} = \Gamma^{(0)\bar{k}}_{ij} = 0$. Using (5.53), the Ricci tensor perturbation (5.44) simplifies to

$$\delta R_{AB} = -\frac{1}{2}\left[\Box^{(0)}h_{AB} + 2h^{CD}R^{(0)}_{CADB}\right], \tag{5.54}$$

where $R^{(0)}_{CABD}$ denotes the background Riemann tensor. Therefore, the trace-reverse Einstein equations (5.41) in this gauge take the form[12]

$$\Box^{(0)}h_{AB} + 2h^{CD}R^{(0)}_{CADB} = -2\kappa_{10}^2\overline{T}^{\mathfrak{lo}}_{AB}, \tag{5.55}$$

where $\overline{T}^{\mathfrak{lo}}_{AB}$ is the (trace-reversed) leading order contribution to the complete stress-energy tensor (5.26).

[11]This comes from imposing $h_{\mu i} = 0$; $h_{ij} = 0$; $h_{00} = -h_{\mu\mu}$; $h_{\mu\mu} = h_{\nu\nu}$; $h_{\mu\nu} = 0$; $h_{0\mu} = 0$ where $\mu, \nu = 1, 2, 3$ and $\mu \neq \nu$.

[12]This result agrees with equation (349) of [35], in units such that $\kappa_{10}^2 = 8\pi$.

5.3 Fluxes from gaugino condensation

In [31] it is shown that gaugino condensation on D7-branes dynamically deforms the supersymmetry conditions of the internal manifold, which have to be treated employing the formalism of *generalized complex geometry.* Through a perturbative approach in $\langle S\rangle$ and assuming all the background fluxes to be vanishing, there it is shown that, in the case of gaugino condensation occurring on a stack of D7-branes (and possibly O7-planes) wrapping a divisor D, the deformed supersymmetry conditions for the fluxes *at first order* in $\langle S\rangle$ are solved by $F_5=0$, $\mathrm{d}\tau=\mathrm{d}\tau^{(0)}$ and by[13]

$$G_3=\frac{i}{4}\mathrm{e}^{-\phi^{(0)}}\partial\theta^m\wedge\iota_m\overline{\Omega}+\frac{i}{4}\mathrm{e}^{-\phi^{(0)}}\left(\nabla_m\phi\right)^{(0)}\overline{\theta}^m\Omega+\frac{i}{4}\mathrm{e}^{-\phi^{(0)}}\left(\nabla^m\theta_m\right)\overline{\Omega},\tag{5.56}$$

where all of the contractions are performed using the background *internal* metric $g^{(0)}_{i\bar{\jmath}}$, the apex $^{(0)}$ denotes the zero-order component in $\langle S\rangle$, ι_m is the interior derivative along ∂_m, Ω is the internal holomorphic 3-form with normalization given by $\frac{i}{8}\mathrm{e}^{-\phi^{(0)}}\Omega\wedge\overline{\Omega}=\sqrt{g_6}\mathrm{d}^6y$, namely

$$\Omega=\frac{2^{\frac{3}{2}}}{3!}\mathrm{e}^{\frac{\phi^{(0)}}{2}}\epsilon_{ijk}\mathrm{d}z^i\wedge\mathrm{d}z^j\wedge\mathrm{d}z^k,\tag{5.57}$$

and finally $\theta=\theta_m\mathrm{d}y^m$ is the $(1,0)$-form specified by the conditions

$$\partial\theta=0;\tag{5.58a}$$

$$\overline{\partial}\theta=-2i\ell_s^4\langle S\rangle\delta^2_D,\tag{5.58b}$$

where δ^2_D is the delta 2-form localized on D. Notice that the structure of (5.56) is harmonic $(1,2)+(3,0)+(0,3)$, that is it contains a IASD $(1,2)+(3,0)$ part, and a ISD part $(0,3)$. This is agreement with [28], as we already explained in §1.3.

The origin of this $(1,0)$-form is the following. One of the supersymmetry conditions for the complex polyform $\mathcal{Z}=\mathcal{Z}_1+\mathcal{Z}_3+\mathcal{Z}_5$ in the case of condensing D7-branes on the $\mathbb{P}^2$, that is for non-vanishing gaugino condensate $\langle S\rangle\neq0$, becomes

$$\mathrm{d}\mathcal{Z}=-2i\ell_s^4\langle S\rangle\delta^2_{\mathbb{P}^2}.\tag{5.59}$$

This is solved by

$$\mathcal{Z}=\theta+\Omega,\tag{5.60}$$

where Ω is the holomorphic 3-form associated with the CY structure of X_0, while θ is a 1-form solving

$$\mathrm{d}\theta=-2i\ell_s^4\langle S\rangle\delta^2_{\mathbb{P}^2}.\tag{5.61}$$

A non-vanishing 1-form contribution $\mathcal{Z}_1$ signals a genuine complex structure deformation, which in this case is manifestly sourced by the gaugino condensate thanks to (5.61). A local solution is found in [31], imposing θ to be a (1,0)-form. This condition stems from

[13] This is equation (5.22) of [31].

the matching with β *deformations*, employed there. Since $\delta^2_{\mathbb{P}^2}$ is a (1,1)-form, then it has to satisfy (5.58a, 5.58b), which in our setup are

$$\partial\theta = 0; \tag{5.62a}$$
$$\overline{\partial}\theta = -2i\ell_s^4\langle S\rangle\delta^2_{\mathbb{P}^2}. \tag{5.62b}$$

These are solved locally by means of the Poincaré-Lelong lemma, finding

$$\theta = -\frac{\ell_s^4}{\pi}\partial w; \tag{5.63}$$
$$w(z) = \langle S\rangle\log h(z) + w_0, \tag{5.64}$$

where $h(z)$ is the holomorphic section of the $\mathbb{P}^2$ line bundle that defines $\mathbb{P}^2$ via $h(z)|_{\mathbb{P}^2} = 0$, and w_0 is a constant. Since X_0 is a negative line bundle over $\mathbb{P}^2$, it does not admit global holomorphic sections, therefore (5.64) is necessarily local. Concretely, this provides four maximally extended, but still local, solutions obtained by taking the local holomorphic sections

$$\begin{aligned} h^{(i)}\left(\xi_{(i)}\right) &= \frac{1}{\ell_s^3}\xi_{(i)} \qquad \textit{on} \quad \mathcal{U}_{(i)}; \\ h^{(4)} &= 1 \qquad \textit{on} \quad \mathcal{U}_{(4)}, \end{aligned} \tag{5.65}$$

which correspond to four local 1-forms $\theta^{(i)}$, $\theta^{(4)}$ defined on $\mathcal{U}_{(i)}, \mathcal{U}_{(4)}$, solving (5.61) on $\mathcal{U}_{(i)}, \mathcal{U}_{(4)}$ only. Notice that $\theta^{(4)} = 0$, which is indeed a (1,0)-form solving (5.58a-5.58b) on $\mathcal{U}_{(4)}$, namely

$$\partial\theta = 0; \tag{5.66a}$$
$$\overline{\partial}\theta = 0. \tag{5.66b}$$

On the other hand,

$$\theta^{(i)} = -\ell_s^4\langle S\rangle\frac{1}{\pi}\frac{\mathrm{d}\xi_{(i)}}{\xi_{(i)}}. \tag{5.67}$$

One can check this solves (5.58b) using the well-known identity (2.132). If we were to extend $\theta^{(i)}$ e.g. to $\mathcal{U}_{(4)}$, using (2.23) we would find

$$\theta^{(i)} = -\ell_s^4\langle S\rangle\frac{3}{\pi}\frac{\mathrm{d}z^i}{z^i}, \tag{5.68}$$

which satisfies

$$\overline{\partial}\theta^{(i)} = 3\ell_s^4\langle S\rangle\delta(z^i)\mathrm{d}z^i\wedge\mathrm{d}\overline{z}^{\overline{\imath}} \neq 0. \tag{5.69}$$

This explicitly shows that $\theta^{(i)}$ only solve (5.58b) on $\mathcal{U}_{(i)}$, and no holomorphic extension is possible.

This result is trivial. Indeed, (5.61) does not admit globally defined 1-form solutions θ, since $\delta^2_{\mathbb{P}^2}$ is non-trivial in cohomology. This seems to be an issue, since were flux (5.56) only locally defined, its physical interpretation would appear obscure to us. One could argue that (5.56) might still be globally defined for classically constant dilaton $(\nabla_m \phi)^{(0)}$, since then θ enters G_3 only through derivative combinations, and patching together the $\theta^{(i)}$ could be a well-defined operation. Unfortunately, one can see immediately that this is not the case, since

$$\nabla^m \theta^{(i)}_m - \nabla^m \theta^{(j)}_m = -3\ell_s^4 \langle S\rangle \left(g^{i\bar{\imath}}|_{z^i=0}\delta(z^i) - g^{j\bar{\jmath}}|_{z^j=0}\delta(z^j)\right) \neq 0. \tag{5.70}$$

This motivates us to seek an appropriated *trivialization* of (5.61) which would admit a $U(3)$-invariant global solution, in order to proceed with our computation of the Einstein equations for the perturbation functions.

5.3.1 Trivializing the complex geometry deformation

Let us illustrate how one could go about solving the issue of absence of global solutions of the defining equation for θ (5.61) by directly modifying in a possibly natural way. Contracting both sides of (5.61) with the Kähler form J yields

$$\nabla^m \theta_m = -2\ell_s^4 \langle S\rangle \delta^{(0)}_{\mathbb{P}^2}, \tag{5.71}$$

where $\delta^{(0)}_{\mathbb{P}^2}$ is given by (2.140), and we used[14] $J \lrcorner \mathrm{d}\theta = i\nabla^m\theta_m$. Local solutions to (5.61) satisfy this equation. Let us assume (5.71) to be more fundamental than (5.61), and let us require the global solution we are looking for to obey (5.71). This identifies the trivialization of $\delta^{2}_{\mathbb{P}^2}$ completely. Indeed, (5.71) does admit global $U(3)$-invariant solutions thanks to the fact that the topological obstruction from cohomology has been evaded by contraction with the Kähler form. Imposing that θ is still a $(1,0)$-form, so that the construction of [31] remains applicable, yields a unique solution (up to a constant) given by

$$\theta = 2\ell_s^4 \langle S\rangle \partial \kappa_{(4)}, \tag{5.72}$$

where $\kappa_{(4)}$ is given by (2.101). This is indeed globally defined thanks to (2.135). Moreover, notice that it is singular in $r^2 = 0$ because $\kappa_{(4)}$ is, see (2.101), once it is interpreted as globally defined, as in non-patchwise defined. Naturally, (5.72) no longer satisfies (5.61). Instead, from direct computation and using (2.135), one finds that (5.72) obeys

$$\mathrm{d}\theta = -2i\ell_s^4 \langle S\rangle \left(\delta^{2}_{\mathbb{P}^2} - \omega\right). \tag{5.73}$$

This suggests a trivialization in cohomology of $\delta^{2}_{\mathbb{P}^2}$ by subtraction of a cohomological counterterm ω:

$$\delta^{2}_{\mathbb{P}^2} \overset{\textit{trivialize}}{\longmapsto} \delta^{2}_{\mathbb{P}^2} - \omega. \tag{5.74}$$

Recall that ω is the compactly-supported and primitive 2-form Poincaré-dual to the $\mathbb{P}^2$ exceptional divisor, see §2.9.1. Clearly, (5.73) is no longer subject to the topological obstruction against global solutions since $[\omega] = [\delta^{2}_{\mathbb{P}^2}]$. By (2.135), a more explicit expression for the trivialized 2-form in (5.74) is given by

$$\delta^{2}_{\mathbb{P}^2} - \omega = -\frac{1}{2}\mathrm{d}\mathrm{d}^c \kappa_{(4)}. \tag{5.75}$$

[14]Recall the definition of the interior product for 2-forms $\alpha_2 \lrcorner \beta_2 = \frac{1}{2}\alpha^{mn}\beta_{mn}$.

Although still being singular in $r^2 = 0$ (since $\kappa_{(4)}$ is), (5.75) is no longer concentrated in $r^2 = 0$. Notice that (5.74) effectively corresponds to subtracting from $\delta^2_{\mathbb{P}^2}$ exactly the cohomologically non-trivial part, represented by the harmonic form ω, so that only its exact component is left behind, see (2.134). Let us point out that, since ω is regular on $\mathbb{P}^2$, approaching $r^2 \sim 0$ it holds $\delta^2_{\mathbb{P}^2} - \omega \sim_0 \delta^2_{\mathbb{P}^2}$. This observation may suggest that the original defining equation for θ (5.61) could be interpreted as correct only in the vicinity of the exceptional divisor $\mathbb{P}^2$, while (5.73) is the correct modification holding everywhere on X_0. However, let us make it clear that this procedure has been engineered in order to find globally defined complex geometry deformations while trying to be as little invasive as possible towards the defining equation for θ (5.61), and that a putative proper physical interpretation for it remains obscure to us.

In the rest of this work, we will use the global solution (5.72) to compute the leading order Einstein equations. Let us make it clear that this is just Since we assume that the dilaton is constant at lowest order, in our model (5.57) becomes

$$G_3^{\mathfrak{lo}} = \frac{i}{4}\mathrm{e}^{-\frac{\phi^{(0)}}{2}}\partial\theta^m \wedge \iota_m\overline{\Omega} + \frac{i}{4}\mathrm{e}^{-\frac{\phi^{(0)}}{2}}\left(\nabla^m\theta_m\right)\overline{\Omega}. \tag{5.76}$$

Already at this level we can see that G_3 consists of a bulk $(1,2)$ contribution, i.e. which is non-vanishing over the whole internal manifold, and by a $(0,3)$ term localized on the 4-cycle $\mathbb{P}^2$ where the D7-branes and the O7-plane are wrapped. This means that G_3 sources a bulk component to the stress-energy tensor, which survives even when considering the Einstein equations away from the $\mathbb{P}^2$. Therefore, we expect non-trivial dynamics in the perturbation functions generated from this flux. Inserting the solution for θ (5.72) in (5.76) we find

$$G_3^{\mathfrak{lo}} = -i\sqrt{2}\,\mathrm{e}^{-\frac{\phi^{(0)}}{2}}\ell_s^4\langle S\rangle\left(\frac{9}{2\pi}\frac{c^4}{r^8}\frac{1}{2!}\overline{z}^{\overline{\imath}}\epsilon_{\overline{\imath}\overline{\jmath}\overline{k}}\partial r^2\wedge \mathrm{d}\overline{z}^{\overline{\jmath}}\wedge\mathrm{d}\overline{z}^{\overline{k}} + \delta^{(0)}_{\mathbb{P}^2}\frac{1}{3!}\epsilon_{\overline{\imath}\overline{\jmath}\overline{k}}\mathrm{d}\overline{z}^{\overline{\imath}}\wedge\mathrm{d}\overline{z}^{\overline{\jmath}}\wedge\mathrm{d}\overline{z}^{\overline{k}}\right). \tag{5.77}$$

This is an $SU(3)$-invariant solution, while it spontaneously breaks the $U(1)$ factor of $U(3)$ (see (2.29)) since it changes sign under the orientifold holomorphic involution $\sigma : z^i \mapsto -z^i$, and $\sigma \in U(1)$. However, this is expected, because only the σ-odd components of G_3 are singlets of the orientifold action, and thus are not projected out of the theory. In particular, this solution shows that $G_3^{\mathfrak{lo}}, F_3^{\mathfrak{lo}}$ and $H^{\mathfrak{lo}}$ all scale like $\langle S\rangle$.

5.3.2 Leading order stress-energy tensor

The first step in order to find the explicit equations for the perturbation functions implied by the Einstein equations (5.55) is to compute the leading order stress-energy tensor $T^{\mathfrak{lo}}_{AB}$. Taking into account the zero-order value of the background forms (5.19), from (5.26) and (5.12) one can recast the lowest order contribution to the stress-energy tensor as

$$\begin{aligned} T^{\mathfrak{lo}}_{AB} = {} & \frac{1}{2\kappa_{10}^2}\left[\mathrm{e}^{2\phi^{(0)}}\left(\nabla_{(A}\tau\right)^{\mathfrak{lo}}\left(\nabla_{B)}\tau^\star\right)^{\mathfrak{lo}} + \mathrm{e}^{\phi^{(0)}}\left|G_3^{\mathfrak{lo}}\right|^2_{AB} + \frac{1}{2}\left|F_5^{\mathfrak{lo}}\right|^2_{AB}\right] + \\ & - \frac{1}{4\kappa_{10}^2}\hat{g}_{AB}\left[\mathrm{e}^{2\phi^{(0)}}\left|(\mathrm{d}\tau)^{\mathfrak{lo}}\right|^2 + \mathrm{e}^{\phi^{(0)}}\left|G_3^{\mathfrak{lo}}\right|^2\right] + T^{\langle S\rangle\mathfrak{lo}}_{AB}, \end{aligned} \tag{5.78}$$

where the apex ${}^{\mathrm{lo}}$ extracts the lowest order contribution, and it is understood that any higher-order contribution to the r.h.s. is to be discarded. Let us stress the fact that, for instance, $\left(|G_3|^2\right)^{\mathrm{lo}} = \left|G_3^{\mathrm{lo}}\right|^2$ exactly thanks to the fact that we assume G_3 to vanish at zero-order, that is $G_3^{(0)} = 0$. We already showed in (5.77) that $G_3^{\mathrm{lo}} \sim \langle S \rangle$. In this subsection, we build upon this result to show that

$$F_5^{\mathrm{lo}} \sim |\langle S \rangle|^2 \,; \tag{5.79a}$$

$$(\mathrm{d}\tau)^{\mathrm{lo}} \sim |\langle S \rangle|^2 \,; \tag{5.79b}$$

$$T_{AB}^{\langle S \rangle \mathrm{lo}} \sim |\langle S \rangle|^2, \tag{5.79c}$$

so that the only contribution to the stress-energy tensor at leading order comes from G_3 and from $T_{AB}^{\langle S \rangle}$, and

$$T_{AB}^{\mathrm{lo}} \sim |\langle S \rangle|^2 \,. \tag{5.80}$$

While doing that, we will explicitly compute the leading order contributions to G_3 and to the stress-energy tensor of our model. Notice that (5.80) implies together with the Einstein equations (5.55) that the linear perturbation h_{AB} and in turn the perturbation functions A_h, B_h, C_h, F_h, G_h scale in general like $|\langle S \rangle|^2$.

Leading order scaling of F_5 and $\mathrm{d}\tau$

Consider the Bianchi identity for F_5 (5.16e) in terms of G_3, keeping it at the lowest order:

$$\mathrm{d}F_5^{\mathrm{lo}} = \frac{i}{2} \mathrm{e}^{\phi^{(0)}} G_3^{\mathrm{lo}} \wedge \overline{G_3}^{\mathrm{lo}}. \tag{5.81}$$

This shows that $F_5^{\mathrm{lo}} \sim |\langle S \rangle|^2$. Using (5.77) one can even compute this equation explicitly, finding

$$\mathrm{d}F_5^{\mathrm{lo}} = 8\, \ell_s^8 \, |\langle S \rangle|^2 \left[-\frac{81}{4\pi^2} \frac{c^8}{r^{12}} + \left(\delta_{\mathbb{P}^2}^{(0)} \right)^2 \right] \mathrm{d}^6 y, \tag{5.82}$$

where we used (5.24).

Let us move on to $\mathrm{d}\tau$. Consider first the equation of motion for F_1 (5.16b) at lowest order:

$$\mathrm{d}\left(\mathrm{e}^{2\phi^{(0)}} \star^{(0)} F_1^{\mathrm{lo}} \right) = -\mathrm{e}^{\phi^{(0)}} H^{\mathrm{lo}} \wedge \star^{(0)} F_3^{\mathrm{lo}}, \tag{5.83}$$

where $\star^{(0)}$ denotes the Hodge dual with respect to the background metric. This shows that $\mathrm{d}C_0$ is of order $|\langle S \rangle|^2$. Now consider the equation of motion for ϕ (5.16a) at lowest order:

$$\mathrm{d} \star^{(0)} \mathrm{d}\phi^{\mathrm{lo}} = -\frac{1}{2} \mathrm{e}^{-\phi^{(0)}} H^{\mathrm{lo}} \wedge \star^{(0)} H^{\mathrm{lo}} + \mathrm{e}^{2\phi^{(0)}} F_1^{\mathrm{lo}} \wedge \star^{(0)} F_1^{\mathrm{lo}} + \frac{1}{2} \mathrm{e}^{\phi^{(0)}} F_3^{\mathrm{lo}} \wedge \star^{(0)} F_3^{\mathrm{lo}}. \tag{5.84}$$

Thanks to the fact that $F_1^{\mathrm{lo}} \sim |\langle S \rangle|^2$, this shows that $\mathrm{d}\phi^{\mathrm{lo}}$ too scales like $|\langle S \rangle|^2$. This implies that at leading order $\mathrm{d}\tau = \mathrm{d}\left(C_0 + i\mathrm{e}^{-\phi}\right) = F_1 - i\mathrm{e}^{-\phi}\mathrm{d}\phi$ scales like $|\langle S \rangle|^2$.

Type IIB leading order stress-energy tensor

In the previous sections we showed that $G_3^{\mathfrak{lo}} \sim \langle S\rangle$, while $F_5^{\mathfrak{lo}}, \mathrm{d}\tau^{\mathfrak{lo}} \sim |\langle S\rangle|^2$, so that $\left|G_3^{\mathfrak{lo}}\right|^2 \sim |\langle S\rangle|^2$, while $\left|F_5^{\mathfrak{lo}}\right|^2, \left|\mathrm{d}\tau^{\mathfrak{lo}}\right|^2 \sim |\langle S\rangle|^4$. On the other hand, one immediately sees from substituting the supersymmetric solution for $G_3^{\mathfrak{lo}}$ (5.77) in the quadratic part of the D7-brane action (5.30) that the lowest order contribution from $T_{AB}^{\lambda\lambda}$ is of order $|\langle S\rangle|^2$. Since $T_{AB}^{\lambda\lambda\lambda\lambda}$ is directly quadratic in the gaugino condensate, this means that $T_{AB}^{\langle S\rangle\mathfrak{lo}}$ receives contributions from both $T_{AB}^{\lambda\lambda}$ and $T_{AB}^{\lambda\lambda\lambda\lambda}$, and it is of order $|\langle S\rangle|^2$. This shows that the only contribution to the stress-energy tensor at lowest order (5.78) comes from $G_3^{\mathfrak{lo}}$ and from $T_{AB}^{\langle S\rangle\mathfrak{lo}}$, and it is of order $|\langle S\rangle|^2$. Thus, (5.78) simplifies to

$$T_{AB}^{\mathfrak{lo}} = \frac{\mathrm{e}^{\phi^{(0)}}}{2\kappa_{10}^2}\left(\left|G_3^{\mathfrak{lo}}\right|^2_{AB} - \frac{1}{2}\hat{g}_{AB}^{(0)}\left|G_3^{\mathfrak{lo}}\right|^2\right) + T_{AB}^{\langle S\rangle\mathfrak{lo}}. \tag{5.85}$$

From (5.77) one can compute

$$\left|G_3^{\mathfrak{lo}}\right|^2 = 16\,\mathrm{e}^{-\phi^{(0)}}\ell_s^8|\langle S\rangle|^2\left[\frac{81}{4\pi^2}\frac{c^8}{r^{12}} + \left(\delta_{\mathbb{P}^2}^{(0)}\right)^2\right]. \tag{5.86}$$

Likewise, the only non-vanishing components of $\left|G_3^{\mathfrak{lo}}\right|^2_{AB}$ are

$$\left|G_3^{\mathfrak{lo}}\right|^2_{ij} = -\frac{36}{\pi}\,\mathrm{e}^{-\phi^{(0)}}\ell_s^8|\langle S\rangle|^2\frac{1}{r^2}\left(1+\frac{r^6}{c^6}\right)^{-\frac{2}{3}}\delta_{\mathbb{P}^2}^{(0)}\frac{\overline{z}_i\overline{z}_j}{r^2}; \tag{5.87a}$$

$$\left|G_3^{\mathfrak{lo}}\right|^2_{i\overline{\jmath}} = 4\,\mathrm{e}^{-\phi^{(0)}}\ell_s^8|\langle S\rangle|^2\left[\frac{81}{4\pi^2}\frac{c^8}{r^{12}} + \left(\delta_{\mathbb{P}^2}^{(0)}\right)^2\right]g_{i\overline{\jmath}}^{(0)}. \tag{5.87b}$$

Notice that from direct computation we found that the supersymmetric solution for G_3 at leading order (5.77) satisfies

$$\left|G_3^{\mathfrak{lo}}\right|^2_{i\overline{\jmath}} = \frac{1}{4}\left|G_3^{\mathfrak{lo}}\right|^2 g_{i\overline{\jmath}}^{(0)}. \tag{5.88}$$

Recalling that $\hat{g}_{i\overline{\jmath}} = \frac{1}{2}g_{i\overline{\jmath}}$, this implies that

$$T_{i\overline{\jmath}}^{\mathfrak{lo}} = T_{i\overline{\jmath}}^{\langle S\rangle\mathfrak{lo}}. \tag{5.89}$$

As for the rest of the components of (5.85), one finds

$$\begin{aligned}
T_{\mu\nu}^{\mathfrak{lo}} &= -4\,\frac{\ell_s^8}{\kappa_{10}^2}|\langle S\rangle|^2\left[\frac{81}{4\pi^2}\frac{c^8}{r^{12}} + \left(\delta_{\mathbb{P}^2}^{(0)}\right)^2\right]\eta_{\mu\nu} + T_{\mu\nu}^{\langle S\rangle\mathfrak{lo}};\\
T_{\mu i}^{\mathfrak{lo}} &= T_{\mu i}^{\langle S\rangle\mathfrak{lo}};\\
T_{ij}^{\mathfrak{lo}} &= -\frac{18}{\pi}\frac{\ell_s^8}{\kappa_{10}^2}|\langle S\rangle|^2\frac{1}{r^2}\left(1+\frac{r^6}{c^6}\right)^{-\frac{2}{3}}\delta_{\mathbb{P}^2}^{(0)}\frac{\overline{z}_i\overline{z}_j}{r^2} + T_{ij}^{\langle S\rangle\mathfrak{lo}};\\
T_{i\overline{\jmath}}^{\mathfrak{lo}} &= T_{i\overline{\jmath}}^{\langle S\rangle\mathfrak{lo}}.
\end{aligned} \tag{5.90}$$

Continuity equation

The differential Bianchi identity for the Einstein tensor together with the Einstein equations imply the continuity equation for the stress-energy tensor

$$\nabla^{(0)A} T^{\mathfrak{lo}}_{AB} = 0. \tag{5.91}$$

Enforcing the continuity equation leads to a constraint on $T^{\langle S\rangle \mathfrak{lo}}_{AB}$. Using the explicit stress-energy tensor (5.90), one finds the equations

$$\nabla^{(0)A} T^{\langle S\rangle \mathfrak{lo}}_{A\mu} = 0; \tag{5.92a}$$

$$\nabla^{(0)A} T^{\langle S\rangle \mathfrak{lo}}_{Ai} + \frac{36}{\pi}\frac{\ell_s^8}{\kappa_{10}^2}|\langle S\rangle|^2 \frac{c^4}{r^8}\left(\delta^{(0)}_{\mathbb{P}^2} - r^2 \delta^{(0)\prime}_{\mathbb{P}^2}\right)\overline{z}_i = 0, \tag{5.92b}$$

where $\delta^{(0)\prime}_{\mathbb{P}^2}$ is the distributional derivative of $\delta^{(0)}_{\mathbb{P}^2}$ with respect to r^2. Let us point out that (5.92b) is nothing but an explicit rewriting of

$$\nabla^{(0)A} T^{\langle S\rangle \mathfrak{lo}}_{Ai} = -\nabla^{(0)A} T^{(IIB)\mathfrak{lo}}_{Ai}. \tag{5.93}$$

Notice that these equations are trivially satisfied outside of the complex cone base, thanks to (5.36), which ensures that $T^{\langle S\rangle \mathfrak{lo}}_{AB} \propto \delta^{(0)}_{\mathbb{P}^2}$. In the following, we will assume that these equations hold everywhere on X_0.

5.4 Equations of motion for the perturbation functions

Let us consider the *trace-reversed* perturbation

$$\bar{h}_{AB} = h_{AB} - \frac{1}{2}\hat{g}^{(0)}_{AB} h. \tag{5.94}$$

Exploiting the fact that the unperturbed metric is Ricci-flat, $R^{(0)}_{AB} = 0$, the Einstein equations (5.55) and the de Donder gauge (5.53) for our $\mathbb{R}^3 \rtimes SO(3) \times U(3)$-symmetric perturbation can be rewritten in terms of the trace-reversed perturbation as

$$\Box^{(0)} \bar{h}_{AB} + 2\bar{h}^{CD} R^{(0)}_{CADB} = -2\kappa_{10}^2 T^{\mathfrak{lo}}_{AB}; \tag{5.95a}$$

$$\nabla^{(0)A} \bar{h}_{AB} = 0. \tag{5.95b}$$

Clearly, the trace-reversed perturbation is still of the form (5.52), with perturbation functions that are related to the ones in h_{AB} by:

$$A = 2A_h - \left(3 + \frac{c^6}{r^6}\right) F_h + \frac{c^6}{r^6} G_h; \tag{5.96a}$$

$$B = B_h; \tag{5.96b}$$

$$C = C_h; \tag{5.96c}$$

$$F = 2A_h - \left(5 + 2\frac{c^6}{r^6}\right) F_h + 2\frac{c^6}{r^6} G_h; \tag{5.96d}$$

$$G = 2A_h - 2\left(3 + \frac{c^6}{r^6}\right) F_h + \left(1 + 2\frac{c^6}{r^6}\right) G_h, \tag{5.96e}$$

where we used the explicit expression for the trace of the perturbation

$$h = \hat{g}^{(0)AB}\, h_{AB} = 4\left[-A_h + \left(3 + \frac{c^6}{r^6}\right) F_h - \frac{c^6}{r^6} G_h\right]. \tag{5.97}$$

Given the simpler form of the de Donder gauge in terms of the trace-reversed perturbation (5.95b), our strategy will be to compute the Einstein equations for the trace-reversed perturbation (5.95a)-(5.95b). Once one finds the solution for the trace-reversed perturbation functions, the original perturbation functions are easily recovered inverting the relations (5.96a)-(5.96e):

$$A_h = \frac{1}{4}\left[5A - \left(3 + \frac{c^6}{r^6}\right) F + \frac{c^6}{r^6} G\right]; \tag{5.98a}$$

$$B_h = B; \tag{5.98b}$$

$$C_h = C; \tag{5.98c}$$

$$F_h = \frac{1}{2}\left[A - \left(1 + \frac{c^6}{r^6}\right) F + \frac{c^6}{r^6} G\right]; \tag{5.98d}$$

$$G_h = \frac{1}{2}\left[A - \left(3 + \frac{c^6}{r^6}\right) F + \left(2 + \frac{c^6}{r^6}\right) G\right]. \tag{5.98e}$$

Let us start recalling the $\mathbb{R}^3 \rtimes SO(3) \times U(3)$-symmetric Ansatz for the trace-reversed metric perturbation:

$$\begin{aligned} \bar{h}_{AB}\mathrm{d}X^A\mathrm{d}X^B &= 2A\,\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + B\,\mathrm{d}r^2\mathrm{d}t + C\,\mathrm{d}^c r^2\mathrm{d}t + \\ &+ \left[2(F-A)\left(1+\frac{c^6}{r^6}\right)^{\frac{1}{3}}\delta_{i\bar{\jmath}} - 2(G-A)\frac{c^6}{r^6}\left(1+\frac{c^6}{r^6}\right)^{-\frac{2}{3}}\frac{\bar{z}_i z_{\bar{\jmath}}}{r^2}\right]\mathrm{d}z^i\mathrm{d}\bar{z}^{\bar{\jmath}}. \end{aligned} \tag{5.99}$$

Computing the Einstein equations and the de Donder gauge (5.95a)-(5.95b) explicitly in local coordinates, we find

$$\mathfrak{h}_1\eta_{\mu\nu} = 8\ell_s^8\,|\langle S\rangle|^2\left[\frac{81}{4\pi^2}\frac{c^8}{r^{12}} + \left(\delta^{(0)}_{\mathbb{P}^2}\right)^2\right]\eta_{\mu\nu} - 2\kappa_{10}^2 T^{\langle S\rangle\mathfrak{lo}}_{\mu\nu}; \tag{5.100a}$$

$$\mathfrak{h}_2\bar{z}_i\eta_{\mu t} = -2\kappa_{10}^2 T^{\langle S\rangle\mathfrak{lo}}_{\mu i}; \tag{5.100b}$$

$$0 = \frac{36}{\pi}\,\ell_s^8\,|\langle S\rangle|^2\frac{1}{r^2}\left(1+\frac{r^6}{c^6}\right)^{-\frac{2}{3}}\delta^{(0)}_{\mathbb{P}^2}\,\frac{\bar{z}_i\bar{z}_j}{r^2} - 2\kappa_{10}^2 T^{\langle S\rangle\mathfrak{lo}}_{ij}; \tag{5.100c}$$

$$\mathfrak{h}_3\delta_{i\bar{\jmath}} + \mathfrak{h}_4\frac{\bar{z}_i z_{\bar{\jmath}}}{r^2} = -2\kappa_{10}^2 T^{\langle S\rangle\mathfrak{lo}}_{i\bar{\jmath}}, \tag{5.100d}$$

where

$$\mathfrak{h}_1 = -4\left(\Delta^{(0)}A + \frac{1}{2}\ddot{A}\right); \tag{5.101}$$

$$\mathrm{Re}\,\mathfrak{h}_2 = \left(\Delta^{(0)}\int B\,\mathrm{d}r^2\right)' + \frac{1}{2}\ddot{B}; \tag{5.102}$$

$$\mathrm{Im}\,\mathfrak{h}_2 = \left(\Delta^{(0)}\int C\,\mathrm{d}r^2\right)' + \frac{1}{2}\ddot{C}; \tag{5.103}$$

$$\begin{aligned}\mathfrak{h}_3 =& \frac{1}{r^8(c^6+r^6)}\left[4c^6(-3c^6+r^6)\left(\tilde{F}-\tilde{G}\right) - 4r^4(c^6+r^6)^2\left(\tilde{F}''-\tilde{G}''\right) + \right.\\ &\left. - 2r^6(c^6+r^6)^{\frac{4}{3}}\left(\Delta^{(0)}\tilde{F} + \frac{1}{2}\ddot{\tilde{F}}\right)\right];\end{aligned} \tag{5.104}$$

$$\begin{aligned}\mathfrak{h}_4 =& \frac{c^6}{r^8(c^6+r^6)^2}\left[16c^6(3c^6+5r^6)\left(\tilde{F}-\tilde{G}\right) - 4r^2(c^6+r^6)(5c^6+3r^6)\left(\tilde{F}'-\tilde{G}'\right) + \right.\\ &\left. - r^6(c^6+r^6)^{\frac{4}{3}}\left(\ddot{\tilde{F}}-\ddot{\tilde{G}}\right) + 2r^6(c^6+r^6)^{\frac{4}{3}}\left(\Delta^{(0)}\tilde{F} + \frac{1}{2}\ddot{\tilde{F}}\right)\right],\end{aligned} \tag{5.105}$$

where $' = \frac{\mathrm{d}}{\mathrm{d}r^2}$, $\dot{} = \frac{\mathrm{d}}{\mathrm{d}t}$, we defined

$$\tilde{F} = F - A; \tag{5.106a}$$

$$\tilde{G} = G - A, \tag{5.106b}$$

and we introduced the Laplacian for the background Eguchi-Hanson metric $\Delta^{(0)} = -2\, g^{(0)i\bar{\jmath}}\partial_i\partial_{\bar{\jmath}}$, whose action on $U(3)$-symmetric functions is given by (2.141), namely

$$\Delta^{(0)} f(r^2) = -\frac{2}{r^4}\left(c^6+r^6\right)^{-\frac{1}{3}}\left[r^2(c^6+r^6)\frac{\mathrm{d}^2}{\mathrm{d}(r^2)^2} + (c^6+3r^6)\frac{\mathrm{d}}{\mathrm{d}r^2}\right] f(r^2). \tag{5.107}$$

The de Donder gauge (5.53), on the other hand, yields the following non-dynamical constraints:

$$-\frac{1}{2}\Delta^{(0)}\int B\,\mathrm{d}r^2 + \dot{A} = 0; \tag{5.108a}$$

$$4c^6(-c^6+r^6)\left(\tilde{F}-\tilde{G}\right) - 4c^6(c^6+r^6)r^2\left(\tilde{F}'-\tilde{G}'\right) - r^8(c^6+r^6)\left[4\tilde{F}'-\dot{B}\right] = 0; \tag{5.108b}$$

$$\dot{C} = 0. \tag{5.108c}$$

Once one specifies the explicit form of $T^{\langle S\rangle \mathfrak{lo}}_{AB}$, from equations (5.100a)-(5.100d) one can extract the corresponding Einstein equations valid on the whole internal manifold X_0.

5.4.1 Gaugino condensate stress-energy tensor constraints

Notice that the (ij) component of the Einstein equations (5.100c) is really just another non-dynamical constraint on the D7-brane gaugino-couplings stress-energy tensor, thanks to our choice of coordinates in order to preserve hermiticity of the internal metric. It effectively determines the (ij) component of the gaugino condensate stress-energy tensor completely in this model, and replacing the ten-dimensional Newton's constant with its expression with respect to the string length (1.13), we get

$$T^{\langle S\rangle \mathfrak{lo}}_{ij} = |\langle S\rangle|^2\,\frac{72}{r^2}\left(1+\frac{r^6}{c^6}\right)^{-\frac{2}{3}}\delta^{(0)}_{\mathbb{P}^2}\,\frac{\bar{z}_i\bar{z}_j}{r^2}, \tag{5.109}$$

This requires the (ij) component of the leading order stress-energy tensor to vanish, and it can be recast into the more compact form

$$T_{ij}^{\langle S\rangle\mathfrak{lo}} = -T_{ij}^{(IIB)\mathfrak{lo}}. \tag{5.110}$$

Combining (5.110) with the (i) component of the continuity equation (5.92b), one finds the new (but dependent) constraint

$$-\dot{T}_{ti}^{\langle S\rangle\mathfrak{lo}} + 2g^{(0)k\bar{\ell}}\partial_k T_{\bar{\ell}i}^{\langle S\rangle\mathfrak{lo}} = 0, \tag{5.111}$$

where $T_{ti}^{\langle S\rangle\mathfrak{lo}}$ denotes the $(\mu = 0, i)$ component of $T_{AB}^{\langle S\rangle\mathfrak{lo}}$. This is trivially satisfied outside of the $\mathbb{P}^2$ base.

As we already explained below (5.39), the divergent terms on the r.h.s. of (5.100a) should cancel in order for the equation to be well-defined. This puts a further constraint on the external components of the gaugino condensate stress-energy tensor:

$$T_{\mu\nu}^{\langle S\rangle\mathfrak{lo}} = 16\pi|\langle S\rangle|^2 \left(\delta_{\mathbb{P}^2}^{(0)}\right)^2 \eta_{\mu\nu} + \tilde{T}_{\mu\nu}^{\langle S\rangle\mathfrak{lo}}, \tag{5.112}$$

where we used (1.13), and the leftover piece $\tilde{T}_{\mu\nu}^{\langle S\rangle\mathfrak{lo}}$ only contains terms proportional to $\delta_{\mathbb{P}^2}^{(0)}$ and not $\left(\delta_{\mathbb{P}^2}^{(0)}\right)^2$.

A gaugino condensate stress-energy tensor Ansatz

Let us show that the gaugino condensate stress-energy tensor Ansatz (5.37) proposed in [49] satisfies all consistency constraints from our explicit analysis (5.109, 5.111, 5.112). Let us start by noticing that in our non-compact setting, ν defined in (5.34) vanishes, so that (5.40) holds and

$$T^{\langle S\rangle\mathfrak{lo}} = T^{\lambda\lambda\mathfrak{lo}}. \tag{5.113}$$

Plugging the global solution for G_3 (5.77) into the explicit formulas (5.37a, 5.37b) we find

$$T_{\mu\nu}^{\langle S\rangle\mathfrak{lo}} = 16\pi|\langle S\rangle|^2 \left(\delta_{\mathbb{P}^2}^{(0)}\right)^2 \eta_{\mu\nu}; \tag{5.114a}$$

$$T_{ij}^{\langle S\rangle\mathfrak{lo}} = |\langle S\rangle|^2 \frac{72}{r^2}\left(1+\frac{r^6}{c^6}\right)^{-\frac{2}{3}} \delta_{\mathbb{P}^2}^{(0)} \frac{\overline{z}_i\overline{z}_j}{r^2}. \tag{5.114b}$$

This straightforwardly satisfies (5.109, 5.111, 5.112), with

$$\tilde{T}_{\mu\nu}^{\langle S\rangle\mathfrak{lo}} = 0. \tag{5.115}$$

A slightly more general Ansatz compatible with (5.109, 5.111, 5.112) would be

$$\tilde{T}_{\mu\nu}^{\langle S\rangle\mathfrak{lo}} = \alpha\,|\langle S\rangle|^2 \frac{8\pi}{c^2}\,\delta_{\mathbb{P}^2}^{(0)}\eta_{\mu\nu}, \tag{5.116}$$

where $\alpha \in \mathbb{R}$, and the numerical normalization has been chosen for later convenience.

5.4.2 Simplifying assumptions

Due to (5.36), the gaugino condensate stress-energy tensor is completely localized on the $\mathbb{P}^2$, therefore its unique role in (5.100a)-(5.100d) is to contribute to fixing some boundary conditions for the solutions. Let us consider the Ansatz for the gaugino condensate stress energy tensor given by (5.112, 5.114b, 5.116). This is a partly literature-motivated assumption, which has the advantage of leading to simpler boundary conditions; one recovers the proposal of [49] setting $\alpha = 0$. Moreover, it is clear from the explicit expressions (5.104) and (5.105) that the l.h.s. of (5.100a)-(5.100d) drastically simplify assuming $\tilde{F} = \tilde{G}$. Using the definitions (5.106a, 5.106b) and the relations (5.96a)-(5.96e), it is trivial to see that this is equivalent to asking

$$F_h = G_h. \tag{5.117}$$

More precisely the relations of the trace-reversed perturbation functions with the proper perturbation functions (5.98a)-(5.98e), under the assumption (5.117), become

$$A_h = \frac{1}{4}\left(2A - 3\tilde{F}\right); \tag{5.118a}$$

$$B_h = B; \qquad C_h = C; \tag{5.118b}$$

$$F_h = G_h = -\frac{1}{2}\tilde{F}. \tag{5.118c}$$

From the non-perturbative form of the perturbated internal metric (5.51b), one sees that this amounts to assuming that the internal geometry does not receive gaugino-condensate corrections up to a conformal factor e^{2F_h}.

Thus, assuming the Ansatz for the gaugino-condensate stress-energy tensor (5.112, 5.114b, 5.116), together with the further requirement (5.117), the Einstein equations (5.100a)-(5.100d) and the de Donder gauge conditions (5.108a)-(5.108c) reduce to the equations of motion

$$\Delta^{(0)}A + \frac{1}{2}\ddot{A} = \ell_s^8 |\langle S\rangle|^2 \left(-\frac{81}{2\pi^2}\frac{c^8}{r^{12}} + \frac{\alpha}{c^2}\,\delta^{(0)}_{\mathbb{P}^2}\right); \tag{5.119a}$$

$$\left(\Delta^{(0)} \int B\,\mathrm{d}r^2\right)' + \frac{1}{2}\ddot{B} = 0; \tag{5.119b}$$

$$\left(\Delta^{(0)} \int C\,\mathrm{d}r^2\right)' + \frac{1}{2}\ddot{C} = 0; \tag{5.119c}$$

$$\Delta^{(0)}\tilde{F} + \frac{1}{2}\ddot{\tilde{F}} = 0; \tag{5.119d}$$

$$-\frac{1}{2}\Delta^{(0)} \int B\,\mathrm{d}r^2 + \dot{A} = 0; \tag{5.119e}$$

$$4\tilde{F}' - \dot{B} = 0; \tag{5.119f}$$

$$\dot{C} = 0. \tag{5.119g}$$

Let us stress that these equations hold *over the whole internal manifold* X_0, namely they already encompass boundary conditions due to the stress-energy tensor[15].

5.5 Solutions

Let us look for solutions of the PDE set (5.119a)-(5.119g). Recall that these describe the dynamics of the trace-reversed perturbations functions, with the assumptions explained above (5.119a).

Let us start by noting that the system (5.119) does not admit trivial solutions, since (5.119a) imposes that $A \neq 0$. Therefore, fluxes from gaugino condensation do source non-trivial metric perturbations. In the following, first we are going to derive the most general time-independent solutions, and we will show that these depend of three real parameters. Secondly, we are going to exhibit a class of time-dependent solutions for (5.119a)-(5.119g), which have the property of growing indefinitely with time. We argue that these solutions are the ones relevant for our discussion, and we show that by comparing with the four-dimensional results from §4 one can fix the perturbed ten-dimensional solution, although not completely. This singles out a class of candidates, depending on four real parameters, for a leading order description of gaugino condensation from ten dimensions at the level of the metric in this model.

5.5.1 Stationary solutions

Let us assume time-independence of the (trace-reversed) perturbation functions A, B, C, F. First, let us consider the equation for A (5.119a). It is convenient to introduce dimensionless counterparts to the above quantities:[16]

$$\begin{aligned} A &=: \frac{\ell_s^8}{c^2}\,|\langle S\rangle|^2\,\mathcal{A}; \qquad & B &=: \frac{\ell_s^8}{c}\,|\langle S\rangle|^2\,\mathcal{B}; \\ \tilde{F} &=: \frac{\ell_s^8}{c^2}\,|\langle S\rangle|^2\,\tilde{\mathcal{F}}; \qquad & C &=: \frac{\ell_s^8}{c}\,|\langle S\rangle|^2\,\mathcal{C}; \end{aligned} \tag{5.120}$$

$$u := \frac{r^2}{c^2}. \tag{5.121}$$

Then, (5.119a) can be recast as

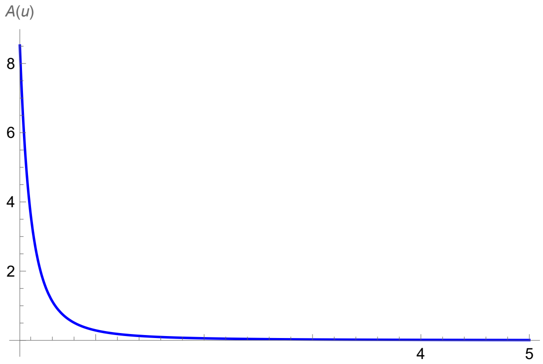
$$\Delta_u^{(0)}\mathcal{A} = -\frac{81}{2\pi^2}\frac{1}{u^6} + \alpha\delta^{(0)}_{\mathbb{P}^2}, \tag{5.122}$$

where we introduced the dimensionless background Hodge-de Rham Laplacian $\Delta_u^{(0)} := c^2\Delta^{(0)}$, where $\Delta^{(0)}$ is defined in (2.141), namely

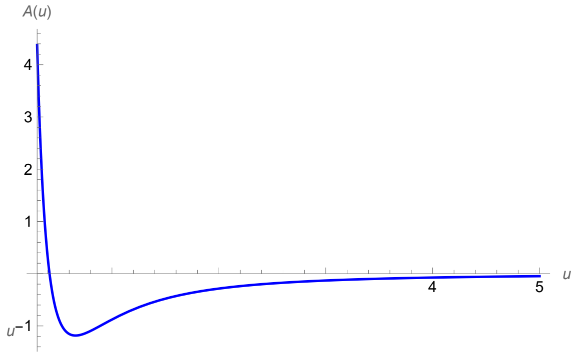
$$\Delta_u^{(0)} f(u) = -\frac{2}{u^2}\left(1+u^3\right)^{-\frac{1}{3}}\left[u(1+u^3)\frac{\mathrm{d}^2}{\mathrm{d}u^2} + (1+3u^3)\frac{\mathrm{d}}{\mathrm{d}u}\right] f(u). \tag{5.123}$$

[15] Naturally, not all boundary conditions are specified by the stress-energy tensor delta-functions. A complete set of boundary conditions would determine the solution to the equations (5.119) uniquely, and in §5.5 we show that this is not the case.

[16] Recall that B and C have dimension ℓ^{-1}, while A, F, G are dimensionless.



(a) $\alpha = 0$ *(Ansatz from [49])*



(b) $\alpha = -\frac{6}{\pi} - 10$

Figure 5.1: *Profile of $\mathcal{A}_s(u)$ defined in (5.124) for two values of α, and $c_1 = 0$. The function always diverges to $+\infty$ in $u \sim 0^+$ and vanishes for $u \sim +\infty$. However, for $\alpha \geq -\frac{6}{\pi}$ it has no stationary points, while for $\alpha < -\frac{6}{\pi}$ it exhibits one in $u_\star = \left(\frac{9}{|6+\pi\alpha|}\right)^{\frac{1}{3}}$.*

Using (2.140), the solution of (5.122) is given by

$$\mathcal{A}_s(u; c_1) = c_1 + \frac{\alpha}{2}\,\kappa_{(4)}(u) + \frac{9}{4\pi^2}\frac{(1+u^3)^{\frac{1}{3}}}{u^3}, \tag{5.124}$$

where $\kappa_{(4)}$ is the local potential for the harmonic form ω Poincaré-dual to the exceptional divisor $\mathbb{P}^2$ on the local patch $\mathcal{U}_{(4)}$ away from the $\mathbb{P}^2$, defined in (2.101), and $c_1 \in \mathbb{R}$ is to be fixed. Notice that $c_1 = \lim_{u\to\infty} \mathcal{A}_s(u; c_1)$, and that the $\delta^{(0)}_{\mathbb{P}^2}$ term in (5.122) is exactly the boundary condition that fixes one of the two integration constants of the homogeneous part of the solution (5.124). A plot of (5.124) is displayed in figure 5.1. This solution displays an unavoidable singularity in $r^2 = 0$, namely reaching the $\mathbb{P}^2$ base, irrespectively of the value of α. Naturally, its physical origin is gaugino condensation occurring on the $\mathbb{P}^2$. More precisely, the logarithmic divergence from $\kappa_{(4)}$ (see (2.105)) is due to the localized contribution from the external component of the gaugino condensate stress-energy tensor (5.116), while its polynomial divergence $\sim u^{-3}$ is essentially due again to the singularity of $\kappa_{(4)}$ in $r^2 = 0$, which makes the global flux solution G_3 in (5.77) singular as well.

Let us move on to B. It is more convenient to solve its non-dynamical constraint (5.119e) and then make sure that it solves its dynamical equation (5.119b). In the time-independent case, (5.119e) reduces to

$$\left(\Delta^{(0)} \int B\,\mathrm{d}r^2\right)' = 0. \tag{5.125}$$

Then, (5.125) admits the only solution $\int B_s\,\mathrm{d}r^2 = const$, namely

$$B_s = 0, \tag{5.126}$$

which also trivially solves (5.119b). Notice that we took into account the fact that no delta-function appears on the r.h.s. of (5.125).

Next, let us consider C. Due to (5.119g), it is forced to always be time-independent. Then, in dimensionless quantities (5.119c) becomes

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\Delta_u^{(0)}\int \mathcal{C}\,\mathrm{d}u\right) = 0. \tag{5.127}$$

The general solution to (5.127) is given by[17]

$$\mathcal{C}_s(u) = c_2\, u^2\left(1+u^3\right)^{-\frac{2}{3}}, \tag{5.128}$$

where $c_2 \in \mathbb{R}$ to be fixed. Once again, the absence of delta-functions on the r.h.s. of (5.127) sends to zero one integration constant, otherwise (5.128) would receive an additional term $+c_3\frac{\mathrm{d}\kappa_{(4)}}{\mathrm{d}u}$. Notice that $c_2 = \lim_{u\to\infty}\mathcal{C}_s(u)$.

Finally, let us consider F. In the time-independent case, (5.119f) is simply

$$\tilde{F}' = 0. \tag{5.129}$$

Then, (5.129) is solved by

$$\tilde{\mathcal{F}}_s = c_3, \tag{5.130}$$

where $c_3 \in \mathbb{R}$ to be fixed. Recalling the definition (5.106a), this is equivalent to $\mathcal{F}_s = c_3 + \mathcal{A}_s$, where $\mathcal{A}_s$ is given by (5.124).

Therefore, the full stationary solution for the trace-reverse perturbation functions is given by (5.124, 5.126, 5.128, 5.130), and it depends on three real parameters c_1, c_2, c_3.

5.5.2 Time-dependent solutions

Let us drop the time-independence assumption from the previous section, and let us introduce the dimensionless time coordinate

$$s := \frac{t}{c}. \tag{5.131}$$

Let us consider the dynamical equation for A (5.119a). We can easily devise two classes of time-dependent solutions, one oscillating and one hyperbolic:

$$\mathcal{A}(u,s) = \mathcal{A}_s(u;c_1) + \beta_+(u)\cos(\sqrt{2}s) + \gamma_+(u)\sin(\sqrt{2}s) + c_2 s; \tag{5.132a}$$

$$\mathcal{A}(u,s) = \mathcal{A}_s(u;c_1) + \beta_-(u)\cosh(\sqrt{2}s) + \gamma_-(u)\sinh(\sqrt{2}s) + c_2 s, \tag{5.132b}$$

where $\mathcal{A}_s(u;c_1)$ is the stationary solution (5.124) depending on the real parameter c_1, $c_2, c_3 \in \mathbb{R}$ are integration constants, and $\beta_\pm$ and $\gamma_\pm$ are eigenfunctions of the dimensionless Hodge-de Rham Laplacian associated to the eigenvalues ± 1, namely

$$\Delta_u^{(0)}\beta_\pm(u) = \pm\beta_\pm(u), \tag{5.133}$$

[17] Using the fact that a particular solution for $\Delta^{(0)} f(r^2) = 1$ is $f_p(r^2) = -\frac{1}{6}(c^6 + r^6)^{\frac{1}{3}}$.

and the same holds for $\gamma_\pm$. Since we look for a ten-dimensional description of the runaway behavior of the ρ chiral field found in §4.5.3, and since at early times $\mathrm{Re}\,\rho$ departs from its initial value scaling like t^2 (see (4.122)), it seems promising to consider the solution increasing with time (5.132b). Thus, here we are not concerned with the physical interpretation of a perturbation solution (5.132a) oscillating with time. Let us stress that *the existence of solutions with cosh and sinh growth in time, like (5.132b), is due to the fact that the internal space X_0 is non-compact.* The spectrum of the Hodge-de Rham Laplacian on a compact manifold is non-negative, therefore β_- and γ_- do not exist on a compact space, and in turn (5.132b) would be ill-defined. In the present case, there is no closed form for β_- and γ_-, but approximations for $u \sim 0$ and $u \sim \infty$ are easily found using the explicit expression (5.123); since they are solutions of a second order linear ODE, they each depend on two free real parameters.

Let us look for the solution for the other perturbations functions $B, C, \tilde{F}$ associated with the hyperbolic solution for A (5.132b). Solving the de Donder condition for B (5.119e), using the input data (5.132b), we find[18]

$$\mathcal{B}(u,s) = -2\sqrt{2}\left(\frac{\mathrm{d}\beta_-(u)}{\mathrm{d}u}\sinh(\sqrt{2}s) + \frac{\mathrm{d}\gamma_-(u)}{\mathrm{d}u}\cosh(\sqrt{2}s)\right) - \frac{c_2}{3}\,u^2(1+u^3)^{-\frac{2}{3}}. \quad (5.134)$$

One can check by direct computation that this also solves its dynamical equation (5.119b).

Since C is decoupled from the other perturbation functions, its general solution is still given by

$$\mathcal{C}(u) = c_3\,u^2(1+u^3)^{-\frac{2}{3}}. \quad (5.135)$$

Finally, $\tilde{F}$ is found solving its de Donder constraint (5.119f) using the input data (5.134). One finds

$$\tilde{\mathcal{F}}(u,s) = -\Big(\beta_-(u)\cosh(\sqrt{2}s) + \gamma_-(u)\sinh(\sqrt{2}s)\Big) + c_4 s + c_5, \quad (5.136)$$

where $c_4, c_5 \in \mathbb{R}$ are integration constants. Once can check by direct computation that this also solves its dynamical equation (5.119d).

Thus, the full solution is given by (5.132b, 5.134, 5.135, 5.136), and it depends on nine real parameters: $c_1, \ldots, c_5$ and four from β_- and γ_-.

5.5.3 Comparison with the four-dimensional analysis

Let us use the results from §4.5.3 to fix some of the free parameters of the time-dependent solution in §5.5.2, as well as the Eguchi-Hanson parameter c, which is defined in (2.63), and which determines the unperturbed background. In order to do that, we will compute the time evolution at early times of the chiral field $\mathrm{Re}\,\rho$, defined in (4.10), associated to the time-dependent ten-dimensional solution found in §5.5.2, and we will match it with the leading order early time evolution of $\mathrm{Re}\,\rho_t$ found in (4.126).

Let us stress that *this is a trivial matching*, in the sense that the number of free parameters exceeds the number of independents constraints, as we will see. Nonetheless,

[18]Using again footnote 17.

this is interesting to carry out, since it gives some insight on how some of the free constants of the ten-dimensional solutions might be fixed. Moreover, let us point out that, due to the fact that our ten-dimensional analysis has been performed at leading order in $|\langle S\rangle|^2$, we expect it to produce correct results only at early times, when the perturbation is small. More precisely, we expect the time-dependent solution in §5.5.2 to be physically sensible only for times

$$t \lessapprox \frac{1}{\ell_s^2\,|\langle S\rangle|}. \tag{5.137}$$

This is in agreement with the characteristic time of early expansion (4.127), coming from the four-dimensional analysis.

The strategy to fix some of the real parameters of the time-dependent solution presented in §5.5.2 is to compute from a ten-dimensional definition the chiral field $\mathrm{Re}\,\rho$, whose dynamics is described by the four-dimensional low-energy EFT (4.88) whose construction is found in chapter 4. More precisely, by comparing the time-dependence of $\mathrm{Re}\,\rho$ from its ten-dimensional definition with the early time expansion of low-energy runaway solutions, which was derived in §4.5.3, equation (4.122), we will be able to fix some free parameters.

Unfortunately, for multiple reasons it is not clear how one should define the four-dimensional chiral field $\mathrm{Re}\,\rho$ in a ten-dimensional setting. Indeed, one could use the non-perturbative definition $\mathrm{Re}\,\rho \sim \int_{\mathbb{P}^2} \mathrm{e}^{-4A}\mathrm{dVol}$ that we derived in (4.10), which includes the warp factor; or one could neglect the warp factor, given that in the analysis of §4 once the D3-branes are not included no warp factor survives (neglecting curvature corrections from D7-branes), so that we would get $\mathrm{Re}\,\rho \sim \int_{\mathbb{P}^2} \mathrm{dVol}$. Moreover, $\mathrm{Re}\,\rho$ is defined up to field redefinitions in the context of its EFT, so that it is not clear how this invariance could be restored from a ten-dimensional definition. Nevertheless, let us proceed. In the following we will adopt both definitions of $\mathrm{Re}\,\rho$, and we will point out the differences between the two approaches and their outcomes.

$\mathrm{Re}\,\rho$ from warped volume

Let us compute the real part of the ρ chiral field using its non-perturbative definition (4.10) employing the perturbed metric (5.51a, 5.51b), with the trace-reversed perturbation solutions (5.132b, 5.134, 5.135, 5.136). Using the relation (5.118a), we find

$$\mathrm{Re}\,\rho_t = \frac{1}{\ell_s^4}\int_{\mathbb{P}^2} \mathrm{e}^{-4A_h(r^2=0,t)}\,\mathrm{dVol}(\mathbb{P}^2) = \frac{1}{\ell_s^4}\int_{\mathbb{P}^2} \mathrm{e}^{-2A(r^2=0,t)}\,\mathrm{dVol}^{(0)}(\mathbb{P}^2), \tag{5.138}$$

where, using (5.24),

$$\mathrm{dVol}(\mathbb{P}^2) = \sqrt{g_6}\,\mathrm{d}^4y = 8\mathrm{e}^{6F_h(r^2=0,t)}\mathrm{d}^4y = \mathrm{e}^{6F_h(r^2=0,t)}\mathrm{dVol}^{(0)}(\mathbb{P}^2). \tag{5.139}$$

Notice that (5.138) is ill defined, since $A_s(r^2)$ in (5.124) diverges in $r^2 = 0$ due to gaugino condensation, as we explained below its expression. How to interpret and how to treat this divergence is a delicate point, to which by no means do we intend to give a definite answer. We could argue, however, that this might be seen as a divergence associated to a quantum effect, which would make it natural to try to renormalize it. In fact, as we explained in §1.3.1, the need for counterterms in the on-shell gaugino condensate action has been established by [50], therefore this procedure could even seem in principle justified to some extent.

Therefore, let us try to sketch a simple renormalization procedure for $\mathrm{Re}\,\rho$ in (5.138). The most straightforward way to do this is to introduce the regulator $\epsilon > 0$ and to define

$$\mathcal{A}_s^\epsilon(u) := \mathcal{A}_s(u+\epsilon; c_1^\epsilon) \tag{5.140}$$

where $c_1^\epsilon \in \mathbb{R}$, and $\mathcal{A}_s(u; c_1)$ is defined in (5.124). In this way, $\mathcal{A}_s^\epsilon(0)$ is well-defined, and the physical solution is obtained in the limit $\epsilon \to 0$, once c_1^ϵ is fixed. Thus, once the regularization procedure described above is applied, (5.138) can be expanded at leading order in the gaugino condensate and for small times, finding

$$\begin{aligned}\mathrm{Re}\,\rho_t =& \frac{\mathrm{Vol}^{(0)}(\mathbb{P}^2)}{\ell_s^4} - 2\ell_s^4|\langle S\rangle|^2 \frac{\mathrm{Vol}^{(0)}(\mathbb{P}^2)}{c^2}\left(\mathcal{A}_s^\epsilon(0) + \beta_-(0)\right) + \\ & - 2\ell_s^4|\langle S\rangle|^2 \frac{\mathrm{Vol}^{(0)}(\mathbb{P}^2)}{c^3}\left(\sqrt{2}\gamma_-(0) + c_2\right) t - 2\ell_s^4|\langle S\rangle|^2 \frac{\mathrm{Vol}^{(0)}(\mathbb{P}^2)}{c^4}\beta_-(0)\, t^2 + o_0(t^2),\end{aligned} \tag{5.141}$$

where $\mathrm{Vol}^{(0)}(\mathbb{P}^2)$ is given by (2.97). Let us point out that, in this framework, the Eguchi-Hanson parameter c controlling $\mathrm{Vol}^{(0)}(\mathbb{P}^2)$ should not receive quantum corrections, since it determines the vacuum solution.

Let us comment on the values at $u = 0$ of the eigenfunctions $\beta_-(0)$ and $\gamma_-(0)$. As we already mentioned below (5.133), they depend on two real parameters each, and they do not admit a closed form representation. However, imposing them to be real-valued, for $u \sim 0$ they can be approximated at leading order by

$$\beta_-(u) \sim_0 a_1 I_0\left(\frac{\sqrt{2}}{3}u^{\frac{3}{2}}\right) + a_2 \mathrm{Re}\,K_0\left(\frac{\sqrt{2}}{3}u^{\frac{3}{2}}\right); \tag{5.142a}$$

$$\gamma_-(u) \sim_0 b_1 I_0\left(\frac{\sqrt{2}}{3}u^{\frac{3}{2}}\right) + b_2 \mathrm{Re}\,K_0\left(\frac{\sqrt{2}}{3}u^{\frac{3}{2}}\right), \tag{5.142b}$$

where $I_n(z)$ and $K_n(z)$ are the modified Bessel functions of I and II kind respectively, and $a_1, a_2, b_1, b_2 \in \mathbb{R}$. Their relevant asymptotics are

$$I_0\left(\frac{\sqrt{2}}{3}u^{\frac{3}{2}}\right) \sim_0 1 + \mathcal{O}(u^3); \tag{5.143a}$$

$$\mathrm{Re}\,K_0\left(\frac{\sqrt{2}}{3}u^{\frac{3}{2}}\right) \sim_0 \frac{3}{2}\log\frac{1}{u} + \mathcal{O}(1). \tag{5.143b}$$

Once a_1, a_2, b_1, b_2 are fixed, the eigenfunctions are completely specified over the whole internal manifold X_0. In order not to introduce divergences which are not physically due to gaugino condensation, let us set

$$a_2 = b_2 = 0. \tag{5.144}$$

Comparing (5.141) with the $\mathbb{P}^2$ expansion found from the four-dimensional EFT (4.126) leads to the following identifications:

$$c = \left(\frac{\operatorname{Re}\rho_0}{2\pi^2}\right)^{\frac{1}{4}} \ell_s; \tag{5.145a}$$

$$c_1^\epsilon = \frac{3}{\pi}\left(\operatorname{Re}\rho_0 - \frac{3}{4\pi}\right) - \frac{\alpha}{4}\left(\frac{3}{\pi}\log\frac{1}{\epsilon} + \frac{3}{2\pi}\log 3 - \frac{1}{\sqrt{3}}\right) - \frac{9}{4\pi^2}\frac{1}{\epsilon^3} + \mathcal{O}(\epsilon); \tag{5.145b}$$

$$c_2 = -\sqrt{2}b_1; \tag{5.145c}$$

$$a_1 = -\frac{3}{\pi}\left(\operatorname{Re}\rho_0 - \frac{3}{4\pi}\right). \tag{5.145d}$$

Here we set the universal modulus a of the four-dimensional analysis to one due to the choice (5.21). The initial value of the $\operatorname{Re}\rho$ field identifies the compactification, therefore it is natural for all other parameters to depend on it. Let us comment these assignments: (5.145a) fixes the Eguchi-Hanson parameter in terms of the initial value of $\operatorname{Re}\rho$, and it is equivalent to setting

$$\frac{\mathrm{Vol}^{(0)}(\mathbb{P}^2)}{\ell_s^4} = \operatorname{Re}\rho_0; \tag{5.146}$$

(5.145b) includes the counterterms needed to cancel the singularity of $\mathcal{A}_s(u)$ in $u = 0$, and the matching specifies the precise form of the finite terms; (5.145c) relates two parameters of the solution in §5.5.2 in order to cancel time-independent quantum corrections; (5.145d) together with (5.144) identifies the eigenfunction $\beta_-(0)$ completely. It is important to note that this appears to be a trivial matching procedure, since the number of free real parameters involved exceeds the number of independent constraints. Nonetheless, it can be regarded as a method to fix some parameters of the ten-dimensional solution found in §5.5.2.

In conclusion, this procedure fixes five out of nine parameters of the time-dependent solution, and the vacuum Eguchi-Hanson parameter c. There remain four real constants to be fixed, so that we still lack a precise ten-dimensional realization of the four-dimensional phenomenon of $\mathbb{P}^2$ expansion. This could be achieved by imposing to reproduce some further physical quantity able to determine the remaining constants.

$\operatorname{Re}\rho$ **from unwarped volume**

Let us define the chiral field $\operatorname{Re}\rho$ neglecting the warping in (4.10). Therefore, using the relation (5.118c) and the decomposition (5.139), we find

$$\operatorname{Re}\rho_t = \frac{1}{\ell_s^4}\int_{\mathbb{P}^2} \mathrm{dVol}(\mathbb{P}^2) = \mathrm{e}^{-3\tilde{F}(r^2=0,t)}\mathrm{Vol}^{(0)}(\mathbb{P}^2). \tag{5.147}$$

Here $\tilde{F}$ is defined by (5.136). Therefore, expanding at leading order in the gaugino condensate and for small times, we find

$$\operatorname{Re}\rho_t = \frac{\mathrm{Vol}^{(0)}(\mathbb{P}^2)}{\ell_s^4}\left\{1 - 3|\langle S\rangle|^2\frac{\ell_s^8}{c^2}\left[c_5 - \beta_-(0) + \left(c_4 - \sqrt{2}\gamma_-(0)\right)\frac{t}{c} - \beta_-(0)\frac{t^2}{c^2} + o_0(t^2)\right]\right\}. \tag{5.148}$$

Like before, we want to compare this expression with the early time evolution (4.122) from the four-dimensional EFT analysis.

Let us assume (5.144) so that the eigenfunctions do not introduce any divergence. It is apparent from (5.148) that neglecting the warp factor in the definition of $\operatorname{Re}\rho$ technically ameliorated the situation, since we no longer have to deal with diverging quantities. The comparison yields the identifications

$$c = \left(\frac{\operatorname{Re}\rho_0}{2\pi^2}\right)^{\frac{1}{4}} \ell_s; \tag{5.149a}$$

$$c_5 = a_1; \tag{5.149b}$$

$$c_4 = \sqrt{2} b_1; \tag{5.149c}$$

$$a_1 = \frac{2a}{\pi}\left(\operatorname{Re}\rho_0 - \frac{3}{4\pi}\right). \tag{5.149d}$$

Here, a_1 and b_1 have been defined in (5.142). Like in the case of $\operatorname{Re}\rho$ defined as warped volume, this procedure fixes five real parameters and the Eguchi-Hanson parameter. However, all of these are finite, since no renormalization procedure has been invoked.

Stationary solution interpretation

Let us note that, in the perspective of matching the ten-dimensional perturbative solutions with the four-dimensional findings, the stationary solutions in §5.5.1 could be interpreted as the ten-dimensional realization of the dS unstable vacuum found in (4.104), shown in figure 4.2. However, this would require fixing its three free real constants by some non-trivial procedure which does not appear clear to us.

Appendix A

Details on the geometry of the $\mathbb{P}^2$ cone

A.1 Chain complexes

Given $n+1$ groups G_i, a chain complex is a sequence of groups and homomorphisms

$$G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} G_2 \xrightarrow{f_3} \dots \xrightarrow{f_n} G_n \tag{A.1}$$

such that $\operatorname{im} f_i \subseteq \ker f_{i+1}$. An exact sequence is a chain complex (A.1) such that $\operatorname{im} f_i = \ker f_{i+1}$. Therefore, a short sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \tag{A.2}$$

is exact iff f is an injective morphism and g is a surjective morphism. This implies that

$$C \simeq B/\operatorname{im} f. \tag{A.3}$$

In such case, if A, B, and C are vector spaces,

$$\dim B = \dim A + \dim C. \tag{A.4}$$

A short exact sequence (A.2) splits iff there exists a homomorphism $h : C \to B$ such that $g \circ h = \mathrm{id}_C$; interpreting C as the equivalence classes of elements of B with respect to the action of $\operatorname{im} f$, this amounts to requiring that an equivalence class $[c] \in C$ is sent through h to the corresponding element $c \in B$ modulo the action of $\operatorname{im} f$. In this case, if A, B and C are abelian groups (for instance, if they are vector spaces), $\operatorname{im} f$ is abelian as well, therefore we have the isomorphism $B \simeq A \oplus C$.

A.2 Relative homology and cohomology

Given a non-compact[1] D-dimensional smooth manifold X with boundary $Y = \partial X$, we define the relative n-chains as n-chains of X up to n-chains of Y:

$$C_n(X, Y; \mathbb{R}) = \frac{C_n(X; \mathbb{R})}{C_n(Y; \mathbb{R})}. \tag{A.5}$$

The boundary operator ∂ naturally extends to the relative chains complex, therefore defining the relative homology groups $H_n(X, Y; \mathbb{R}) = \ker \partial_{n+1} / \mathrm{im}\, \partial_n$. A relative n-cycle is a n-chain of X $\alpha \in C_n(X)$ such that its boundary is zero up to an $(n-1)$-chain of Y, namely $\partial\alpha \in C_{n-1}(A)$. The relative cohomology groups $H^n(X, Y; \mathbb{R})$ are defined as the dual of $H_n(X, Y; \mathbb{R})$.

The relation between the relative homology groups and the absolute homology groups is described as follows. It is well defined the short exact sequence of real n-chains

$$0 \longrightarrow C_n(Y; \mathbb{R}) \xrightarrow{i} C_n(X; \mathbb{R}) \xrightarrow{j} C_n(X, Y; \mathbb{R}) \longrightarrow 0 \tag{A.6}$$

where $i : Y \hookrightarrow X$ is the natural inclusion, and $j : X \twoheadrightarrow X/Y$ is the natural projection[2]. Dualizing (A.6) we get a short exact sequence for n-cocycles

$$0 \longrightarrow C^n(X, Y; \mathbb{R}) \xrightarrow{j^\star} C^n(X; \mathbb{R}) \xrightarrow{i^\star} C^n(Y; \mathbb{R}) \longrightarrow 0. \tag{A.7}$$

With a little work, this implies the existence of a long exact sequence of cohomology groups:

$$\begin{aligned} &0 \longrightarrow H^0(X, Y; \mathbb{R}) \longrightarrow \dots \\ &\dots \longrightarrow H^n(X, Y; \mathbb{R}) \xrightarrow{j^\star} H^n(X; \mathbb{R}) \xrightarrow{i^\star} H^n(Y; \mathbb{R}) \xrightarrow{\delta} H^{n+1}(X, Y; \mathbb{R}) \longrightarrow \dots \end{aligned} \tag{A.8}$$

where $i^\star$ and $j^\star$ are the lift of i and j to cohomology, and δ is the dual operator of ∂ lifted to cohomology. The analogous sequence for homology groups is obtained from dualizing (A.8):

$$\begin{aligned} &\dots \longrightarrow H_n(Y; \mathbb{R}) \xrightarrow{i_\star} H_n(X; \mathbb{R}) \xrightarrow{j_\star} H_n(X, Y; \mathbb{R}) \xrightarrow{\partial} H_{n-1}(Y; \mathbb{R}) \longrightarrow \dots \\ &\dots \longrightarrow H_0(X, Y; \mathbb{R}) \longrightarrow 0. \end{aligned} \tag{A.9}$$

The long exact sequences (A.8) and (A.9) hold equally well in the case of integral coefficients, i.e. substituting $\mathbb{R}$ with $\mathbb{Z}$. More details can be found in [14].

[1] Meaning non-compact and with boundary.

[2] By an abuse of notation, we denote by the same symbol the pointwise maps and their lift to chain complexes.

A.3 Spin structure

Proposition A.3.1. *The* $\mathbb{P}^2$ *base in* $\mathcal{O}_{\mathbb{P}^2}(-3)$ *is not Spin.*

Proof. The divisor E is Spin if and only if its normal bundle $\mathcal{N}_E$ is Spin [10]. On the other hand, if $\mathbb{E}_{\mathbb{C}}$ is a complex vector bundle, and if $\mathbb{E}_{\mathbb{R}}$ is the associated real vector bundle, then the second Stiefel-Whitney class of the latter is given by the first Chern class of the former mod 2, namely

$$w_2(\mathbb{E}_{\mathbb{R}}) = c_1(\mathbb{E}_{\mathbb{C}}) \mod 2. \tag{A.10}$$

By the adjuction formula, $\mathcal{N}_E \simeq \mathcal{O}_{\mathcal{K}_E}(E)_{|E}$. Moreover, by the bijection induced by the first Chern class between line bundles over E (up to isomorphisms) and divisors (up to linear equivalences), we get $[c_1(\mathcal{N}_E)] = [E]$ in $H^2(E;\mathbb{Z})$. Therefore, for any 2-cycle $C \subset E$ one has

$$\int_C c_1(\mathcal{N}_E) = \int_C [E] = C \cdot E \tag{A.11}$$

and we showed there is only one independent homology class of curves in $\mathbb{P}^2$, that is $[C] = [C_i] = [\mathbb{P}^1]$, so that using (2.14) we get

$$\int_C c_1(\mathcal{N}_E) = -3. \tag{A.12}$$

This is odd, therefore $w_2(E) \neq 0$ and E is not Spin. □

A.4 Hodge theorem, Lefschetz decomposition and Kähler identities

On a compact[3] Kähler manifold M, the Hodge theorem guarantees that each cohomology class contains exactly one harmonic representative, making $H^k(M)$ isomorphic to the $b_k(M)$-dimensional vectorial space of harmonic k-form on M, denotes $\mathcal{H}^k(M)$. However, in general for M non-compact[4] this does not hold. What fails here is that $\mathrm{d}^\dagger$ is no longer the L^2-dual operator of d, namely

$$(\mathrm{d}\omega_p, \lambda_{p+1}) \neq (\omega_p, \mathrm{d}^\dagger \lambda_{p+1}). \tag{A.13}$$

Some consequences of this are that not all harmonic forms ω are such that $\mathrm{d}\omega = \mathrm{d}^\dagger\omega = 0$, and that equations in cohomology translate in pointwise equations for harmonic forms up to exact and co-exact terms. Moreover, in the non-compact case one has to deal with both relative and absolute (co)homologies, which suggests one should work with two separate sets of harmonic forms. Therefore, one does not expect every cohomology class to have a harmonic representative, nor it to be unique.

[3] Meaning compact and without boundary.
[4] Meaning non-compact and with boundary.

In the case of a *compact* Kähler m-fold M, it is well known that its cohomology groups $H^\bullet(M)$ are generated by its primitive cohomology, as specified by

$$H^k(M) = \bigoplus_{\substack{s\geq 0 \\ 0\leq k-2s\leq m}} L_+^s \left(PH^{k-2s}(M)\right), \tag{A.14}$$

where L_+ is the raising operator of the Lefschetz $SU(2)$, which acts on harmonic forms by wedging them with the Kähler form[5], i.e.

$$\begin{aligned} L_+ :&\mathcal{H}^p(M) \to \mathcal{H}^{p+2}(M) \\ &\alpha \mapsto J \wedge \alpha \end{aligned} \tag{A.17}$$

where $\mathcal{H}^p(M)$ is the vector space of harmonic p-forms on M, and the primitive cohomology groups are defined as

$$PH^{m-r}(M) = \ker\left(L_+^{r+1} : \mathcal{H}^{m-r}(M) \to \mathcal{H}^{m+r+2}(M)\right). \tag{A.18}$$

Equivalently stated, a harmonic p-form is primitive iff $p \leq m$ and it is annihilated by wedging it with J^{m-p+1}. Notice that for $p = 0, 1$ all harmonic p-forms are primitive.

One can show that for a $\omega_p \in \mathcal{H}^p(M)$, $2 \leq p \leq m$, $J^{m-p+1} \wedge \omega_p = 0$ is equivalent to $\iota_J\omega_p = 0$, where ι_J is the interior product with the Kähler form, i.e.

$$\iota_J\omega_p = \frac{1}{2\cdot(p-2)!} J^{mn}\omega_{mnr_1\ldots r_{p-2}} \mathrm{d}x^{r_1} \wedge \ldots \wedge \mathrm{d}x^{r_{p-2}}. \tag{A.19}$$

Therefore, one could also define primitive forms as harmonic p-forms ω_p, $p \leq m$, such that $\iota_J\omega_p = 0$[6]. For this reason, given a closed p-form $\omega_p \in H^p(M)$, $p \geq 2$, we will refer to

$$\iota_J\omega_p = 0 \tag{A.20}$$

as the *primitivity condition.* Note that it suffices to define L_+ on harmonic forms thanks to the Hodge theorem. In this context, (A.14) is called *Lefschetz decomposition* of $H^k(M)$, and it means that a harmonic k-form on M can be uniquely decomposed into a linear combination of primitive forms wedged with the suitable power of J. In other words, all

[5]Note that wedging a harmonic $p-$form α_p with another harmonic 2-form β does not yield a harmonic form in general, but it does if β is the Kähler form. Indeed, $\mathrm{d}(\alpha_p \wedge J) = 0$ by Kählerity; on the other hand, by direct computation, for a generic p-form α_p it holds

$$\mathrm{d}^\dagger(J \wedge \alpha_p) = \mathrm{d}^c\alpha_p + J \wedge \mathrm{d}^\dagger\alpha_p \tag{A.15}$$

in the convention $\mathrm{d}^\dagger = (-1)^{D(p+1)+1} \star \mathrm{d}\star$, where $D = 2m$ is the dimension of the (possibly quasi-complex) manifold. Since M is a complex manifold, $\mathrm{d}\alpha_p = 0$ is equivalent to $\partial\alpha_p = \overline{\partial}\alpha_p = 0$, which implies $\mathrm{d}^c\alpha_p = 0$, and by harmonicity it also holds $\mathrm{d}^\dagger\alpha_p = 0$, therefore

$$\mathrm{d}^\dagger(J \wedge \alpha_p) = 0, \tag{A.16}$$

which shows that $J \wedge \alpha_p$ is harmonic too.

[6]For $p = 0, 1$ it is assumed always satisfied.

the homology groups of M are generated as linear combinations of Lefschetz multiples of spin $0 \leq \ell \leq \frac{m-p}{2}$. This is a consequence of the hard Lefschetz theorem, and it is compatible with the Hodge (p, q) decomposition. In the non-compact case, in general the Lefschetz theory does not apply. Therefore, we do not expect a Lefschetz decomposition to exist at all, nor to be unique. Nonetheless, primitive representatives of cohomology classes can be of interest.

In this thesis we see by direct computation that the compactly supported cohomology[7] of $\mathcal{O}_{\mathbb{P}^2}(-3)$ admits harmonic representatives for all cohomology classes, and that the Lefschetz decomposition appears to hold for them. Equivalently, we assume the Hodge theorem and the Lefschetz theory to apply to $\mathcal{O}_{\mathbb{P}^2}(-3)$ as well (in the sense of the previous statement), and we do not run into any trouble[8].

Finally, let us mention some technical results, which are of great aid in dealing with direct computations on Kähler manifolds. Let $\alpha_p \in \mathcal{H}^P(M)$, and let us define the operators

$$L_+(\alpha_p) = \alpha_p \wedge J \tag{A.21a}$$
$$L_-(\alpha_p) = L_+^\dagger(\alpha_p) = \iota_J \alpha_p \tag{A.21b}$$
$$L_3(\alpha_p) = \frac{p-m}{2}\alpha_p \tag{A.21c}$$

One can show from direct computations that L_+, L_- and L_3 generate the $SU(2)$ algebra ([23]):

$$[L_+, L_-] = 2L_3 \tag{A.22a}$$
$$[L_3, L_\pm] = \pm L_\pm \tag{A.22b}$$

Moreover, they obey the following commutation rules with respect to the Hodge dual ([25]):

$$L_+ \star = \star L_- \tag{A.23a}$$
$$L_- \star = \star L_+ \tag{A.23b}$$
$$[L_3, \star] = 0. \tag{A.23c}$$

Defining as usual the angular momentum operator $\mathbf{L}^2 = L_+L_- + L_3^2 - L_3$, one easily finds from these relations that

$$[\mathbf{L}^2, \star] = 0. \tag{A.24}$$

This means that the each element of the same Lefschetz spin multiplet transforms in the same representation of the Hodge duality.

[7]Recall that in this case the absolute and the compactly supported cohomologies are isomorphic up to cyclic terms.

[8]This does seem to be guaranteed by theorem (4.8) of [2].

Furthermore, the following identities hold for any Kähler manifold M (in particular for M non-compact):

$$\left[L_+, \overline{\partial}\right] = \left[L_+, \partial\right] = 0 \tag{A.25a}$$
$$\left[L_-, \overline{\partial}^\dagger\right] = \left[L_-, \partial^\dagger\right] = 0 \tag{A.25b}$$
$$\left[L_+, \overline{\partial}^\dagger\right] = -i\partial \tag{A.25c}$$
$$\left[L_+, \partial^\dagger\right] = i\overline{\partial} \tag{A.25d}$$
$$\left[L_-, \overline{\partial}\right] = -i\partial^\dagger \tag{A.25e}$$
$$\left[L_-, \partial\right] = i\overline{\partial}^\dagger \tag{A.25f}$$

These can be equivalently recast in the following form:

$$\left[L_+, \mathrm{d}\right] = \left[L_+, \mathrm{d}^c\right] = 0 \tag{A.26a}$$
$$\left[L_-, \mathrm{d}^\dagger\right] = \left[L_-, \mathrm{d}^{c\dagger}\right] = 0 \tag{A.26b}$$
$$\left[L_-, \mathrm{d}\right] = -\mathrm{d}^{c\dagger} \tag{A.26c}$$
$$\left[L_+, \mathrm{d}^\dagger\right] = \mathrm{d}^c \tag{A.26d}$$
$$\left[L_-, \mathrm{d}^c\right] = \mathrm{d}^\dagger \tag{A.26e}$$
$$\left[L_+, \mathrm{d}^{c\dagger}\right] = -\mathrm{d} \tag{A.26f}$$

These are known as the Kähler identities.

This allows us to prove the following simple, but convenient technical fact:

Lemma A.4.1. *Let M be a Kähler manifold, $\omega_p \in \Omega^p(M)$ a p-form on M. If ω_p is closed ($\mathrm{d}\omega_p = 0$) and satisfies the primitivity condition (A.20), then it is harmonic, i.e.*

$$\Delta\omega_p = 0, \qquad \Delta = \mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger\mathrm{d}. \tag{A.27}$$

Proof. A p-form satisfying $\mathrm{d}\omega_p = \mathrm{d}^\dagger\omega_p = 0$ is harmonic even if M is non-compact (although we already noted that in the non-compact case not all harmonic forms satisfy these relations). Let us assume ω_p is closed and it obeys the primitivity condition. Then $\mathrm{d}\omega_p = 0$ by hypothesis. Being M complex, this is equivalent to $\partial\omega_p = \overline{\partial}\omega_p = 0$. On the other hand, using again the fact that M is a complex manifold to split the differential in holomorphic and antiholomorphic parts and the Kähler identities (A.25e, A.25f),

$$\mathrm{d}^\dagger\omega_p = \partial^\dagger\omega_p + \overline{\partial}^\dagger\omega_p = \left[L_-, i\overline{\partial}\right]\omega_p - \left[L_-, i\partial\right]\omega_p = -i\overline{\partial}(\iota_J\omega_p) + i\partial(\iota_J\omega_p) = 0. \tag{A.28}$$

□

A.5 Harmonic forms and other representatives

A.5.1 Primitive 2-form derivation

Let us look for harmonic representatives, if any, for the $(1,1)$-cohomology class Poincaré-dual to the resolved divisor $\mathbb{P}^2$ Recall that $\mathbb{P}^2 = \{r^2 = 0\}$[9], which is $U(3)$-invariant, therefore let us consider a $U(3)$-symmetric Ansatz for the dual 2-form[10]:

$$\omega = iA(r^2)\mathrm{d}z^i \wedge \mathrm{d}\bar{z}_i + iB(r^2)\bar{z}_i z_{\bar{j}}\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{j}}. \tag{A.29}$$

Closure easily implies $B = A'$, so that

$$\omega = \frac{1}{2}\mathrm{d}\left(A(r^2)\mathrm{d}^c r^2\right). \tag{A.30}$$

We showed in lemma A.4.1 that we can now just enforce primitivity (A.20) to find automatically harmonic and primitive representatives for the cohomology. In the $p = 2$ case, the primitivity condition becomes

$$g^{i\bar{j}}\omega_{i\bar{j}} = 0. \tag{A.31}$$

Using (2.67) this takes the form

$$\left(3 + \frac{c^6}{r^6}\right)A + \left(1 + \frac{c^6}{r^6}\right)A' = 0. \tag{A.32}$$

This can be recast as

$$\frac{\mathrm{d}}{\mathrm{d}r^2}\log\left(r^2 A(r^2)\right) = -\frac{2}{r^2\left(1 + \frac{c^6}{r^6}\right)}, \tag{A.33}$$

which is solved by

$$A(r^2) = \frac{a}{r^2\left(1 + \frac{r^6}{c^6}\right)^{\frac{2}{3}}}, \tag{A.34}$$

where $a \in \mathbb{R}$. This solution can be recast in the form

$$\omega = \frac{1}{2}\mathrm{d}\mathrm{d}^c\kappa_{(4)}(r^2) \tag{A.35}$$

where we introduced the local potential

$$\kappa_{(4)}(r^2) = \int A(r^2)\mathrm{d}r^2. \tag{A.36}$$

[9]From the homogenous coordinates point of view, $\mathbb{P}^2 : Z^4 = 0$, which is $U(3)$-invariant because Z^4 is.

[10]We consider the Ansatz on $\mathcal{U}_{(4)}$ since this is the maximal $U(3)$-invariant coordinate neighborhood. The i is needed to make ω real, i.e. $\overline{\omega} = \omega$, which is a necessary condition to being Poincaré-dual to a divisor.

We found a family of linearly dependent primitive harmonic $(1,1)$-forms. In order to fix the normalization and single out the dual form to $\mathbb{P}^2$, we impose the intersection products involving the non-compact holomorphic curves[11] (2.15), e.g. choosing $\tilde{C}_3$:

$$\int_{\tilde{C}_3} \omega = 1. \tag{A.37}$$

On the other hand, recalling that $\tilde{C}_3 = \{z^1 = z^2 = 0\}$ on $\mathcal{U}_{(4)}$, that $\partial X_0 = S^5/\mathbb{Z}_3$, decomposing $z^3 = r\mathrm{e}^{i\theta_3}$ and using Stokes' theorem we get

$$\begin{aligned} \int_{\tilde{C}_3} \omega &= \frac{1}{2}\int_{\partial\tilde{C}_3} A(r^2)\mathrm{d}^c r^2 = \frac{1}{2}\int_{\partial\tilde{C}_3} A(|z^3|^2)\frac{\overline{z}_3\mathrm{d}z^3 - z_{\overline{3}}\mathrm{d}\overline{z}^{\overline{3}}}{i} \\ &= \frac{2\pi}{3}\left(\lim_{r^2\to\infty} r^2A(r^2) - \lim_{r^2\to 0} r^2A(r^2)\right) \\ &= -\frac{2\pi a}{3}. \end{aligned} \tag{A.38}$$

Thus, imposing $[\omega] = [E]$ fixes

$$a = -\frac{3}{2\pi}, \tag{A.39}$$

and we find

$$A(r^2) = -\frac{3}{2\pi}\frac{1}{r^2\left(1 + \frac{r^6}{c^6}\right)^{\frac{2}{3}}}. \tag{A.40}$$

A.5.2 Testing the Lefschetz decomposition

In §A.5.1 we showed by direct computation that $H^{1,1}(X_0, Y_0; \mathbb{Z})$ does have one linearly independent cohomology class with harmonic and primitive representative, as one would expect from the Hodge theorem and the Lefschetz decomposition:

$$H^{1,1}(X_0;\mathbb{R}) = PH^{1,1}(X_0;\mathbb{R}) \oplus \left(PH^0(X_0;\mathbb{R})\right) J. \tag{A.41}$$

One can test the Lefschetz decomposition even further in this non-compact setting, as we now show. Let us consider the Hodge dual of $\omega \in H^{2,2}(X_0, Y_0; \mathbb{Z})$. By direct computation, using the convention

$$\epsilon_{i_1 i_2 i_3 \bar{\imath}_1 \bar{\imath}_2 \bar{\imath}_3} = 3!\, i\, g_{[i_1|\bar{\imath}_1} g_{|i_2|\bar{\imath}_2} g_{|i_3]\bar{\imath}_3} \tag{A.42}$$

and the definition

$$\star\,\alpha_p = \frac{1}{p!(D-p)!}\alpha_{m_1\ldots m_p}\tilde{\epsilon}^{m_1\ldots m_p}{}_{n_1\ldots n_{D-p}}\mathrm{d}x^{n_1}\wedge\ldots\wedge\mathrm{d}x^{n_{D-p}}, \tag{A.43}$$

where $\tilde{\epsilon}_{m_1\ldots m_D} = \sqrt{g_6}\epsilon_{m_1\ldots m_D}$, and $D = \dim_{\mathbb{R}}(M)$, we find

$$\star\,\alpha_2 = (\iota_J\alpha_2)\frac{1}{2}J\wedge J - \alpha_2\wedge J \tag{A.44}$$

[11]We choose the non-compact curves so that we can perform the explicit calculation in the (z^i) coordinates.

for any $(1,1)$-form α_2. For the primitive ω, this yields

$$\star\,\omega = -\omega \wedge J, \tag{A.45}$$

while for J, using $\iota_J J = 3$, we get

$$\star\,J = \frac{1}{2} J \wedge J. \tag{A.46}$$

This is exactly what one expects from the Lefschetz theory. Indeed, the Lefschetz decomposition for this space would look like

$$H^{2,2}(X_0;\mathbb{R}) = J \wedge PH^{1,1}(X_0;\mathbb{R}) \oplus \left(PH^0(X_0;\mathbb{R})\right) J^2. \tag{A.47}$$

This means that a harmonic $(2,2)$-form α_4 can be decomposed uniquely as

$$\alpha_4 = a\,\frac{1}{2} J \wedge J + b\,\omega \wedge J. \tag{A.48}$$

Here, $\frac{1}{2}J \wedge J$ belongs to the $\ell = \frac{3}{2}$ multiplet $(1, J, J^2, J^3)$, while $\omega \wedge J$ belongs to the $\ell = \frac{1}{2}$ multiplet $(\omega, J \wedge \omega)$. In the case of $\alpha_4 = \star\omega$[12], the preservation of Lefschetz spin under Hodge dual (A.24) implies $a = 0$. One can also see this using (A.23) and the equivalent primitivity condition $\omega \wedge J^2 = 0$, so that

$$0 = \star L_- \omega = L_+ \star \omega = a\,\frac{1}{2} J^3 + b\,\omega \wedge J^2 = a\,\frac{1}{2} J^3. \tag{A.49}$$

On the other hand, for $\alpha_4 = \star J$, the preservation of Lefschetz spin under Hodge dual sets $b = 0$. The value of the proportionality constants is found by integration, in particular for $\star\omega = b\,\omega \wedge J$ it holds

$$b = \frac{\int_{X_0} \omega \wedge \star\omega}{\int_{X_0} \omega^2 \wedge J} = -\frac{1}{6\pi c^2} \int_{X_0} \|\omega\|^2 \mathrm{dVol}(X_0) = -1, \tag{A.50}$$

where we used (2.14), (2.90), the definition $\|\omega\|^2 = \iota_\omega \omega$, and the direct computations

$$\omega \wedge \star\omega = \|\omega\|^2 \,\mathrm{dVol}(X_0) \tag{A.51a}$$

$$\|\omega\|^2 = \frac{27}{2\pi^2 c^4} \left(1 + \frac{r^6}{c^6}\right)^{-2} \tag{A.51b}$$

$$\mathrm{dVol}(X_0) = \sqrt{g_6}\mathrm{d}^6 x = 8 r^5 \mathrm{d}r \wedge \mathrm{d}\Omega_5 \tag{A.51c}$$

$$\int_{\partial X_0} \mathrm{d}\Omega_5 = \int_{S^5/\mathbb{Z}_3} \mathrm{d}\Omega_5 = \frac{\pi^3}{3} \tag{A.51d}$$

$$\int_{X_0} \|\omega\|^2 \,\mathrm{dVol}(X_0) = 6\pi c^2 \tag{A.51e}$$

[12] Recall that α_p is harmonic iff $\star\alpha_p$ is.

In passing, we also showed that ω is a L_2-normalizable harmonic 2-form of $\mathcal{O}_{\mathbb{P}^2}(-3)$, since

$$\|\omega\|^2 \sim_\infty \frac{1}{r^{12}}. \tag{A.52}$$

As for $\star J = a\,\frac{1}{2} J \wedge J$, the proportionality constant is found by

$$a = \frac{\int_{X_0} J \wedge \star J}{\int_{X_0} \frac{1}{2} J^3} = \frac{1}{3} \lim_{r^2 \to \infty} \frac{\int_{X_0} \|J\|^2 \, \mathrm{dvol}(X_0)}{8\, r^6 \mathrm{Vol}(S^5/\mathbb{Z}_3)} = \frac{\|J\|^2}{3} = 1, \tag{A.53}$$

where we used $\frac{1}{3!} J^3 = \mathrm{dVol}(X_0)$ and $\|J\|^2 = \iota_J J = 3$.

A.5.3 Scalar delta function for $\mathbb{P}^2$: two derivations

Starting from the delta 2-form concentrated on $\mathbb{P}^2$, one naturally defines a scalar delta-function localized on $\mathbb{P}^2$ as

$$\delta^{(0)}_{\mathbb{P}^2} = J \lrcorner \delta^2_{\mathbb{P}^2} = -i g^{i\bar{\jmath}} \delta^2_{\mathbb{P}^2\, i\bar{\jmath}}, \tag{A.54}$$

where we used the more democratic notation for 2-forms $J \lrcorner \alpha_2 := \iota_J \alpha_2 = \iota_{\alpha_2} J = \frac{1}{2} J^{mn} \alpha_{mn}$. One would intuitively guess that $\delta^{(0)}_{\mathbb{P}^2} \propto \delta(r^2)$, and this is (almost) correct, as we are going to show by computing this explicitly. There are multiple ways one can go about this. Let us start with the explicit form of $\delta^2_{\mathbb{P}^2}$ on $\mathcal{U}_{(i)}$ in (2.127). It allows us to write, dropping the coordinate pedix $_{(i)}$,

$$J \lrcorner \delta^2_{\mathbb{P}^2} = \frac{1}{2} g^{\xi\bar{\xi}}|_{\xi=0}\, \delta(\xi), \tag{A.55}$$

where $g^{\xi\bar{\xi}}|_{\xi=0}$ is the $(\xi\bar{\xi})$ component of the inverse metric evaluated (but not pulled-back) at $\xi = 0$, found in (2.86b). Therefore, we find

$$\delta^{(0)}_{\mathbb{P}^2} = \frac{9}{2} \frac{c^4}{(1+\rho^2)^3}\, \delta(\xi) \qquad on \quad \mathcal{U}_{(i)}. \tag{A.56}$$

This is still somewhat unsatisfactory, since it uses locally defined coordinates and it does not seems clear how $\delta(\xi)$ should transform under coordinate change, while we do expect this to be globally defined (i.e. that the transformations of each factor cancel out). For this reason, it seems we are after a globally defined expression for $\delta^{(0)}_{\mathbb{P}^2}$. Such expression should involve only r^2, which is the sole $U(3)$-symmetric global coordinate at our disposal. This is indeed correct, as we now show. Let us consider (2.134), and let us contract both sides with the Kähler form, so that by primitivity of ω we get

$$J \lrcorner \delta^2_{\mathbb{P}^2} = -\frac{1}{2} J \lrcorner \mathrm{d}\mathrm{d}^c \kappa_{(4)} = \frac{1}{2} \left(-2 g^{i\bar{\jmath}} \partial_i \partial_{\bar{\jmath}}\, \kappa_{(4)} \right). \tag{A.57}$$

On the other hand, recalling the general expression for the Hodge-de Rham operator $\Delta = \mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger \mathrm{d}$ in local coordinates on a p-form,

$$(\Delta \omega_p)_{m_1 \dots m_p} = -\nabla^k \nabla_k \omega_{m_1 \dots m_p} - p\, R_{k[m_1} \omega^k{}_{m_2 \dots m_p]} - \frac{1}{2} p(p-1) R_{jk[m_1 m_2} \omega^{jk}{}_{m_3 \dots m_p]}, \tag{A.58}$$

its action on scalars reduces to[13]

$$\Delta f(z,\bar{z}) = -2g^{i\bar{\jmath}}\partial_i\partial_{\bar{\jmath}} f(z,\bar{z}). \tag{A.59}$$

Using (2.67) one can even explicitly compute this in the case of a $U(3)$-symmetric function, finding

$$\Delta f(r^2) = -2\left[r^2\left(1+\frac{c^6}{r^6}\right)^{\frac{2}{3}}\frac{\mathrm{d}^2}{\mathrm{d}\,(r^2)^2} + \frac{3+\frac{c^6}{r^6}}{\left(1+\frac{c^6}{r^6}\right)^{\frac{1}{3}}}\frac{\mathrm{d}}{\mathrm{d}r^2}\right] f(r^2). \tag{A.60}$$

Therefore, we proved the relation

$$J \lrcorner \delta^2_{\mathbb{P}^2} = \frac{1}{2}\Delta\kappa_{(4)}, \tag{A.61}$$

which gives us a way to compute

$$\delta^{(0)}_{\mathbb{P}^2} = \frac{1}{2}\Delta\kappa_{(4)}. \tag{A.62}$$

This looks reasonable, since a straightforward computation using (2.101) and (A.60) shows that $\Delta\kappa_{(4)} = 0$ for $r^2 \neq 0$, while $\kappa_{(4)}$ is singular in $r^2 = 0$ with a logarithmic divergence (2.105), therefore we can expect a second derivative combination of $\kappa_{(4)}$ to define some distribution localized in $r^2 = 0$[14].

Let us consider a test function $\varphi \in \mathcal{S}(\mathbb{R}^6)$, then using (2.101) and (A.60)

$$\langle\Delta\kappa_{(4)}|\varphi\rangle = -2\left[\langle\frac{\mathrm{d}}{\mathrm{d}r^2}A(r^2)|r^2\left(1+\frac{c^6}{r^6}\right)^{\frac{2}{3}}\varphi\rangle + \langle A(r^2)|\frac{3+\frac{c^6}{r^6}}{\left(1+\frac{c^6}{r^6}\right)^{\frac{1}{3}}}\varphi\rangle\right]. \tag{A.63}$$

We expect the first term to yield a singular distribution together with a term perfectly cancelling the second. Indeed, we compute (using $\mathrm{d}^6\mathbf{x} = r^5\mathrm{d}r\wedge\mathrm{d}\Omega_5 = r^4\frac{\mathrm{d}r^2}{2}\wedge\mathrm{d}\Omega_5$)

$$\begin{aligned}
&\langle\frac{\mathrm{d}}{\mathrm{d}r^2}A(r^2)|r^2\left(1+\frac{c^6}{r^6}\right)^{\frac{2}{3}}\varphi\rangle = \int_{\mathbb{C}^3/\mathbb{Z}_3}\left(\frac{\mathrm{d}}{\mathrm{d}r^2}A(r^2)\right)r^2\left(1+\frac{c^6}{r^6}\right)^{\frac{2}{3}}\varphi(\mathbf{x})\,\mathrm{d}^6\mathbf{x} =\\
&= \frac{1}{2}\int_{S^5/\mathbb{Z}_3}\mathrm{d}\Omega_5\int_0^\infty\left(\frac{\mathrm{d}}{\mathrm{d}r^2}A(r^2)\right)r^2\left(1+\frac{c^6}{r^6}\right)^{\frac{2}{3}}\varphi(\mathbf{x})\,\mathrm{d}r^2\\
&:= -\frac{1}{2}\int_{S^5/\mathbb{Z}_3}\mathrm{d}\Omega_5\int_0^\infty A(r^2)\frac{\mathrm{d}}{\mathrm{d}r^2}\left[r^2\left(1+\frac{c^6}{r^6}\right)^{\frac{2}{3}}\varphi(\mathbf{x})\right]\mathrm{d}r^2\\
&= -\frac{1}{2}\int_{S^5/\mathbb{Z}_3}\mathrm{d}\Omega_5\int_0^\infty A(r^2)\left[\frac{3+\frac{c^6}{r^6}}{\left(1+\frac{c^6}{r^6}\right)^{\frac{1}{3}}}r^4\varphi(\mathbf{x}) + r^6\left(1+\frac{c^6}{r^6}\right)^{\frac{2}{3}}\frac{\mathrm{d}}{\mathrm{d}r^2}\varphi(\mathbf{x})\right]\mathrm{d}r^2\\
&= -\langle A(r^2)|\frac{3+\frac{c^6}{r^6}}{\left(1+\frac{c^6}{r^6}\right)^{\frac{1}{3}}}\varphi\rangle + \frac{3}{4\pi}c^4\int_{S^5/\mathbb{Z}_3}\mathrm{d}\Omega_5\int_0^\infty\frac{\mathrm{d}}{\mathrm{d}r^2}\varphi(\mathbf{x})\mathrm{d}r^2.
\end{aligned} \tag{A.64}$$

[13]Recall that the connection symbols Γ^r_{mn} on a Kähler manifold are pure in their indices.
[14]For instance, this is motivated by $\frac{1}{2}\mathrm{dd}^c\log r^2 = \frac{2\pi}{3}\delta^2_{\mathbb{P}^2} + J_{FS}$ (direct computation).

Here, we compute separately

$$\begin{aligned} &\int_{S^5/\mathbb{Z}_3} \mathrm{d}\Omega_5 \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}r^2}\varphi(\mathbf{x})\mathrm{d}r^2 = \int_{S^5/\mathbb{Z}_3} \mathrm{d}\Omega_5 \left(\lim_{R\to\infty} \varphi(R\,\hat{\mathbf{x}}) - \lim_{\epsilon\to 0} \varphi(\epsilon\,\hat{\mathbf{x}}) \right) \\ &= -\frac{\pi^3}{3}\,\varphi(0) \end{aligned} \tag{A.65}$$

where we used $\mathrm{Vol}(S^5/\mathbb{Z}_3) = \frac{\pi^3}{3}$. On the other hand, notice that

$$\langle \frac{6}{\pi^3}\frac{\delta(r^2)}{r^4}|\varphi\rangle = \varphi(0), \tag{A.66}$$

namely that on $\mathbb{C}^3/\mathbb{Z}_3$ it holds

$$\delta(\mathbf{x}) = \frac{6}{\pi^3}\frac{\delta(r^2)}{r^4}. \tag{A.67}$$

This shows that

$$\langle \Delta\kappa_{(4)}|\varphi\rangle = \langle \frac{3}{\pi}\frac{c^4}{r^4}\,\delta(r^2)|\varphi\rangle, \tag{A.68}$$

that is, in the distributional sense,

$$\Delta\kappa_{(4)} = \frac{3}{\pi}\frac{c^4}{r^4}\,\delta(r^2) = \frac{9}{\pi}\,c^4\,\delta(r^6). \tag{A.69}$$

We conclude that

$$\delta^{(0)}_{\mathbb{P}^2} = \frac{9}{2\pi}\,c^4\,\delta(r^6), \tag{A.70}$$

which is indeed proportional to $\delta(r^2)$, as we already guessed. Unlike (A.56), this expression is manifestly global, since it only depends on the global coordinate r^2; it also does carry the right dimension ℓ^{-2}. Let us point out that this is consistent with (A.56). In order to make the connection between the two expressions, we just need to use the distributional identity

$$\delta\left(|\xi|^2\right) = \pi\,\delta(\xi), \tag{A.71}$$

so that, on $\mathcal{U}_{(i)}$, using (2.26)

$$\delta(r^6) = \delta\left(|\xi|^2\,(1+\rho^2)^3\right) = \frac{\delta\left(|\xi|^2\right)}{(1+\rho^2)^3} = \frac{\pi}{(1+\rho^2)^3}\,\delta(\xi). \tag{A.72}$$

In passing, by (A.71) we also showed the correct transformation law of $\delta(\xi)$ under (2.22):

$$\delta\left(\xi_{(j)}\right) = \frac{\delta\left(\xi_{(i)}\right)}{\left|u^j_{(i)}\right|^6}. \tag{A.73}$$

A.6 Derivation of the Eguchi-Hanson geometry

We look for a $U(3)$-symmetric, Calabi-Yau and Ricci-flat metric[15], therefore we start on $\mathcal{U}_{(4)}$, with coordinates (z^i). The most generic Ansatz including $U(3)$ symmetry is

$$\begin{aligned} \mathrm{d}s_6^2 &= g_{i\bar{\jmath}}\mathrm{d}z^i\mathrm{d}\bar{z}^{\bar{\jmath}} \\ &= \left(A(r^2)\delta_{i\bar{\jmath}} + B(r^2)\frac{\bar{z}_i z_{\bar{\jmath}}}{r^2}\right)\mathrm{d}z^i\mathrm{d}\bar{z}^{\bar{\jmath}}. \end{aligned} \tag{A.74}$$

Let us impose Kählerity:

$$\partial_i g_{j\bar{k}} = \partial_j g_{i\bar{k}}; \tag{A.75}$$

one finds

$$\partial_i g_{j\bar{k}} = A'\bar{z}_i\delta_{j\bar{k}} + \left(\frac{B}{r^2}\right)'\bar{z}_i\bar{z}_j z_{\bar{k}} + \frac{B}{r^2}\bar{z}_j\delta_{i\bar{k}}, \tag{A.76}$$

therefore Kählerity is equivalent to

$$B = r^2 A'. \tag{A.77}$$

As for the curvature 2-form $\mathcal{R} = -i\partial\bar{\partial}\log\sqrt{g_6}$, one way to achieve vanishing first Chern class is to make g_6 globally defined on $\mathcal{O}_{\mathbb{P}^2}(-3)$. Let us forget this for a moment and look simply for a Ricci-flat metric. This amounts to imposing

$$\partial_i\partial_{\bar{\jmath}}\log\sqrt{g_6} = 0. \tag{A.78}$$

On the other hand, $U(3)$ symmetry constraints g_6 to be a function of r^2 alone, and for a generic smooth function $f(r^2)$ it holds

$$\partial_i\partial_{\bar{\jmath}}f(r^2) = f''(r^2)\bar{z}_i z_{\bar{\jmath}} + f'(r^2)\delta_{i\bar{\jmath}}, \tag{A.79}$$

therefore (A.78) is solved iff $g_6 = \text{const}$. Then, let us choose

$$\det(g_{i\bar{\jmath}}) = 1. \tag{A.80}$$

Notice that this also implies that g_6 is globally defined, and $\mathcal{R}$ is exact, namely $\mathcal{O}_{\mathbb{P}^2}(-3)$ is Calabi-Yau as we expect.

In order to impose (A.80), we compute the determinant by means of the Levi-Civita tensor:

$$\begin{aligned} \det g_{i\bar{\jmath}} &= \frac{1}{3!}\epsilon^{i_1 i_2 i_3}\epsilon^{\bar{\jmath}_1\bar{\jmath}_2\bar{\jmath}_3}\left(A\delta_{i_1\bar{\jmath}_1} + A'\bar{z}_{i_1}z_{\bar{\jmath}_1}\right)\left(A\delta_{i_2\bar{\jmath}_2} + A'\bar{z}_{i_2}z_{\bar{\jmath}_2}\right)\left(A\delta_{i_3\bar{\jmath}_3} + A'\bar{z}_{i_3}z_{\bar{\jmath}_3}\right) \\ &= A^2(A + r^2A'). \end{aligned} \tag{A.81}$$

[15]Recall that a Calabi-Yau space is a Kähler manifold with trivial first Chern class. This implies that it admits a Ricci-flat metric in the same Kähler class. Equivalently, a Calabi-Yau space is a Kähler n-fold with exactly $SU(n)$-holonomy.

Then (A.80) becomes

$$A^2(A + r^2 A') = 1, \tag{A.82}$$

and multiplying by $3r^4$ both sides this can be recast in

$$\frac{\mathrm{d}}{\mathrm{d}r^2}\left[\left(r^2 A(r^2)\right)^3\right] = 3r^4, \tag{A.83}$$

which is solved by

$$A(r^2) = \left(1 + \frac{c^6}{r^6}\right)^{\frac{1}{3}}, \tag{A.84}$$

where $c \in \mathbb{R}$ controls the size the resolution.

Appendix B

Gravitational waves in general relativity

B.1 Gauge choice, Einstein equations and degrees of freedom

Let us review how the story goes in general relativity when dealing with gravitational waves (see e.g. [16]).[1] First of all, gravitational waves are linear perturbations $h_{\mu\nu}$ around Minkowski empty spacetime,[2]

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{B.1}$$

This means that gravitational waves are the solution of the linearized Einstein equations around Minkowski, which in a generic gauge take the form

$$\Box h_{\mu\nu} + \partial_\mu \partial_\nu h - \partial_\rho \partial_\mu h^\rho{}_\nu - \partial_\rho \partial_\nu h^\rho{}_\mu = 0, \tag{B.2}$$

where $h = \eta^{\mu\nu} h_{\mu\nu}$, and indices are raised with the background metric $\eta_{\mu\nu}$. In order to find physical solutions, one needs to fix the gauge first. One convenient way to do so is to choose the so-called de Donder gauge, or harmonic gauge,

$$\Box^{(g)} \tilde{x}^\mu = 0, \tag{B.3}$$

where $\Box^{(g)}$ is the Laplace-Beltrami operator built with the perturbed metric (B.1) and linearized in $h_{\mu\nu}$. This means that we are choosing harmonic coordinates, which we can go to by means of an infinitesimal coordinate change

$$\tilde{x}^\mu = x^\mu + \xi^\mu, \tag{B.4}$$

where the vector field ξ^μ is such that

$$\Box \xi^\mu = -\Box^{(g)} x^\mu. \tag{B.5}$$

[1]Only in this section, we will denote with greek indices $\mu, \nu, \ldots = 0, \ldots, 3$ the four-dimensional spacetime indices and with latin indices $i, j, \ldots = 1, 2, 3$ the spacial indices.

[2]See appendix B.2 for a more formal introduction to the subject.

Notice that this PDE admits infinite solutions, unless we specify some initial conditions in order to single out a specific solution of the homogeneous equation. We will come back to this later. The de Donder gauge (B.3) is equivalent to asking $\eta^{\mu\nu}\Gamma^{\rho}_{\mu\nu}[g] = 0$, or even more explicitly

$$\partial_\mu h^{\mu\nu} = \frac{1}{2}\partial^\nu h, \tag{B.6}$$

which is a non-dynamical constraint on the initial values of the perturbation components, effectively lowering the degrees of freedom thereof. We will need to impose this together with the Einstein equations. Thus, in the de Donder gauge (B.6), the Einstein equations (B.2) simplify to

$$\Box h_{\mu\nu} = 0 \tag{B.7a}$$

$$\partial_\mu h^{\mu\nu} - \frac{1}{2}\partial^\nu h = 0 \tag{B.7b}$$

One usually considers the *trace-reversed perturbation*

$$\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \tag{B.8}$$

in terms of which the Einstein equations in the de Donder gauge become

$$\Box\overline{h}_{\mu\nu} = 0 \tag{B.9a}$$

$$\partial_\mu\overline{h}^{\mu\nu} = 0 \tag{B.9b}$$

Let us go back to the issue of fixing the gauge. We already noted that the de Donder gauge condition (B.3) does not fix the gauge vector ξ^μ completely since (B.5), which is a linear second-order PDE, admits solutions of the form

$$\xi^\mu = \xi^\mu_0 + \tilde{\xi}^\mu, \tag{B.10}$$

where $\tilde{\xi}^\mu$ is the particular solution of the PDE (B.3), while ξ^μ_0 is a solution of the homogeneous PDE $\Box\xi^\mu_0 = 0$. The arbitrariness in the choice of ξ^μ_0 is referred to as *residual gauge invariance*. In order to completely fix the gauge, one should impose some initial conditions such that ξ^μ_0 is singled out unambiguously. Since these are four homogeneous wave equation in flat spacetime, and since it is well known (see e.g. [41]) that each one of them requires a *(second order) degree of freedom*[3] to be fixed in order to admit a single solution, one needs to impose four independent (second order) initial data constraints to fix the residual gauge invariance. This is done, once again, by fixing four degrees of freedom of the perturbation $h_{\mu\nu}$. The standard choice in cosmology for these four constraints is $h = h_{0i} = 0$, so that the full gauge fixing condition takes the form

$$\partial_\mu h^{\mu\nu} = 0 \tag{B.11a}$$

$$h = 0 \tag{B.11b}$$

$$h_{0i} = 0. \tag{B.11c}$$

[3]That is, the initial value of the functions $\xi^\mu_0(0, x^i)$ and of their time derivatives $\partial_0\xi^\mu_0(0, x^i)$.

This is the so-called *traceless-transverse* gauge. Indeed, one can show[4] that the conditions (B.11a-B.11c) single out a unique solution ξ^μ for (B.5), therefore fixing the gauge completely. To sum things up, the linearized Einstein equations supplemented with a *complete* gauge fixing are

$$\Box h_{\mu\nu} = 0 \tag{B.12a}$$

$$\partial_\mu h^{\mu\nu} = 0 \tag{B.12b}$$

$$h = 0 \tag{B.12c}$$

$$h_{0i} = 0 \tag{B.12d}$$

The last crucial point one should establish before actually solving these equations is the counting of physical degrees of freedom of the solution. This is most easily done looking at the above equations. Equation (B.12a) yields ten degrees of freedom, the de Donder gauge condition takes away four of them, and likewise the residual gauge fixing conditions take away another four[5]. Thus one is left with two physical degrees of freedom, as expected.

B.2 Perturbation theory in general relativity

As explained in the introduction of [1], a formal treatment of linear perturbation theory in general relativity requires the introduction of a triplet $(\mathcal{M}_4^{(0)}, \mathcal{M}_4, \psi)$, where $\mathcal{M}_4^{(0)}$ and $\mathcal{M}_4$ are four-dimensional Lorentzian manifolds, and $\psi : \mathcal{M}_4^{(0)} \to \mathcal{M}_4$ is an arbitrary diffeomorphism. In the case at hand, $\mathcal{M}_4^{(0)} = (X_4, \eta_{\mu\nu})$ is the unperturbed spacetime equipped with the background metric, and $\mathcal{M}_4 = (X_4, g_{\mu\nu})$ is the perturbed spacetime equipped with the perturbed metric. The atlas on $\mathcal{M}_4$ is defined such that the representative of ψ in local coordinates is the identity.[6] The gauge freedom of perturbation theory in general relativity is precisely the freedom of replacing the *gauge function* ψ with another diffeomorphism $\tilde{\psi}$.[7] A gauge choice is the choice of a specific diffeomorphism ψ. Therefore, a gauge transformation taking from the gauge ψ to the gauge $\tilde{\psi}$ is identified with the diffeomorphism $\Phi : \mathcal{M}_4^{(0)} \to \mathcal{M}_4^{(0)}$ defined by $\Phi = \psi^{-1} \circ \tilde{\psi}$. Then, by definition two gauge choices ψ and $\tilde{\psi}$ are related by

$$\tilde{\psi} = \psi \circ \Phi. \tag{B.13}$$

Let us call λ the perturbative expansion parameter of the system, then for $\lambda \to 0$ the perturbed manifold collapses onto the unperturbed one, and no perturbative gauge

[4]See the appendix B.3.

[5]This counting is equivalent to taking into account the differential Bianchi identity to reduce the number of dynamical Einstein equations down to six, and considering how many non-dynamical constraints are left for the physical degrees of freedom that this procedure has singled out.

[6]More precisely, this is to say that a local chart of $\mathcal{M}_4$, φ, is defined in terms of ψ and of a local chart on $\mathcal{M}_4^{(0)}$, $\varphi^{(0)}$, by $\varphi = \varphi^{(0)} \circ \psi^{-1}$.

[7]Notice that this, in turn, corresponds to replacing the local charts $\varphi = \varphi^{(0)} \circ \psi^{-1}$ of $\mathcal{M}_4$ with $\tilde{\varphi} = \varphi^{(0)} \circ \tilde{\psi}^{-1}$.

freedom is left. This shows that Φ should be an infinitesimal diffeomorphism, with local representative $\hat{\Phi}^\mu(x) = x^\mu + \xi^\mu$, where the vector field ξ is of order λ. At this point, notice that a gauge transformation is completely identified with the vector field $\xi \sim \mathcal{O}(\lambda)$, and one can think of $\hat{\Phi}^\mu(x) = x^\mu + \xi^\mu$ as an infinitesimal coordinate transformation of $\mathcal{M}_4^{(0)}$ [8]

$$\tilde{x}^\mu = x^\mu + \xi^\mu. \tag{B.14}$$

On the other hand, one defines the *perturbation of the metric tensor* as the section of $T^\star\mathcal{M}_4^{(0)} \otimes T^\star\mathcal{M}_4^{(0)}$ given by[9]

$$h_{\mu\nu} = \psi^\star g_{\mu\nu} - \eta_{\mu\nu}. \tag{B.15}$$

By definition of perturbative expansion parameter, $h_{\mu\nu} \sim \mathcal{O}(\lambda)$. Moreover, it is clear that the metric perturbation depends on the gauge choice. Under a gauge transformation taking ψ to $\tilde{\psi}$, it transforms to $\tilde{h}_{\mu\nu} = \tilde{\psi}^\star g_{\mu\nu} - \eta_{\mu\nu} = \Phi^\star\psi^\star g_{\mu\nu} - \eta_{\mu\nu}$, where we made use of (B.13). Interpreting $\Phi(x) = x + \xi$ as an infinitesimal change of coordinate, one can verify by direct computation that the action of the pullback of Φ on a $(0,2)-$rank tensor field T on $\mathcal{M}_4^{(0)}$ corresponds to a change of coordinates under Φ, and that

$$\Phi^\star T_{\mu\nu} = T_{\mu\nu} + \mathcal{L}_\xi T_{\mu\nu}, \tag{B.16}$$

where $\mathcal{L}_\xi\cdot$ is the Lie derivative along the flux of ξ, and we neglect higher order terms in λ since we are interested in linear perturbations. Therefore, using (B.16), under gauge transformations the perturbation to the metric tensor transforms as $\tilde{h}_{\mu\nu} = h_{\mu\nu} + \mathcal{L}_\xi\left(\psi^\star g_{\mu\nu}\right)$, and since we are working at the linear approximation it holds $\mathcal{L}_\xi\left(\psi^\star g_{\mu\nu}\right) = \mathcal{L}_\xi\eta_{\mu\nu}$, where we used (B.15). Thus, the gauge transformations of the metric pertubation are

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} + \mathcal{L}_\xi\eta_{\mu\nu} \tag{B.17}$$
$$= h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}, \tag{B.18}$$

as one would expect.

B.3 Cauchy problem for gravitational waves

In this section, greek indices $\mu, \nu, \ldots = 0, \ldots, 3$ will denote four-dimensional spacetime indices, latin indices $i, j, \ldots = 1, 2, 3$ will denote the three-dimensional spacial indices, t will denote the time coordinate and $\mathbf{x}$ will denote the spacial components of the spacetime coordinates x^μ, namely (x^i). We are interested in pinpointing a single solution of the homogeneous equation for the gauge vector in flat spacetime

$$\Box\xi^\mu = 0. \tag{B.19}$$

[8]Here the slightly more mathematically involved nature of this approach to perturbation theory pays off, making it clear as to why one should consider infinitesimal coordinate transformations of the order of the perturbation parameter when dealing with gauge transformations of the perturbation tensor.

[9]From here on, we will denote by the same symbol a function and its local representative.

This is achieved once the four second order initial data $(\xi^\mu(0,\mathbf{x}), \partial_0\xi^\mu(0,\mathbf{x}))$ are fixed. To ease up the notation, let us call these

$$\xi^\mu(0,\mathbf{x}) = f_1^\mu(\mathbf{x}); \tag{B.20a}$$
$$\partial_0\xi^\mu(0,\mathbf{x}) = f_2^\mu(\mathbf{x}). \tag{B.20b}$$

We are now going to show that $f_1^\mu(\mathbf{x})$ and $f_2^\mu(\mathbf{x})$ are uniquely determined by the four conditions

$$\tilde{h}(t,\mathbf{x}) = 0; \tag{B.21a}$$
$$\tilde{h}_{0i}(t,\mathbf{x}) = 0; \tag{B.21b}$$

once they are supplemented with the boundary condition

$$\lim_{|\mathbf{x}|^2\to\infty} f_1^\mu(\mathbf{x}) = \lim_{|\mathbf{x}|^2\to\infty} f_2^\mu(\mathbf{x}) = 0. \tag{B.22}$$

This is due to the fact that physical perturbations $h_{\mu\nu}$ vanish at spacial infinity, so that asymptotically there is no gauge invariance, meaning $\xi^\mu(t,\mathbf{x})$ must vanish at spacial infinity too. We already showed in appendix B.2 that the metric perturbation around Minskowski transforms under gauge symmetry as

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu. \tag{B.23}$$

The conditions (B.21a)-(B.21b) imply the four initial data constraints

$$\tilde{h}(0,\mathbf{x}) = 0 \tag{B.24a}$$
$$\tilde{h}_{0i}(0,\mathbf{x}) = 0 \tag{B.24b}$$
$$\partial_0\tilde{h}(0,\mathbf{x}) = 0 \tag{B.24c}$$
$$\partial_0\tilde{h}_{0i}(0,\mathbf{x}) = 0. \tag{B.24d}$$

Using (B.23) and (B.19), these conditions can be easily recast into the following PDEs for the initial data f_1^μ and f_2^μ:

$$h(0,\mathbf{x}) + 2f_2^0(\mathbf{x}) + 2\partial_i f_1^i(\mathbf{x}) = 0 \tag{B.25a}$$
$$h_{0i}(0,\mathbf{x}) + f_2^i(\mathbf{x}) - \partial_i f_1^0(\mathbf{x}) = 0 \tag{B.25b}$$
$$\partial_0 h(0,\mathbf{x}) + 2\nabla^2 f_1^0(\mathbf{x}) + 2\partial_i f_2^i(\mathbf{x}) = 0 \tag{B.25c}$$
$$\partial_0 h_{0i}(0,\mathbf{x}) + \nabla^2 f_1^i(\mathbf{x}) - \partial_i f_2^0(\mathbf{x}) = 0 \tag{B.25d}$$

where $\nabla^2 = \partial_i\partial^i$ is the three-dimensional Laplacian operator. Acting on (B.25b) with ∂_i and contracting the spacial index, we get $\partial_i f_2^i(\mathbf{x}) = \nabla^2 f_1^0(\mathbf{x}) - \partial^i h_{0i}(0,\mathbf{x})$. Plugging this in (B.25c) one gets

$$\nabla^2 f_1^0(\mathbf{x}) = \frac{1}{2}\left[\partial^i h_{0i}(0,\mathbf{x}) - \frac{1}{2}\partial_0 h(0,\mathbf{x})\right]. \tag{B.26}$$

One can show (see [41]) that ∇^2 is invertible on the space of functions vanishing at spacial infinity, therefore this equation admits a unique solution, which determines $f_1^0(\mathbf{x})$. Then (B.25b) determines $f_2^i(\mathbf{x})$. Similarly, (B.25a) can be recast as $\partial_i f_1^i(\mathbf{x}) = -f_2^0(\mathbf{x}) - \frac{1}{2}h(0,\mathbf{x})$; on the other hand, acting on (B.25d) with ∂_i and contracting the spacial index, one gets $\partial^i\partial_0 h_{0i}(0,\mathbf{x}) + \nabla^2\partial_i f_1^i(\mathbf{x}) - \nabla^2 f_2^0(\mathbf{x}) = 0$, so plugging in this (B.25a) one gets

$$\nabla^2 f_2^0(\mathbf{x}) = \frac{1}{2}\left[\partial^i\partial_0 h_{0i}(0,\mathbf{x}) - \frac{1}{2}\nabla^2 h(0,\mathbf{x})\right], \tag{B.27}$$

which uniquely determines $f_2^0(\mathbf{x})$. Then by the same token (B.25d) determines $f_1^i(\mathbf{x})$. We showed that the system (B.25a)-(B.25d) admits a unique solution for $f_1^\mu(\mathbf{x})$ and $f_2^\mu(\mathbf{x})$. The rest of the conditions implied by (B.21a)-(B.21b), namely $\partial_0^n\tilde{h}(0,\mathbf{x}) = 0$ and $\partial_0^n\tilde{h}_{0i}(0,\mathbf{x}) = 0$, are redundant once the Einstein equation $\Box h_{\mu\nu} = 0$ is taken into account. Indeed, for instance $\partial_0^2\tilde{h}(0,\mathbf{x}) = \nabla^2\tilde{h}(0,\mathbf{x}) = 0$ thanks to $\tilde{h}(0,\mathbf{x}) = 0$, and similarly $\partial_0^3\tilde{h}(0,\mathbf{x}) = \nabla^2\partial_0\tilde{h}(0,\mathbf{x}) = 0$ thanks to $\partial_0\tilde{h}(0,\mathbf{x}) = 0$. This concludes the proof.

Appendix C

About the minimal ten-dimensional Ansatz

Let us consider the generic non-perturbative $\mathbb{R}^3 \rtimes SO(3) \times U(3)$-symmetric Ansatz[1]

$$\hat{g}_{AB}\mathrm{d}X^A\mathrm{d}X^B = -2D\mathrm{d}t^2 + \mathrm{e}^{2A}\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + B\mathrm{d}r^2\mathrm{d}t + C\mathrm{d}^c r^2\mathrm{d}t + \mathrm{e}^{-2A}g_{i\overline{\jmath}}\mathrm{d}z^i\mathrm{d}\overline{z}^{\overline{\jmath}} \quad \text{(C.1)}$$

$$\begin{aligned} g_{i\overline{\jmath}}\mathrm{d}z^i\mathrm{d}\overline{z}^{\overline{\jmath}} = \mathrm{e}^{2F}\left(1+\frac{c^6}{r^6}\right)^{\frac{1}{3}}\mathrm{d}z^i\mathrm{d}\overline{z}_i - \mathrm{e}^{2G}\frac{c^6}{r^6}\left(1+\frac{c^6}{r^6}\right)^{-\frac{2}{3}}\frac{\partial r^2\overline{\partial}r^2}{r^2} + \\ + H\left(\partial r^2\right)^2 + \overline{H}\left(\overline{\partial}r^2\right)^2 \end{aligned} \quad \text{(C.2)}$$

where the perturbation functions A, B, C, D, F, G, H are functions of (t, r^2) and they are of order $|\langle S\rangle|^2$. The goal of this section is to show that one can always set $D = 0$ and $H = 0$ via coordinate redefinition, up to $\mathcal{O}(|\langle S\rangle|^4)$ corrections.

First, let us show that, at leading order, one can always trade $D = 0$ with a redefinition of B. Externally, (C.1) is more explicitly $\hat{g}_{AB}\mathrm{d}X^A\mathrm{d}X^B = -(2D + \mathrm{e}^{2A})\mathrm{d}t^2 + 2A\mathrm{d}^3x + \ldots$, where d^3x denotes the euclidean three-dimensional metric. We look for a $\mathbb{R}^3 \rtimes SO(3) \times U(3)$-symmetric time redefinition

$$\tilde{t} = t + \delta t(t, r^2), \quad \text{(C.3)}$$

where $\delta t(t, r^2)$ is real and order $|\langle S\rangle|^2$. Under this time redefinition one finds

$$-(2D+\mathrm{e}^{2A})\mathrm{d}t^2 + B\mathrm{d}r^2\mathrm{d}t^2 + C\mathrm{d}^c r^2\mathrm{d}t = -\left(1+2\tilde{A}\right)\mathrm{d}\tilde{t}^2 + \tilde{B}\mathrm{d}r^2\mathrm{d}\tilde{t} + C\mathrm{d}^c r^2\mathrm{d}t + \mathcal{O}(|\langle S\rangle|^4) \quad \text{(C.4)}$$

where

$$\tilde{A} = A + D - \dot{\delta t} \quad \text{(C.5a)}$$

$$\tilde{B} = B - 2\delta t' \quad \text{(C.5b)}$$

[1]In this section, we drop the pedix $_h$ for the sake of simplicity.

where $\dot{f}(t, r^2)$ denotes the derivative of f with respect to time, and $f'(t, r^2)$ denotes the derivative of f with respect to r^2. Therefore, imposing $A = \tilde{A}$, we find that the time redefinitions we are after are

$$\delta t(t, r^2) = \int D \, \mathrm{d}t + c(r^2) \tag{C.6}$$

where $c(r^2)$ is some function of r^2 alone. Then B is redefined according to (C.5b).

Let us move on to show that hermiticity of the internal metric ($H = 0$) can be achieved at the cost of redefining B, C and G. As a side remark, note that internal hermiticity up to internal coordinate redefinitions would be guaranteed if the perturbation functions were functions of the internal coordinates alone given that X_0 is a complex manifold. Let us consider a $\mathbb{R}^3 \rtimes SO(3) \times U(3)$-symmetric internal coordinate redefinition

$$\tilde{z}^i = z^i \left(1 + \delta z(t, r^2)\right), \tag{C.7}$$

where $\delta z(t, r^2)$ is complex and order $|\langle S \rangle|^2$. Up to $\mathcal{O}(|\langle S \rangle|^4)$ corrections, the perturbation functions are transformed to

$$\tilde{H} = H - \left(1 + \frac{c^6}{r^6}\right)^{-\frac{2}{3}} \left(\mathrm{Re}\delta z' - i\mathrm{Im}\dot{\delta z}\right) \tag{C.8a}$$

$$\tilde{B} = B - \left(1 + \frac{c^6}{r^6}\right)^{-\frac{2}{3}} \mathrm{Re}\dot{\delta z} \tag{C.8b}$$

$$\tilde{C} = C - \left(1 + \frac{c^6}{r^6}\right)^{-\frac{2}{3}} \mathrm{Im}\dot{\delta z} \tag{C.8c}$$

$$\tilde{G} = G - \mathrm{Re}\delta z + \frac{r^8}{c^6}\mathrm{Re}\delta z' \tag{C.8d}$$

Imposing $\tilde{H} = 0$, we find the internal coordinate redefinition we are after:

$$\mathrm{Re}\delta z = \int \left(1 + \frac{c^6}{r^6}\right)^{\frac{2}{3}} \mathrm{Re}H \mathrm{d}r^2 + c_1(t) \tag{C.9a}$$

$$\mathrm{Im}\delta z = -\int \left(1 + \frac{c^6}{r^6}\right)^{\frac{2}{3}} \mathrm{Im}H \mathrm{d}t + c_2(r^2) \tag{C.9b}$$

where $c_1(t)$ and $c_2(r^2)$ are arbitrary functions.

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