# Secondary constructions on generalized bent functions

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**Abstract.** In this paper, we construct generalized bent Boolean functions in n + 2 variables from 4 generalized Boolean functions in n variables. We also show that the direct sum of two generalized bent Boolean functions is generalized bent. Finally, we identify a set of affine functions in which every function is generalized bent.

**Key words:** Generalized Boolean functions; generalized bent functions; Walsh–Hadamard transform.

### 1 Introduction

In the recent years several authors have proposed generalizations of Boolean functions [8, 11, 12] and studied the effect of Walsh–Hadamard transform on these classes. As in the Boolean case, in the generalized setup the functions which have flat spectra with respect to the Walsh–Hadamard transform are said to be generalized bent and are of special interest. The classical notion of bent was invented by Rothaus [10].

The generalization due to Schmidt [11] is defined as follows:

**Definition 1.** [11, Schmidt] A function from  $\mathbb{Z}_2^n$  to  $\mathbb{Z}_q$  ( $\mathbb{Z}_q$  is ring of integers modulo q), for any positive integer  $q \ge 2$ , is called generalized Boolean function on n variables, where  $\mathbb{Z}_2^n = \{\mathbf{x} = (x_n, \dots, x_1) : x_\ell \in \mathbb{Z}_2, \ell = 1, 2, \dots, n\}$  denotes an n-dimensional vector space over  $\mathbb{Z}_2$  with the standard operations. The set of such functions denoted by  $\mathcal{GB}_n^n$ .

The generalized bent Boolean functions are used for constructing the constant amplitude codes for the q valued version of multicode Code Division Multiple Access (MC-CDMA). For q = 4, Schmidt [11] studied the relations between generalized bent functions, constant amplitude codes, and  $\mathbb{Z}_4$ -linear codes. For some problems concerning cyclic codes, Kerdock codes, and Delsarte-Goethals codes, Schmidt's generalization of Boolean function seems more natural than the generalization due to Kumar, Scholtz and Welch [8]. Solé and Tokareva [12] investigated systematically the links between Boolean bent functions [10], generalized bent Boolean functions [11], and quaternary bent functions [8]. Schmidt generalized the classical notion of Maiorana-McFarland class of bent functions, for

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2 B. K. Singh

q = 4. Recently, Stanica, Gangopadhyay and Singh [13] studied several properties generalized bent Boolean functions, characterized generalized bent Boolean functions symmetric with respect to two variables. They further generalized the classical notion of Maiorana-McFarland class of bent functions for any even positive integer q. The collections of the functions in this class is denoted by GMMF. They also provide the analogous of Dillon partial spreads type bent functions [7] in generalized setup and this class of generalized bent functions is termed as generalized Dillon's class (GD). In the same paper, authors provide an analogous of GPS class [1, 4, 5] in generalized setup which they refer as generalized spread class (GS) and proved that  $GD \cup GMMF \subseteq GS$ .

#### 1.1 Preliminaries

Let us denote the set of integers, real numbers and complex numbers by  $\mathbb{Z}$ ,  $\mathbb{R}$ and  $\mathbb{C}$ , respectively. By '+' we denote the addition over  $\mathbb{Z}$ ,  $\mathbb{R}$  and  $\mathbb{C}$ , whereas ' $\oplus$ ' denotes the addition over  $\mathbb{Z}_2^n$ , for all  $n \ge 1$ . Addition modulo q is denoted by '+' and it is understood from the context. For any  $\mathbf{x} = (x_n, \ldots, x_1)$  and  $\mathbf{y} = (y_n, \ldots, y_1)$  in  $\mathbb{Z}_2^n$ , the scalar (or inner) product is defined by  $\mathbf{x} \cdot \mathbf{y} :=$  $x_n y_n \oplus \cdots \oplus x_2 y_2 \oplus x_1 y_1$ . The conjugate of a bit b denoted by  $\overline{b}$ . If  $z = a + b i \in \mathbb{C}$ , then  $|z| = \sqrt{a^2 + b^2}$  denotes the absolute value of z, and  $\overline{z} = a - b i$  denotes the complex conjugate of z, where  $i^2 = -1$ , and  $a, b \in \mathbb{R}$ . Re[z] denotes the real part of z.  $\mathbb{R}i = \{ai : a \in \mathbb{R}\}$ , denotes the set of purely imaginary numbers.

Let  $\zeta = e^{2\pi i/q}$  be the complex *q*-primitive root of unity. The (generalized) Walsh-Hadamard transform of  $f \in \mathcal{GB}_n^q$  at any point  $\mathbf{u} \in \mathbb{Z}_2^n$  is the complex valued function

$$\mathcal{H}_f(\mathbf{u}) = 2^{-\frac{n}{2}} \sum_{\mathbf{x} \in \mathbb{Z}_2^n} \zeta^{f(\mathbf{x})} (-1)^{\mathbf{u} \cdot \mathbf{x}}.$$

A function  $f \in \mathcal{GB}_n^q$  is a generalized bent function if  $|\mathcal{H}_f(\mathbf{u})| = 1$ , for all  $\mathbf{u} \in \mathbb{Z}_2^n$ . Generalized bent functions exists for even integers and odd integers both whereas the bent Boolean functions (q = 2) exists only for even integers [10].

The sum

$$\mathcal{C}_{f,g}(\mathbf{u}) = \sum_{\mathbf{x} \in \mathbb{Z}_2^n} \zeta^{f(\mathbf{x}) - g(\mathbf{x} \oplus \mathbf{u})}$$

is the crosscorrelation between  $f, g \in \mathcal{GB}_n^q$  at  $\mathbf{u} \in \mathbb{Z}_2^n$ . The autocorrelation of  $f \in \mathcal{GB}_n^q$  at  $\mathbf{u} \in \mathbb{Z}_2^n$  is  $\mathcal{C}_{f,f}(\mathbf{u})$  above, which we denote by  $\mathcal{C}_f(\mathbf{u})$ .

If q = 2 (in Definition 1), we obtain the classical Boolean functions on n variables, whose set will be denoted by  $\mathcal{B}_n$ . The Walsh-Hadamard transform of any function  $f \in \mathcal{B}_n$  at  $\mathbf{u} \in \mathbb{Z}_2^n$  is defined by

$$W_f(\mathbf{u}) = \sum_{\mathbf{x} \in \mathbb{Z}_2^n} (-1)^{f(\mathbf{x}) + \mathbf{u} \cdot \mathbf{x}}.$$

A function  $f \in \mathcal{B}_n$  for even *n* is bent if and only if  $W_f(\mathbf{u}) = \pm 2^{\frac{n}{2}}$ , for all  $\mathbf{u} \in \mathbb{Z}_2^n$ .

There are several ways to construct bent Boolean functions in  $\mathcal{B}_{n+m}$  starting from bent functions in  $\mathcal{B}_n$  and  $\mathcal{B}_m$  [10]. Direct sum of two Boolean bent functions [6, pp. 81] is Boolean bent. Preneel et al. [9] constructed bent functions in n+2variables from 4 bent functions in n variables. The construction due to Preneel et al. [9] is given in the following

**Proposition 1.** [9, Theorem 7] The concatenation  $f \in \mathcal{B}_{n+2}$  of 4 bent functions  $f_{\ell} \in \mathcal{B}_n$  ( $\ell = 0, 1, 2, 3$ ) is bent if and only if

$$W_{f_0}(\mathbf{u})W_{f_1}(\mathbf{u})W_{f_2}(\mathbf{u})W_{f_3}(\mathbf{u}) = -2^{2n}, \text{ for all } \mathbf{u} \in \mathbb{Z}_2^n.$$

**Proposition 2.** [9, Corollary 2] The order of the  $f_{\ell}$ 's has no importance, i.e., suppose  $f = f_0 ||f_1||f_2||f_3$  with  $f_{\ell} \in \mathcal{B}_n$ .

(i) If  $f, f_0, f_1$  and  $f_2$  are bent, then  $f_3$  is bent. (ii) If  $f_0 = f_1$ , then  $f_2 = 1 \oplus f_3$ , and if  $f_0 = f_1 = f_2$ , then  $f_3 = 1 \oplus f_1$ .

## 2 Properties of Walsh–Hadamard transform on generalized Boolean functions

We gather in the current section several properties of the Walsh–Hadamard transform on generalized Boolean functions discussed in [13] are similar to the Boolean function case.

#### Theorem 1. We have:

(i) Let  $f \in \mathcal{GB}_n^q$ . The inverse of the Walsh-Hadamard transform is given by

$$\zeta^{f(\mathbf{y})} = 2^{-\frac{n}{2}} \sum_{\mathbf{u} \in \mathbb{Z}_2^n} \mathcal{H}_f(\mathbf{u})(-1)^{\mathbf{u} \cdot \mathbf{y}}.$$

(ii) If  $f, g \in \mathcal{GB}_n^q$ , then

$$\sum_{\mathbf{u}\in\mathbb{Z}_2^n} \mathcal{C}_{f,g}(\mathbf{u})(-1)^{\mathbf{u}\cdot\mathbf{x}} = 2^n \mathcal{H}_f(\mathbf{x}) \overline{\mathcal{H}_g(\mathbf{x})},$$
$$\mathcal{C}_{f,g}(\mathbf{u}) = \sum_{\mathbf{x}\in\mathbb{Z}_2^n} \mathcal{H}_f(\mathbf{x}) \overline{\mathcal{H}_g(\mathbf{x})}(-1)^{\mathbf{u}\cdot\mathbf{x}}.$$

Further,  $C_{f,g}(\mathbf{u}) = \overline{C_{g,f}(\mathbf{u})}$ , for all  $\mathbf{u} \in \mathbb{Z}_2^n$ , which implies that  $C_f(\mathbf{u})$  is always real.

(iii) Taking the particular case f = g we obtain

$$\mathcal{C}_f(\mathbf{u}) = \sum_{\mathbf{x} \in \mathbb{Z}_2^n} |\mathcal{H}_f(\mathbf{x})|^2 (-1)^{\mathbf{u} \cdot \mathbf{x}}.$$
 (1)

(iv) If  $f \in \mathcal{GB}_n^q$ , then f is a generalized bent function if and only if

$$\mathcal{C}_f(\mathbf{u}) = \begin{cases} 2^n & \text{if } \mathbf{u} = 0, \\ 0 & \text{if } \mathbf{u} \neq 0. \end{cases}$$

(v) Moreover, the (generalized) Parseval's identity holds

$$\sum_{\mathbf{x}\in\mathbb{Z}_2^n} |\mathcal{H}_f(\mathbf{x})|^2 = 2^n.$$
 (2)

The properties of these transforms for q = 2 can be derived from the previous theorem. For more results on Boolean functions, the interested reader can consult [2, 3, 6].

#### **3** Direct sum of two generalized bent Boolean functions

**Theorem 2.** Suppose  $f_1$  and  $f_2$  are two arbitrary generalized Boolean functions on  $\mathbb{Z}_2^r$  and  $\mathbb{Z}_2^s$  respectively. Then a function  $g: \mathbb{Z}_2^r \times \mathbb{Z}_2^s \to \mathbb{Z}_q$  expressed as

$$g(\mathbf{x}, \mathbf{y}) = f_1(\mathbf{x}) + f_2(\mathbf{y}), \text{ for all } \mathbf{x} \in \mathbb{Z}_2^r, \mathbf{y} \in \mathbb{Z}_2^s,$$

is generalized bent if and only if  $f_1$  and  $f_2$  both are generalized bents.

*Proof.* Let  $(\mathbf{u}, \mathbf{v}) \in \mathbb{Z}_2^r \times \mathbb{Z}_2^s$  be arbitrary. We compute,

$$\mathcal{H}_{g}(\mathbf{u}, \mathbf{v}) = 2^{-\frac{r+s}{2}} \sum_{(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}_{2}^{r} \times \mathbb{Z}_{2}^{s}} \zeta^{g(\mathbf{x}, \mathbf{y})}(-1)^{\mathbf{u} \cdot \mathbf{x} + \mathbf{v} \cdot \mathbf{y}}$$
$$= 2^{-\frac{r}{2}} \sum_{\mathbf{x} \in \mathbb{Z}_{2}^{r}} \zeta^{f_{1}(\mathbf{x})}(-1)^{\mathbf{u} \cdot \mathbf{x}} 2^{-\frac{s}{2}} \sum_{\mathbf{y} \in \mathbb{Z}_{2}^{s}} \zeta^{f_{2}(\mathbf{y})}(-1)^{\mathbf{v} \cdot \mathbf{y}}$$
$$= \mathcal{H}_{f_{1}}(\mathbf{u}) \mathcal{H}_{f_{2}}(\mathbf{v}).$$
(3)

Suppose  $f_1$  and  $f_2$  are two arbitrary generalized bent Boolean functions on  $\mathbb{Z}_2^r$ and  $\mathbb{Z}_2^s$  respectively, then we have  $|\mathcal{H}_{f_1}(\mathbf{u})| = 1$  and  $|\mathcal{H}_{f_2}(\mathbf{v})| = 1$ . Therefore, from (3),  $|\mathcal{H}_g(\mathbf{u}, \mathbf{v})| = |\mathcal{H}_{f_1}(\mathbf{u})| |\mathcal{H}_{f_2}(\mathbf{v})| = 1$ , for all  $(\mathbf{u}, \mathbf{v}) \in \mathbb{Z}_2^r \times \mathbb{Z}_2^s$ , this implies that g is generalized bent Boolean function.

Conversely, we assume g is generalized bent Boolean function, our aim is to show that the functions  $f_1$  and  $f_2$  are generalized bent Boolean functions. Let us suppose that  $f_1$  is not bent, then there exists  $\mathbf{u} \in \mathbb{Z}_2^r$  such that  $|\mathcal{H}_{f_1}(\mathbf{u})| = \ell \neq 1$ . Therefore, from (3),  $|\mathcal{H}_{f_2}(\mathbf{v})| = \frac{1}{\ell}$  for every  $\mathbf{v} \in \mathbb{Z}_2^s$ . This implies

$$\sum_{\mathbf{v}\in\mathbb{Z}_2^s} |\mathcal{H}_{f_2}(\mathbf{v})|^2 = \frac{2^s}{\ell^2} \neq 2^s.$$

Which is a contradiction. This completes the proof.

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# 4 Generalized bent functions in $\mathcal{GB}_{n+2}^q$ through 4 functions in $\mathcal{GB}_n^q$

Let  $\mathbf{v} = (v_r, \ldots, v_1)$ . We define

$$f_{\mathbf{v}}(x_{n-r},\ldots,x_1) = f(x_n = v_r,\ldots,x_{n-r+1} = v_1,x_{n-r},\ldots,x_1).$$

Let  $\mathbf{u} = (u_r, \ldots, u_1) \in \mathbb{Z}_2^r$  and  $\mathbf{w} = (w_{n-r}, \ldots, w_1) \in \mathbb{Z}_2^{n-r}$ . We define the vector concatenation by

$$\mathbf{uw} := (u_r, \dots, u_1, w_{n-r}, \dots, w_1)$$

**Lemma 1.** Let  $\mathbf{u} \in \mathbb{Z}_2^r$ ,  $\mathbf{w} \in \mathbb{Z}_2^{n-r}$  and f be an n-variable generalized Boolean function. Then

$$\mathcal{C}_f(\mathbf{u}\mathbf{w}) = \sum_{\mathbf{v}\in\mathbb{Z}_2^r} \mathcal{C}_{f_{\mathbf{v}},f_{\mathbf{v}\oplus\mathbf{u}}}(\mathbf{w}).$$

In particular, for 
$$r = 1$$
,  $C_f(0\mathbf{w}) = C_{f_0}(\mathbf{w}) + C_{f_1}(\mathbf{w})$ , and  $C_f(1\mathbf{w}) = 2Re[C_{f_0,f_1}(\mathbf{w})]$ .

**Theorem 3.** A function  $f \in \mathcal{GB}_{n+2}^q$  expressed as

$$f(z, y, \mathbf{x}) = f_0(\mathbf{x})(1 \oplus z)(1 \oplus y) + f_1(\mathbf{x})(1 \oplus z)y + f_2(\mathbf{x})(1 \oplus y)z + f_3(\mathbf{x})yz,$$

where  $f_{\ell} \in \mathcal{GB}_n^q$ ,  $(\ell = 0, 1, 2, 3)$ , is generalized bent if and only if

- (a)  $\sum_{\ell=0}^{3} C_{f_{\ell}}(\mathbf{u}) = 0$ , for all  $\mathbf{u} \in \mathbb{Z}_{2}^{n} \setminus \{0\}$ , and
- (b)  $C_{f_0,f_1}(\mathbf{u}) + C_{f_2,f_3}(\mathbf{u}), C_{f_0,f_2}(\mathbf{u}) + C_{f_1,f_3}(\mathbf{u}), C_{f_0,f_3}(\mathbf{u}) + C_{f_1,f_2}(\mathbf{u}) \in \mathbb{R}_l$ , for all  $\mathbf{u} \in \mathbb{Z}_2^n$ .

*Proof.* Let  $F_{\ell}$  ( $\ell \in \mathbb{Z}_2$ ) be the restriction of f on the hyperplane { $\ell$ } ×  $\mathbb{Z}_2 \times \mathbb{Z}_2^n \equiv \mathbb{Z}_2^{n+1}$ . Then  $F_0(y, \mathbf{x}) = f(0, y, \mathbf{x}) = f_0(\mathbf{x})(1 \oplus y) + f_1(\mathbf{x})y$  and  $F_1(y, \mathbf{x}) = f(1, y, \mathbf{x}) = f_2(\mathbf{x})(1 \oplus y) + f_3(\mathbf{x})y$ . Now,

$$\begin{aligned} \mathcal{C}_{F_0,F_1}(0,\mathbf{u}) &= \sum_{(y,\mathbf{x})\in\mathbb{Z}_2\times\mathbb{Z}_2^n} \zeta^{F_0(y,\mathbf{x})-F_1((y,\mathbf{x})\oplus(0,\mathbf{u}))} \\ &= \sum_{(y,\mathbf{x})\in\mathbb{Z}_2\times\mathbb{Z}_2^n} \zeta^{F_0(y,\mathbf{x})-F_1((y,\mathbf{x}\oplus\mathbf{u}))} \\ &= \sum_{\mathbf{x}\in\mathbb{Z}_2^n} \zeta^{F_0(0,\mathbf{x})-F_1((0,\mathbf{x}\oplus\mathbf{u}))} + \sum_{\mathbf{x}\in\mathbb{Z}_2^n} \zeta^{F_0(1,\mathbf{x})-F_1((1,\mathbf{x}\oplus\mathbf{u}))} \qquad (4) \\ &= \sum_{\mathbf{x}\in\mathbb{Z}_2^n} \zeta^{f_0(\mathbf{x})-f_2(\mathbf{x}\oplus\mathbf{u})} + \sum_{\mathbf{x}\in\mathbb{Z}_2^n} \zeta^{f_1(\mathbf{x})-f_3(\mathbf{x}\oplus\mathbf{u})} \\ &= \mathcal{C}_{f_0,f_2}(\mathbf{u}) + \mathcal{C}_{f_1,f_3}(\mathbf{u}). \end{aligned}$$

Similarly we compute,

$$\mathcal{C}_{F_0,F_1}(1,\mathbf{u}) = \sum_{\substack{(y,\mathbf{x})\in\mathbb{Z}_2\times\mathbb{Z}_2^n\\\mathbf{x}\in\mathbb{Z}_2^n}} \zeta^{F_0(y,\mathbf{x})-F_1((1\oplus y,\mathbf{x}\oplus \mathbf{u}))} \\
= \sum_{\mathbf{x}\in\mathbb{Z}_2^n} \zeta^{F_0(0,\mathbf{x})-F_1((1,\mathbf{x}\oplus \mathbf{u}))} + \sum_{\mathbf{x}\in\mathbb{Z}_2^n} \zeta^{F_0(1,\mathbf{x})-F_1((0,\mathbf{x}\oplus \mathbf{u}))} \\
= \sum_{\mathbf{x}\in\mathbb{Z}_2^n} \zeta^{f_0(\mathbf{x})-f_3(\mathbf{x}\oplus \mathbf{u})} + \sum_{\mathbf{x}\in\mathbb{Z}_2^n} \zeta^{f_1(\mathbf{x})-f_2(\mathbf{x}\oplus \mathbf{u})} \\
= \mathcal{C}_{f_0,f_3}(\mathbf{u}) + \mathcal{C}_{f_1,f_2}(\mathbf{u}).$$
(5)

#### 6 B. K. Singh

Using Lemma 1 for r = 1, we have

$$\mathcal{C}_f(0, a, \mathbf{u}) = \mathcal{C}_{F_0}(a, \mathbf{u}) + \mathcal{C}_{F_1}(a, \mathbf{u}), a \in \mathbb{Z}_2, \mathbf{u} \in \mathbb{Z}_2^n, \text{ and}$$
(6)

$$\mathcal{C}_f(1, a, \mathbf{u}) = \mathcal{C}_{F_0, F_1}(a, \mathbf{u}) + \overline{\mathcal{C}_{F_0, F_1}(a, \mathbf{u})}, a \in \mathbb{Z}_2, \mathbf{u} \in \mathbb{Z}_2^n.$$
(7)

Further, using Lemma 1 in (6), we have

$$\mathcal{C}_f(0,0,\mathbf{u}) = \mathcal{C}_{f_0}(\mathbf{u}) + \mathcal{C}_{f_1}(\mathbf{u}) + \mathcal{C}_{f_2}(\mathbf{u}) + \mathcal{C}_{f_3}(\mathbf{u}), \text{ and}$$
(8)

$$\mathcal{C}_{f}(0,1,\mathbf{u}) = \mathcal{C}_{F_{0}}(1,\mathbf{u}) + \mathcal{C}_{F_{1}}(1,\mathbf{u})$$
  
$$= \mathcal{C}_{f_{0},f_{1}}(\mathbf{u}) + \overline{\mathcal{C}_{f_{0},f_{1}}(\mathbf{u})} + \mathcal{C}_{f_{2},f_{3}}(\mathbf{u}) + \overline{\mathcal{C}_{f_{2},f_{3}}(\mathbf{u})}, \qquad (9)$$
  
$$= 2Re\left[\mathcal{C}_{f_{0},f_{1}}(\mathbf{u}) + \mathcal{C}_{f_{2},f_{3}}(\mathbf{u})\right].$$

Combining (4) and (7), we have

$$\mathcal{C}_{f}(1,0,\mathbf{u}) = \mathcal{C}_{F_{0},F_{1}}(0,\mathbf{u}) + \overline{\mathcal{C}_{F_{0},F_{1}}(0,\mathbf{u})}$$
$$= \mathcal{C}_{f_{0},f_{2}}(\mathbf{u}) + \mathcal{C}_{f_{1},f_{3}}(\mathbf{u}) + \overline{\mathcal{C}_{f_{0},f_{2}}(\mathbf{u}) + \mathcal{C}_{f_{1},f_{3}}(\mathbf{u})}$$
$$= 2Re\left[\mathcal{C}_{f_{0},f_{2}}(\mathbf{u}) + \mathcal{C}_{f_{1},f_{3}}(\mathbf{u})\right].$$
(10)

Similarly on combining (5) and (7), we have

$$\mathcal{C}_f(1,1,\mathbf{u}) = 2Re\left[\mathcal{C}_{f_0,f_3}(\mathbf{u}) + \mathcal{C}_{f_1,f_2}(\mathbf{u})\right].$$
(11)

Suppose  $f \in \mathcal{GB}_{n+2}^q$  such that conditions (a) and (b) holds, then from (8), (9), (10) and (11) we have  $\mathcal{C}_f(b, a, \mathbf{u}) = 0$ , for all  $(b, a, \mathbf{u}) \neq (0, 0, \mathbf{0})$  and  $\mathcal{C}_f(0, 0, \mathbf{0}) = 2^{n+2}$ . Therefore f is generalized bent.

Conversely, if f is generalized bent, then  $C_f(b, a, \mathbf{u}) = 0$ , for all  $(b, a, \mathbf{u}) \neq (0, 0, \mathbf{0})$  and  $C_f(0, 0, \mathbf{0}) = 2^{n+2}$ . Applying (8), (9), (10) and (11) with the above conditions we have (a) and (b).

*Example 1.* Let  $g \in \mathcal{GB}_n^q$  be any generalized bent function. If  $f \in \mathcal{GB}_{n+2}^q$  expressed as  $f = g \|g\|g\|(g + \frac{q}{2})$ , then f is generalized bent.

*Proof.* Here  $f_0 = f_1 = f_2 = g$  and  $f_3 = g + \frac{q}{2}$ . Since g is generalized bent, i.e.,  $C_{f_\ell}(\mathbf{u}) = 2^n \phi_{\{\mathbf{0}\}}(\mathbf{u})$ . Therefore, we have the condition (a) of Theorem 3. The condition (b) of Theorem 3 follows from the fact  $C_{h,h+\frac{q}{2}}(\mathbf{u}) + C_h(\mathbf{u}) = 0$  for all  $\mathbf{u} \in \mathbb{Z}_2^n$ .

In the following theorem, we construct a generalized bent function in  $\mathcal{GB}_{n+2}^q$  from two generalized bent functions in  $\mathcal{GB}_n^q$ .

**Theorem 4.** Suppose  $f \in \mathcal{GB}_{n+2}^q$  is expressed as

$$f(z, y, \mathbf{x}) = f_{y \oplus 1}(\mathbf{x}) + \left(\frac{q}{2}\right) yz, \text{ for all } y, z, \in \mathbb{Z}_2, \mathbf{x} \in \mathbb{Z}_2^n,$$

where  $f_{y\oplus 1} \in \mathcal{GB}_n^q$   $(y \in \mathbb{Z}_2)$ . Then f is generalized bent if and only if  $f_0$  and  $f_1$  both are generalized bent.

*Proof.* Let  $(b, a, \mathbf{u}) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2^n$  be arbitrary. We compute,

$$\begin{aligned} \mathcal{H}_{f}(b, a, \mathbf{u}) &= 2^{-\frac{n+2}{2}} \sum_{(z, y, \mathbf{x}) \in \mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2}^{n}} \zeta^{f(z, y, \mathbf{x})}(-1)^{\mathbf{u} \cdot \mathbf{x} + ya + zb} \\ &= 2^{-\frac{n+2}{2}} \sum_{\mathbf{x} \in \mathbb{Z}_{2}^{n}} \sum_{y \in \mathbb{Z}_{2}} \sum_{z \in \mathbb{Z}_{2}} \zeta^{f_{y \oplus 1}(\mathbf{x}) + \left(\frac{q}{2}\right)yz}(-1)^{\mathbf{u} \cdot \mathbf{x} + ya + zb} \\ &= 2^{-\frac{n+2}{2}} \sum_{\mathbf{x} \in \mathbb{Z}_{2}^{n}} \sum_{y \in \mathbb{Z}_{2}} \zeta^{f_{y \oplus 1}(\mathbf{x})}(-1)^{\mathbf{u} \cdot \mathbf{x} \oplus ya} \sum_{z \in \mathbb{Z}_{2}} (-1)^{(y \oplus b)z} \\ &= 2^{-\frac{n+2}{2}} \sum_{\mathbf{x} \in \mathbb{Z}_{2}^{n}} \sum_{y \in \mathbb{Z}_{2}} \zeta^{f_{y \oplus 1}(\mathbf{x})}(-1)^{\mathbf{u} \cdot \mathbf{x} \oplus ya} \left( 2\phi_{\{b\}}(y) \right) \\ &= 2^{-\frac{n}{2}} \sum_{\mathbf{x} \in \mathbb{Z}_{2}^{n}} \zeta^{f_{b \oplus 1}(\mathbf{x})}(-1)^{\mathbf{u} \cdot \mathbf{x} \oplus ba} \\ &= (-1)^{ab} \mathcal{H}_{f_{b \oplus 1}}(\mathbf{u}). \end{aligned}$$

Since  $f_0$  and  $f_1$  are generalized bent, that is,  $|\mathcal{H}_{f_0}(\mathbf{u})| = |\mathcal{H}_{f_1}(\mathbf{u})| = 1$ , for all  $\mathbf{u} \in \mathbb{Z}_2^n$ . This implies that  $|\mathcal{H}_f(b, a, \mathbf{u})| = |\mathcal{H}_{f_{b\oplus 1}}(\mathbf{u})| = 1$ , for all  $\mathbf{u} \in \mathbb{Z}_2^n$  and  $a, b \in \mathbb{Z}_2$ , and hence, we have the result.

Remark 1. The functions constructed in Theorem 4 are not symmetric with respect to y and z if  $f_0 \neq f_1$ . This result follows from the fact that  $f(1, 0, \mathbf{x}) = f_1(\mathbf{x})$  and  $f(0, 1, \mathbf{x}) = f_0(\mathbf{x})$ .

#### 5 Existence of generalized bent functions in affine set

Suppose  $q = 2^h$ , h is a positive integer. Then an affine function  $f_{\lambda} \in \mathcal{GB}_n^q$  [11] is expressed as  $f_{\lambda}(\mathbf{x}) = \lambda_0 + \sum_{i=1}^n \lambda_i x_i, \lambda_i \in \mathbb{Z}_q$ . Let  $\mathcal{A}_n^q$  denotes the set of all the affine functions on n variables. In the classical notion (q = 2), all the affine functions are either balanced or constant and therefore, they are not bent. In the generalized set up, we prove that there exists a set of affine functions in which every function is generalized bent. Further, we identify another affine set in which no function is generalized bent.

**Theorem 5.** Let q be a positive integer such that  $q = 0 \mod 4$ . Then an affine function  $f_{\lambda} \in \mathcal{A}_n^q$  is generalized bent if and only if

$$\prod_{i=1}^{n} \left( 1 + (-1)^{u_i} \cos\left(\frac{2\pi\lambda_i}{q}\right) \right) = 1, \text{ for all } \mathbf{u} \in \mathbb{Z}_2^n.$$
(12)

*Proof.* Let q be a positive integer such that  $q = 0 \mod 4$ . Suppose  $f_{\lambda} \in \mathcal{A}_n^q$  be arbitrary. Then it is expressed as  $f_{\lambda}(\mathbf{x}) = \lambda_0 + \sum_{i=1}^n \lambda_i x_i, \lambda_i \in \mathbb{Z}_q$ . The

7

## 8 B. K. Singh

Walsh-Hadamard transform of  $f_{\lambda}$  at  $\mathbf{u} \in \mathbb{Z}_2^n$  is

$$\begin{aligned} \mathcal{H}_{f_{\lambda}}(\mathbf{u}) &= 2^{-\frac{n}{2}} \sum_{\mathbf{x} \in \mathbb{Z}_{2}^{n}} \zeta^{f_{\lambda}(\mathbf{x})}(-1)^{\mathbf{u} \cdot \mathbf{x}} \\ &= \zeta^{\lambda_{0}} 2^{-\frac{n}{2}} \sum_{\mathbf{x} \in \mathbb{Z}_{2}^{n}} \zeta^{\sum_{i=1}^{n} (\lambda_{i} x_{i} + \left(\frac{q}{2}\right) u_{i} x_{i})} \\ &= \zeta^{\lambda_{0}} 2^{-\frac{n}{2}} \prod_{i=1}^{n} \sum_{x_{i} \in \mathbb{Z}_{2}} \zeta^{(\lambda_{i} + \left(\frac{q}{2}\right) u_{i}) x_{i}} \\ &= \zeta^{\lambda_{0}} 2^{-\frac{n}{2}} \prod_{i=1}^{n} (1 + \zeta^{k_{i}}), \quad k_{i} = \lambda_{i} + \left(\frac{q}{2}\right) u_{i} \\ &= \zeta^{\lambda_{0}} 2^{-\frac{n}{2}} \prod_{i=1}^{n} \left(1 + \cos\left(\frac{2\pi k_{i}}{q}\right) + i \sin\left(\frac{2\pi k_{i}}{q}\right)\right) \\ &= \zeta^{\lambda_{0}} 2^{-\frac{n}{2}} \prod_{i=1}^{n} \left(1 + (-1)^{u_{i}} \cos\left(\frac{2\pi \lambda_{i}}{q}\right) + i(-1)^{u_{i}} \sin\left(\frac{2\pi \lambda_{i}}{q}\right)\right), \end{aligned}$$

which implies that

$$|\mathcal{H}_{f_{\lambda}}(\mathbf{u})|^{2} = 2^{-n} \prod_{i=1}^{n} 2\left(1 + (-1)^{u_{i}} \cos\left(\frac{2\pi\lambda_{i}}{q}\right)\right)$$
$$= \prod_{i=1}^{n} \left(1 + (-1)^{u_{i}} \cos\left(\frac{2\pi\lambda_{i}}{q}\right)\right).$$

Therefore,  $f_\lambda$  is generalized bent if and only if

$$\prod_{i=1}^{n} \left( 1 + (-1)^{u_i} \cos\left(\frac{2\pi\lambda_i}{q}\right) \right) = 1, \text{ for all } \mathbf{u} \in \mathbb{Z}_2^n.$$
(13)

It is to be noted that for any i = 1, 2, ..., n,  $\lambda_i = \frac{q}{4}, \frac{3q}{4}$  and  $\lambda_0 \in \mathbb{Z}_q$  are the solutions of the equation (13). Thus we have the following proposition

**Proposition 3.** Let q be a positive integer such that  $q = 0 \mod 4$ . Consider a set  $\mathcal{A}'$  of affine functions in  $\mathcal{A}_n^q$  defined by

$$\mathcal{A}' = \{ f_{\lambda} \in \mathcal{A}_n^q : \lambda_0 \in \mathbb{Z}_q, \lambda_\ell = \frac{q}{4} \text{ or } \frac{3q}{4}, \text{ for all } \ell \in \{1, 2, \dots, n\} \}.$$

Then every function in  $\mathcal{A}'$  is generalized bent.

Remark 2. Define a set  $\mathcal{S} \subset \mathcal{A}_n^q$  as follows

$$\mathcal{S} = \{ f_{\lambda} \in \mathcal{A}_{n}^{q} : \lambda_{0} \in \mathbb{Z}_{q}, \text{ and for some } \ell \in \{1, 2, \dots, n\}, \lambda_{\ell} = 0 \text{ or } \frac{q}{2} \}.$$

Then no function in  ${\mathcal S}$  is generalized bent.

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