# Secondary constructions on generalized bent functions 

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#### Abstract

In this paper, we construct generalized bent Boolean functions in $n+2$ variables from 4 generalized Boolean functions in $n$ variables. We also show that the direct sum of two generalized bent Boolean functions is generalized bent. Finally, we identify a set of affine functions in which every function is generalized bent.


Key words: Generalized Boolean functions; generalized bent functions; Walsh-Hadamard transform.

## 1 Introduction

In the recent years several authors have proposed generalizations of Boolean functions $[8,11,12]$ and studied the effect of Walsh-Hadamard transform on these classes. As in the Boolean case, in the generalized setup the functions which have flat spectra with respect to the Walsh-Hadamard transform are said to be generalized bent and are of special interest. The classical notion of bent was invented by Rothaus [10].

The generalization due to Schmidt [11] is defined as follows:
Definition 1. [11, Schmidt] A function from $\mathbb{Z}_{2}^{n}$ to $\mathbb{Z}_{q}\left(\mathbb{Z}_{q}\right.$ is ring of integers modulo $q$ ), for any positive integer $q \geq 2$, is called generalized Boolean function on $n$ variables, where $\mathbb{Z}_{2}^{n}=\left\{\mathbf{x}=\left(x_{n}, \ldots, x_{1}\right): x_{\ell} \in \mathbb{Z}_{2}, \ell=1,2, \ldots, n\right\}$ denotes an $n$-dimensional vector space over $\mathbb{Z}_{2}$ with the standard operations. The set of such functions denoted by $\mathcal{G B}_{n}^{q}$.

The generalized bent Boolean functions are used for constructing the constant amplitude codes for the $q$ valued version of multicode Code Division Multiple Access (MC-CDMA). For $q=4$, Schmidt [11] studied the relations between generalized bent functions, constant amplitude codes, and $\mathbb{Z}_{4}$-linear codes. For some problems concerning cyclic codes, Kerdock codes, and Delsarte-Goethals codes, Schmidt's generalization of Boolean function seems more natural than the generalization due to Kumar, Scholtz and Welch [8]. Solé and Tokareva [12] investigated systematically the links between Boolean bent functions [10], generalized bent Boolean functions [11], and quaternary bent functions [8]. Schmidt generalized the classical notion of Maiorana-McFarland class of bent functions, for

[^0]$q=4$. Recently, Stanica, Gangopadhyay and Singh [13] studied several properties generalized bent Boolean functions, characterized generalized bent Boolean functions symmetric with respect to two variables. They further generalized the classical notion of Maiorana-McFarland class of bent functions for any even positive integer $q$. The collections of the functions in this class is denoted by GMMF. They also provide the analogous of Dillon partial spreads type bent functions [7] in generalized setup and this class of generalized bent functions is termed as generalized Dillon's class (GD). In the same paper, authors provide an analogous of GPS class $[1,4,5]$ in generalized setup which they refer as generalized spread class (GS) and proved that $G D \cup G M M F \subseteq G S$.

### 1.1 Preliminaries

Let us denote the set of integers, real numbers and complex numbers by $\mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$, respectively. By ' + ' we denote the addition over $\mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$, whereas ' $\oplus$ ' denotes the addition over $\mathbb{Z}_{2}^{n}$, for all $n \geq 1$. Addition modulo $q$ is denoted by ' + ' and it is understood from the context. For any $\mathbf{x}=\left(x_{n}, \ldots, x_{1}\right)$ and $\mathbf{y}=\left(y_{n}, \ldots, y_{1}\right)$ in $\mathbb{Z}_{2}^{n}$, the scalar (or inner) product is defined by $\mathbf{x} \cdot \mathbf{y}:=$ $x_{n} y_{n} \oplus \cdots \oplus x_{2} y_{2} \oplus x_{1} y_{1}$. The conjugate of a bit $b$ denoted by $\bar{b}$. If $z=a+b \imath \in \mathbb{C}$, then $|z|=\sqrt{a^{2}+b^{2}}$ denotes the absolute value of $z$, and $\bar{z}=a-b \imath$ denotes the complex conjugate of $z$, where $\imath^{2}=-1$, and $a, b \in \mathbb{R}$. Re[z] denotes the real part of $z . \mathbb{R}_{\imath}=\{a \imath: a \in \mathbb{R}\}$, denotes the set of purely imaginary numbers.

Let $\zeta=e^{2 \pi \imath / q}$ be the complex $q$-primitive root of unity. The (generalized) Walsh-Hadamard transform of $f \in \mathcal{G B}_{n}^{q}$ at any point $\mathbf{u} \in \mathbb{Z}_{2}^{n}$ is the complex valued function

$$
\mathcal{H}_{f}(\mathbf{u})=2^{-\frac{n}{2}} \sum_{\mathbf{x} \in \mathbb{Z}_{2}^{n}} \zeta^{f(\mathbf{x})}(-1)^{\mathbf{u} \cdot \mathbf{x}}
$$

A function $f \in \mathcal{G B}_{n}^{q}$ is a generalized bent function if $\left|\mathcal{H}_{f}(\mathbf{u})\right|=1$, for all $\mathbf{u} \in \mathbb{Z}_{2}^{n}$. Generalized bent functions exists for even integers and odd integers both whereas the bent Boolean functions $(q=2)$ exists only for even integers [10].

The sum

$$
\mathcal{C}_{f, g}(\mathbf{u})=\sum_{\mathbf{x} \in \mathbb{Z}_{2}^{n}} \zeta^{f(\mathbf{x})-g(\mathbf{x} \oplus \mathbf{u})}
$$

is the crosscorrelation between $f, g \in \mathcal{G B}_{n}^{q}$ at $\mathbf{u} \in \mathbb{Z}_{2}^{n}$. The autocorrelation of $f \in \mathcal{G B}_{n}^{q}$ at $\mathbf{u} \in \mathbb{Z}_{2}^{n}$ is $\mathcal{C}_{f, f}(\mathbf{u})$ above, which we denote by $\mathcal{C}_{f}(\mathbf{u})$.

If $q=2$ (in Definition 1), we obtain the classical Boolean functions on $n$ variables, whose set will be denoted by $\mathcal{B}_{n}$. The Walsh-Hadamard transform of any function $f \in \mathcal{B}_{n}$ at $\mathbf{u} \in \mathbb{Z}_{2}^{n}$ is defined by

$$
W_{f}(\mathbf{u})=\sum_{\mathbf{x} \in \mathbb{Z}_{2}^{n}}(-1)^{f(\mathbf{x})+\mathbf{u} \cdot \mathbf{x}}
$$

A function $f \in \mathcal{B}_{n}$ for even $n$ is bent if and only if $W_{f}(\mathbf{u})= \pm 2^{\frac{n}{2}}$, for all $\mathbf{u} \in \mathbb{Z}_{2}^{n}$.

There are several ways to construct bent Boolean functions in $\mathcal{B}_{n+m}$ starting from bent functions in $\mathcal{B}_{n}$ and $\mathcal{B}_{m}$ [10]. Direct sum of two Boolean bent functions [ 6, pp. 81] is Boolean bent. Preneel et al. [9] constructed bent functions in $n+2$ variables from 4 bent functions in $n$ variables. The construction due to Preneel et al. [9] is given in the following
Proposition 1. [9, Theorem 7] The concatenation $f \in \mathcal{B}_{n+2}$ of 4 bent functions $f_{\ell} \in \mathcal{B}_{n} \quad(\ell=0,1,2,3)$ is bent if and only if

$$
W_{f_{0}}(\mathbf{u}) W_{f_{1}}(\mathbf{u}) W_{f_{2}}(\mathbf{u}) W_{f_{3}}(\mathbf{u})=-2^{2 n}, \text { for all } \mathbf{u} \in \mathbb{Z}_{2}^{n}
$$

Proposition 2. [9, Corollary 2] The order of the $f_{\ell}$ 's has no importance, i.e., suppose $f=f_{0}\left\|f_{1}\right\| f_{2} \| f_{3}$ with $f_{\ell} \in \mathcal{B}_{n}$.
(i) If $f, f_{0}, f_{1}$ and $f_{2}$ are bent, then $f_{3}$ is bent.
(ii) If $f_{0}=f_{1}$, then $f_{2}=1 \oplus f_{3}$, and if $f_{0}=f_{1}=f_{2}$, then $f_{3}=1 \oplus f_{1}$.

## 2 Properties of Walsh-Hadamard transform on generalized Boolean functions

We gather in the current section several properties of the Walsh-Hadamard transform on generalized Boolean functions discussed in [13] are similar to the Boolean function case.

Theorem 1. We have:
(i) Let $f \in \mathcal{G B}_{n}^{q}$. The inverse of the Walsh-Hadamard transform is given by

$$
\zeta^{f(\mathbf{y})}=2^{-\frac{n}{2}} \sum_{\mathbf{u} \in \mathbb{Z}_{2}^{n}} \mathcal{H}_{f}(\mathbf{u})(-1)^{\mathbf{u} \cdot \mathbf{y}}
$$

(ii) If $f, g \in \mathcal{G B}_{n}^{q}$, then

$$
\begin{array}{r}
\sum_{\mathbf{u} \in \mathbb{Z}_{2}^{n}} \mathcal{C}_{f, g}(\mathbf{u})(-1)^{\mathbf{u} \cdot \mathbf{x}}=2^{n} \mathcal{H}_{f}(\mathbf{x}) \overline{\mathcal{H}_{g}(\mathbf{x})}, \\
\mathcal{C}_{f, g}(\mathbf{u})=\sum_{\mathbf{x} \in \mathbb{Z}_{2}^{n}} \mathcal{H}_{f}(\mathbf{x}) \overline{\mathcal{H}_{g}(\mathbf{x})}(-1)^{\mathbf{u} \cdot \mathbf{x}}
\end{array}
$$

Further, $\mathcal{C}_{f, g}(\mathbf{u})=\overline{\mathcal{C}_{g, f}(\mathbf{u})}$, for all $\mathbf{u} \in \mathbb{Z}_{2}^{n}$, which implies that $\mathcal{C}_{f}(\mathbf{u})$ is always real.
(iii) Taking the particular case $f=g$ we obtain

$$
\begin{equation*}
\mathcal{C}_{f}(\mathbf{u})=\sum_{\mathbf{x} \in \mathbb{Z}_{2}^{n}}\left|\mathcal{H}_{f}(\mathbf{x})\right|^{2}(-1)^{\mathbf{u} \cdot \mathbf{x}} \tag{1}
\end{equation*}
$$

(iv) If $f \in \mathcal{G B}_{n}^{q}$, then $f$ is a generalized bent function if and only if

$$
\mathcal{C}_{f}(\mathbf{u})= \begin{cases}2^{n} & \text { if } \mathbf{u}=0 \\ 0 & \text { if } \mathbf{u} \neq 0\end{cases}
$$

(v) Moreover, the (generalized) Parseval's identity holds

$$
\begin{equation*}
\sum_{\mathbf{x} \in \mathbb{Z}_{2}^{n}}\left|\mathcal{H}_{f}(\mathbf{x})\right|^{2}=2^{n} \tag{2}
\end{equation*}
$$

The properties of these transforms for $q=2$ can be derived from the previous theorem. For more resuts on Boolean functions, the interested reader can consult $[2,3,6]$.

## 3 Direct sum of two generalized bent Boolean functions

Theorem 2. Suppose $f_{1}$ and $f_{2}$ are two arbitrary generalized Boolean functions on $\mathbb{Z}_{2}^{r}$ and $\mathbb{Z}_{2}^{s}$ respectively. Then a function $g: \mathbb{Z}_{2}^{r} \times \mathbb{Z}_{2}^{s} \rightarrow \mathbb{Z}_{q}$ expressed as

$$
g(\mathbf{x}, \mathbf{y})=f_{1}(\mathbf{x})+f_{2}(\mathbf{y}), \text { for all } \mathbf{x} \in \mathbb{Z}_{2}^{r}, \mathbf{y} \in \mathbb{Z}_{2}^{s}
$$

is generalized bent if and only if $f_{1}$ and $f_{2}$ both are generalized bents.
Proof. Let $(\mathbf{u}, \mathbf{v}) \in \mathbb{Z}_{2}^{r} \times \mathbb{Z}_{2}^{s}$ be arbitrary. We compute,

$$
\begin{align*}
\mathcal{H}_{g}(\mathbf{u}, \mathbf{v}) & =2^{-\frac{r+s}{2}} \sum_{(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}_{2}^{r} \times \mathbb{Z}_{2}^{s}} \zeta^{g(\mathbf{x}, \mathbf{y})}(-1)^{\mathbf{u} \cdot \mathbf{x}+\mathbf{v} \cdot \mathbf{y}} \\
& =2^{-\frac{r}{2}} \sum_{\mathbf{x} \in \mathbb{Z}_{2}^{r}} \zeta^{f_{1}(\mathbf{x})}(-1)^{\mathbf{u} \cdot \mathbf{x}} 2^{-\frac{s}{2}} \sum_{\mathbf{y} \in \mathbb{Z}_{2}^{s}} \zeta^{f_{2}(\mathbf{y})}(-1)^{\mathbf{v} \cdot \mathbf{y}}  \tag{3}\\
& =\mathcal{H}_{f_{1}}(\mathbf{u}) \mathcal{H}_{f_{2}}(\mathbf{v}) .
\end{align*}
$$

Suppose $f_{1}$ and $f_{2}$ are two arbitrary generalized bent Boolean functions on $\mathbb{Z}_{2}^{r}$ and $\mathbb{Z}_{2}^{s}$ respectively, then we have $\left|\mathcal{H}_{f_{1}}(\mathbf{u})\right|=1$ and $\left|\mathcal{H}_{f_{2}}(\mathbf{v})\right|=1$. Therefore, from (3), $\left|\mathcal{H}_{g}(\mathbf{u}, \mathbf{v})\right|=\left|\mathcal{H}_{f_{1}}(\mathbf{u})\right|\left|\mathcal{H}_{f_{2}}(\mathbf{v})\right|=1$, for all $(\mathbf{u}, \mathbf{v}) \in \mathbb{Z}_{2}^{r} \times \mathbb{Z}_{2}^{s}$, this implies that $g$ is generalized bent Boolean function.

Conversely, we assume $g$ is generalized bent Boolean function, our aim is to show that the functions $f_{1}$ and $f_{2}$ are generalized bent Boolean functions. Let us suppose that $f_{1}$ is not bent, then there exists $\mathbf{u} \in \mathbb{Z}_{2}^{r}$ such that $\left|\mathcal{H}_{f_{1}}(\mathbf{u})\right|=\ell \neq 1$. Therefore, from (3), $\left|\mathcal{H}_{f_{2}}(\mathbf{v})\right|=\frac{1}{\ell}$ for every $\mathbf{v} \in \mathbb{Z}_{2}^{s}$. This implies

$$
\sum_{\mathbf{v} \in \mathbb{Z}_{2}^{s}}\left|\mathcal{H}_{f_{2}}(\mathbf{v})\right|^{2}=\frac{2^{s}}{\ell^{2}} \neq 2^{s} .
$$

Which is a contradiction. This completes the proof.

## 4 Generalized bent functions in $\mathcal{G B}_{n+2}^{q}$ through 4 functions in $\mathcal{G B}_{n}^{q}$

Let $\mathbf{v}=\left(v_{r}, \ldots, v_{1}\right)$. We define

$$
f_{\mathbf{v}}\left(x_{n-r}, \ldots, x_{1}\right)=f\left(x_{n}=v_{r}, \ldots, x_{n-r+1}=v_{1}, x_{n-r}, \ldots, x_{1}\right)
$$

Let $\mathbf{u}=\left(u_{r}, \ldots, u_{1}\right) \in \mathbb{Z}_{2}^{r}$ and $\mathbf{w}=\left(w_{n-r}, \ldots, w_{1}\right) \in \mathbb{Z}_{2}^{n-r}$. We define the vector concatenation by

$$
\mathbf{u w}:=\left(u_{r}, \ldots, u_{1}, w_{n-r}, \ldots, w_{1}\right)
$$

Lemma 1. Let $\mathbf{u} \in \mathbb{Z}_{2}^{r}$, $\mathbf{w} \in \mathbb{Z}_{2}^{n-r}$ and $f$ be an $n$-variable generalized Boolean function. Then

$$
\begin{gathered}
\mathcal{C}_{f}(\mathbf{u w})=\sum_{\mathbf{v} \in \mathbb{Z}_{2}^{r}} \mathcal{C}_{f_{\mathbf{v}}, f_{\mathbf{v} \oplus \mathbf{u}}}(\mathbf{w}) \\
\text { In particular, forr }=1, \mathcal{C}_{f}(0 \mathbf{w})=\mathcal{C}_{f_{0}}(\mathbf{w})+\mathcal{C}_{f_{1}}(\mathbf{w}), \operatorname{and} \mathcal{C}_{f}(1 \mathbf{w})=2 \operatorname{Re}\left[\mathcal{C}_{f_{0}, f_{1}}(\mathbf{w})\right] .
\end{gathered}
$$

Theorem 3. A function $f \in \mathcal{G B}_{n+2}^{q}$ expressed as

$$
f(z, y, \mathbf{x})=f_{0}(\mathbf{x})(1 \oplus z)(1 \oplus y)+f_{1}(\mathbf{x})(1 \oplus z) y+f_{2}(\mathbf{x})(1 \oplus y) z+f_{3}(\mathbf{x}) y z
$$

where $f_{\ell} \in \mathcal{G B}_{n}^{q},(\ell=0,1,2,3)$, is generalized bent if and only if
(a) $\sum_{\ell=0}^{3} \mathcal{C}_{f_{\ell}}(\mathbf{u})=0$, for all $\mathbf{u} \in \mathbb{Z}_{2}^{n} \backslash\{0\}$, and
(b) $\mathcal{C}_{f_{0}, f_{1}}(\mathbf{u})+\mathcal{C}_{f_{2}, f_{3}}(\mathbf{u}), \mathcal{C}_{f_{0}, f_{2}}(\mathbf{u})+\mathcal{C}_{f_{1}, f_{3}}(\mathbf{u}), \mathcal{C}_{f_{0}, f_{3}}(\mathbf{u})+\mathcal{C}_{f_{1}, f_{2}}(\mathbf{u}) \in \mathbb{R} \imath$, for all $\mathbf{u} \in \mathbb{Z}_{2}^{n}$.

Proof. Let $F_{\ell}\left(\ell \in \mathbb{Z}_{2}\right)$ be the restriction of $f$ on the hyperplane $\{\ell\} \times \mathbb{Z}_{2} \times$ $\mathbb{Z}_{2}^{n} \equiv \mathbb{Z}_{2}^{n+1}$. Then $F_{0}(y, \mathbf{x})=f(0, y, \mathbf{x})=f_{0}(\mathbf{x})(1 \oplus y)+f_{1}(\mathbf{x}) y$ and $F_{1}(y, \mathbf{x})=$ $f(1, y, \mathbf{x})=f_{2}(\mathbf{x})(1 \oplus y)+f_{3}(\mathbf{x}) y$. Now,

$$
\begin{align*}
\mathcal{C}_{F_{0}, F_{1}}(0, \mathbf{u}) & =\sum_{(y, \mathbf{x}) \in \mathbb{Z}_{2} \times \mathbb{Z}_{2}^{n}} \zeta^{F_{0}(y, \mathbf{x})-F_{1}((y, \mathbf{x}) \oplus(0, \mathbf{u}))} \\
& =\sum_{(y, \mathbf{x}) \in \mathbb{Z}_{2} \times \mathbb{Z}_{2}^{n}} \zeta^{F_{0}(y, \mathbf{x})-F_{1}((y, \mathbf{x} \oplus \mathbf{u}))} \\
& =\sum_{\mathbf{x} \in \mathbb{Z}_{2}^{n}} \zeta^{F_{0}(0, \mathbf{x})-F_{1}((0, \mathbf{x} \oplus \mathbf{u}))}+\sum_{\mathbf{x} \in \mathbb{Z}_{2}^{n}} \zeta^{F_{0}(1, \mathbf{x})-F_{1}((1, \mathbf{x} \oplus \mathbf{u}))}  \tag{4}\\
& =\sum_{\mathbf{x} \in \mathbb{Z}_{2}^{n}} \zeta^{f_{0}(\mathbf{x})-f_{2}(\mathbf{x} \oplus \mathbf{u})}+\sum_{\mathbf{x} \in \mathbb{Z}_{2}^{n}} \zeta^{f_{1}(\mathbf{x})-f_{3}(\mathbf{x} \oplus \mathbf{u})} \\
& =\mathcal{C}_{f_{0}, f_{2}}(\mathbf{u})+\mathcal{C}_{f_{1}, f_{3}}(\mathbf{u})
\end{align*}
$$

Similarly we compute,

$$
\begin{align*}
\mathcal{C}_{F_{0}, F_{1}}(1, \mathbf{u}) & =\sum_{(y, \mathbf{x}) \in \mathbb{Z}_{2} \times \mathbb{Z}_{2}^{n}} \zeta^{F_{0}(y, \mathbf{x})-F_{1}((1 \oplus y, \mathbf{x} \oplus \mathbf{u}))} \\
& =\sum_{\mathbf{x} \in \mathbb{Z}_{2}^{n}} \zeta^{F_{0}(0, \mathbf{x})-F_{1}((1, \mathbf{x} \oplus \mathbf{u}))}+\sum_{\mathbf{x} \in \mathbb{Z}_{2}^{n}} \zeta^{F_{0}(1, \mathbf{x})-F_{1}((0, \mathbf{x} \oplus \mathbf{u}))}  \tag{5}\\
& =\sum_{\mathbf{x} \in \mathbb{Z}_{2}^{n}} \zeta^{f_{0}(\mathbf{x})-f_{3}(\mathbf{x} \oplus \mathbf{u})}+\sum_{\mathbf{x} \in \mathbb{Z}_{2}^{n}} \zeta^{f_{1}(\mathbf{x})-f_{2}(\mathbf{x} \oplus \mathbf{u})} \\
& =\mathcal{C}_{f_{0}, f_{3}}(\mathbf{u})+\mathcal{C}_{f_{1}, f_{2}}(\mathbf{u}) .
\end{align*}
$$

Using Lemma 1 for $r=1$, we have

$$
\begin{align*}
& \mathcal{C}_{f}(0, a, \mathbf{u})=\mathcal{C}_{F_{0}}(a, \mathbf{u})+\mathcal{C}_{F_{1}}(a, \mathbf{u}), a \in \mathbb{Z}_{2}, \mathbf{u} \in \mathbb{Z}_{2}^{n}, \text { and }  \tag{6}\\
& \mathcal{C}_{f}(1, a, \mathbf{u})=\mathcal{C}_{F_{0}, F_{1}}(a, \mathbf{u})+\overline{\mathcal{C}_{F_{0}, F_{1}}(a, \mathbf{u})}, a \in \mathbb{Z}_{2}, \mathbf{u} \in \mathbb{Z}_{2}^{n} . \tag{7}
\end{align*}
$$

Further, using Lemma 1 in (6), we have

$$
\begin{equation*}
\mathcal{C}_{f}(0,0, \mathbf{u})=\mathcal{C}_{f_{0}}(\mathbf{u})+\mathcal{C}_{f_{1}}(\mathbf{u})+\mathcal{C}_{f_{2}}(\mathbf{u})+\mathcal{C}_{f_{3}}(\mathbf{u}), \text { and } \tag{8}
\end{equation*}
$$

$$
\begin{align*}
\mathcal{C}_{f}(0,1, \mathbf{u}) & =\mathcal{C}_{F_{0}}(1, \mathbf{u})+\mathcal{C}_{F_{1}}(1, \mathbf{u}) \\
& =\mathcal{C}_{f_{0}, f_{1}}(\mathbf{u})+\overline{\mathcal{C}_{f_{0}, f_{1}}(\mathbf{u})}+\mathcal{C}_{f_{2}, f_{3}}(\mathbf{u})+\overline{\mathcal{C}_{f_{2}, f_{3}}(\mathbf{u})},  \tag{9}\\
& =2 \operatorname{Re}\left[\mathcal{C}_{f_{0}, f_{1}}(\mathbf{u})+\mathcal{C}_{f_{2}, f_{3}}(\mathbf{u})\right] .
\end{align*}
$$

Combining (4) and (7), we have

$$
\begin{align*}
\mathcal{C}_{f}(1,0, \mathbf{u}) & =\mathcal{C}_{F_{0}, F_{1}}(0, \mathbf{u})+\overline{\mathcal{C}_{F_{0}, F_{1}}(0, \mathbf{u})} \\
& =\mathcal{C}_{f_{0}, f_{2}}(\mathbf{u})+\mathcal{C}_{f_{1}, f_{3}}(\mathbf{u})+\overline{\mathcal{C}_{f_{0}, f_{2}}(\mathbf{u})+\mathcal{C}_{f_{1}, f_{3}}(\mathbf{u})}  \tag{10}\\
& =2 \operatorname{Re}\left[\mathcal{C}_{f_{0}, f_{2}}(\mathbf{u})+\mathcal{C}_{f_{1}, f_{3}}(\mathbf{u})\right] .
\end{align*}
$$

Similarly on combining (5) and (7), we have

$$
\begin{equation*}
\mathcal{C}_{f}(1,1, \mathbf{u})=2 \operatorname{Re}\left[\mathcal{C}_{f_{0}, f_{3}}(\mathbf{u})+\mathcal{C}_{f_{1}, f_{2}}(\mathbf{u})\right] . \tag{11}
\end{equation*}
$$

Suppose $f \in \mathcal{G B}_{n+2}^{q}$ such that conditions ( $a$ ) and (b) holds, then from (8), (9), (10) and (11) we have $\mathcal{C}_{f}(b, a, \mathbf{u})=0$, for all $(b, a, \mathbf{u}) \neq(0,0, \mathbf{0})$ and $\mathcal{C}_{f}(0,0, \mathbf{0})=$ $2^{n+2}$. Therefore $f$ is generalized bent.

Conversely, if $f$ is generalized bent, then $\mathcal{C}_{f}(b, a, \mathbf{u})=0$, for all $(b, a, \mathbf{u}) \neq$ $(0,0, \boldsymbol{0})$ and $\mathcal{C}_{f}(0,0, \mathbf{0})=2^{n+2}$. Applying (8), (9), (10) and (11) with the above conditions we have ( $a$ ) and ( $b$ ).

Example 1. Let $g \in \mathcal{G B}_{n}^{q}$ be any generalized bent function. If $f \in \mathcal{G B}_{n+2}^{q}$ expressed as $f=g\|g\| g \|\left(g+\frac{q}{2}\right)$, then $f$ is generalized bent.

Proof. Here $f_{0}=f_{1}=f_{2}=g$ and $f_{3}=g+\frac{q}{2}$. Since $g$ is generalized bent, i.e., $\mathcal{C}_{f_{\ell}}(\mathbf{u})=2^{n} \phi_{\{\mathbf{0}\}}(\mathbf{u})$. Therefore, we have the condition (a) of Theorem 3. The condition (b) of Theorem 3 follows from the fact $\mathcal{C}_{h, h+\frac{q}{2}}(\mathbf{u})+\mathcal{C}_{h}(\mathbf{u})=0$ for all $\mathbf{u} \in \mathbb{Z}_{2}^{n}$.

In the following theorem, we construct a generalized bent function in $\mathcal{G} \mathcal{B}_{n+2}^{q}$ from two generalized bent functions in $\mathcal{G B}_{n}^{q}$.
Theorem 4. Suppose $f \in \mathcal{G B}_{n+2}^{q}$ is expressed as

$$
f(z, y, \mathbf{x})=f_{y \oplus 1}(\mathbf{x})+\left(\frac{q}{2}\right) y z, \text { for all } y, z, \in \mathbb{Z}_{2}, \mathbf{x} \in \mathbb{Z}_{2}^{n}
$$

where $f_{y \oplus 1} \in \mathcal{G B}_{n}^{q}\left(y \in \mathbb{Z}_{2}\right)$. Then $f$ is generalized bent if and only if $f_{0}$ and $f_{1}$ both are generalized bent.

Proof. Let $(b, a, \mathbf{u}) \in \mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2}^{n}$ be arbitrary. We compute,

$$
\begin{aligned}
\mathcal{H}_{f}(b, a, \mathbf{u}) & =2^{-\frac{n+2}{2}} \sum_{(z, y, \mathbf{x}) \in \mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2}^{n}} \zeta^{f(z, y, \mathbf{x})}(-1)^{\mathbf{u} \cdot \mathbf{x}+y a+z b} \\
& =2^{-\frac{n+2}{2}} \sum_{\mathbf{x} \in \mathbb{Z}_{2}^{n}} \sum_{y \in \mathbb{Z}_{2}} \sum_{z \in \mathbb{Z}_{2}} \zeta^{f_{y \oplus 1}(\mathbf{x})+\left(\frac{q}{2}\right) y z}(-1)^{\mathbf{u} \cdot \mathbf{x}+y a+z b} \\
& =2^{-\frac{n+2}{2}} \sum_{\mathbf{x} \in \mathbb{Z}_{2}^{n}} \sum_{y \in \mathbb{Z}_{2}} \zeta^{f_{y \oplus 1}(\mathbf{x})}(-1)^{\mathbf{u} \cdot \mathbf{x} \oplus y a} \sum_{z \in \mathbb{Z}_{2}}(-1)^{(y \oplus b) z} \\
& =2^{-\frac{n+2}{2}} \sum_{\mathbf{x} \in \mathbb{Z}_{2}^{n}} \sum_{y \in \mathbb{Z}_{2}} \zeta^{f_{y \oplus 1}(\mathbf{x})}(-1)^{\mathbf{u} \cdot \mathbf{x} \oplus y a}\left(2 \phi_{\{b\}}(y)\right) \\
& =2^{-\frac{n}{2}} \sum_{\mathbf{x} \in \mathbb{Z}_{2}^{n}} \zeta^{f_{b \oplus 1}(\mathbf{x})}(-1)^{\mathbf{u} \cdot \mathbf{x} \oplus b a} \\
& =(-1)^{a b} \mathcal{H}_{f_{b \oplus 1}}(\mathbf{u}) .
\end{aligned}
$$

Since $f_{0}$ and $f_{1}$ are generalized bent, that is, $\left|\mathcal{H}_{f_{0}}(\mathbf{u})\right|=\left|\mathcal{H}_{f_{1}}(\mathbf{u})\right|=1$, for all $\mathbf{u} \in \mathbb{Z}_{2}^{n}$. This implies that $\left|\mathcal{H}_{f}(b, a, \mathbf{u})\right|=\left|\mathcal{H}_{f_{b \oplus 1}}(\mathbf{u})\right|=1$, for all $\mathbf{u} \in \mathbb{Z}_{2}^{n}$ and $a, b \in \mathbb{Z}_{2}$, and hence, we have the result.

Remark 1. The functions constructed in Theorem 4 are not symmetric with respect to $y$ and $z$ if $f_{0} \neq f_{1}$. This result follows from the fact that $f(1,0, \mathbf{x})=$ $f_{1}(\mathbf{x})$ and $f(0,1, \mathbf{x})=f_{0}(\mathbf{x})$.

## 5 Existence of generalized bent functions in affine set

Suppose $q=2^{h}, h$ is a positive integer. Then an affine function $f_{\lambda} \in \mathcal{G} \mathcal{B}_{n}^{q}$ [11] is expressed as $f_{\lambda}(\mathbf{x})=\lambda_{0}+\sum_{i=1}^{n} \lambda_{i} x_{i}, \lambda_{i} \in \mathbb{Z}_{q}$. Let $\mathcal{A}_{n}^{q}$ denotes the set of all the affine functions on $n$ variables. In the classical notion ( $q=2$ ), all the affine functions are either balanced or constant and therefore, they are not bent. In the generalized set up, we prove that there exists a set of affine functions in which every function is generalized bent. Further, we identify another affine set in which no function is generalized bent.

Theorem 5. Let $q$ be a positive integer such that $q=0 \bmod 4$. Then an affine function $f_{\lambda} \in \mathcal{A}_{n}^{q}$ is generalized bent if and only if

$$
\begin{equation*}
\prod_{i=1}^{n}\left(1+(-1)^{u_{i}} \cos \left(\frac{2 \pi \lambda_{i}}{q}\right)\right)=1, \text { for all } \mathbf{u} \in \mathbb{Z}_{2}^{n} \tag{12}
\end{equation*}
$$

Proof. Let $q$ be a positive integer such that $q=0 \bmod 4$. Suppose $f_{\lambda} \in \mathcal{A}_{n}^{q}$ be arbitrary. Then it is expressed as $f_{\lambda}(\mathbf{x})=\lambda_{0}+\sum_{i=1}^{n} \lambda_{i} x_{i}, \lambda_{i} \in \mathbb{Z}_{q}$. The

Walsh-Hadamard transform of $f_{\lambda}$ at $\mathbf{u} \in \mathbb{Z}_{2}^{n}$ is

$$
\begin{aligned}
\mathcal{H}_{f_{\lambda}}(\mathbf{u}) & =2^{-\frac{n}{2}} \sum_{\mathbf{x} \in \mathbb{Z}_{2}^{n}} \zeta^{f_{\lambda}(\mathbf{x})}(-1)^{\mathbf{u} \cdot \mathbf{x}} \\
& =\zeta^{\lambda_{0}} 2^{-\frac{n}{2}} \sum_{\mathbf{x} \in \mathbb{Z}_{2}^{n}} \zeta^{\sum_{i=1}^{n}\left(\lambda_{i} x_{i}+\left(\frac{q}{2}\right) u_{i} x_{i}\right)} \\
& =\zeta^{\lambda_{0}} 2^{-\frac{n}{2}} \prod_{i=1}^{n} \sum_{x_{i} \in \mathbb{Z}_{2}} \zeta^{\left(\lambda_{i}+\left(\frac{q}{2}\right) u_{i}\right) x_{i}} \\
& =\zeta^{\lambda_{0}} 2^{-\frac{n}{2}} \prod_{i=1}^{n}\left(1+\zeta^{k_{i}}\right), \quad k_{i}=\lambda_{i}+\left(\frac{q}{2}\right) u_{i} \\
& =\zeta^{\lambda_{0}} 2^{-\frac{n}{2}} \prod_{i=1}^{n}\left(1+\cos \left(\frac{2 \pi k_{i}}{q}\right)+\imath \sin \left(\frac{2 \pi k_{i}}{q}\right)\right) \\
& =\zeta^{\lambda_{0}} 2^{-\frac{n}{2}} \prod_{i=1}^{n}\left(1+(-1)^{u_{i}} \cos \left(\frac{2 \pi \lambda_{i}}{q}\right)+\imath(-1)^{u_{i}} \sin \left(\frac{2 \pi \lambda_{i}}{q}\right)\right),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\left|\mathcal{H}_{f_{\lambda}}(\mathbf{u})\right|^{2} & =2^{-n} \prod_{i=1}^{n} 2\left(1+(-1)^{u_{i}} \cos \left(\frac{2 \pi \lambda_{i}}{q}\right)\right) \\
& =\prod_{i=1}^{n}\left(1+(-1)^{u_{i}} \cos \left(\frac{2 \pi \lambda_{i}}{q}\right)\right)
\end{aligned}
$$

Therefore, $f_{\lambda}$ is generalized bent if and only if

$$
\begin{equation*}
\prod_{i=1}^{n}\left(1+(-1)^{u_{i}} \cos \left(\frac{2 \pi \lambda_{i}}{q}\right)\right)=1, \text { for all } \mathbf{u} \in \mathbb{Z}_{2}^{n} \tag{13}
\end{equation*}
$$

It is to be noted that for any $i=1,2, \ldots, n, \lambda_{i}=\frac{q}{4}, \frac{3 q}{4}$ and $\lambda_{0} \in \mathbb{Z}_{q}$ are the solutions of the equation (13). Thus we have the following proposition

Proposition 3. Let $q$ be a positive integer such that $q=0 \bmod 4$. Consider a set $\mathcal{A}^{\prime}$ of affine functions in $\mathcal{A}_{n}^{q}$ defined by

$$
\mathcal{A}^{\prime}=\left\{f_{\lambda} \in \mathcal{A}_{n}^{q}: \lambda_{0} \in \mathbb{Z}_{q}, \lambda_{\ell}=\frac{q}{4} \text { or } \frac{3 q}{4}, \text { for all } \ell \in\{1,2, \ldots, n\}\right\} .
$$

Then every function in $\mathcal{A}^{\prime}$ is generalized bent.
Remark 2. Define a set $\mathcal{S} \subset \mathcal{A}_{n}^{q}$ as follows

$$
\mathcal{S}=\left\{f_{\lambda} \in \mathcal{A}_{n}^{q}: \lambda_{0} \in \mathbb{Z}_{q}, \text { and for some } \ell \in\{1,2, \ldots, n\}, \lambda_{\ell}=0 \text { or } \frac{q}{2}\right\} .
$$

Then no function in $\mathcal{S}$ is generalized bent.

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[^0]:    * Research supported by CSIR, INDIA.

