

A conjecture about Gauss sums and bentness of binomial Boolean functions

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Abstract

In this note, the polar decomposition of binary fields of even extension degree is used to reduce the evaluation of the Walsh transform of binomial Boolean functions to that of Gauss sums. In the case of extensions of degree four times an odd number, an explicit formula involving a Kloosterman sum is conjectured, proved with further restrictions, and supported by extensive experimental data in the general case. In particular, the validity of this formula is shown to be equivalent to a simple and efficient characterization for bentness previously conjectured by Mesnager.

Keywords. Boolean functions, bent functions, Walsh spectrum, exponential sums, Gauss sums, Kloosterman sums.

1 Introduction

Bent functions are Boolean functions defined over an extension of even degree and achieving optimal non-linearity. They are of both combinatorial and cryptographic interest. Unfortunately, characterizing bentness of an arbitrary Boolean function is a difficult problem, and even the less general question of providing simple and efficient criteria within infinite families of functions in a specific polynomial form is still challenging.

For a Boolean function f defined over \mathbb{F}_{2^n} with $n = 2m$ and given in polynomial form, a classical characterization for bentness is that its Walsh transform $\widehat{\chi}_f$ values are only $2^{\pm m}$. Nevertheless, such a characterization is neither concise nor efficient: the best algorithm to compute the full Walsh spectrum has complexity $O(n2^n)$, which is asymptotically optimal. Whence the need to restrict to functions in a given form and to look for more efficient criteria. Unfortunately, only a few infinite families of Boolean functions with a simple and efficient criterion for bentness are known.

The most classical family is due to Dillon [7] and is made of monomial functions:

$$f_a(x) = \text{Tr}_1^n \left(ax^{r(2^m-1)} \right) ,$$

where $n = 2m$, $a \in \mathbb{F}_{2^n}^*$ and r is co-prime with $2^m + 1$. Such functions are bent (and even hyper-bent: for any r coprime with $(2^n - 1)$ the function $f_a(x^r)$ is also bent) if and only

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if the Kloosterman sum $K_m(a)$ associated with a is equal to zero [7, 15, 3]. Not only does such a criterion gives a concise and elegant characterization for bentness, but using the connection between Kloosterman sums and elliptic curves [14, 13] it also allows to check for bentness in polynomial time [16, 1]. Further results on Kloosterman sums involving p -adic arithmetic [12, 11, 20] lead to even faster generation of zeros of Kloosterman sums and so of (hyper-)bent functions.

Mesnager [19, 18] proved a similar criterion for a family Boolean functions in binomial form:

$$f_{a,b}(x) = \text{Tr}_1^n \left(ax^{r(2^m-1)} \right) + \text{Tr}_1^2 \left(bx^{\frac{2^n-1}{3}} \right) ,$$

where $n = 2m$, $a \in \mathbb{F}_{2^n}^*$, $b \in \mathbb{F}_4^*$ and r is co-prime with $2^m + 1$ (but also $r = 3$ which divides $2^m + 1$ [17]). When the extension degree n is twice an odd number, that is when m is odd, $f_{a,b}$ is (hyper-)bent if and only if $K_m(a) = 4$. Moreover, (hyper-)bent functions in this family can be quickly generated as techniques used to generate zeros of Kloosterman sums can be transposed to the value 4 [9].

Unfortunately, the proof of the aforementioned characterization does not extend to the case where m is even. Nevertheless, it is easy to show that $K_m(a) = 4$ is still a necessary condition for $f_{a,b}$ to be bent in this latter case (but note that $f_{a,b}$ can no longer be hyper-bent). Further experimental evidence gathered by Flori, Mesnager and Cohen [9] supported the conjecture that it should also be a sufficient condition: for m up to 16, $f_{a,b}$ is bent if and only if $K_m(a) = 4$.

In this note, the polar decomposition of fields of even extension degree $n = 2^\nu m$ with m odd is used to reduce the evaluation of the Walsh transform of $f_{a,b}$ at $\omega \in \mathbb{F}_{2^n}^*$ to that of a Gauss sum of the form

$$\sum_{u \in U} \psi_n (b \text{Tr}_m^n (\omega u)) \chi \left(\text{Tr}_1^n \left(au^{2^{2^\nu-1}m-1} \right) \right) , \quad (1)$$

where $\mathbb{F}_{2^n}^*$ is decomposed as $\mathbb{F}_{2^n}^* \simeq U \times \mathbb{F}_{2^m}^*$, ψ_n is a cubic multiplicative character and χ a quadratic additive character.

In the case of extensions of degree four times an odd number, that is when n is four times an odd number m , an explicit formula involving the Kloosterman sum $K_{n/2}(a)$ is proved for ω lying in the subfield $\mathbb{F}_{2^{n/2}}$, and conjectured and supported by extensive experimental evidence when $\omega \in \mathbb{F}_{2^n}$. In particular, the validity of this formula would prove the following conjecture for extensions of degree four times an odd number (and give hope to prove the conjecture for n of any 2-adic valuation):

Conjecture 1. *Let $n = 4m$ with m odd, $a \in \mathbb{F}_{2^{n/2}}^*$ and $b \in \mathbb{F}_4^*$. The function $f_{a,b}$ is bent if and only if $K_{n/2}(a) = 4$.*

2 Notation

2.1 Field trace

Definition 2 (Field trace). *For extension degrees m and n such that m divides n , the field trace from \mathbb{F}_{2^n} down to \mathbb{F}_{2^m} is denoted by $\text{Tr}_m^n(x)$.*

2.2 Polar decomposition

Definition 3 (Extension degrees). *Let $n \geq 2$ be an even integer and $\nu \geq 1$ denote its 2-adic valuation. We denote by m_i for $0 \leq i \leq \nu$ the integer $n/2^i$, e.g. $m_0 = n$ and $m_\nu = m$ in the introduction.*

For $0 \leq i < \nu$, the multiplicative group $\mathbb{F}_{2^{m_i}}^*$ can be split using the so-called polar decomposition

$$\mathbb{F}_{2^{m_i}}^* \simeq U_{i+1} \times \mathbb{F}_{2^{m_{i+1}}}^* ,$$

where $U_{i+1} \subset \mathbb{F}_{2^{m_i}}^*$ is the subgroup of $(2^{m_{i+1}} + 1)$ -th roots of unity and $\mathbb{F}_{2^{m_{i+1}}}^*$ the subgroup of $(2^{m_{i+1}} - 1)$ -th roots of unity. Repeating this construction yields the following decomposition.

Lemma 4 (Polar decomposition). *Let $\nu \geq 1$ and denote by U denote the image of $U_1 \times \cdots \times U_\nu$ within $\mathbb{F}_{2^{m_0}}^*$. Then $\mathbb{F}_{2^{m_0}}^*$ decomposes as*

$$\begin{aligned} \mathbb{F}_{2^{m_0}}^* &\simeq U_1 \times \cdots \times U_\nu \times \mathbb{F}_{2^{m_\nu}}^* \\ &\simeq U \times \mathbb{F}_{2^{m_\nu}}^* . \end{aligned}$$

2.3 Hilbert's Theorem 90

Definition 5. *For $1 \leq i \leq \nu$ and $j \in \mathbb{F}_2$, let $\mathcal{T}_{m_i}^j$ be the set*

$$\mathcal{T}_{m_i}^j = \left\{ x \in \mathbb{F}_{2^{m_i}}, \text{Tr}_1^{m_i}(x^{-1}) = j \right\}$$

of elements of $\mathbb{F}_{2^{m_i}}$ whose inverses have trace j (defining 0^{-1} to be 0).

Hilbert's Theorem 90 [8] implies that the function $x \mapsto x + x^{-1}$ is 2-to-1 from $U_i \setminus \{1\}$ to $\mathcal{T}_{m_i}^1$ and from $\mathbb{F}_{2^{m_i}}^* \setminus \{1\}$ to $\mathcal{T}_{m_i}^0 \setminus \{0\}$ (and both 0 and 1 are sent onto 0).

2.4 Dickson polynomials

Definition 6. *We denote by D_3 the third Dickson polynomial of the first kind $D_3(x) = x^3 + x$.*

A notable property of D_3 is that $D_3(x + x^{-1}) = x^3 + x^{-3}$. It implies in particular that D_3 induces a permutation of $\mathcal{T}_{m_1}^0$ when m_1 is odd and of $\mathcal{T}_{m_1}^1$ when m_1 is even [8, Propositions 5, 6 and Theorem 7].

2.5 Characters

Definition 7 (Additive character). *Denote by χ the non-principal quadratic additive character of \mathbb{F}_2 .*

Together with the field trace, χ can be used to construct all quadratic additive characters of $\mathbb{F}_{2^{m_i}}$ for any $0 \leq i \leq \nu$.

Definition 8 (Multiplicative character). *The non-principal cubic multiplicative character ψ_{m_i} of $\mathbb{F}_{2^{m_i}}$ for any $0 \leq i < \nu$ is defined for $x \in \mathbb{F}_{2^{m_i}}$ as*

$$\psi_{m_i}(x) = x^{\frac{2^{m_i}-1}{3}} .$$

Note that if x lies in a subextension, that is $x \in \mathbb{F}_{2^{m_{i+j}}}$ with $0 \leq i+j < \nu$, then

$$\psi_{m_i}(x) = \psi_{m_{i+j}}(x)^{2^j} .$$

Remark that 3 divides $2^{m_\nu} + 1$ and is coprime with $2^{m_\nu} - 1$ and $2^{m_i} + 1$ for $0 \leq i < \nu$. Therefore the function $x \mapsto x^3$ is a permutation of $\mathbb{F}_{2^{m_\nu}}^*$ and U_i for $1 \leq i < \nu$, and 3-to-1 on U_ν . In particular, the multiplicative character ψ_{m_0} is trivial everywhere on $\mathbb{F}_{2^{m_0}}^*$ but on U_ν .

2.6 Walsh transform

Definition 9. *The Walsh transform of a Boolean function f at $\omega \in \mathbb{F}_{2^{m_0}}$ is*

$$\widehat{\chi}_f(\omega) = \sum_{x \in \mathbb{F}_{2^{m_0}}} \chi(f(x) + \text{Tr}_1^{m_0}(\omega x)) .$$

It is well-known that a Boolean function f is bent if and only if its Walsh transform only takes the values $2^{\pm m_1}$.

2.7 Kloosterman sums

Definition 10. *For $a \in \mathbb{F}_{2^{m_1}}$, the Kloosterman sum $K_{m_1}(a)$ is*

$$K_{m_1}(a) = \sum_{x \in \mathbb{F}_{2^{m_1}}} \chi\left(\text{Tr}_1^{m_1}(ax + x^{-1})\right) .$$

The following identities (proved using the map from Section 2.3) are well-known:

$$\begin{aligned} \sum_{u_1 \in U_1} \chi(\text{Tr}_1^{m_0}(au_1)) &= 1 + 2 \sum_{t \in \mathcal{T}_{m_1}^1} \chi(\text{Tr}_1^{m_1}(at)) \\ &= 1 - 2 \sum_{t \in \mathcal{T}_{m_1}^0} \chi(\text{Tr}_1^{m_1}(at)) \\ &= 1 - K_{m_1}(a) . \end{aligned}$$

2.8 Cubic sums

Definition 11. *For $a, b \in \mathbb{F}_{2^{m_1}}$, the cubic sum $C_{m_1}(a, b)$ is*

$$C_{m_1}(a, b) = \sum_{x \in \mathbb{F}_{2^{m_1}}} \chi\left(\text{Tr}_1^{m_1}(ax^3 + bx)\right) .$$

The possible values of $C_{m_1}(a, b)$ were determined by Carlitz [2] together with simple criteria involving a and b .

The most important consequence of Carlitz's results in our context is that $C_{m_1}(a, a) = \sum_{x \in \mathbb{F}_{2^{m_1}}} \chi(\text{Tr}_1^{m_1}(aD_3(x))) = 0$ if and only if

- $\text{Tr}_1^{m_1}(\alpha) = 0$ for $\alpha \in \mathbb{F}_{2^{m_1}}^*$ such that $a = \alpha^3$ when m_1 is odd (in that case a is always a cube),
- and when there exists $\alpha \in \mathbb{F}_{2^{m_1}}^*$ such that $a = \alpha^3$ (that is a is a cube or equivalently $\psi_{m_1}(a) = 1$) and $\text{Tr}_2^{m_1}(\alpha) \neq 0$ (that is the cube root's half-trace is non zero) when m_1 is even.

Charpin et al. later deduced that both in the odd case [4] and in the even case [5, 6] these conditions are equivalent to $K_{m_1}(a) \equiv 1 \pmod{3}$.

For completeness, the other possible values for $C_{m_1}(a, a)$ when m_1 is even follow:

- When a is a cube and $\text{Tr}_2^{m_1}(\alpha) = 0$, then $C_{m_1}(a, a) = 2^{m_2+1} \chi(\text{Tr}_1^{m_1}(u_0^3))$, where u_0 is any solution to $u^4 + u = \alpha^4$, that is $u_0 = \sum_{i=0}^{(m_2-3)/2} \alpha^{4^{2^*i+2}} + \gamma$ for any $\gamma \in \mathbb{F}_4$.
- When a is not a cube, then $C_{m_1}(a, a) = -2^{m_2} \chi(\text{Tr}_1^{m_1}(au_0^3))$, where u_0 is the unique solution to $u^4 + u/a = 1$, that is $u_0 = \psi_{m_1}(a) \sum_{i=0}^{m_2-1} a^{4^i} a^{(4^i-1)/3}$.

Finally, Carlitz also proved the following result on $C_{m_1}(a, 0)$ when $m_1 = 2m_2$ is even:

$$C_{m_1}(a, 0) = \begin{cases} (-1)^{m_2+1} 2^{m_2+1} & \text{if } \psi_{m_1}(a) = 1, \\ (-1)^{m_2} 2^{m_2} & \text{if } \psi_{m_1}(a) \neq 1. \end{cases}$$

2.9 Binomial functions

The binomial Boolean functions $f_{a,b}$ studied in this note are defined over $\mathbb{F}_{2^{m_0}}$.

Definition 12. For $\nu \geq 1$, $a \in \mathbb{F}_{2^{m_0}}^*$ and $b \in \mathbb{F}_4^*$, we denote by $f_{a,b}$ the binomial function

$$f_{a,b}(x) = \text{Tr}_1^{m_0}(ax^{2^{m_1}-1}) + \text{Tr}_1^2(b\psi_{m_0}(x)) . \quad (2)$$

We also define $f_a = f_{a,0}$ (corresponding to Dillon's monomial) and $g_b(x) = \text{Tr}_1^2(b\psi_{m_0}(x))$.

3 Preliminaries

3.1 Field of definition of the coefficients

First notice that it is enough to know how to evaluate the Walsh transform of functions $f_{a,b}$ for $a \in \mathbb{F}_{2^{m_1}}^*$.

Lemma 13. Let $a \in \mathbb{F}_{2^{m_0}}^*$ be written as $a = \alpha\tilde{a}$ with $\alpha \in U_1$ and $\tilde{a} \in \mathbb{F}_{2^{m_1}}^*$ using the polar decomposition of $\mathbb{F}_{2^{m_0}}^*$. Let $\tilde{\alpha} \in U_1$ be a square root of α and $\beta \in \mathbb{F}_4^*$ be $\beta = \psi_{m_0}(\alpha)^{-1}$. Then

$$\widehat{\chi}_{f_{a,b}}(\omega) = \widehat{\chi}_{f_{\tilde{a},\beta b}}(\tilde{\alpha}\omega) .$$

Proof. Indeed, $x \mapsto \tilde{\alpha}x$ induces a permutation of $\mathbb{F}_{2^{m_0}}$, $\tilde{\alpha}^{2^{m_1}-1} = \tilde{\alpha}^{-2} = \alpha^{-1}$, and $\psi_{m_0}(\tilde{\alpha}) = \psi_{m_0}(\alpha)^{-1}$, so that

$$\begin{aligned}\widehat{\chi_{f_{a,b}}}(\omega) &= \sum_{x \in \mathbb{F}_{2^{m_0}}} \chi(f_{a,b}(x) + \text{Tr}_1^{m_0}(\omega x)) \\ &= \sum_{x \in \mathbb{F}_{2^{m_0}}} \chi(f_{a,b}(\tilde{\alpha}x) + \text{Tr}_1^{m_0}(\omega \tilde{\alpha}x)) \\ &= \sum_{x \in \mathbb{F}_{2^{m_0}}} \chi(f_{\tilde{a},\beta b}(x) + \text{Tr}_1^{m_0}(\omega \tilde{\alpha}x)) \\ &= \widehat{\chi_{f_{\tilde{a},\beta b}}}(\tilde{\alpha}\omega) .\end{aligned}\quad \square$$

From now on we can suppose that $a \in \mathbb{F}_{2^{m_1}}^*$ without loss of generality.

3.2 Polar decomposition

The polar decomposition yields the following expression for $f_{a,b}$.

Lemma 14. *For $\nu \geq 1$, $a \in \mathbb{F}_{2^{m_0}}^*$ and $b \in \mathbb{F}_4^*$, and $x \in \mathbb{F}_{2^{m_0}}^*$, $f_{a,b}(x)$ is*

$$f_{a,b}(x) = f_{a,b}(u) = f_a(u_1) + g_b(u_\nu) . \quad (3)$$

Proof. Notice that $f_{a,b}(x) = f_a(x) + g_b(x)$. Moreover f_a is trivial on $\mathbb{F}_{2^{m_1}}^*$ and g_b is trivial everywhere but on U_ν as noted in Section 2.5. \square

We now split the sum expressing the Walsh transform of $f_{a,b}$ at $\omega \in \mathbb{F}_{2^{m_0}}$ using the polar decomposition of $\mathbb{F}_{2^{m_0}}^*$ as $\mathbb{F}_{2^{m_0}}^* \simeq U \times \mathbb{F}_{2^{m_\nu}}^*$. We write $x \in \mathbb{F}_{2^{m_0}}^*$ as $x = uy$ for $u \in U$, and $y \in \mathbb{F}_{2^{m_\nu}}^*$.

Lemma 15. *For $\nu \geq 1$, $a \in \mathbb{F}_{2^{m_1}}^*$ and $b \in \mathbb{F}_4^*$, the Walsh transform of $f_{a,b}$ at $\omega \in \mathbb{F}_{2^{m_0}}$ is, for $\omega = 0$:*

$$\widehat{\chi_{f_{a,b}}}(0) = 1 + (2^{m_\nu} - 1) \sum_{u \in U} \chi(f_{a,b}(u)) , \quad (4)$$

and for $\omega \neq 0$:

$$\widehat{\chi_{f_{a,b}}}(\omega) = 1 - \sum_{u \in U} \chi(f_{a,b}(u)) + 2^{m_\nu} \sum_{u \in U, \text{Tr}_{m_\nu}^{m_0}(\omega u) = 0} \chi(f_{a,b}(u)) . \quad (5)$$

Proof. Using the polar decomposition, the Walsh transform of $f_{a,b}$ at $\omega \in \mathbb{F}_{2^{m_0}}$ can indeed be written

$$\begin{aligned}\widehat{\chi_{f_{a,b}}}(\omega) &= \sum_{x \in \mathbb{F}_{2^{m_0}}} \chi(f_{a,b}(x) + \text{Tr}_1^{m_0}(\omega x)) \\ &= 1 + \sum_{x \in \mathbb{F}_{2^{m_0}}^*} \chi(f_{a,b}(x) + \text{Tr}_1^{m_0}(\omega x))\end{aligned}$$

$$= 1 + \sum_{(u,y) \in U \times \mathbb{F}_{2^{m_\nu}}^*} \chi(f_{a,b}(uy)) \chi(\text{Tr}_1^{m_0}(\omega uy)) .$$

Note that 3 divides $2^{m_\nu} + 1$ so that $\frac{2^{m_0}-1}{3} = (2^{m_\nu} - 1) \frac{2^{m_\nu}+1}{3} \prod_{i=1}^{\nu-1} (2^{m_i} + 1)$ and $f_{a,b}(uy) = f_{a,b}(u)$. Therefore

$$\widehat{\chi f_{a,b}}(\omega) = 1 + \sum_{u \in U} \chi(f_{a,b}(u)) \sum_{y \in \mathbb{F}_{2^{m_\nu}}^*} \chi(\text{Tr}_1^{m_\nu}(\text{Tr}_{m_\nu}^{m_0}(\omega u) y)) .$$

The sum ranging over $\mathbb{F}_{2^{m_\nu}}^*$ is equal to -1 when $\text{Tr}_{m_\nu}^{m_0}(\omega u) \neq 0$ and $2^{m_1} - 1$ when $\text{Tr}_{m_\nu}^{m_0}(\omega u) = 0$. In particular, when $\omega = 0$, the trace is 0 for all $u \in U$. \square

To go further, the cases $\nu = 1$ and $\nu > 1$ have to be dealt with separately.

4 Odd case

In this section, it is supposed that $\nu = 1$, i.e. m_1 is odd and $U = U_1$, which is the case that Mesnager settled [19, 18] with the following proposition. We recall the main ingredients and results of her work as similar ideas will be used for the even case.

Proposition 16 ([19, 18]). *For $\nu = 1$, $a \in \mathbb{F}_{2^{m_1}}^*$ and $b \in \mathbb{F}_4^*$, the Walsh transform of $f_{a,b}$ at $\omega \in \mathbb{F}_{2^{m_0}}$ is, for $\omega = 0$:*

$$\widehat{\chi f_{a,b}}(0) = \begin{cases} 1 + \frac{2^{m_1}-1}{3} (1 - K_{m_1}(a) - 4C_{m_1}(a, a)) & \text{if } b = 1, \\ 1 + \frac{2^{m_1}-1}{3} (1 - K_{m_1}(a) + 2C_{m_1}(a, a)) & \text{if } b \neq 1, \end{cases} \quad (6)$$

and for $\omega \neq 0$:

$$\widehat{\chi f_{a,b}}(\omega) = \begin{cases} 1 + 2^{m_1} \chi(f_{a,b}(w_1^{-1})) + \frac{1}{3} (1 - K_{m_1}(a) - 4C_{m_1}(a, a)) & \text{if } b = 1, \\ 1 + 2^{m_1} \chi(f_{a,b}(w_1^{-1})) + \frac{1}{3} (1 - K_{m_1}(a) + 2C_{m_1}(a, a)) & \text{if } b \neq 1. \end{cases} \quad (7)$$

Proof. For $\omega \neq 0$, $\text{Tr}_{m_1}^{m_0}(\omega u_1) = 0$ if and only if $u_1 = w_1^{-1}$, so that

$$\sum_{u_1 \in U_1, \text{Tr}_{m_1}^{m_0}(\omega u_1) = 0} \chi(f_{a,b}(u_1)) = \chi(f_{a,b}(w_1^{-1})) .$$

The only difficulty lies in the computation of $\sum_{u_1 \in U_1} \chi(f_{a,b}(u_1))$ which can be done by splitting the sum on U_1 according to the value of $\psi_{m_0}(u_1)$:

$$\begin{aligned} \sum_{u_1 \in U_1} \chi(f_{a,b}(u_1)) &= \sum_{u_1 \in U_1} \chi(f_a(u_1)) \chi(g_b(u_1)) \\ &= \sum_{u_1 \in U_1, b\psi_{m_0}(u_1)=1} \chi(f_a(u_1)) - \sum_{u_1 \in U_1, b\psi_{m_0}(u_1) \neq 1} \chi(f_a(u_1)) \\ &= 2 \sum_{u_1 \in U_1, b\psi_{m_0}(u_1)=1} \chi(f_a(u_1)) - \sum_{u_1 \in U_1} \chi(f_a(u_1)) . \end{aligned}$$

As noted in Section 2.7 the second sum is

$$\sum_{u_1 \in U_1} \chi(f_a(u_1)) = 1 - K_{m_1}(a) .$$

As far as the first one is concerned, let us denote it $S_1(a, b, \omega)$. As m_1 is odd, using properties of the Dickson polynomial D_3 given in Section 2.4, one can show that for $b = 1$:

$$S_1(a, b, \omega) = \frac{1}{3} (1 - K_{m_1}(a) + 2C_{m_1}(a, a)) .$$

As $S_1(a, b, \omega)$ takes the same value for both $b \neq 1$, one deduces that for $b \neq 1$:

$$S_1(a, b, \omega) = \frac{1}{3} (1 - K_{m_1}(a) - C_{m_1}(a, a)) . \quad \square$$

Results of Carlitz [2] on $C_{m_1}(a, a)$ when m_1 is odd yield a concise and easy to compute the Walsh transform of $f_{a,b}$ at any $\omega \in \mathbb{F}_{2^{m_0}}$.

Together with Charpin et al. results [5, 6] and the Hasse–Weil bound on $K_{m_1}(a)$, these formulae prove that $f_{a,b}$ is (hyper-)bent if and only if $K_{m_1}(a) = 4$ as was noted by Mesnager [19, 18].

Theorem 17 ([19, 18]). *For $\nu = 1$, $a \in \mathbb{F}_{2^m}^*$ and $b \in \mathbb{F}_4^*$, the function $f_{a,b}$ is bent if and only if $K_{m_1}(a) = 4$.*

5 Even case

5.1 General extension degree

In this section, it is supposed that $\nu > 1$, i.e. both m_0 and m_1 are even. The main difference with the case $\nu = 1$ is that 3 does now divide $2^{m_1} - 1$ (in fact $2^{m_\nu} + 1$) rather than $2^{m_1} + 1$, and $\psi_{m_0}(u)$ does not depend on the value of u_1 (but only on that of u_ν).

In particular, the computation of $\sum_{u \in U} f_{a,b}(u)$ becomes straightforward.

Lemma 18. *For $\nu > 1$, $a \in \mathbb{F}_{2^{m_1}}^*$ and $b \in \mathbb{F}_4^*$,*

$$\sum_{u \in U} \chi(f_{a,b}(u)) = -\frac{2^{2^{\nu-1}m_\nu} - 1}{3(2^{m_\nu} - 1)} (1 - K_{m_1}(a)) . \quad (8)$$

Proof. Splitting U as $U \simeq U_1 \times \cdots \times U_\nu$, the sum can be rewritten:

$$\begin{aligned} \sum_{u \in U} \chi(f_{a,b}(u)) &= \prod_{j=2}^{\nu-1} (2^{m_j} + 1) \sum_{u_1 \in U_1} \chi(f_a(u_1)) \sum_{u_\nu \in U_\nu} \chi(g_b(u_\nu)) \\ &= \prod_{j=2}^{\nu-1} (2^{m_j} + 1) \frac{2^{m_\nu} + 1}{3} (1 - K_{m_1}(a)) \sum_{c \in \mathbb{F}_4^*} \chi(\text{Tr}_1^2(bc)) \end{aligned}$$

$$= - \prod_{j=2}^{\nu-1} (2^{m_j} + 1) \frac{2^{m_\nu} + 1}{3} (1 - K_{m_1}(a)) .$$

Finally, using the identity $(2^{2^j m_\nu} + 1)(2^{2^j m_\nu} - 1) = (2^{2^{j+1} m_\nu} - 1)$, the product of the $(2^{m_j} + 1)$'s is

$$\prod_{j=2}^{\nu} (2^{m_j} + 1) = \prod_{j=2}^{\nu} (2^{2^{\nu-j} m_\nu} + 1) = \frac{2^{2^{\nu-1} m_\nu} - 1}{2^{m_\nu} - 1} . \quad \square$$

The value of the Walsh transform at $\omega = 0$ given by Equation (4) can now be simplified.

Lemma 19. *For $\nu > 1$, $a \in \mathbb{F}_{2^{m_1}}^*$ and $b \in \mathbb{F}_4^*$, the Walsh transform of $f_{a,b}$ at $\omega = 0$ is*

$$\widehat{\chi}_{f_{a,b}}(0) = 1 - \frac{2^{m_1} - 1}{3} (1 - K_{m_1}(a)) . \quad (9)$$

As noted by Mesnager [19, 18], the Hasse–Weil bound on $K_{m_1}(a)$ implies that, if $f_{a,b}$ is bent, then $\widehat{\chi}_{f_{a,b}}(0) = 2^{m_1}$ and $K_{m_1}(a) = 4$.

Proposition 20 ([19, 18]). *For $\nu > 1$, $a \in \mathbb{F}_{2^{m_1}}^*$ and $b \in \mathbb{F}_4^*$, if the function $f_{a,b}$ is bent, then $K_{m_1}(a) = 4$.*

Finally, the value of the Walsh transform at $\omega \neq 0$ given by Equation (5) is simplified as follows.

Lemma 21. *For $\nu > 1$, $a \in \mathbb{F}_{2^{m_1}}^*$ and $b \in \mathbb{F}_4^*$, the Walsh transform of $f_{a,b}$ at $\omega \in \mathbb{F}_{2^{m_0}}^*$ is*

$$\widehat{\chi}_{f_{a,b}}(\omega) = 1 + \frac{2^{2^{\nu-1} m_\nu} - 1}{3(2^{m_\nu} - 1)} (1 - K_{m_1}(a)) + 2^{m_\nu} \sum_{u \in U, \text{Tr}_{m_\nu}^{m_0}(\omega u) = 0} \chi(f_{a,b}(u)) . \quad (10)$$

5.2 Descending to an odd degree extension

To simplify further Equation (10), the sum over $u \in U$ can be split into smaller sums according to the extension $\mathbb{F}_{2^{m_i}}$ (with $1 \leq i \leq \nu$) where $\text{Tr}_{m_i}^{m_0}(u\omega)$ becomes 0, giving the following expression.

Proposition 22. *For $\nu > 1$, $a \in \mathbb{F}_{2^{m_1}}^*$ and $b \in \mathbb{F}_4^*$, and $\omega \in \mathbb{F}_{2^{m_0}}^*$, denote by $S_\nu(a, b, \omega)$ the sum*

$$S_\nu(a, b, \omega) = \sum_{\text{Tr}_{m_{\nu-1}}^{m_0}(u\omega) \neq 0, \text{Tr}_{m_\nu}^{m_0}(u\omega) = 0, b\psi_{m_0}(u_\nu) = 1} \chi(f_a(u_1)) . \quad (11)$$

The Walsh transform of $f_{a,b}$ at $\omega \neq 0$ is

$$\widehat{\chi}_{f_{a,b}}(\omega) = 1 - \frac{2 \cdot 2^{(2^{\nu-1}-1)m_\nu} - 1}{3} (1 - K_{m_1}(a))$$

$$\begin{aligned}
& - \frac{2 \cdot 2^{(2^{\nu-1}-1)m_\nu} (2^{m_\nu-1} - 1)}{3} \chi(f_a(w_1)) \\
& + 2^{m_\nu+1} S_\nu(a, b, \omega) .
\end{aligned} \tag{12}$$

Proof. The sum over U can be divided into subsums σ_i over U_i : $\sum_{u \in U, \text{Tr}_{m_\nu}^{m_0}(\omega u)=0} \chi(f_{a,b}(u)) = \sum_{i=1}^\nu \sigma_i$ with

$$\begin{aligned}
\sigma_i = & \sum_{\substack{\text{Tr}_{m_{i-1}}^{m_0}(u_1 \cdots u_{i-1} w_1 \cdots w_{i-1}) \neq 0, \\ \text{Tr}_{m_i}^{m_0}(u_1 \cdots u_i w_1 \cdots w_i) = 0, \\ u_{i+1} \in U_{i+1}, \dots, u_\nu \in U_\nu}} \chi(f_a(u_1)) \chi(g_b(u_\nu)) .
\end{aligned}$$

The first sum σ_1 can be simplified as Equation (8):

$$\begin{aligned}
\sigma_1 = & \prod_{j=2}^{\nu-1} (2^{m_j} + 1) \chi(f_a(w_1^{-1})) \sum_{u_\nu \in U_\nu} \chi(g_b(u_\nu)) \\
= & - \prod_{j=2}^{\nu-1} (2^{m_j} + 1) \frac{2^{m_\nu} + 1}{3} \chi(f_a(w_1^{-1})) \\
= & - \frac{2^{2^{\nu-1}m_\nu} - 1}{3(2^{m_\nu} - 1)} \chi(f_a(w_1^{-1})) .
\end{aligned} \tag{13}$$

The last sum σ_ν can be split according to the value of $\psi_{m_0}(u_\nu)$ as in Section 4:

$$\begin{aligned}
\sigma_\nu = 2 \sum_{\substack{\text{Tr}_{m_{\nu-1}}^{m_0}(u\omega) \neq 0, \\ \text{Tr}_{m_\nu}^{m_0}(u\omega) = 0, \\ b\psi_{m_0}(u_\nu) = 1}} \chi(f_a(u_1)) - \sum_{\substack{\text{Tr}_{m_{\nu-1}}^{m_0}(u\omega) \neq 0, \\ \text{Tr}_{m_\nu}^{m_0}(u\omega) = 0}} \chi(f_a(u_1)) ,
\end{aligned} \tag{14}$$

where the first term is $2S_\nu(a, b, \omega)$ and the second term is

$$\begin{aligned}
- \sum_{\substack{\text{Tr}_{m_{\nu-1}}^{m_0}(u\omega) \neq 0, \\ \text{Tr}_{m_\nu}^{m_0}(u\omega) = 0}} \chi(f_a(u_1)) & = - \prod_{j=2}^{\nu-1} 2^{m_j} \sum_{u_1 \neq w_1^{-1}} \chi(f_a(u_1)) \\
& = - \prod_{j=2}^{\nu-1} 2^{m_j} \left(1 - \chi(f_a(w_1^{-1})) - K_{m_1}(a) \right) \\
& = -2^{2(2^{\nu-2}-1)m_\nu} \left(1 - \chi(f_a(w_1^{-1})) - K_{m_1}(a) \right) ,
\end{aligned} \tag{15}$$

as the product of the 2^{m_j} 's is

$$\prod_{j=2}^{\nu-1} 2^{m_j} = \prod_{j=2}^{\nu-1} 2^{2^{\nu-j}m_\nu} = 2^{2^{\nu-2} \sum_{j=0}^{\nu-3} 2^{-j}m_\nu} = 2^{2^{\nu-2}(1-2^{-\nu+2})m_\nu} . \tag{16}$$

For $\nu > 2$, the intermediate sums σ_i for $2 < i < \nu$ are:

$$\begin{aligned}\sigma_i &= \prod_{j=2}^{i-1} 2^{m_j} \prod_{j=i+1}^{\nu-1} (2^{m_j} + 1) \sum_{u_1 \neq w_1^{-1}} \chi(f_a(u_1)) \sum_{u_\nu \in U_\nu} \chi(g_b(u_\nu)) \\ &= - \prod_{j=2}^{i-1} 2^{m_j} \prod_{j=i+1}^{\nu-1} (2^{m_j} + 1) \frac{2^{m_\nu} + 1}{3} \left(1 - \chi(f_a(w_1^{-1})) - K_{m_1}(a)\right) .\end{aligned}$$

Fortunately, a simpler expression for the sum of the products of 2^{m_j} 's and $(2^{m_j} + 1)$'s for $1 < j < \nu$ can be devised. Indeed, for $k \geq 3$ and any rational number m , the sum that we denote by $\Sigma(m, k)$ is

$$\Sigma(m, k) = \sum_{i=2}^{k-1} \left(\prod_{j=2}^{i-1} 2^{2^{k-j}m} \prod_{j=i+1}^k (2^{2^{k-j}m} + 1) \right) = \frac{2^{2(2^{k-2}-1)m} - 1}{2^m - 1} . \quad (17)$$

The proof goes by induction on k . For $k = 3$, the identity states $2^m + 1 = \frac{2^{2m}-1}{2^m-1}$. Let us now suppose that Equation (17) is verified up to some $k \geq 3$ for all rational numbers m 's. The sum for $k + 1$ is

$$\Sigma(m, k + 1) = (2^m + 1) \Sigma(2m, k) + (2^m + 1) \prod_{j=2}^{k-1} 2^{2^{k-j}(2m)}$$

By induction and a variation of Equation (16), the identity is proved for $k + 1$:

$$\begin{aligned}\Sigma(m, k + 1) &= (2^m + 1) \frac{2^{2(2^{k-2}-1)(2m)} - 1}{2^{2m} - 1} + (2^m + 1) 2^{4(2^{k-2}-1)m} \\ &= \frac{2^{4(2^{k-2}-1)m} - 1}{2^m - 1} + \frac{(2^{2m} - 1) 2^{4(2^{k-2}-1)m}}{2^m - 1} \\ &= \frac{2^{2(2^{k-1}-1)m} - 1}{2^m - 1} .\end{aligned}$$

Setting $k = \nu$ and $m = m_\nu$ in Equation (17) yields

$$\sum_{i=2}^{\nu-1} \sigma_i = - \frac{2^{2(2^{\nu-2}-1)m_\nu} - 1}{3(2^{m_\nu} - 1)} \left(1 - \chi(f_a(w_1^{-1})) - K_{m_1}(a)\right) . \quad (18)$$

Note that for $\nu = 2$, both sides of the above equality are zero. Therefore, for any $\nu > 1$, Equations (13), (14), (15) and (18), lead to the following expression for the Walsh transform at $\omega \neq 0$:

$$\begin{aligned}\widehat{\chi_{f_{a,b}}}(\omega) &= 1 + \frac{2^{2^{\nu-1}m_\nu} - 1}{3(2^{m_\nu} - 1)} (1 - K_{m_1}(a)) \\ &\quad - 2^{m_\nu} \frac{2^{2^{\nu-1}m_\nu} - 1}{3(2^{m_\nu} - 1)} \chi(f_a(w_1^{-1}))\end{aligned}$$

$$\begin{aligned}
& - 2^{m_\nu} \frac{2^{2(2^{\nu-2}-1)m_\nu} - 1}{3(2^{m_\nu} - 1)} \left(1 - \chi \left(f_a(w_1^{-1}) \right) - K_{m_1}(a) \right) \\
& - 2^{m_\nu} 2^{2(2^{\nu-2}-1)m_\nu} \left(1 - \chi \left(f_a(w_1^{-1}) \right) - K_{m_1}(a) \right) \\
& + 2^{m_\nu+1} S_\nu(a, b, \omega) \ ,
\end{aligned}$$

which gives the announced expression by gathering independently the terms in $\chi \left(f_a(w_1^{-1}) \right)$ and $(1 - K_{m_1}(a))$. \square

Unfortunately, making the remaining sum $S_\nu(a, b, \omega)$ explicit is a hard problem. Doing so is equivalent to evaluating a Gauss sum as in Equation (1): an exponential sum involving a multiplicative character and an additive character. In the next section, we manage to tackle the case $\nu = 2$ when $\omega \in \mathbb{F}_{2^{m_1}}^*$ (that is $w_1 = 1$) and conjecture a partial formula when $\omega \notin \mathbb{F}_{2^{m_1}}^*$.

5.3 Four times an odd number

From now on, it is supposed that $\nu = 2$, i.e. m_0 is four times the odd number m_2 .

For $\text{Tr}_{m_2}^{m_0}(u\omega)$ to be zero with $u_1 \neq w_1^{-1}$, u_2 must be the polar part of $(\omega_2 \text{Tr}_{m_1}^{m_0}(u_1\omega_1))^{-1}$ so that the sum of Equation (11) becomes

$$S_2(a, b, \omega) = \sum_{u_1 \neq w_1^{-1}, \psi_{m_0}(w_2 \text{Tr}_{m_1}^{m_0}(u_1\omega_1))=b} \chi \left(\text{Tr}_1^{m_0} \left(au_1^{-2} \right) \right) \ . \quad (19)$$

5.3.1 The subfield case

We now restrict to the case $w_1 = 1$, that is $\omega \in \mathbb{F}_{2^{m_1}}^*$ rather than $\omega \in \mathbb{F}_{2^{m_0}}^*$.

Lemma 23. *For $a \in \mathbb{F}_{2^{m_1}}^*$ and $b \in \mathbb{F}_4^*$, and $\omega \in \mathbb{F}_{2^{m_1}}^*$, define $\gamma \in \mathbb{F}_4^*$ by $\gamma = b\psi_{m_1}(w_2)$. Then*

$$S_2(a, b, \omega) = 2 \sum_{t \in \mathcal{T}_{m_1}^1, \psi_{m_1}(t)=\gamma} \chi \left(\text{Tr}_1^{m_1} (at) \right) \ . \quad (20)$$

Proof. As $w_1 = 1$, both the multiplicative and additive characters act on the the same inputs so that we can use the function $u_1 \mapsto u_1 + u_1^{-1}$ to transform the sum over U_1 of Equation (19) into a sum over $\mathcal{T}_{m_1}^1$:

$$\begin{aligned}
S_2(a, b, \omega) &= \sum_{u_1 \neq 1, \psi_{m_1}(u_1^2 + u_1^{-2})=b\psi_{m_1}(w_2)} \chi \left(\text{Tr}_1^{m_1} \left(a \left(u_1^{-2} + u_1^2 \right) \right) \right) \\
&= \sum_{u_1 \neq 1, \psi_{m_1}(u_1 + u_1^{-1})=b\psi_{m_1}(w_2)} \chi \left(\text{Tr}_1^{m_1} \left(a \left(u_1 + u_1^{-1} \right) \right) \right) \\
&= 2 \sum_{t \in \mathcal{T}_{m_1}^1, \psi_{m_1}(t)=b\psi_{m_1}(w_2)} \chi \left(\text{Tr}_1^{m_1} (at) \right) \ . \quad \square
\end{aligned}$$

Remark that the sum in Equation (20) can be seen as a first step toward generalizing the sum computed in Section 4 in the odd case: rather than involving u_1 directly, it involves its trace $t = \text{Tr}_{m_1}^{m_0}(u_1)$.

As is customary, the sum over $\mathcal{T}_{m_1}^1$ can be evaluated using sums over all of $\mathbb{F}_{2^{m_1}}$:

$$S_2(a, b, \omega) = \sum_{x \in \mathbb{F}_{2^{m_1}}^*, \psi_{m_1}(x) = \gamma} \chi(\text{Tr}_1^{m_1}(ax)) - \sum_{x \in \mathbb{F}_{2^{m_1}}^*, \psi_{m_1}(x) = \gamma} \chi\left(\text{Tr}_1^{m_1}(ax + x^{-1})\right). \quad (21)$$

The first sum is easily seen to be a cubic sum whereas the computation of the second sum is more involved.

Proposition 24. *For $\nu = 2$, $a \in \mathbb{F}_{2^{m_1}}^*$ and $\gamma \in \mathbb{F}_4^*$. Define $\alpha \in \mathbb{F}_4^*$ by $\alpha = \psi_{m_1}(a)$. The following equality holds:*

$$\sum_{x \in \mathbb{F}_{2^{m_1}}^*, \psi_{m_1}(x) = \gamma} \chi(\text{Tr}_1^{m_1}(ax)) = \begin{cases} \frac{2^{m_2+1}-1}{3} & \text{if } \gamma = \alpha^{-1}, \\ \frac{-2^{m_2}-1}{3} & \text{if } \gamma \neq \alpha^{-1}. \end{cases} \quad (22)$$

Proof. Let $c \in \mathbb{F}_{2^{m_1}}^*$ be such that $\psi_{m_1}(c) = \gamma$. We make the change of variables $x = cx$ to transform the sum into a cubic sum:

$$\begin{aligned} \sum_{x \in \mathbb{F}_{2^{m_1}}^*, \psi_{m_1}(x) = \gamma} \chi(\text{Tr}_1^{m_1}(ax)) &= \sum_{x \in \mathbb{F}_{2^{m_1}}^*, \psi_{m_1}(x) = \psi_{m_1}(c)} \chi(\text{Tr}_1^{m_1}(ax)) \\ &= \sum_{x \in \mathbb{F}_{2^{m_1}}^*, \psi_{m_1}(x) = 1} \chi(\text{Tr}_1^{m_1}(acx)) \\ &= \frac{1}{3} \sum_{x \in \mathbb{F}_{2^{m_1}}^*} \chi\left(\text{Tr}_1^{m_1}(acx^3)\right) \\ &= \frac{1}{3} (C_{m_1}(ac, 0) - 1). \end{aligned}$$

Carlitz's results [2] give explicit values for this cubic sum when m_1 is even and m_2 is odd. \square

Proposition 25. *For $\nu = 2$, $a \in \mathbb{F}_{2^{m_1}}^*$ and $\gamma \in \mathbb{F}_4^*$. Define $\alpha \in \mathbb{F}_4^*$ by $\alpha = \psi_{m_1}(a)$. The following equality holds:*

$$\sum_{x \in \mathbb{F}_{2^{m_1}}^*, \psi_{m_1}(x) = \gamma} \chi\left(\text{Tr}_1^{m_1}(ax + x^{-1})\right) = \begin{cases} \frac{1}{3} (2C_{m_1}(a, a) + K_{m_1}(a) - 1) & \text{if } \gamma = \alpha, \\ \frac{1}{3} (-C_{m_1}(a, a) + K_{m_1}(a) - 1) & \text{if } \gamma \neq \alpha. \end{cases} \quad (23)$$

Proof. First remark that summing over the three possible values of γ yields

$$\sum_{x \in \mathbb{F}_{2^{m_1}}^*} \chi\left(\text{Tr}_1^{m_1}(ax + x^{-1})\right) = K_{m_1}(a) - 1.$$

Moreover, making the change of variable $x = (ax)^{-1}$ shows that the sum takes the same value for γ and $\alpha^{-1}\gamma^{-1}$. In particular, it takes the same value for $\alpha\beta$ and $\alpha\beta^2$, where $\beta \in \mathbb{F}_4^*$ is a primitive third root of unity, that is for the elements of \mathbb{F}_4^* different from α , and this value can be deduced from the value for $\gamma = \alpha$ which we now compute.

Denote by r a square root of a . The change of variable $x = rx$ and properties of the Dickson polynomial D_3 when m_1 is even show that for $\gamma = \alpha = \psi_{m_1}(r^{-1})$:

$$\begin{aligned}
\sum_{x \in \mathbb{F}_{2^{m_1}}^*, \psi_{m_1}(x) = \alpha} \chi\left(\mathrm{Tr}_1^{m_1}(ax + x^{-1})\right) &= \sum_{x \in \mathbb{F}_{2^{m_1}}^*, \psi_{m_1}(x) = 1} \chi\left(\mathrm{Tr}_1^{m_1}\left(r(x + x^{-1})\right)\right) \\
&= \frac{1}{3} \sum_{x \in \mathbb{F}_{2^{m_1}}^*} \chi\left(\mathrm{Tr}_1^{m_1}\left(r(x^3 + x^{-3})\right)\right) \\
&= \frac{1}{3} \sum_{x \in \mathbb{F}_{2^{m_1}}^*} \chi\left(\mathrm{Tr}_1^{m_1}\left(rD_3(x + x^{-1})\right)\right) \\
&= \frac{1}{3} \left(2 \sum_{t \in \mathcal{T}_{m_1}^0} \chi\left(\mathrm{Tr}_1^{m_1}(rD_3(t))\right) - 1 \right) \\
&= \frac{1}{3} \left(2C_{m_1}(r, r) - 2 \sum_{t \in \mathcal{T}_{m_1}^1} \chi\left(\mathrm{Tr}_1^{m_1}(rt)\right) - 1 \right) \\
&= \frac{1}{3} (2C_{m_1}(r, r) + K_{m_1}(r) - 1) \\
&= \frac{1}{3} (2C_{m_1}(a, a) + K_{m_1}(a) - 1) . \quad \square
\end{aligned}$$

Equations (22) and (23) give the following expression for $S_2(a, b, \omega)$.

Theorem 26. *For $\nu = 2$, $a \in \mathbb{F}_{2^{m_1}}^*$ and $b \in \mathbb{F}_4^*$, and $\omega \in \mathbb{F}_{2^{m_1}}^*$, let $\gamma = b\psi_{m_1}(w_2)$. Then the sum $S_2(a, b, \omega)$ is*

$$S_2(a, b, \omega) = \begin{cases} \frac{1}{3} (2^{m_2+1} - 2C_{m_1}(a, a) - K_{m_1}(a)) & \text{if } \gamma = \alpha \text{ and } \gamma = \alpha^{-1}, \\ \frac{1}{3} (-2^{m_2} - 2C_{m_1}(a, a) - K_{m_1}(a)) & \text{if } \gamma = \alpha \text{ and } \gamma \neq \alpha^{-1}, \\ \frac{1}{3} (2^{m_2+1} + C_{m_1}(a, a) - K_{m_1}(a)) & \text{if } \gamma \neq \alpha \text{ and } \gamma = \alpha^{-1}, \\ \frac{1}{3} (-2^{m_2} + C_{m_1}(a, a) - K_{m_1}(a)) & \text{if } \gamma \neq \alpha \text{ and } \gamma \neq \alpha^{-1}. \end{cases} \quad (24)$$

Carlitz's results [2] recalled in Section 2.8 can be used to make the cubic sum $C_{m_1}(a, a)$ explicit. In the particular case where $K_{m_1}(a) \equiv 1 \pmod{3}$, which is equivalent to $C_{m_1}(a, a) = 0$ and implies that a is a cube, the expression for $S_2(a, b, \omega)$ gets very concise, as does Equation (12) for the Walsh transform.

Corollary 27. For $\nu = 2$, $a \in \mathbb{F}_{2^{m_1}}^*$ with $K_{m_1}(a) \equiv 1 \pmod{3}$ and $b \in \mathbb{F}_4^*$, and $\omega \in \mathbb{F}_{2^{m_1}}^*$, let $\gamma = b\psi_{m_1}(w_2)$. Then the sum $S_2(a, b, \omega)$ is

$$S_2(a, b, \omega) = \frac{2^{m_2+1} - K_{m_1}(a)}{3} - \text{Tr}_1^2(\gamma) 2^{m_2} . \quad (25)$$

and the Walsh transform at $\omega \neq 0$ is

$$\widehat{\chi_{f_{a,b}}}(\omega) = \chi\left(\text{Tr}_1^2(\gamma)\right) 2^{m_1} + \frac{4 - K_{m_1}(a)}{3} . \quad (26)$$

Note that Corollary 27 shows that for $\omega \in \mathbb{F}_{2^{m_1}}^*$ the Walsh transform of $f_{a,b}$ at ω is that of a bent function if and only if $K_{m_1}(a) = 4$.

5.4 A conjectural general formula

The techniques used in the previous section do not apply to the general case where $w_1 \neq 1$, i.e. $\omega \in \mathbb{F}_{2^{m_0}}^*$. The main reason being that the multiplicative and additive characters of $\mathbb{F}_{2^{m_1}}$ act on different values, e.g. $r = \text{Tr}_{m_1}^{m_0}(w_1 u_1)$, or $s = \text{Tr}_{m_1}^{m_0}(w_1^{-1} u_1)$, for one of them, and $t = \text{Tr}_{m_1}^{m_0}(u_1)$ for the other one. Considering $v = \text{Tr}_{m_1}^{m_0}(w_1)$, these values are related by $r + s = vt$. Moreover, the sum $S_2(a, b, \omega)$ takes the same value for w_1 and w_1^{-1} , so there is hope to introduce enough symmetry to reduce the case $w_1 \neq 1$ to the case $w_1 = 1$. Unfortunately, we could not devise a way to do so.

Yet, experimental evidence presented in more details in Section 5.5 suggests that the following conjecture, which relates the value of $S_2(a, b, \omega)$ for $w_1 = 1$ and $w_1 \neq 1$, is true.

Conjecture 28. For $\nu = 2$, $a \in \mathbb{F}_{2^{m_1}}^*$ with $K_{m_1}(a) \equiv 1 \pmod{3}$ and $b \in \mathbb{F}_4^*$, and $\omega \in \mathbb{F}_{2^{m_0}}^*$, let $\gamma = b\psi_{m_1}(w_2)$. There exists a Boolean function $h_{a,b}(\omega)$ such that the sum $S_2(a, b, \omega)$ is

$$S_2(a, b, \omega) = \frac{2^{m_2+1} - K_{m_1}(a)}{3} - 2f_a(w_1^{-1}) \frac{2^{m_2+1} - 1}{3} - h_{a,b}(\omega) \chi\left(f_a(w_1^{-1})\right) 2^{m_2} . \quad (27)$$

The Walsh transform at $\omega \neq 0$ is then

$$\widehat{\chi_{f_{a,b}}}(\omega) = \chi\left(h_{a,b}(\omega) f_a(w_1^{-1})\right) 2^{m_1} + \frac{4 - K_{m_1}(a)}{3} . \quad (28)$$

In particular, this conjecture implies Conjecture 1: if $K_{m_1}(a) = 4$, then $f_{a,b}$ is bent. (And Corollary 27 already does so when $\omega \in \mathbb{F}_{2^{m_1}}^*$.)

5.5 Experimental data

The computation of $S_2(a, b, \omega)$ was implemented in C and assembly¹, using AVX extensions for the arithmetic of $\mathbb{F}_{2^{m_0}}^*$, PARI/GP [21] to compute the Kloosterman sums $K_{m_1}(a)$, and Pthreads [10] for parallelization.

The computational cost of verifying Conjecture 28 can be somewhat leveraged using elementary properties of $S_2(a, b, \omega)$:

¹The source code is available at <https://github.com/jpflori/expsums>.

- it only depends on the cyclotomic class of $a \in \mathbb{F}_{2^{m_1}}^*$,
- it is the same for w_1 and w_1^{-1} ,
- the inner value can be computed at the same time for u_1 and u_1^{-1} .

Whatsoever, there are:

- 3 values of $\gamma \in \mathbb{F}_4^*$,
- $\tilde{O}(2^{m_1})$ values of $a \in \mathbb{F}_{2^{m_1}}^*$,
- 2^{m_1-1} values of $w_1 \in U_1 \setminus \{1\}$,
- $\tilde{O}(2^{m_1})$ operations in $\mathbb{F}_{2^{m_0}}$ for each triple (γ, a, w_1) .

Therefore, checking the conjectured formula for $S_2(a, b, \omega)$ over $\mathbb{F}_{2^{m_0}}$ has time complexity $\tilde{O}(2^{3m_1})$ which quickly becomes overcostly (and is comparable to that of computing the Walsh spectrum for every cyclotomic class of $a \in \mathbb{F}_{2^{m_1}}^*$ which has time complexity $\tilde{O}(2^{3m_1})$ as well but space complexity $\tilde{O}(2^{m_1})$).

Still, we checked Conjecture 28

- completely for $m_2 = 3, 5, 7, 9$,
- for i up to 3405 where $a = z^i$ and z is a primitive element of $\mathbb{F}_{2^{m_1}}$ for $m_2 = 11$.

Finally, assuming $K_{m_1}(a) \equiv 1 \pmod{3}$ and Conjecture 28 is correct, Parseval's equality yields the following relation:

$$\begin{aligned} \sum_{x \in \mathbb{F}_{2^{m_0}}^*} \chi \left(h_{a,b}(\omega) f_a(w_1^{-1}) \right) &= \frac{2^{m_1} - 1}{3} (K_{m_1}(a) - 1) \\ &= \widehat{\chi}_{f_{a,b}}(0) - 1 . \end{aligned}$$

This is supported by experimental evidence that there are exactly $2^{m_1-1} + (5/6)(K_{m_1}(a) - 4) + 3$ (respectively $2^{m_1-1} - (1/6)(K_{m_1}(a) - 4)$) values of $w_1 \in U_1$ such that $h_{a,b}(\omega) f_a(w_1^{-1})$ is zero when $\gamma = 1$ (respectively $\gamma \neq 1$).

6 Further research and open problems

Hopefully, Conjecture 28 can be proved using similar techniques as the ones used by Mesnager [19, 18] and in this note. Otherwise, more involved techniques could be tried, e.g. considering a whole family of sums as a whole and their geometric structure. Another possibility would be to directly treat the general Gauss sums of Equations (1) and (11) without focussing on the case $\nu = 2$.

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