Collapsing sponges: Post-quantum security of the sponge construction

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March 27, 2017

Abstract

We investigate the post-quantum security of hash functions based on the sponge construction. A crucial property for hash functions in the post-quantum setting is the collapsing property (a strengthening of collision-resistance). We show that the sponge construction is collapsing (and in consequence quantum collision-resistant) under suitable assumptions about the underlying block function. In particular, if the block function is a random function or a (non-invertible) random permutation, the sponge construction is collapsing.

Contents

1 Introduction

Hash functions are one of the central primitives in cryptography. One of the most important properties of a hash function H is collision-resistance. That is, it is infeasible to find $x \neq x'$ with $H(x) = H(x')$. Intuitively, collision-resistance guarantees some kind of computational injectivity – given $H(x)$, the value x is effectively determined. Of course, information-theoretically, x is not determined, but in many situations, we can treat the preimage x as unique, because we will never see another value with the same hash. For example, collision-resistant hashes can be used to extend the message space of signature schemes (by signing the hash of the message), or to create a commitment schemes (e.g., sending $H(x\|r)$ for random r commits us to x; the sender cannot change his mind about x because he cannot find another preimage).

In the post-quantum setting,[1] however, it was shown by Unruh [Unr16a] that collision-resistance is weaker than expected: For example, the commitment scheme sketched in the previous paragraph is not binding: it is possible for an attacker to send a hash h, then to be

[1] We mean a situation in which the protocols and primitives that are studied are classical, but the attacker can perform quantum computations.

given a value x, and then to send a random value r such that $h = H(x\|r)$, thus opening the commitment to any desired value – even if H is collision-resistant against quantum adversaries.[2] This contradicts the intuitive requirement that $H(x)$ determines x.

Fortunately, Unruh [Unr16a] also presented an strengthened security definition for post-quantum secure hash functions: collapsing hash functions. Roughly speaking, a hash function is collapsing if, given a superposition of values m, measuring $H(m)$ has the same effect as measuring m (at least from the point of view of a computationally limited observer). Collapsing hash functions serve as a drop-in replacement for collision-resistant ones in the post-quantum setting: Unruh showed that several natural classical commitment schemes (namely the scheme sketched above, and the statistically-hiding schemes from [HM96]) become post-quantum secure when using a collapsing hash function instead of a collision-resistant one. (And the collapsing property also directly implies collision-resistance.)

In light of these results, it is desirable to find hash functions that are collapsing. Unruh [Unr16a] showed that the random oracle is collapsing. (That is, a hash function $H(x) := \mathcal{O}(x)$ is collapsing when $\mathcal{O}$ is a random oracle.) However, this has little relevance for real-world hash functions: A practical hash function is typically constructed by iteratively applying some elementary building block (e.g., a "compression function") in order to hash large messages. So even if we are willing to model the elementary building block as a random oracle, the overall hash function construction should arguably not be modeled as a random oracle.[3]

For hash functions based on the Merkle-Damgård (MD) construction (such as SHA2 [Nat15]), Unruh [Unr16b] showed: If the compression function is collapsing, so is the hash function resulting from the MD construction. In particular, if we model the compression function as a random oracle (as is commonly done in the analysis of practical hash functions), we have that hash functions based on the MD construction are collapsing (and thus suitable for use in a post-quantum setting).

However, not all hash functions are constructed using MD. Another popular construction is the sponge construction [Ber+07], underlying for example the current international hash function standard SHA3 [NIS14], but also other hash functions such as Quark [Aum+10], Photon [GPP11], Spongent [Bog+13], and Gluon [Ber+12]. The sponge construction builds a hash function H from a block function[4] $\mathbf{f}$. In the classical setting, we know that the sponge construction is collision-resistant if the block function $\mathbf{f}$ is modeled as a random oracle, or a random permutation, or an invertible random permutation [Ber+08].[5] However, their proof does not carry over to the post-quantum setting: their proof relies on the fact that queries performed by the adversary to the block function are classical (i.e., not in superposition between different values). As first argued in [Bon+11], random oracles and related objects should be modeled as functions that can be queried in superposition of different inputs. Thus, we do not know whether the sponge construction (and thus hash functions like SHA3) is collapsing (or at least collision-resistant).

In the present paper we tackle the question whether the sponge construction is collapsing. We show:

- If the block function $\mathbf{f}$ is collapsing when restricted to the left and right half of its output, respectively, and if it is hard to find a zero-preimage of $\mathbf{f}$ (restricted to the right half of its output), then the sponge construction is collapsing.

[2] More precisely, [Unr16a] shows that relative to certain oracles, a collision-resistant hash function exists that allows such attacks. In particular, this means that there cannot be a relativizing proof that the commitment scheme is binding assuming a collision-resistant hash function.

[3] For example, hash functions using the Merkle-Damgård construction are not well modeled as a random oracle. If we use $MAC(k, m) := H(k\|m)$ as a message authentication code (MAC) with key k, we have that MAC is secure (unforgeable) when H is a random oracle, but easily broken when H is a hash function built using the Merkle-Damgård construction.

[4] It is not called a compression function, since the domain and range of $\mathbf{f}$ are identical.

[5] [Ber+08] shows that the sponge construction is indifferentiable from a random oracle *in the classical setting*. Together with the fact that the random oracle is collision-resistant, collision-resistance of the sponge construction follows.

- If the block function **f** is a random oracle or a random permutation, then the sponge construction is collapsing.

It should be stressed that we *do not* show that the sponge construction is collapsing (or even collision-resistant) if the block function **f** is an *efficiently invertible* random permutation. This means that the present result cannot be directly used to show the security of, say, SHA3, because SHA3 uses an efficiently invertible permutation as block function. Our results apply to hash functions where the block function is not (efficiently) invertible, e.g., Gluon [Ber+12]. But we believe that our results are also a first step towards understanding the sponge construction for invertible block functions, and towards showing the post-quantum security of SHA3.

Open questions / future work. Beyond the results of this paper, there are a few natural avenues for future research:

- *Efficiently invertible permutations.* Many hash functions (e.g., SHA3) use the sponge construction with an invertible permutation. Our results do not apply to those. How can we prove the collapsing property (or at least quantum collision-resistance) in that setting?

- *Other hash functions.* [Unr16b] has covered hash functions based on the Merkle-Damgård construction, we have covered hash functions based on the sponge construction. Which other common constructions of hash functions are collapsing?

- *Tightness of the results.* Are our concrete security bounds tight? If not, can they be improved? In particular, it would be interesting to improve our analysis of the squeezing phase (i.e., the second half of the sponge construction) since our analysis only takes into account the first output block and thus probably looses a lot in terms of concrete security in the case of hash functions with long output.

- *Other properties of the sponge construction.* Classically, we get many properties of the sponge construction for free from the fact that the sponge construction is indifferentiable from a random oracle [Ber+08]. For example, it immediately follows that the sponge construction is a pseudo-random function (PRF). Can we show those properties in the quantum setting, too? E.g., is the sponge construction a PRF secure against quantum adversaries? Is it secure even against quantum adversaries that have superposition access to the PRF (as in [Zha12])?

Organization. In Section 2 ("Preliminaries") we recall some standard notation and definitions. In Section 3 ("Collapsing hash functions"), we recall the definition of collapsing hash functions and some important properties of that definition. In Section 4 ("Sponge construction") we recall the sponge construction. In Section 5 ("Sponges are collapsing"), we present our main result – that the sponge construction is collapsing. In Section 5.1 ("Using random oracles or random permutations") we specialize this result to the case where the block function is a random function or a random permutation. In Section 6 ("Concrete security results"), we state and prove the results from Section 5 with concrete security bounds.

2 Preliminaries

$x \oplus y$ denotes the bitwise XOR of bitstrings x and y. Let $\operatorname{range} f$ denote the range of the function f, and $\operatorname{im} f$ its image (i.e., $\operatorname{im} f$ contains all values that the function f actually attains, while $\operatorname{range} f$ refers to the set into which f maps according to the declaration of f).

By $\{0,1\}^n$ we denote the set of all bitstrings of length n. By $(\{0,1\}^n)^+$ we denote the set of non-empty bitstrings that consists of blocks of length n. (I.e., bitstrings of length kn for some $k \geq 1$.) $|m|$ denotes the length of a bitstring, but in abuse of notation, for elements

$m \in (\{0,1\}^n)^+$, $|m|$ denotes the number of n-bit blocks in m. (But for sets M, $|M|$ is the cardinality of M.)

We call a real-valued function $f \geq 0$ *negligible* iff for any polynomial $p > 0$, we have $f < 1/p$ eventually. We call $f \leq 1$ *overwhelming* if $1 - f$ is negligible.

A quantum state $|\Psi\rangle$ is an element of the complex Hilbert space $\mathbb{C}^M$ for some set M. (E.g., M could be the set of all bitstrings, or of fixed length bit strings, or $\{0,1\}$.) We assume a canonical orthonormal basis of that Hilbert space, called the *computational basis*, its elements are written $|m\rangle$ for $m \in M$. Operations on quantum states are represented by unitary operations U on the quantum states, or by projective measurements. A projective measurement is specified by a family of projectors $\{P_i\}$ that are mutually orthogonal and sum up to 1, one projector P_i for each possibly measurement outcome i. For any quantum state $|\Psi\rangle$, let $\langle\Psi|$ denote the adjoint of $|\Psi\rangle$. In particular, $|\Psi\rangle\langle\Psi|$ denotes the orthogonal projector onto $|\Psi\rangle$. We assume familiarity with these concepts throughout the technical parts of this paper. For an introduction, we refer the reader to a textbook on quantum computation or quantum information such as [NC10].

For algorithms A (classical or quantum), we use the notation $(a, b, c) \leftarrow A(d, e, f)$ to denote that A is executed with inputs d, e, f, and the output triple is assigned to the variables a, b, c. We use capital letters to denote quantum registers, i.e., subsystems of the quantum state of the whole system. E.g., $(y, M') \leftarrow A(x, M)$ means the algorithm A has classical input x, and gets the quantum register M, and it outputs classical output y and a new quantum register M' (the register M is consumed by A). By $x \overset{\$}{\leftarrow} M$ we mean that x is chosen uniformly at random from the set M. If $\mathcal{M}$ denotes a projective measurement, we write $x \leftarrow \mathcal{M}(M)$ to denote that x is assigned the outcome of measuring the register M with $\mathcal{M}$. Note that in this case, the register M is not discarded, instead it contains the post-measurement state. We write $A^{\mathcal{O}}$ when an algorithm A has access to the function $\mathcal{O}$ as an oracle. That is, $A^{\mathcal{O}}$ can evaluate the unitary $U_{\mathcal{O}} : |x, y\rangle \to |x, y \oplus \mathcal{O}(x)\rangle$ in a single step. In results with asymptotic statements (referring, e.g., to quantum-polynomial-time adversaries), we assume a security parameter that is implicitly provided to all adversaries, and that all parameters and functions may implicitly depend on.

A game is a sequence of steps (such as $(y, M') \leftarrow A(x, M)$ or $x \overset{\$}{\leftarrow} M$ or $x \leftarrow \mathcal{M}(M)$). We write $\Pr[C : G]$ to denote the probability that the condition C holds after executing the steps in game G. (E.g., $\Pr[b = 1 : b \leftarrow A()]$ denote the probability that the bit b returned by $A()$ is 1.)

Definition 1 (Zero-preimage-resistance) *We call a function* $f^{\mathcal{O}} : \{0,1\}^d \to \{0,1\}^e$ zero-preimage-resistant *iff for any quantum-polynomial-time adversary* $A^{\mathcal{O}}$*, we have that* $\Pr[f^{\mathcal{O}}(x) = 0^e : x \leftarrow A^{\mathcal{O}}()]$ *is negligible.*

3 Collapsing hash functions

In this section, we recall the notion of collapsing hash functions H from [Unr16a]. We describe both the underlying intuition, as well as the formal definitions.

A *hash function* is a function $H^{\mathcal{O}} : X \to Y$ for some range X and domain Y. (Typically, Y consists of fixed length bitstrings, and X consists of fixed length bitstrings or $\{0,1\}^*$.) H can be depend on an oracle $\mathcal{O}$. (Typically, $\mathcal{O}$ will be a random function, a random function, or simply be missing if we are in the standard model. Unless specified otherwise, we make no assumptions about the distribution of $\mathcal{O}$.)

As mentioned in the introduction, intuitively, we wish that $H(m)$ uniquely identifies m in some sense. In the classical setting, this naturally leads to the requirement that it is hard to find $m \neq m'$ with $H(m) = H(m')$. Then we can treat $H(m)$ as if it had only a single preimage (even though, of course, a compressing H will have many preimages, we just cannot find them). In the quantum setting, there is another interpretation of the requirement that $H(m)$ identifies m. Namely, if we are given a register M that contains a superposition of many values m, then measuring $H(m)$ on that register should – intuitively – fully determine m. That

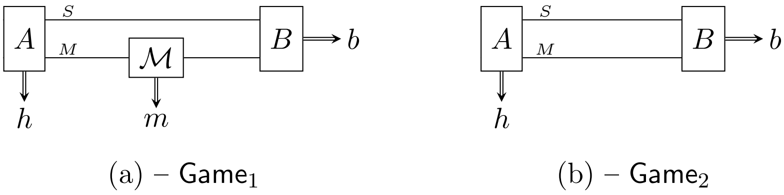


Figure 1: Games from the definition of collapsing hash functions. $\mathcal{M}$ represents a measurement in the computational basis. (A, B) is assumed to satisfy the property that $\mathcal{M}$ always returns m with $H(m) = h$. We require that the probability of $b = 1$ is negligibly close in both games.

is, the effect on the register M should be the same, no matter whether we measure just the hash $H(m)$ or the whole message m. One can see that for any compressing function H, it is impossible that measuring $H(m)$ and m has information-theoretically the same effect on the state.[6] However, what we can hope that for a computationally limited adversary, the two situations are indistinguishable. In other words, we require that no quantum-polynomial-time adversary can distinguish whether we measure $H(m)$ or m. This property is then useful in proofs, because we can replace $H(m)$-measurements by m-measurements and vice versa.

We can slightly simplify this condition if we require that the register M already contains a superposition of values m that all have the same hash $H(m)$. In this case, measuring $H(m)$ has no effect on the state, so we can state the requirement as: If M contains a superposition of messages m with the same $H(m) = h$, then no quantum-polynomial-time adversary can distinguish whether we measure M in the computational basis, or whether we do not measure it at all.

Or slightly more formally: We let the adversary A produce a register M and a hash value h (subject to the promise that measuring M would lead to an m with $H(m) = h$). The adversary additionally keeps an internal state in register S. Then we either measure M in the computational basis (Game_1, depicted in Figure 1 (a)), or we do not perform any such measurement (Game_2, depicted in Figure 1 (b)). Finally, we give registers S (the internal state) and M (the potentially measured message register) to the adversary's second part B. We call H *collapsing* if no quantum-polynomial-time (A, B) can distinguish Game_1 and Game_2.

This is formalized by the following definition:

Definition 2 (Collapsing [Unr16a]) *For algorithms A, B, consider the following games:*

$$\begin{array}{ll} \mathsf{Game}_1: & (S,M,h) \leftarrow A^{\mathcal{O}}(),\ m \leftarrow \mathcal{M}(M),\ b \leftarrow B^{\mathcal{O}}(S,M) \\ \mathsf{Game}_2: & (S,M,h) \leftarrow A^{\mathcal{O}}(),\ \qquad\qquad\qquad b \leftarrow B^{\mathcal{O}}(S,M) \end{array}$$

Here S, M are quantum registers. $\mathcal{M}(M)$ is a measurement of M in the computational basis.

For a set $\mathbf{M}$*, we call an adversary (A, B)* valid on $\mathbf{M}$ for $H^{\mathcal{O}}$ *iff* $\Pr[H^{\mathcal{O}}(m) = h \land m \in \mathbf{M}] = 1$ *when we run $(S, M, h) \leftarrow A^{\mathcal{O}}()$ and measure M in the computational basis as m. If we omit "on* $\mathbf{M}$*", we assume* $\mathbf{M}$ *to be the domain of $H^{\mathcal{O}}$.*

A function H is collapsing *(on* $\mathbf{M}$*) iff for any quantum-polynomial-time adversary (A, B) that is valid for $H^{\mathcal{O}}$ (on* $\mathbf{M}$*), $\big|\Pr[b = 1 : \mathsf{Game}_1] - \Pr[b = 1 : \mathsf{Game}_2]\big|$ is negligible.*

The definition follows [Unr16a], except that we made the oracle $\mathcal{O}$ explicit (which was implicit in [Unr16a]).

[6] E.g., M could contain $\sum_m 2^{-|m|}|m\rangle$. Then measuring $H(m)$ will lead to the state $\sum_{m \text{ s.t. } H(m)=h} \frac{1}{|H^{-1}(\{m\})|}|m\rangle$ which is almost orthogonal for large $|H^{-1}(\{m\})|$ to the state $|m\rangle$ we get when measuring m.

Miscellaneous facts. The following properties of collapsing hash functions will be useful throughout this paper. They are immediate consequences of their concrete-security variants in Section 3.1 below.

Lemma 3 *If $H^{\mathcal{O}}$ is injective, then $H^{\mathcal{O}}$ is collapsing.*

Theorem 4 *If $\mathcal{O} : \{0,1\}^e \to \{0,1\}^d$ is a random function with superlogarithmic d, then $H^{\mathcal{O}} := \mathcal{O}$ is collapsing.*

Lemma 5 *If $G^{\mathcal{O}} \circ H^{\mathcal{O}}$ is collapsing, and $G^{\mathcal{O}}$ is quantum-polynomial-time computable, then $H^{\mathcal{O}}$ is collapsing.*

Lemma 6 *If $G^{\mathcal{O}}$ and $H^{\mathcal{O}}$ are collapsing, and $H^{\mathcal{O}}$ is quantum-polynomial-time computable, then $G^{\mathcal{O}} \circ H^{\mathcal{O}}$ is collapsing.*

3.1 Definitions for concrete security

Definition 2 allows us to state the results of this paper in asymptotic terms (namely, that the sponge construction is collapsing). However, when stating concrete security results, one can achieve tighter results by directly analyzing the security of t parallel evaluations of the hash function (see [Unr16b]). This leads to the following definition:

Definition 7 (Collapsing – concrete security) *For algorithms A, B, and an integer t, consider the following games:*

$$\begin{aligned}
\mathsf{Game}_1 : \quad & (S, M_1, \dots, M_t, h_1, \dots, h_t) \leftarrow A^{\mathcal{O}}(), \\
& m_1 \leftarrow \mathcal{M}(M_1), \ \dots, \ m_t \leftarrow \mathcal{M}(M_t), \\
& b \leftarrow B^{\mathcal{O}}(S, M_1, \dots, M_t) \\
\mathsf{Game}_2 : \quad & (S, M_1, \dots, M_t, h_1, \dots, h_t) \leftarrow A^{\mathcal{O}}(), \\
& b \leftarrow B^{\mathcal{O}}(S, M_1, \dots, M_t)
\end{aligned}$$

Here $S, M_1, \dots, M_t$ are quantum registers. $\mathcal{M}(M_i)$ is a measurement of M_i in the computational basis.

*For a set $\mathbf{M}$, we call an adversary (A, B) t-*valid on $\mathbf{M}$ for $H^{\mathcal{O}}$ *iff* $\Pr[\forall i.\ H^{\mathcal{O}}(m_i) = c_i \wedge m_i \in \mathbf{M}] = 1$ *when we run $(S, M_1, \dots, M_t, h_1, \dots, h_t) \leftarrow A^{\mathcal{O}}()$ and measure all M_i in the computational basis as m_i. If we omit "on $\mathbf{M}$", we assume $\mathbf{M}$ to be the domain of $H^{\mathcal{O}}$.*

We call $\bigl|\Pr[b = 1 : \mathsf{Game}_1] - \Pr[b = 1 : \mathsf{Game}_2]\bigr|$ *the* collapsing-advantage *of (A, B) against H.*

This definition is from [Unr16b], with the only difference that now adversaries and hash functions may depend on an oracle $\mathcal{O}$, instead of depending on a public parameter.

The two definitions of collapsing hash functions are equivalent in the following sense:

Lemma 8 ([Unr16b]) *$H^{\mathcal{O}}$ is collapsing (according to Definition 2) on $\mathbf{M}$ iff for any quantum-polynomial-time $(A^{\mathcal{O}}, B^{\mathcal{O}})$ that is t-valid on $\mathbf{M}$, the collapsing-advantage (according to Definition 7) is negligible.*

Miscellaneous facts. We restate the facts from Section 3, with concrete security bounds. Unless specified otherwise, these facts were proven in [Unr16b] (in a setting without oracle $\mathcal{O}$, but all proofs from [Unr16b] relativize).

All results in this paper hold both if runtime is measured in computation steps, and when time is measured in the number oracle queries.

Lemma 9 *If $H^{\mathcal{O}}$ is injective, and $(A^{\mathcal{O}}, B^{\mathcal{O}})$ is a t-valid adversary with collapsing-advantage ε against $H^{\mathcal{O}}$, then $\varepsilon = 0$.*

Theorem 10 *If $\mathcal{O} : \{0,1\}^e \to \{0,1\}^d$ is a random function, and $(A^{\mathcal{O}}, B^{\mathcal{O}})$ is t-valid for $\mathcal{O}$ and has collapsing-advantage ε, then $\varepsilon \in O(tq^{3/2}2^{-d/2})$.*

This was shown for the case $t = 1$ in [Unr16a]. For general t, this theorem then follows by using the fact that the collapsing property parallel composes (fully analogous to the parallel composition of collapse-binding commitments shown in [Unr16a]).

Lemma 11 *Fix oracle functions $G^{\mathcal{O}}$ and $H^{\mathcal{O}}$. Let (A', B') be a t-valid adversary against some function $H^{\mathcal{O}}$ with runtime τ and collapsing-advantage ε against $H^{\mathcal{O}}$.*

Then there is an adversary (A', B') that is t-valid for $G^{\mathcal{O}} \circ H^{\mathcal{O}}$, has runtime $\tau + t\tau_G$, and collapsing-advantage ε against $G^{\mathcal{O}} \circ H^{\mathcal{O}}$.

Here τ_G is the time required for computing $G^{\mathcal{O}}$.

Proof. Let A' perform the following steps: Run $(S, M_1, \dots, M_t, h_1, \dots, h_t) \leftarrow A^{\mathcal{O}}()$, and return $\big(S, M_1, \dots, M_t, G^{\mathcal{O}}(h_1), \dots, G^{\mathcal{O}}(h_t)\big)$.

Let B' be identical to B.

Since (A, B) is is t-valid for $H^{\mathcal{O}}$, the t-validity of (A', B') for $G^{\mathcal{O}} \circ H^{\mathcal{O}}$ follows directly from the construction of (A', B') and the definition of t-validity.

The runtime of (A', B') is $\tau + t\tau_G$, since there are t additional calls to $G^{\mathcal{O}}$ compared with (A, B).

Since (A', B') differs from (A, B) only in the classical outputs $h_1, \dots, h_t$, and since the games Game_1 and Game_2 from Definition 7 do not use these classical outputs, we have that $\Pr[b = 1 : \mathsf{Game}_1 \text{ using } A, B] = \Pr[b = 1 : \mathsf{Game}_1 \text{ using } A', B']$ and $\Pr[b = 1 : \mathsf{Game}_2 \text{ using } A, B] = \Pr[b = 1 : \mathsf{Game}_2 \text{ using } A', B']$. Thus (A', B') has collapsing-advantage

$$\begin{aligned}&\Big|\Pr[b = 1 : \mathsf{Game}_1 \text{ using } A', B'] - \Pr[b = 1 : \mathsf{Game}_2 \text{ using } A', B']\Big| \\ &\quad = \Big|\Pr[b = 1 : \mathsf{Game}_1 \text{ using } A, B] - \Pr[b = 1 : \mathsf{Game}_2 \text{ using } A, B]\Big| = \varepsilon. \qquad \square\end{aligned}$$

Lemma 12 *Fix oracle functions $G^{\mathcal{O}}$ and $H^{\mathcal{O}}$. If there is a τ-time adversary (A, B), t-valid for $G^{\mathcal{O}} \circ H^{\mathcal{O}}$, with collapsing-advantage ε against $G^{\mathcal{O}} \circ H^{\mathcal{O}}$, then there are:*

- *a $(\tau + O(t\tau_H))$-time adversary (A', B'), t-valid for $G^{\mathcal{O}}$ on $\operatorname{im} H^{\mathcal{O}}$, with some collapsing-advantage ε' against $G^{\mathcal{O}}$,*
- *a $(\tau + O(t\tau_H))$-time adversary (A'', B''), t-valid for $H^{\mathcal{O}}$, with some collapsing-advantage ε'' against $H^{\mathcal{O}}$.*

such that $\varepsilon \le \varepsilon' + \varepsilon''$.

Here τ_H is an upper bound on the time for evaluating $H^{\mathcal{O}}$ (on the messages that A outputs on the registers M_j).

In [Unr16b], this lemma had an additional $O(t\ell_{mid})$ in the runtime of (A'', B'') where ℓ_{mid} denotes the length of the output of $H^{\mathcal{O}}$. Since this is always dominated by $O(t\tau_H)$, we omit this term here.

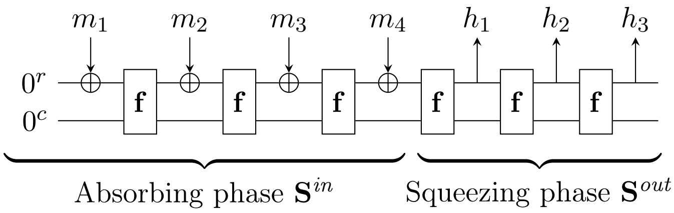

m_1
m_2
m_3
m_4
h_1
h_2
h_3
0^r
0^c
f
f
f
f
f
f
Absorbing phase $\mathbf{S}^{in}$
Squeezing phase $\mathbf{S}^{out}$

Figure 2: The sponge construction $\mathbf{S}$ with a four block input $m_1\|m_2\|m_3\|m4$ and a three block output $h_1\|h_2\|h_3$. The application of the padding function is not depicted (we assume $m_1\|m_2\|m_3\|m_4 = pad(m)$).

4 Sponge construction

In this section, we review the sponge construction introduced by [Ber+07]. The sponge construction has two internal parameters r, c called the *rate* and the *capacity*, respectively. The internal state has $r + c$ bits. We refer to the first part of the state as the left state, and to the second part of the state as the right state. Underlying the sponge construction is a block function $\mathbf{f}$ that inputs and outputs $r + c$ bits. To hash a message m, the message is first padded to a non-zero multiple of the rate r. That is, we use some injective *padding function pad* to get $k \geq 1$ message blocks $m_1\|\dots\|m_k = pad(m)$.[7] Then we XOR m_1 to the left state, apply $\mathbf{f}$ to the (whole) state, XOR m_2 to the left state, apply $\mathbf{f}$ to the state, ..., apply $\mathbf{f}$ to the state, XOR m_k to the left state. The steps performed so far are referred to as the *absorbing phase* (denoted in this paper with $\mathbf{S}^{in}$). Now we start with the *squeezing phase* $\mathbf{S}^{out}$: We apply apply $\mathbf{f}$ to the state, read the left state as h_1, apply $\mathbf{f}$ to the state, read the left state as h_2, ... We continue to do so until $h_1\|h_2\|\dots$ contains $\geq n$ bits (where n is a parameter specifying the desired output length), and return the first n bits of $h_1\|h_2\|\dots$. The whole process described here (padding, absorbing phase, squeezing phase) is the sponge construction, referred to as $\mathbf{S}$ in this paper. The sponge construction is illustrated in Figure 2 for the special case of $k = 4$ and $n = 3r$ (four input blocks and three output blocks). The following definition makes the above explanation precise:

Definition 13 (Sponge construction) *Fix integers $c > 0$ (the* capacity*) and $r > 0$ (the* rate*), and $n > 0$ (the* output length*). Fix $\mathbf{f} : \{0,1\}^{r+c} \to \{0,1\}^{r+c}$ (the* block function*) and $pad : \{0,1\}^* \to (\{0,1\}^r)^+$.*

For $m_1, \dots, m_k \in \{0,1\}^r$, let

$$\mathbf{S}^{in}_{c,r,\mathbf{f}}(m_1\|\dots\|m_k) := \mathbf{f}\big(\mathbf{S}^{in}_{c,r,\mathbf{f}}(m_1\|\dots\|m_{k-1})\big) \oplus (m_k\|0^c)$$
$$\mathbf{S}^{in}_{c,r,\mathbf{f}}(m_1) := m_1\|0^c$$

(We call $\mathbf{S}^{in}$ the absorbing phase.*)*

For $s \in \{0,1\}^{r+c}$, let

$$\mathbf{S}^{out}_{c,r,\mathbf{f},n}(s) = \begin{cases} s'\|\mathbf{S}^{out}_{c,r,\mathbf{f},n-|s'|}(\mathbf{f}(s)) & (n > 0) \\ \textit{empty word} & (n = 0) \end{cases}$$

where s' consists of the first $\min\{n, r\}$ bits of $\mathbf{f}(s)$. (We call $\mathbf{S}^{out}$ the squeezing phase.*)*

Let $\mathbf{S}_{c,r,\mathbf{f},pad,n} := \mathbf{S}^{out}_{c,r,\mathbf{f},n} \circ \mathbf{S}^{in}_{c,r,\mathbf{f}} \circ pad$. We call $\mathbf{S}_{c,r,\mathbf{f},pad,n}$ the sponge construction.

Usually, $c, r, \mathbf{f}, pad, n$ will be clear from the context. Then we omit them and simply write $\mathbf{S}^{in}, \mathbf{S}^{out}, \mathbf{S}$.

Notation: The sponge construction operates on a state of size $r + c$, and we will often need to refer to the two halves of that state separately: For any $s \in \{0,1\}^{r+c}$, let s^{left} denote the first

[7] The original construction requires that the last block of $pad(m)$ is non-zero, this is important for other properties than collision-resistance/collapsing. In this work, we do not put any such requirement on pad. We do, however, assume that pad outputs at least one block.

r bits of s, and let s^{right} refer to the last c bits of s. If f is a function with $r+c$ bit output, then we write f^{left} for the function defined by $f^{\mathsf{left}}(x) := f(x)^{\mathsf{left}}$. And f^{right} analogously.

In [Ber+08], it was shown that the sponge construction is indifferentiable from a random oracle, assuming that $\mathbf{f}$ is a random oracle or an (invertible) random permutation. From this collision-resistance follows. However, the proof from [Ber+08] works only in the classical case. If the adversary has superposition access to $\mathbf{f}$, their proof breaks down because it needs to track on which inputs $\mathbf{f}$ has been queried. As far as we know, no results concerning the post-quantum security of the sponge constructions are known (besides what we show in this paper).

5 Sponges are collapsing

In this section, we show that the sponge constrution is collapsing, under certain assumptions about the block function $\mathbf{f}$. We only state the qualitative results here, more precise statements with concrete security bounds will be given in Section 6.

The results in this section hold for all distributions of the oracle $\mathcal{O}$ (including the case that there is no oracle $\mathcal{O}$). The specific cases of random functions and random permutations are covered in Section 5.1. Since all adversaries $(A, B, A', B', \dots)$ and the block function $\mathbf{f}$ have oracle access to $\mathcal{O}$ throughout the section, we omit the oracle $\mathcal{O}$ from our notation for increased readability (i.e., we write $A, \mathbf{f}$ instead of $A^{\mathcal{O}}, \mathbf{f}^{\mathcal{O}}$). Throughout this section, we assume that $\mathbf{f}^{\mathcal{O}}$ can be computed in quantum-polynomial-time (given oracle access to $\mathcal{O}$).

We will analyze the sponge construction in three parts. First, we analyze the security of the absorbing phase $\mathbf{S}^{in}$, then we analyze the security of the squeezing phase $\mathbf{S}^{out}$, and finally we conclude security of the whole sponge $\mathbf{S}$, consisting of padding, absorbing, and squeezing.

First, we analyze the absorbing phase. For the absorbing phase (without padding or squeezing) to be collapsing, we will need two properties of $\mathbf{f}^{\mathsf{right}}$:

- $\mathbf{f}^{\mathsf{right}}$ is collapsing. This is the main property required from the block function $\mathbf{f}$. If we would restrict $\mathbf{S}^{in}$ to fixed length messages, then we could show the collapsing property of $\mathbf{S}^{in}$ based on that property alone.
- $\mathbf{f}^{\mathsf{right}}$ is zero-preimage-resistant (see Definition 1). To see why we need this property, consider a block function $\mathbf{f}$ where the adversary can find, e.g., $x, y \in \{0,1\}^r$ with $f(x\|0^c) = y\|0^c$. Then we can see that $\mathbf{S}^{in}(x\|y) = 0^{c+r}$, and thus $\mathbf{S}^{in}(x\|y\|z) = z\|0^c = \mathbf{S}^{in}(z)$ for any $z \in \{0,1\}^r$. Thus $\mathbf{S}^{in}$ would not be collision-resistant, and in particular not collapsing.

We state the result formally:

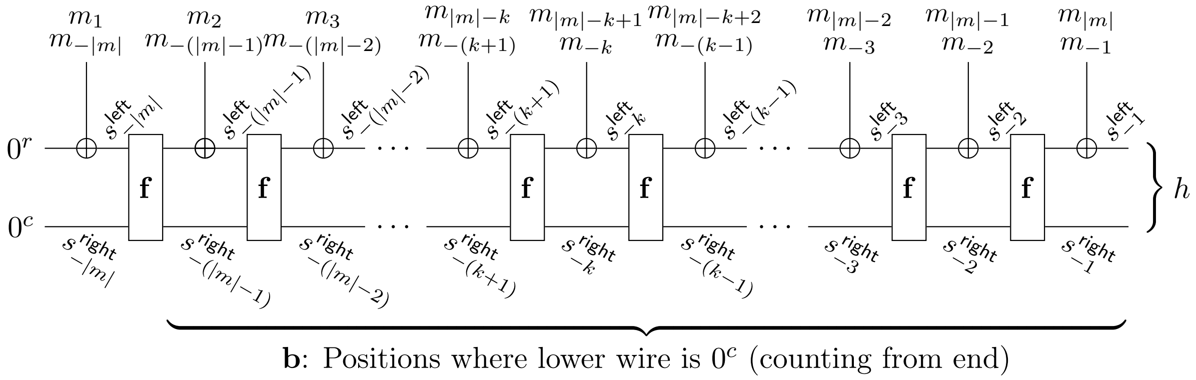
Lemma 14 (Absorbing phase is collapsing) *Assume that $\mathbf{f}^{\mathsf{right}}$ is collapsing, and that $\mathbf{f}^{\mathsf{right}}$ is zero-preimage-resistant. Then $\mathbf{S}^{in}$ is collapsing.*

Note that this lemma does not explicitly state anything about the size of r and c. But of course, $\mathbf{f}^{\mathsf{right}}$ can only be collapsing and zero-preimage-resistant is the capacity c is superlogarithmic.

We only give a detailed proof sketch for Lemma 14. Since Lemma 14 is an immediate corollary of its concrete security variant Lemma 22, we give the full proof directly for Lemma 22 in Section 6 below.

Proof sketch. Consider a quantum-polynomial-time adversary (A, B) where A outputs a hash h, and a superposition of messages m on the register M, and B expects M back and outputs a guess b. We need to show that the two games in Definition 2 (see also Figure 1) are indistinguishable, i.e., the probability of $b = 1$ is approximately the same in both games. Since the domain of $\mathbf{S}^{in}$ is $(\{0,1\}^r)^+$, we can assume that (A, B) is valid on $(\{0,1\}^r)^+$, i.e., M contains a superposition of messages $m \in (\{0,1\}^r)^+$ with $\mathbf{S}^{in}(m) = h$.

To show that the two games are indistinguishable, we start with Game_2 from Definition 2 (Figure 1 (b)) and transform it step by step into Game_1.



m_1
$m_{-|m|}$
m_2
$m_{-(|m|-1)}$
m_3
$m_{-(|m|-2)}$
$m_{|m|-k}$
$m_{-(k+1)}$
$m_{|m|-k+1}$
m_{-k}
$m_{|m|-k+2}$
$m_{-(k-1)}$
$m_{|m|-2}$
m_{-3}
$m_{|m|-1}$
m_{-2}
$m_{|m|}$
m_{-1}
$s^{\mathsf{left}}_{-|m|}$
$s^{\mathsf{left}}_{-(|m|-1)}$
$s^{\mathsf{left}}_{-(|m|-2)}$
$s^{\mathsf{left}}_{-(k+1)}$
s^{left}_{-k}
$s^{\mathsf{left}}_{-(k-1)}$
s^{left}_{-3}
s^{left}_{-2}
s^{left}_{-1}
0^r
0^c
$\mathbf{f}$
h
$s^{\mathsf{right}}_{-|m|}$
$s^{\mathsf{right}}_{-(|m|-1)}$
$s^{\mathsf{right}}_{-(|m|-2)}$
$s^{\mathsf{right}}_{-(k+1)}$
s^{right}_{-k}
$s^{\mathsf{right}}_{-(k-1)}$
s^{right}_{-3}
s^{right}_{-2}
s^{right}_{-1}
$\mathbf{b}$: Positions where lower wire is 0^c (counting from end)

Figure 3: Values occurring in the computation of $h = \mathbf{S}^{in}(m)$, using the notation from the proof sketch of Lemma 14.

Game 1 $(M, h) \leftarrow A()$. $b \leftarrow B(M)$. *(Same as* Game_2 *from Definition 2.)*

Note that we keep the register S (the state of (A, B)) implicit in this proof sketch, to improve readability.

Now, in each successive game, we measure more and more information about the message m contained in M, until in the final game, we measure m completely (like in Game_1 from Definition 2). In order to refer to the different values derived from m that we measure, we will need to use a lot of notation to refer to various intermediate values occurring in the computation of $\mathbf{S}^{in}(m)$. To make it easier to follow the proof, all relevant notation has been depicted in Figure 3, which shows an evaluation of $\mathbf{S}^{in}(m)$.

Since (A, B) is valid, we know that $\mathbf{S}^{in}(m) = h$ where h is the classical output of A. That is, $h = \mathbf{S}^{in}(m)$ has already been measured.[8] In the computation of $\mathbf{S}^{in}(m)$, let s_{-2} refer to the state that goes into the last application of $\mathbf{f}$ (see Figure 3), and let m_{-1} refer to the last block of the input message m (i.e., $m_{-1} = m_{|m|}$). Then $h = \mathbf{f}(s_{-2}) \oplus (m_{-1} \| 0^c)$ by definition of $\mathbf{S}^{in}$, and thus $h^{\mathsf{right}} = \mathbf{f}^{\mathsf{right}}(s_{-2})$. Now since by assumption, $\mathbf{f}^{\mathsf{right}}$ is collapsing, we can reason: Since $\mathbf{f}^{\mathsf{right}}$ is collapsing, and we have measured the output h^{right} of $\mathbf{f}^{\mathsf{right}}(s_{-2})$, it follows that we can additionally measure s_{-2}, and the adversary B will not be able to notice the difference. That is, we get the following game with only negligibly different $\Pr[b = 1]$:

Game 2$_{\text{attempt}}$ $(M, h) \leftarrow A()$. *Measure s_{-2}.* $b \leftarrow B(M)$.

This would indeed work, if we knew that $|m| \geq 2$. However, it could be that $|m| = 1$. In this case, we have $s_{-2} = \bot$ (i.e., s_{-2} does not occur in the computation). Worse, if M contains a superposition of messages m, some of length $|m| = 1$, others of length $|m| \geq 2$, then measuring s_{-2} will reveal whether $|m| \geq 2$ or $|m| = 1$. We cannot guarantee that this measurement will not change the quantum state of M in a noticeable way. Then $\Pr[b = 1 : \text{Game 1}] \not\approx \Pr[b = 1 : \text{Game 2}_{\text{attempt}}]$. The collapsing property of $\mathbf{f}^{\mathsf{right}}$ does not help here, because to apply that property, we need to know that h^{right} is indeed the output of $\mathbf{f}^{\mathsf{right}}$ (which is not the case when $|m| = 1$).

A similar problem also occurs in later games: Let s_{-k} denote the k-th state from the end, with h being s_{-1}, the input to the last $\mathbf{f}$ being s_{-2}, the input to the previous $\mathbf{f}$ being s_{-3}, etc., see Figure 3. (We count backwards because this will make notation easier, since our games will start measuring states from the end.) When we have measured some right state s^{right}_{-k}, we want

[8] In this proof sketch, when we use the expression "measure a" where a is some expression depending on the message m (e.g., a could be $\mathbf{S}^{in}(m)$), then we mean that we measure the register M, but not with a complete measurement, but with a measurement that gives outcome a (e.g., $\mathbf{S}^{in}(m)$) when M contains $|m\rangle$. Formally, that measurement would consist of the projectors P_i defined by $P_i := \sum_{m \text{ s.t. } a=i} |m\rangle\langle m|$. E.g., if we "measure $\mathbf{S}^{in}(m)$", the projectors are $P_i := \sum_{m \text{ s.t. } \mathbf{S}^{in}(m)=i} |i\rangle\langle i|$.

to argue that we can measure the previous state $s_{-(k+1)}$ because $s_{-k}^{\mathsf{right}} = \mathbf{f}^{\mathsf{right}}(s_{-(k+1)})$. Again, this will not be possible because we do not know whether s_{-k} is not already the first state of the computation of $\mathbf{S}^{in}(m)$. (That is, we do not know whether $|m| = k$.)

To get around this problem, we need a mechanism to decide whether a state s_{-k} is the initial state, i.e., whether s_{-k} is $s_{-|m|}$. How do we do that? By construction of $\mathbf{S}^{in}$, the initial state $s_{-|m|}$ satisfies $s_{-|m|}^{\mathsf{right}} = 0^c$. Thus, we might try to decide whether s_{-k} is the initial state by checking whether $s_{-k}^{\mathsf{right}} = 0^c$. For example, if we want to measure s_{-2}, we do so only when $h^{\mathsf{right}} = s_{-1}^{\mathsf{right}} \neq 0^c$. This approach is basically sound, but what happens when a state in the middle has $s_{-k}^{\mathsf{right}} = 0^c$? We would be mislead, and the proof would break down.

To avoid this problem, we will first measure at which positions this bad case happens. Let $\mathbf{b}$ be the set of all indices $k < |m|$ such that $s_{-k}^{\mathsf{right}} = 0^c$. (That is, the indices of all states in which we observe a 0^c in the right part, but which are not the initial state.) Once we know the set $\mathbf{b}$, then we can decide whether s_{-k} is the initial state or not. Namely, s_{-k} is the initial state (i.e., $k = |m|$) iff $s_{-k}^{\mathsf{right}} = 0^c$ and $k \notin \mathbf{b}$.

So the first step in our sequence of games is to measure the set $\mathbf{b}$:

Game 2 $(M, h) \leftarrow A()$. *Measure* $\mathbf{b}$. $b \leftarrow B(M)$.

We assumed that $\mathbf{f}^{\mathsf{right}}$ is zero-preimage-resistant. This implies that with overwhelming probability, $s_{-k}^{\mathsf{right}} = \mathbf{f}^{\mathsf{right}}(s_{-(k+1)}) \neq 0^c$ for all $k < |m|$. Thus $\mathbf{b} = \varnothing$ with overwhelming probability. Therefore measuring $\mathbf{b}$ has only negligible effect on the quantum state. Thus

$$\Pr[\text{Game } 1] \approx \Pr[\text{Game } 2].$$

(We use the shorthand $\Pr[\text{Game } 1]$ for $\Pr[b = 1 : \text{Game } 1]$. And $\approx$ denotes a negligible difference.)

Now we can proceed with measuring more and more states from the computation of $\mathbf{S}^{in}(m)$. First, we measure s_{-1}:

Game 4_1 $(M, h) \leftarrow A()$. *Measure* $\mathbf{b}$ *and* s_{-1}. $b \leftarrow B(M)$.

(Note: The numbering of games in this proof sketch has gaps so that the game numbers here match the game numbers from the full proof of Lemma 22.) Since $s_{-1} = h$ by definition, and since h is already measured by A, the additional measurement does not change the quantum state, and we have:

$$\Pr[\text{Game } 2] = \Pr[\text{Game } 4_1].$$

Now we add a measurement whether s_{-2} is defined:

Game 5_1 $(M, h) \leftarrow A()$. *Measure* $\mathbf{b}$ *and* s_{-1} *and whether* $s_{-2} = \bot$.[9] $b \leftarrow B(M)$.

Given $\mathbf{b}$ and s_{-1} we can already tell whether $s_{-2} = \bot$. Namely, $s_{-2} = \bot$ iff $s_{-1}^{\mathsf{right}} = 0^c$ and $1 \notin \mathbf{b}$. Thus measuring whether $s_{-2} = \bot$ has no effect on the quantum state, and we get

$$\Pr[\text{Game } 4_1] = \Pr[\text{Game } 5_1].$$

Now, finally, we can do what we already intended to do in Game 2_{attempt}: We measure s_{-2}, and use the collapsing property of $\mathbf{f}^{\mathsf{right}}$ to show that this measurement does not noticeably disturb the quantum state:

Game 4_2 $(M, h) \leftarrow A()$. *Measure* $\mathbf{b}$, *and* s_{-1}, *and whether* $s_{-2} = \bot$, *and* s_{-2}. $b \leftarrow B(M)$.

[9] Measuring "whether $s_{-2} = \bot$" means a measurement on M defined by projectors P and $1 - P$ where $P := \sum_{m \text{ s.t. } s_{-2}=\bot} |m\rangle\langle m|$.

In case that we measured that $s_{-2} = \bot$, measuring s_{-2} in Game 4_2 has no effect on the quantum state (since we know that the outcome will be $\bot$). And in case that we measured that $s_{-2} \neq \bot$, we know that $s_{-1}^{\mathsf{right}} = \mathbf{f}^{\mathsf{right}}(s_{-2})$ (as already discussed above), and thus measuring s_{-2} can be noticed with at most noticeable probability by a quantum-polynomial-time adversary. Thus

$$\Pr[\text{Game } 5_1] \approx \Pr[\text{Game } 4_2].$$

And then we continue by adding a measurement whether $s_{-3} \neq \bot$:

Game 5_2 $(M, h) \leftarrow A()$. *Measure* $\mathbf{b}$*, and* s_{-1}*, and whether* $s_{-2} = \bot$*, and* s_{-2}*, and whether* $s_{-3} = \bot$*.* $b \leftarrow B(M)$*.*

Since $s_{-3} = \bot$ iff $s_{-2} = \bot$ or $s_{-2}^{\mathsf{right}} = 0^c$ and $2 \notin \mathbf{b}$, measuring whether $s_{-3} = \bot$ holds has no effect on the quantum state. Thus we get

$$\Pr[\text{Game } 4_2] = \Pr[\text{Game } 5_2].$$

And then we measure s_{-3}:

Game 4_3 $(M, h) \leftarrow A()$. *Measure* $\mathbf{b}$*, and* s_{-1}*, and whether* $s_{-2} = \bot$*, and* s_{-2}*, and whether* $s_{-3} = \bot$*, and* s_{-3}*.* $b \leftarrow B(M)$*.*

Using that $\mathbf{f}^{\mathsf{right}}$ is collapsing, we get

$$\Pr[\text{Game } 5_2] \approx \Pr[\text{Game } 4_3].$$

We continue in this way, alternatively adding a measurement whether the next state $s_{-k} = \bot$, and then adding a measurement of s_{-k}, each time using the collapsing property of $\mathbf{f}^{\mathsf{right}}$. After ℓ such steps, where ℓ is a polynomial upper bound on the length of m, we get the following game:

Game 4_ℓ $(M, h) \leftarrow A()$. *Measure* $\mathbf{b}$*, measure whether* $s_{-1}, \dots, s_{-\ell} = \bot$*, measure* $s_{-1}, \dots, s_{-\ell}$*.* $b \leftarrow B(M)$*.*

Since in each of the steps, we accrue only a negligible distinguishing probability between consecutive games, we get:

$$\Pr[\text{Game } 4_1] \approx \Pr[\text{Game } 4_\ell].$$

(Formally, this argument is not correct, because the sum of polynomially many possibly different negligible functions is not necessarily negligible. However, this is easily fixed by bounding all differences $\Pr[\text{Game } 5_k] - \Pr[\text{Game } 4_{k+1}]$ simultaneously using a reduction that randomly chooses k (a standard technique). Details are given in the full proof of Lemma 22.)

In Game 4_ℓ, we measure $s_{-1}, \dots, s_{-\ell}$. From these, we can compute $|m|$ (since $s_{-k} = \bot$ for $k > |m|$). Furthermore, each message block m_{-k} (the k-th message block from the end) can be computed as follows: We have $s_{-k} = \mathbf{f}(s_{-(k+1)}) \oplus (m_{-k} \| 0^c)$, and thus $m_{-k} = s_{-k}^{\mathsf{left}} \oplus \mathbf{f}(s_{-(k+1)})^{\mathsf{left}}$. Except for $m_{-|m|}$, which can be computed as $m_{-|m|} = s_{-|m|}^{\mathsf{left}}$. (Cf. Figure 3.) Finally, we can compute m as $m = m_{-|m|} \| \dots \| m_{-1}$.

Since we can compute m from the measurements performed in Game 4_ℓ, it follows that those measurements are equivalent (in their effect on the quantum state) to a measurement of m. Thus

$$\Pr[\text{Game } 4_\ell] = \Pr[\text{Game } 6]$$

for the following final game:

Game 6 $(M, h) \leftarrow A()$. *Measure* m*.* $b \leftarrow B(M)$*.*

Altogether, we have shown

$$\Pr[\text{Game 1}] \approx \Pr[\text{Game 6}].$$

And Game 1 and Game 6 are identical to the games Game_2 and Game_1 from Definition 2, respectively. Since (A, B) was an arbitrary quantum-polynomial-time adversary that is valid for $\mathbf{S}^{in}$, it follows by Definition 2 that $\mathbf{S}^{in}$ is collapsing. □

Next, we show that the squeezing phase is collapsing. This one is a relatively trivial consequence of the fact that $\mathbf{f}^{\mathsf{left}}$ is collapsing.

Lemma 15 (Squeezing phase is collapsing) *Assume that $n \geq r$ (i.e., the requested output length of $\mathbf{S}^{out}$ is greater-equal to the rate). Assume that $\mathbf{f}^{\mathsf{left}}$ is collapsing. Then $\mathbf{S}^{out}$ is collapsing.*

A concrete security variant of this lemma is given in Lemma 23.

Proof. Let $G(x)$ return the first r bits of x. If $n \geq r$, then $G(\mathbf{S}^{out}(s)) = \mathbf{f}^{\mathsf{left}}(s)$. Thus the lemma follows directly from Lemma 5. □

And finally we get that the sponge construction as a whole is collapsing. This is a simple corollary from the fact that both the absorbing and the squeezing phase are collapsing.

Theorem 16 (Sponge construction is collapsing) *Assume that $n \geq r$ (i.e., the requested output length of $\mathbf{S}$ is greater-equal to the rate). Assume that $\mathbf{f}^{\mathsf{left}}$ and $\mathbf{f}^{\mathsf{right}}$ are collapsing, and that $\mathbf{f}^{\mathsf{right}}$ is zero-preimage-resistant. Assume that pad is injective. Then $\mathbf{S}$ is collapsing.*

A concrete security variant of this theorem is given in Theorem 24.

Proof. By Lemma 14, $\mathbf{S}^{in}$ is collapsing, and by Lemma 15, $\mathbf{S}^{out}$ is collapsing. Then by Lemma 6, $\mathbf{S}^{out} \circ \mathbf{S}^{in}$ is collapsing. Since *pad* is injective, by Lemma 3, *pad* is collapsing. Thus by Lemma 6, $\mathbf{S} = (\mathbf{S}^{out} \circ \mathbf{S}^{in}) \circ pad$ is collapsing. □

5.1 Using random oracles or random permutations

In the preceding section, we have reduced the security of the sponge construction $\mathbf{S}$ to certain properties of the block function $\mathbf{f}$. In many cases, however, a hash function is designed by constructing a block function heuristically, and we cannot say what specific security properties the block function has. Instead, we model the block function as, e.g., a random function or a random permutation. The present section specifies the security of the sponge construction under those circumstances. (Concrete security statements are deferred to Section 6.1.)

We start by deriving the required properties of a random block function:

Lemma 17 *If $\mathcal{O} : \{0,1\}^{r+c} \to \{0,1\}^{r+c}$ is a random function, and $\mathbf{f}^{\mathcal{O}}(x) := \mathcal{O}(x)$, and r, c are superlogarithmic, then $\mathbf{f}^{\mathsf{left}}$ and $\mathbf{f}^{\mathsf{right}}$ are collapsing.*

This is an immediate corollary of its concrete security variant Lemma 25, for which we give a full proof in Section 6.1.

Proof sketch. By Theorem 10, a random function (with superlogarithmic output length) is collapsing. Since $\mathbf{f}^{\mathsf{left}} = \mathcal{O}^{\mathsf{left}}$ has superlogarithmic output length r, and $\mathbf{f}^{\mathsf{left}} = \mathcal{O}^{\mathsf{left}}$ is a uniformly random function if $\mathcal{O}$ is, it follows that $\mathbf{f}^{\mathsf{left}}$ is collapsing. (A slight technicality is that the adversary does not only get access to $\mathbf{f}^{\mathsf{left}} = \mathcal{O}^{\mathsf{left}}$, but additionally to $\mathcal{O}^{\mathsf{right}}$, but it is easy to see that this does not help him since $\mathcal{O}^{\mathsf{right}}$ is distributed independently from $\mathcal{O}^{\mathsf{left}}$.)

Analogously we get that $\mathbf{f}^{\mathsf{right}}$ is collapsing. □

Lemma 18 *If $\mathcal{O} : \{0,1\}^{r+c} \to \{0,1\}^{r+c}$ is a random function, and $\mathbf{f}^{\mathcal{O}}(x) := \mathcal{O}(x)$, and c is superlogarithmic, then $\mathbf{f}^{\mathsf{right}}$ is zero-preimage-resistant.*

This is an immediate corollary of its concrete security variant Lemma 18, for which we give a full proof in Section 6.1. It is a simple consequence of the hardness of quantum searching in unstructured data [Boy+98].

With these two lemmas, security of the sponge using a random block function follows:

Theorem 19 *If $\mathcal{O} : \{0,1\}^{r+c} \to \{0,1\}^{r+c}$ is a random function, and $\mathbf{f}^{\mathcal{O}}(x) := \mathcal{O}(x)$, and r, c are superlogarithmic, then $\mathbf{S}$ is collapsing.*

A concrete security variant (with security bounds in terms of the number of oracle-queries) is given in Theorem 27.

Proof. Immediate from Theorem 16, using Lemma 17 and Lemma 18 to show that its preconditions are satisfied. □

And since random functions and random permutations are known to be indistinguishable, we readily derive the security of the sponge construction also for block functions that are random permutations.

Theorem 20 *If $\mathcal{O} : \{0,1\}^{r+c} \to \{0,1\}^{r+c}$ is a random permutation, and $\mathbf{f}^{\mathcal{O}}(x) := \mathcal{O}(x)$, and r, c are superlogarithmic, then $\mathbf{S}$ is collapsing.*

A concrete security variant (with security bounds in terms of the number of oracle-queries) is given in Theorem 28.

Proof. Zhandry [Zha15] shows that no adversary making a polynomial number of queries can distinguish a random permutation from a random function with more than negligible probability (assuming that the output length is superlogarithmic). Thus the advantage of the adversary attacking the collapsing property of $\mathbf{S}$ when $\mathcal{O}$ is a random permutation can only be negligibly higher than the advantage of the same adversary attacking the collapsing property of $\mathbf{S}$ when $\mathcal{O}$ is a random function. The latter advantage is negligible by Theorem 19, thus the former advantage is negligible, too. Hence $\mathbf{S}$ is collapsing when $\mathcal{O}$ is a random permutation. □

On invertible permutations. Theorem 28 tells us that the sponge construction is collapsing if the block function $\mathbf{f}$ is a random permutation. There is a caveat, though: If $\mathbf{f}$ is a permutation that we can efficiently invert, the theorem does not tell us anything. More specifically, if $\mathbf{f}$ is a random permutation, and the adversary has access to $\mathcal{O} = (\mathbf{f}, \mathbf{f}^{-1})$,[10] then Theorem 28 does not apply.

In fact, if $\mathbf{f}$ is efficiently invertible, we cannot apply our main result Theorem 16 at all: In that case $\mathbf{f}^{\mathsf{right}}$ is not zero-preimage-resistant (to find a zero-preimage, we simply invoke $\mathbf{f}^{-1}(y\|0^c)$ for an arbitrary $y \in \{0,1\}^r$).

And $\mathbf{f}^{\mathsf{right}}$ is also not collapsing: We get a collision of $\mathbf{f}^{\mathsf{right}}$ by computing $x := \mathbf{f}^{-1}(y\|z)$ and $x' := \mathbf{f}^{-1}(y'\|z)$ for arbitrary y, y', z with $y \neq y'$. So $\mathbf{f}^{\mathsf{right}}$ is not collision-resistant. And since collapsing functions are collision-resistant [Unr16a], $\mathbf{f}^{\mathsf{right}}$ is not collapsing.

Similarly, $\mathbf{f}^{\mathsf{left}}$ is not collapsing, either.

So, none of the preconditions of Theorem 16 are satisfied, and we cannot derive that the sponge construction is collapsing (for efficiently invertible $\mathbf{f}$).

Does that mean that the sponge construction is not collapsing in that setting? No. At least it is not obvious how one would use $\mathbf{f}^{-1}$ to break the collapsing property. For example, if we try to find a two-block collision $(m_1\|m_2, m_1'\|m_2')$ for $\mathbf{S}$, then we need that $\mathbf{f}(m_1\|0^c) \oplus m_2\|0^c = \mathbf{f}(m_1'\|0^c) \oplus m_2'\|0^c$. How can we solve this equation by using $\mathbf{f}^{-1}$? Unclear. For other kinds of collisions that we could think of, we fail similarly.

In fact, we conjecture that the sponge construction is still collapsing in this setting:

[10] Formally, we mean by $\mathcal{O} = (\mathbf{f}, \mathbf{f}^{-1})$: $\mathcal{O}(0, \cdot)$ is a random permutation, $\mathcal{O}(1, \cdot)$ is its inverse, and $\mathbf{f}(x) := \mathcal{O}(0, x)$.

Conjecture 21 *If $\mathcal{O} : \{0,1\} \times \{0,1\}^{r+c} \to \{0,1\}^{r+c}$ is an invertible random permutation (see footnote 10), and $\mathbf{f}^{\mathcal{O}}(x) := \mathcal{O}(0,x)$, and r, c are superlogarithmic, then* $\mathbf{S}$ *is collapsing.*

Showing this conjecture is quite important because it will give evidence for the post-quantum security of SHA3. (Only in the idealized invertible random permutation model. But then, classically, we also have evidence for the collision-resistance of the sponge construction only in that idealized model [Ber+08].)

6 Concrete security results

In this section, we prove the concrete security variants of the results from Section 5. The proofs follow the lines sketched in Section 5, except that we need to be more explicit about the adversaries constructed during the reductions.

The reader interested only in the qualitative fact that the sponge construction is secure, but not in the precise bounds, can safely skip this section.

As in Section 5, we omit $\mathcal{O}$ from $A^{\mathcal{O}}, \mathbf{f}^{\mathcal{O}}$, etc.

Lemma 22 *Let (A, B) be a τ-time adversary, t-valid against* $\mathbf{S}$ *on $(\{0,1\}^r)^+$, with collapsing-advantage ε. Assume that A outputs messages of length at most ℓr bits.*

Then there are:

- *a $(\tau + t\ell\tau_{\mathbf{f}} + O(t\ell c))$-time adversary A' that finds a zero-preimage[11] of $\mathbf{f}^{\mathsf{right}}$ with probability ε, and*
- *a $\big(\tau + O(\ell\tau_{\mathbf{f}})\big)$-time adversary (A'', B''), t-valid against $\mathbf{f}^{\mathsf{right}}$, with collapsing-advantage ε''*

such that $\varepsilon \leq \sqrt{\varepsilon'} + (\ell - 1)\varepsilon''$.

Here $\tau_{\mathbf{f}}$ is the time required a single classical invocation of $\mathbf{f}$*. If time is measured in $\mathcal{O}$-queries (instead of computation steps), the term $O(t\ell c)$ in the runtime of A' can be omitted.*

Proof. We prove this lemma using a sequence of games. The first game is identical to Game_2 from Definition 7.

Game 1 $(S, M_1, \dots, M_t, h_1, \dots, h_t) \leftarrow A()$. $b \leftarrow B(S, M_1, \dots, M_t)$.

To state the second game, we need to introduce some notation. (We recommend the reader to compare with Figure 3. The notation there is slightly different, e.g., we write there s_{-1} and here more precisely $s_{-1}(m)$, but the figure can be helpful to get an overview which parts of the message the different values defined below refer to.)

For some $m \in (\{0,1\}^r)^+$, we denote by $|m|$ the number of r-bit blocks in m. (That is, if m has d bits, then $|m| = d/r$.) By m_i we denote the i-th block of m. We denote by m_{-i} the i-th block of m from the end, i.e., $m_{-i} = m_{|m|-i+1}$ and $m = m_{-|m|} \| \dots \| m_{-1}$.

For $i = 1, \dots, |m|$, let $s_{-i}(m) := \mathbf{S}^{in}(m_1, \dots, m_{|m|-i+1})$. That is, $s_{-i}(m)$ is the i-th state during the evaluation of $\mathbf{S}^{in}(m)$, counted from the end.

In particular, $s_{-1}(m) = \mathbf{S}^{in}(m)$, and $s_{-|m|}(m) = m_1 \| 0^c$. (Since we consider only $|m| \in (\{0,1\}^r)^+$, the border case $|m| = 0$ does not occur.) We define $s_{-i}(m) := \bot$ for $i > |m|$.

Let $\mathbf{b}(m) := \{i : s_{-i}(m)^{\mathsf{right}} = 0^c \wedge i < |m|\}$. Note that we always have $s_{-|m|}^{\mathsf{right}} = 0^c$ by definition, but due to the condition $i < |m|$, we never have $|m| \in \mathbf{b}(m)$.

Notice that $i \in \mathbf{b}(m)$ implies that $s_{-(i+1)}(m)$ is a zero-preimage of $\mathbf{f}^{\mathsf{right}}$.

The intuitive meaning of $\mathbf{b}(m)$ is that it tells us whether (and where) at some point during the computation of $\mathbf{S}^{in}(m)$, there is a state s with $s^{\mathsf{right}} = 0^c$. This is important to know, because

[11] By *zero-preimage* we mean any value x with $\mathbf{f}^{\mathsf{right}}(x) = 0^c$.

later we will interpret such a state as an indication that we have reached the beginning of the computation of $\mathbf{S}^{in}$. $\mathbf{b}(m)$ tells us where there are exceptions to this rule.

Finally, in the following, for a function f, let $\mathcal{M}_f(M)$ denote a measurement of the register M that measures $f(m)$ when $|m\rangle$ is the state of M. More formally, $\mathcal{M}_f(M)$ consists of the projectors $\{P_i\}_{i\in\operatorname{range} f}$ with $P_i := \sum_{m \text{ s.t. } f(m)=i} |m\rangle\langle m|$. For example, if f is the identity, then $\mathcal{M}_f(M)$ is a complete measurement in the computational basis, and if f is a constant function, then $\mathcal{M}_f(M)$ does not do anything. We write $x \leftarrow \mathcal{M}_f(M)$ to denote that the outcome of $\mathcal{M}_f(M)$ is stored in x.

Note that all measurements $\mathcal{M}_f$ commute, even for different functions f. We will use this fact implicitly by treating sequences of measurements as equal when they differ only in their ordering.

Game 2 *$(S, M_1, \dots, M_t, h_1, \dots, h_t) \leftarrow A()$. $\mathbf{b}_j \leftarrow \mathcal{M}_{\mathbf{b}}(M_j)$ for $j = 1, \dots, t$. $b \leftarrow B(S, M_1, \dots, M_t)$.*

That is, compared to Game 1, we additionally measure $\mathbf{b}(m)$ for each of the messages m that are in the registers $M_1, \dots, M_t$. We will need this below in order to decide whether a given message block is the first message block (when working our way backwards through the message). Namely, m_{-i} is the beginning of the message m, iff $s_{-i}(m)^{\mathsf{right}} = 0^c$ and $i \notin \mathbf{b}(m)$.

For any game Game X, let Pr[Game X] denote the probability that $b = 1$ in Game X. (I.e., Pr[Game X] := $\Pr[b = 1 : \text{Game } X]$.)

We will now bound $\big|\Pr[\text{Game 1}] - \Pr[\text{Game 2}]\big|$. Let ρ denote the state of $S, M_1, \dots, M_t$ after executing $(S, M_1, \dots, M_t, h_1, \dots, h_t) \leftarrow A()$. Let ρ' denote the state after additionally performing the measurement "$\mathbf{b}_j \leftarrow \mathcal{M}_{\mathbf{b}}(M_j)$ for $j = 1, \dots, t$".[12] By Lemma 29, the trace distance between ρ and ρ' is bounded by $\sqrt{\varepsilon_0'}$, where $1 - \varepsilon_0' := \Pr[(\mathbf{b}_1, \dots, \mathbf{b}_t) = (\varnothing, \dots, \varnothing) : \text{Game 2}]$. And ρ and ρ' are the state before executing $b \leftarrow B(S, M_1, \dots, M_t)$ in Game 1 and Game 2, respectively. Thus the probability of $b = 1$ in these two games can differ at most by the trace distance between ρ and ρ', hence

$$\Big|\Pr[\text{Game 1} - \text{Game 2}]\Big| \le \sqrt{\varepsilon_0'}. \tag{1}$$

Since $\Pr[(\mathbf{b}_1, \dots, \mathbf{b}_t) = (\varnothing, \dots, \varnothing) : \text{Game 2}] = 1 - \varepsilon_0'$, we have $\Pr[\exists j.\ \mathbf{b}_j \ne \varnothing] = \varepsilon_0'$ in the following game:

Game 3 *$(S, M_1, \dots, M_t, h_1, \dots, h_t) \leftarrow A()$. $m_j \leftarrow \mathcal{M}(M_j)$ for $j = 1, \dots, t$. $\mathbf{b}_j := \mathbf{b}(m_j)$ for $j = 1, \dots, t$. $h_j' := \mathbf{S}^{in}(m_j)$ for $j = 1, \dots, t$.*

By definition of $\mathbf{b}(m)$, we have that $\mathbf{b}(m) \ne \varnothing$ implies that $s_{-(i+1)}(m)$ is a zero-preimage of $\mathbf{f}^{\mathsf{right}}$ for some i. Thus, if $\exists j.\ \mathbf{b}_j \ne \varnothing$ in Game 3, then one of the $s_{-i}(m_j)$ is a zero-preimage of $\mathbf{f}^{\mathsf{right}}$. Thus with probability $\ge \varepsilon_0'$, one of the computations $\mathbf{S}^{in}(m_j)$ in Game 3 performs a call $\hat{s} := \mathbf{f}(s)$ such that $\hat{s}^{\mathsf{right}} = 0^c$.

Consider the following algorithm A': It computes $(S, M_1, \dots, M_t, h_1, \dots, h_t) \leftarrow A()$, measures $m_j \leftarrow \mathcal{M}(M_j)$ for $j = 1, \dots, t$, and then computes $\mathbf{S}^{in}(m_j)$ for $j = 1, \dots, t$. If in one of the invocations $\hat{s} := \mathbf{f}(s)$ performed by $\mathbf{S}^{in}$ we have $\hat{s}^{\mathsf{right}} = 0$, A' returns s.

Then A' outputs a zero-preimage of $\mathbf{f}^{\mathsf{right}}$ with some probability $\varepsilon' \ge \varepsilon_0'$. A' has runtime $\le \tau + t\ell\tau_{\mathbf{f}} + O(t\ell c)$. And from (1), we get

$$\Big|\Pr[\text{Game 1} - \text{Game 2}]\Big| \le \sqrt{\varepsilon'}. \tag{2}$$

For the next game, we introduce one more function, d_i, defined by: $d_{-i}(m) := 1$ if $s_{-i}(m) \ne \bot$, and $d_{-i}(m) := 0$ otherwise. Equivalently, $d_{-i}(m) = 1$ iff $|m| \ge i$.

The next game is a variation of Game 2 and is parametrized by an integer $k = 1, \dots, \ell$.

[12] Strictly speaking, these are t measurements $\mathcal{M}_{\mathbf{b}}(M_1), \dots, \mathcal{M}_{\mathbf{b}}(M_t)$. However, since the measurements $\mathcal{M}_{\mathbf{b}}(M_j)$ commute, one can see them as a single measurement on $M_1, \dots, M_t$ with outcome $\mathbf{b}_1, \dots, \mathbf{b}_t$.

Game 4_k *$(S, M_1, \dots, M_t, h_1, \dots, h_t) \leftarrow A()$. $\mathbf{b}_j \leftarrow \mathcal{M}_{\mathbf{b}}(M_j)$ for $j = 1, \dots, t$. $s_{ij} \leftarrow \mathcal{M}_{s_{-i}}(M_j)$ for $i = 1, \dots, k$ and $j = 1, \dots, t$. $d_{ij} \leftarrow \mathcal{M}_{d_{-i}}(M_j)$ for $i = 1, \dots, k$ and $j = 1, \dots, t$. $b \leftarrow B(S, M_1, \dots, M_t)$.*

In this game, we measure additionally $s_{-1}(m), \dots, s_{-k}(m)$ for each of the messages m on registers $M_1, \dots, M_t$. That is, we measure a *suffix* of length k of the list of states occurring during the calculation of $\mathbf{S}^{in}(m)$. Thus, in each successive game Game 4_k (for increasing k), we measure one more state from the calculation (starting from the end). We also measure $d_{-1}(m), \dots, d_{-k}(m)$, but this measurement has no effect since $d_{-i}(m)$ is determined by $s_{-i}(m)$ by definition.

Notice also that $s_{-k}(m) = \bot$ if $k > |m|$ (by definition of s_{-k}). Thus, measuring $s_{-1}(m), \dots, s_{-k}(m)$ implicitly also measures whether $|m| \leq k$, and if so, what the value of $|m|$ is.

We first consider the case $k = 1$: The only difference between Game 4_1 and Game 2 are the measurements $s_{1j} \leftarrow \mathcal{M}_{s_{-1}}(M_j)$ and $d_{1j} \leftarrow \mathcal{M}_{d_{-1}}(M_j)$. But as mentioned above, $s_{-1}(m) = \mathbf{S}^{in}(m)$. Thus $s_{1j} \leftarrow \mathcal{M}_{s_{-1}}(M_j)$ is the same as $s_{1j} \leftarrow \mathcal{M}_{\mathbf{S}^{in}}(M_j)$. Since (A, B) is a t-valid adversary, we have by definition of t-validity that measuring $\mathcal{M}_{\mathbf{S}^{in}}$ on the registers M_j returns h_j with probability 1 (i.e., $s_{1j} = h_j$). Thus the measurement $s_{1j} \leftarrow \mathcal{M}_{s_{-1}}(M_j)$ has no effect on the state that is measured. Hence we can omit it without changing the distribution of b. Furthermore, $s_{1j}(m) \neq \bot$ for all $m \in (\{0,1\}^r)^+$. Thus we always have $d_{1j}(m) = 1$, thus the measurement $d_{1j} \leftarrow \mathcal{M}_{d_{-1}}(M_j)$ has no effect on b either. Thus

$$\Pr[\text{Game } 4_1] = \Pr[\text{Game } 2]. \tag{3}$$

Game 5_k *$(S, M_1, \dots, M_t, h_1, \dots, h_t) \leftarrow A()$. $\mathbf{b}_j \leftarrow \mathcal{M}_{\mathbf{b}}(M_j)$ for $j = 1, \dots, t$. $s_{ij} \leftarrow \mathcal{M}_{s_{-i}}(M_j)$ for $i = 1, \dots, k$ and $j = 1, \dots, t$. $d_{ij} \leftarrow \mathcal{M}_{d_{-i}}(M_j)$ for $i = 1, \dots, k+1$ and $j = 1, \dots, t$. $b \leftarrow B(S, M_1, \dots, M_t)$.*

Thus, in comparison with Game 4_k, we additionally measure $d_{-(k+1)}(m)$ for the messages m on the registers M_j. In other words, we measure whether $s_{-k}(m) \neq \bot$, which in turn is the same as measuring whether $|m| \geq k + 1$.

Notice further that for any $m \in (\{0,1\}^r)^+$, the following holds: If $d_{-(k+1)}(m) = 0$, we have $|m| \leq k$. In case $|m| < k$, we have $s_{-k}(m) = \bot$. In case $|m| = k$, we have $s_{-k}(m)^{\mathsf{right}} = 0^c$ (because $s_{-|m|}(m) = m_1 \| 0^c$ as mentioned above) and $k \notin \mathbf{b}(m)$ (by definition of $\mathbf{b}$). Thus

$$\forall m \in (\{0,1\}^r)^+. \qquad d_{-(k+1)}(m) = 0 \qquad \Longrightarrow \qquad s_{-k}(m) = \bot \vee \big(s_{-k}(m)^{\mathsf{right}} = 0^c \wedge k \notin \mathbf{b}(m)\big).$$

Furthermore, the converse holds, too: If $s_{-k}(m) = \bot$, then $|m| < k$, hence $d_{-(k+1)}(m) = 0$. And if $s_{-k}(m)^{\mathsf{right}} = 0^c \wedge k \notin \mathbf{b}(m)$, by definition of $\mathbf{b}(m)$, we have $s_{-k}(m)^{\mathsf{right}} = 0^c \wedge \neg\big(s_{-k}(m)^{\mathsf{right}} = 0^c \wedge k < |m|\big)$, thus $k \geq |m|$, thus $s_{-(k+1)}(m) = \bot$ thus $d_{-(k+1)}(m) = 0$. So, altogether we have

$$\forall m \in (\{0,1\}^r)^+. \qquad d_{-(k+1)}(m) = 0 \qquad \Longleftrightarrow \qquad s_{-k}(m) = \bot \vee \big(s_{-k}(m)^{\mathsf{right}} = 0^c \wedge k \notin \mathbf{b}(m)\big).$$

Thus, in the context of Game 5_k (and using that $m \in (\{0,1\}^r)^+$ since (A, B) is t-valid on $(\{0,1\}^r)^+$), we have for all $j = 1, \dots, t$:

$$d_{k+1,j} = 0 \qquad \Longleftrightarrow \qquad s_{kj} = \bot \vee \big(s{-}kj^{\mathsf{right}} = 0^c \wedge k \notin \mathbf{b}_j\big).$$

Thus, the outcome $d_{k+1,j} \in \{0,1\}$ of the measurement $d_{k+1,j} \leftarrow \mathcal{M}_{d_{-(k+1)}}(M_j)$ is determined by the prior measurement outcomes $s_{kj}, \mathbf{b}_j$. Hence $d_{k+1,j} \leftarrow \mathcal{M}_{d_{-(k+1)}}(M_j)$ has no effect on the state and thus omitting it does not change the distribution of b in Game 5_k. Since this measurement is the only difference between Game 4_k and Game 5_k, we have

$$\Pr[\text{Game } 4_k] = \Pr[\text{Game } 5_k]. \tag{4}$$

Let Game $5_\$$ denote the game in which we pick $k \xleftarrow{\$} \{1,\dots,\ell-1\}$, and then execute Game 5_k.

Let Game $4_{\$+1}$ denote the game in which we pick $k \xleftarrow{\$} \{1,\dots,\ell-1\}$, and then execute Game 4_{k+1}.

We will now bound $\bigl|\Pr[\text{Game } 4_{\$+1}]-\Pr[\text{Game } 5_\$]\bigr|$. For this, consider the following algorithms A'', B''. $A''()$ performs the following steps:

- Pick $k \xleftarrow{\$} \{1,\dots,\ell-1\}$.
- Run $(S, M_1,\dots,M_t,h_1,\dots,h_t) \leftarrow A()$.
- Measure $\mathbf{b}_j \leftarrow \mathcal{M}_{\mathbf{b}}(M_j)$ for $j=1,\dots,t$.
- Measure $s_{ij} \leftarrow \mathcal{M}_{s_{-i}}(M_j)$ for $i=1,\dots,k$ and $j=1,\dots,t$.
- Measure $d_{ij} \leftarrow \mathcal{M}_{d_{-i}}(M_j)$ for $i=1,\dots,k+1$ and $j=1,\dots,t$.
- For $j=1,\dots,t$: Initialize M''_j with $|0^{r+c}\rangle$. If $d_{k+1,j}=1$, apply U to the registers M_j, M''_j where U is the unitary with $U|m\rangle|y\rangle = |m\rangle|y\oplus s_{-(k+1)}(m)\rangle$.
- Combine the registers $S, M_1,\dots,M_t$ and classical values k, $(d_{k+1,j})_j$ into a single register S''.
- Let $h''_j := s_{kj}^{\mathsf{right}}$ if $s_{kj}\neq\bot$, and $h''_i := \mathbf{f}(0^{r+c})^{\mathsf{right}}$ otherwise.
- Return $(S'', M''_1,\dots,M''_t,h''_1,\dots,h''_t)$.

And $B''(S'', M''_1,\dots,M''_t)$ performs the following steps:

- Split the register S'' into registers $S, M_1,\dots,M_t$ and classical values k, $(d_{kj})_j$.
- For $j=1,\dots,t$: If $d_{k+1,j}=1$, apply U to the registers M_j, M''_j. (U was defined in the description of A''. Note that U is self-inverse.)
- $b \leftarrow B(S, M_1,\dots,M_t)$.
- Return b.

First, note that (A'',B'') is t-valid for $\mathbf{f}^{\mathsf{right}}$: If $d_{k+1,j}=0$, then M''_j contains $|0^{r+c}\rangle$ and we have $h''_j = \mathbf{f}^{\mathsf{right}}(0^{r+c})$. If $d_{k+1,j}=1$, then M_j contains a superposition of values m with $s_{kj}=s_{-k}(m)\neq\bot$. For these m we have $s_{-k}(m)=\mathbf{f}(s_{-(k+1)}(m))\oplus(m_{-k}\|0^c)$ by definition of s_{-k} and thus

$$\mathbf{f}^{\mathsf{right}}(s_{-(k+1)}(m)) = \bigl(s_{-k}(m)\oplus(m_{-k}\|0^c)\bigr)^{\mathsf{right}} = s_{-k}(m)^{\mathsf{right}} = s_{kj}^{\mathsf{right}} = h''_j.$$

Furthermore, since we applied U to M_j, M''_j, we have that M''_j contains $s_{-(k+1)}(m)$ when M_j contains m. Thus M''_j contains a superposition of values x with $\mathbf{f}^{\mathsf{right}}(x)=h''_j$. So in both cases $d_{k+1,j}=0$ and $d_{k+1,j}=1$, M''_j contains a superposition of values x with $\mathbf{f}^{\mathsf{right}}(x)=h''_j$. Thus (A'',B'') is t-valid for $\mathbf{f}^{\mathsf{right}}$.

(A'',B'') has runtime $\tau+O(t\ell\tau_{\mathbf{f}})$. Namely, τ is the time for executing A and B. Each application U takes time $O(\ell\tau_{\mathbf{f}})$, and there are t of them. Each measurement $\mathcal{M}_{\mathbf{b}}(M_j)$, takes time $O(\ell\tau_{\mathbf{f}})$, and there are t of them. Each measurement $\mathcal{M}_{s_{-i}}(M_j)$, takes time $O(\ell\tau_{\mathbf{f}})$, and there are kt of them, measuring them individually would take time $O(kt\ell\tau_{\mathbf{f}})$. However, $\mathcal{M}_{s_{-1}}(M_j),\dots,\mathcal{M}_{s_{-k}}(M_j)$ can be performed in time $O(\ell\tau_{\mathbf{f}})$ when executed together: one performs a single evaluation of $\mathbf{S}^{in}$ on M_j (in superposition) while keeping all intermediate states in ancilla qubits. Then one measures those intermediate states and uncomputes $\mathbf{S}^{in}$. The effect is to measure $\mathcal{M}_{s_{-1}}(M_j),\dots,\mathcal{M}_{s_{-k}}(M_j)$ in time $O(\ell\tau_{\mathbf{f}})$. Since there are t registers M_j, the total time is $O(t\ell\tau_{\mathbf{f}})$. Similarly, the measurements $\mathcal{M}_{d_{-i}}(M_j)$ can be performed in time $O(t\ell\tau_{\mathbf{f}})$. All other computations can be performed in time $O(t\ell c)$ which is dominated by $O(t\ell\tau_{\mathbf{f}})$. Thus the total time spent by (A'',B'') is $\tau+O(t\ell\tau_{\mathbf{f}})$.

Let Game''_1 and Game''_2 be the games from Definition 7 for adversary (A'',B''). We define ε'' as the collapsing-advantage of (A'',B'') against $\mathbf{f}^{\mathsf{right}}$. In other words, $\varepsilon'' := \bigl|\Pr[\mathsf{Game}''_1]-\Pr[\mathsf{Game}''_2]\bigr|$.

Then (A'',B'') has the properties claimed in the statement of this lemma.

Consider Game''_2. We claim that $\Pr[\mathsf{Game}''_2]=\Pr[\text{Game } 5_\$]$. This is because A'',B'' in Game''_2 perform exactly the steps executed in Game $5_\$$, except for some packaging and subsequent unpackaging of registers inside S'', and for the applications of U to M_j, M''_j. But the latter has no effect because the two applications of U cancel each other out. $\Pr[\mathsf{Game}''_2]=\Pr[\text{Game } 5_\$]$.

Consider Game''_1. Here, in comparison with Game''_2, we have an additional measurement of M''_j in the computational basis. In the case $d_{k+1,j}=1$, this measurement occurs between two applications of U (for a $|0\rangle$-initialized M''_j. The sequence of steps U, $\mathcal{M}(M''_j)$, U is equivalent to the measurement $\mathcal{M}_{s_{-(k+1)}}(M_j)$. And in the case $d_{k+1,j} = 0$, this measurement measures the $|0\rangle$-initialized M''_j and has thus no effect on the state, thus it is equivalent to measuring $\mathcal{M}_{s_{-(k+1)}}(M_j)$ which also has no effect on the state (because it always returns outcome $\bot$ when $d_{k+1,j} = 0$). Thus, the additional measurement in Game''_1 is equivalent to a an additional measurement $\mathcal{M}_{s_{-(k+1)}}(M_j)$. Thus $\Pr[\mathsf{Game}''_1] = \Pr[\text{Game } 4_{\$+1}]$ (note that Game $4_{\$+1}$ differs from $\Pr[\text{Game } 5_\$]$ only by this one additional measurement).

It follows that

$$\bigl|\Pr[\text{Game } 4_{\$+1}] - \Pr[\text{Game } 5_\$]\bigr| = \bigl|\Pr[\mathsf{Game}''_1] - \Pr[\mathsf{Game}''_2]\bigr| = \varepsilon''. \tag{5}$$

By construction of Game $5_\$$, we have $\Pr[\text{Game } 5_\$] = \frac{1}{\ell-1}\sum_{k=1}^{\ell-1}\Pr[\text{Game } 5_k]$. And by construction of Game $4_{\$+1}$, we have $\Pr[\text{Game } 4_{\$+1}] = \frac{1}{\ell-1}\sum_{k=1}^{\ell-1}\Pr[\text{Game } 4_{k+1}]$. Thus

$$\varepsilon'' \overset{(5)}{\geq} \bigl|\Pr[\text{Game } 4_{\$+1}] - \Pr[\text{Game } 5_\$]\bigr| = \frac{1}{\ell-1}\Bigl|\sum_{k=1}^{\ell-1}\Pr[\text{Game } 4_{k+1}] - \sum_{k=1}^{\ell-1}\Pr[\text{Game } 5_k]\Bigr|$$

$$\overset{(4)}{=} \frac{1}{\ell-1}\Bigl|\sum_{k=1}^{\ell-1}\Pr[\text{Game } 4_{k+1}] - \sum_{k=1}^{\ell-1}\Pr[\text{Game } 4_k]\Bigr| = \frac{1}{\ell-1}\bigl|\Pr[\text{Game } 4_\ell] - \Pr[\text{Game } 4_1]\bigr|.$$

Hence

$$\bigl|\Pr[\text{Game } 4_\ell] - \Pr[\text{Game } 4_1]\bigr| \leq (\ell-1)\varepsilon''. \tag{6}$$

We now investigate Game 4_ℓ. For any $m \in (\{0,1\}^r)^+$ with $|m| \leq \ell$, we have: $|m|$ is the largest i such that $s_{-i}(m) \neq \bot$. And $s_{-i}(m) = \mathbf{f}(s_{-(i+1)}(m)) \oplus (m_{-i}\|0^c)$, hence $m_{-i} = \mathbf{f}(s_{-(i+1)}(m))^{\mathsf{left}} \oplus s_{-i}(m)^{\mathsf{left}}$. Thus there is a function γ (depending on $\mathbf{f}$) such that $\gamma\bigl(\mathbf{B}(m), s_{-1}(m), \dots, s_{-\ell}(m), d_{-1}(m), \dots, d_{-\ell}(m)\bigr) = m$. And trivially, there is a function γ' (depending on $\mathbf{f}$) such that $\gamma'(m) = (B(m), s_{-1}(m), \dots, s_{-\ell}(m), d_{-1}(m), \dots, d_{-\ell}(m))$. Thus measuring $B(m), s_{-1}(m), \dots, s_{-\ell}(m), d_{-1}(m), \dots, d_{-\ell}(m)$ is equivalent to measuring m (in its effects on the measured state), assuming $|m| \leq \ell$. Thus, the measurements $\mathbf{b}_j \leftarrow \mathcal{M}_{\mathbf{b}}(M_j)$ and $s_{ij} \leftarrow \mathcal{M}_{s_{-i}}(M_j)$ and $d_{ij} \leftarrow \mathcal{M}_{d_{-i}}(M_j)$ for $i = 1, \dots, \ell$ and $j = 1, \dots, t$ performed in Game 4_ℓ have the same effect on the state as just performing the measurements $m_j \leftarrow \mathcal{M}(M_j)$ for $j = 1, \dots, t$. Thus

$$\Pr[\text{Game } 4_\ell] = \Pr[\text{Game } 6] \tag{7}$$

with the following game:

Game 6 $(S, M_1, \dots, M_t, h_1, \dots, h_t) \leftarrow A()$. *$m_j \leftarrow \mathcal{M}(M_j)$ for $j = 1, \dots, t$.* $b \leftarrow B(S, M_1, \dots, M_t)$

We can now relate the initial and the final game:

$$\bigl|\Pr[\text{Game } 1] - \Pr[\text{Game } 6]\bigr| \overset{(2),(3),(6),(7)}{\leq} \sqrt{\varepsilon'} + (\ell-1)\varepsilon''.$$

Since Game 6 is identical to Game_1 from Definition 7, and Game 1 is identical to Game_2 from the same definition, it follows that the collapsing-advantage ε of (A, B) is $\bigl|\Pr[\text{Game } 1] - \Pr[\text{Game } 6]\bigr|$. Thus $\varepsilon \leq \sqrt{\varepsilon'} + (\ell-1)\varepsilon''$ and the lemma follows. □

Lemma 23 *Assume $n \geq r$.*

Let (A, B) be a t-valid adversary against $\mathbf{S}^{out}$ with runtime τ and collapsing-advantage ε.

Then there is an adversary (A', B') that is t-valid for $\mathbf{f}^{\mathsf{left}}$, has runtime $\tau + O(tr)$, and collapsing-advantage ε.

If time is measured in $\mathcal{O}$-queries (instead of computation steps), the runtime of (A', B') is τ.

Proof. Let $G(x)$ return the first r bits of x. If $n \geq r$, then $G(\mathbf{S}^{out}(s)) = \mathbf{f}^{\mathsf{left}}(s)$. Thus the lemma follows directly from Lemma 11. □

Theorem 24 *Assume $n \geq r$. Assume that pad is injective. Assume a τ-time adversary (A, B), t-valid for $\mathbf{S}$, with collapsing-advantage ε against $\mathbf{S}$. Then there are:*

- *a $(\tau + O(t\tau_{pad}) + O(t\tau_{\mathbf{S}^{in}}))$-time adversary A_1 that finds a zero-preimage of $\mathbf{f}^{\mathsf{right}}$ with probability ε_1, and*
- *a $\big(\tau + O(t\tau_{pad}) + O(t\tau_{\mathbf{S}^{in}})\big)$-time adversary (A_2, B_2), t-valid against $\mathbf{f}^{\mathsf{right}}$, with collapsing-advantage ε_2, and*
- *a $(\tau + O(t\tau_{pad}) + O(t\tau_{\mathbf{S}^{in}}))$-time adversary (A_3, B_3) that is t-valid for $\mathbf{f}^{\mathsf{left}}$, with collapsing-advantage ε_3,*

such that $\varepsilon \leq \sqrt{\varepsilon_1} + (\ell - 1)\varepsilon_2 + \varepsilon_3$.

Here τ_{pad} refers to the maximum runtime of pad. (For values m that A may output on the M_j.) And $\tau_{\mathbf{S}^{in}}$ refers to the maximum runtime of $\mathbf{S}^{in}$ (on outputs of pad).

Proof. Since $\mathbf{S} = (\mathbf{S}^{out} \circ \mathbf{S}^{in}) \circ pad$, we can apply Lemma 12 to get:

- a $(\tau + O(t\tau_{pad}))$-time adversary (A_4, B_4), that is t-valid on $\operatorname{im} pad \subseteq (\{0,1\}^r)^+$ and has collapsing-advantage ε_4 against $\mathbf{S}^{out} \circ \mathbf{S}^{in}$,
- a $(\tau + O(t\tau_{pad}))$-time adversary (A_5, B_5), that is t-valid and has collapsing-advantage ε_5 against pad,

such that $\varepsilon \leq \varepsilon_4 + \varepsilon_5$. Since pad is injective, we have $\varepsilon_5 = 0$ (Lemma 9). Thus $\varepsilon \leq \varepsilon_4$. We can now apply Lemma 12 to $\mathbf{S}^{out} \circ \mathbf{S}^{in}$ and (A_4, B_4). This gives us:

- a $(\tau + O(t\tau_{pad}) + O(t\tau_{\mathbf{S}^{in}}))$-time adversary (A_6, B_6), t-valid with collapsing-advantage ε_6 against $\mathbf{S}^{out}$,
- a $(\tau + O(t\tau_{pad}) + O(t\tau_{\mathbf{S}^{in}}))$-time adversary (A_7, B_7), t-valid on $(\{0,1\}^r)^+$ with collapsing-advantage ε_7 against $\mathbf{S}^{in}$,

where $\varepsilon_4 \leq \varepsilon_6 + \varepsilon_7$. Since $\varepsilon \leq \varepsilon_4$, we have $\varepsilon \leq \varepsilon_6 + \varepsilon_7$. We now apply Lemma 22 to (A_7, B_7). We get:

- a $(\tau + O(t\tau_{pad}) + O(t\tau_{\mathbf{S}^{in}}) + t\ell\tau_{\mathbf{f}} + O(t\ell c)) = (\tau + O(t\tau_{pad}) + O(t\tau_{\mathbf{S}^{in}}))$-time adversary A_1 that finds a zero-preimage of $\mathbf{f}^{\mathsf{right}}$ with probability ε_1, and
- a $\big(\tau + O(t\tau_{pad}) + O(t\tau_{\mathbf{S}^{in}}) + O(t\ell\tau_{\mathbf{f}})\big) = \big(\tau + O(t\tau_{pad}) + O(t\tau_{\mathbf{S}^{in}})\big)$-time adversary (A_2, B_2), t-valid against $\mathbf{f}^{\mathsf{right}}$, with collapsing-advantage ε_2

such that $\varepsilon_7 \leq \sqrt{\varepsilon_1} + (\ell - 1)\varepsilon_2$. And we apply Lemma 23 to (A_6, B_6). We get:

- a $(\tau + O(t\tau_{pad}) + O(t\tau_{\mathbf{S}^{in}}) + O(tr)) = (\tau + O(t\tau_{pad}) + O(t\tau_{\mathbf{S}^{in}}))$-time adversary (A_3, B_3) that is t-valid for $\mathbf{f}^{\mathsf{left}}$, with collapsing-advantage $\varepsilon_3 = \varepsilon_6$.

We have

$$\varepsilon \leq \varepsilon_6 + \varepsilon_7 \leq \varepsilon_3 + \sqrt{\varepsilon_1} + (\ell - 1)\varepsilon_2.$$

□

6.1 Using random oracles or random permutations

Lemma 25 *If $\mathbf{f} : \{0,1\}^{r+c} \to \{0,1\}^{r+c}$ is a random function, and $(A^{\mathbf{f}}, B^{\mathbf{f}})$ is t-valid for $\mathbf{f}^{\mathsf{left}}$ (or for $\mathbf{f}^{\mathsf{right}}$), makes $\leq q$ queries to $\mathbf{f}$, and has collapsing-advantage ε, then $\varepsilon \in O(tq^{3/2}2^{-r/2})$ (or $\varepsilon \in O(tq^{3/2}2^{-c/2})$).*

Proof. Assume first that $A^{\mathbf{f}}$ is t-valid for $\mathbf{f}^{\mathsf{left}}$.

Let $\mathbf{h} : \{0,1\}^{r+c} \to \{0,1\}^r$ be a random function. Let $(\hat{A}^{\mathbf{h}}, \hat{B}^{\mathbf{h}})$ perform the following steps: It picks a random function $\mathbf{g} : \{0,1\}^{r+c} \to \{0,1\}^c$ (we do not care about the runtime of $\hat{A}, \hat{B}$, so it is not a problem that picking $\mathbf{g}$ takes exponential time). Then it simulates $(A^{\hat{\mathbf{f}}}, B^{\hat{\mathbf{f}}})$ where $\hat{\mathbf{f}}$ is the function defined by $\hat{\mathbf{f}}(x) := \mathbf{h}(x) \| \mathbf{g}(x)$. Note that an oracle query to $\hat{\mathbf{f}}$ can be implemented using a single oracle query to $\mathbf{h}$. So $\hat{A}$ still performs q oracle queries.

The joint distribution of $\hat{\mathbf{f}}$ and $\mathbf{h}$ is the same as that of $\mathbf{f}$ and $\mathbf{f}^{\mathsf{left}}$. Thus the advantage of $(\hat{A}^{\mathbf{h}}, \hat{B}^{\mathbf{h}})$ against $\mathbf{h}$ is the same as the advantage ε of $(A^{\mathbf{f}}, B^{\mathbf{f}})$ against $\mathbf{f}^{\mathsf{left}}$. And $(\hat{A}^{\mathbf{h}}, \hat{B}^{\mathbf{h}})$ is t-valid for $\mathbf{h}$.

Thus by Theorem 10, $\varepsilon \in O(tq^{3/2}2^{-r/2})$.

When $A^{\mathbf{f}}$ is t-valid for $\mathbf{f}^{\mathsf{right}}$, an analogous proof gives us $\varepsilon \in O(tq^{3/2}2^{-c/2})$. □

Lemma 26 *Let $\mathbf{f} : \{0,1\}^{r+c} \to \{0,1\}^{r+c}$ be a random function. Then a q-query adversary finds a zero-preimage of $\mathbf{f}^{\mathsf{right}}$ with probability $\leq q2^{-c/2+2}$.*

Proof. Assume an algorithm $A^{\mathbf{f}}$ that finds a zero-preimage of $\mathbf{f}^{\mathsf{right}}$ with some probability ε. Let $F : \{0,1\}^{r+c} \to \{0,1\}$ be a random function where each $F(z)$ is independently chosen, with $\Pr[F(z) = 1] = \gamma := 2^{-c}$. Let B^F be the following algorithm: It picks functions $\mathbf{f}_1, \mathbf{f}_0 : \{0,1\}^{r+c} \to \{0,1\}^{r+c}$, where each $\mathbf{f}_0(x)$ is independently and uniformly chosen from $\{0,1\}^r \times (\{0,1\}^c \setminus \{0^c\})$ and each $\mathbf{f}_1(x)$ is independently and uniformly chosen from $\{0,1\}^r \times \{0^c\}$. Define $\hat{\mathbf{f}}(x) := \mathbf{f}_{F(x)}(x)$. Then B^F runs $x \leftarrow A^{\hat{\mathbf{f}}}$ and returns x. A query to $\hat{\mathbf{f}}$ can be implemented using 2 queries to F. Thus B^F performs $\leq 2q$ queries to F. (We do not care about the runtime of B^F. Thus the exponential time required for implementing $\mathbf{f}_0, \mathbf{f}_1$ is not a problem.)

Note that $\hat{\mathbf{f}} : \{0,1\}^{r+c} \to \{0,1\}^{r+c}$ is a uniformly random function (for F distributed as described above). Thus $A^{\hat{\mathbf{f}}}$ finds a zero-preimage of $\mathbf{f}^{\mathsf{right}}$ with probability ε, i.e., an x with $\mathbf{f}(x)^{\mathsf{right}} = 0^c$. Such an x then satisfies $F(x) = 1$. Thus B^F finds a 1-preimage of F with probability ε. [ARU14, full version, Lemma 37] shows that a $2q$-query adversary can find a 1-preimage in F with probability at most $4q\sqrt{\gamma}$. Thus $\varepsilon \leq 4q\sqrt{\gamma} = q2^{-c/2+2}$. □

Theorem 27 *Let $\mathbf{f} : \{0,1\}^{r+c} \to \{0,1\}^{r+c}$ be a random function. Let (A, B) be an adversary that is t-valid against $\mathbf{S}$ and makes $\leq q$ queries to $\mathbf{f}$. Then (A, B) has collapsing-advantage*

$$O\Big((q + t\ell)^{1/2}2^{-c/4} + \ell t(q + t\ell)^{3/2}2^{-c/2} + t(q + t\ell)^{3/2}2^{-r/2}\Big).$$

Proof. Let ε denote the collapsing-advantage of (A, B). By Theorem 24, there are:

- a $(q + O(t\ell))$-query adversary A_1 that finds a zero-preimage of $\mathbf{f}^{\mathsf{right}}$ with probability ε_1, and
- a $(q + O(t\ell))$-query adversary (A_2, B_2), t-valid against $\mathbf{f}^{\mathsf{right}}$, with collapsing-advantage ε_2, and
- a $(q + O(t\ell))$-query adversary (A_3, B_3) that is t-valid for $\mathbf{f}^{\mathsf{left}}$, with collapsing-advantage ε_3,

with

$$\varepsilon \leq \sqrt{\varepsilon_1} + (\ell - 1)\varepsilon_2 + \varepsilon_3. \tag{8}$$

By Lemma 26, $\varepsilon_1 \in O((q + t\ell)2^{-c/2})$. By Lemma 25, $\varepsilon_2 \in O(t(q + t\ell)^{3/2}2^{-c/2})$ and $\varepsilon_3 \in O(t(q + t\ell)^{3/2}2^{-r/2})$.

□

Theorem 28 *Let $\mathbf{f} : \{0,1\}^{r+c} \to \{0,1\}^{r+c}$ be a random permutation. Let (A, B) be an adversary that is t-valid against $\mathbf{S}$ and makes $\leq q$ queries to $\mathbf{f}$. Then (A, B) has collapsing-advantage*

$$O\Big((q + t\ell)^{1/2}2^{-c/4} + \ell t(q + t\ell)^{3/2}2^{-c/2} + t(q + t\ell)^{3/2}2^{-r/2} + q^3 2^{-c}\Big).$$

Proof. It was shown by [Zha15] that a q-query adversary cannot distinguish a random function on $r + c$ bits from a random permutation on $r + c$ bits with probability greater than $O(q^3 2^{-(r+c)})$.

Thus the collapsing-advantage of $(A^{\mathbf{f}}, B^{\mathbf{f}})$ changes at most by $O(q^3 2^{-(r+c)})$ when we replace the random permutation $\mathbf{f}$ by a random function $\mathbf{f}$. Thus the advantage of $(A^{\mathbf{f}}, B^{\mathbf{f}})$ is at most $O(q^3 2^{-(r+c)})$ larger than the one given in Theorem 27. □

A Technical lemmas

Lemma 29 *Let ρ be a quantum state (density operator of trace 1). Let $\mathcal{M}$ be a projective measurement consisting of projectors $P_1,\dots,P_n$. Assume that applying $\mathcal{M}$ to ρ gives outcome 1 with probability $\geq 1-\varepsilon$.*

Let ρ' be the result of applying $\mathcal{M}$ to ρ (and discarding the result).

Then the trace distance between ρ and ρ' is $\leq \sqrt{\varepsilon}$. (I.e., no algorithm can distinguish those states better than with probability $\sqrt{\varepsilon}$.)

Proof. Without loss of generality, we can assume that ρ is a pure state $\rho = |\Psi\rangle\langle\Psi|$. (The general case of the lemma is then obtained by considering a purification $|\Psi\rangle\langle\Psi|$ of the mixed state ρ.)

Then $\rho' = \sum_i P_i|\Psi\rangle\langle\Psi|P_i$. Let F denote the fidelity and let TD denote the trace distance. By [NC10, (9.60)], we have $F(|\Psi\rangle\langle\Psi|,\rho') = \sqrt{\langle\Psi|\rho'|\Psi\rangle}$. Thus,

$$F(\rho,\rho')^2 = \langle\Psi|\rho'|\Psi\rangle = \sum_i \langle\Psi|P_i|\Psi\rangle\langle\Psi|P_i|\Psi\rangle = \sum_i \bigl|\langle\Psi|P_i|\Psi\rangle\bigr|^2 \geq \bigl|\langle\Psi|P_1|\Psi\rangle\bigr|^2 \geq 1-\varepsilon.$$

Thus

$$\mathrm{TD}(\rho,\rho') \overset{(*)}{\leq} \sqrt{1-F(\rho,\rho')^2} \leq \sqrt{1-(1-\varepsilon)} = \sqrt{\varepsilon}.$$

Here $(*)$ uses [NC10, (9.101)]. □

Index

Symbol index

$\langle\Psi\|$	Conjugate transpose of $\|\Psi\rangle$	4
$\|\Psi\rangle$	Vector in a Hilbert space (usually a quantum state)	4
$F(\rho,\rho')$	Fidelity between ρ and ρ'	22
$\mathrm{TD}(\rho,\rho')$	Trace distance between ρ and ρ'	22
$x \leftarrow A$	x is assigned the output of algorithm A	4
$\mathcal{M}$	A measurement	
$\mathbf{M}$	Message space	5
$x \overset{\$}{\leftarrow} S$	x chosen uniformly from set S/according to distribution S	4
$\mathbf{S}^{in}_{c,r,\mathbf{f}}$	Absorbing phase of sponge construction	8
$\mathcal{O}$	Some oracle (e.g., random oracle)	4
x^{right}	Last c bits of x	9
x^{left}	First r bits of x	8
$\operatorname{im} P$	Image of function/map P	3
$\mathbf{f}$	Block function for sponge construction	8
$\mathbf{S}^{out}_{c,r,\mathbf{f},n}$	Squeezing phase of sponge construction	8
$\operatorname{range} f$	Range of function f	3
$\mathbf{S}_{c,r,\mathbf{f},pad,n}$	Sponge construction (capacity c, bit rate r, block function $\mathbf{f}$, output length n)	8
pad	Padding function	8

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