Combining Forward-Security and Leakage-Resilience, Revisited

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Abstract. We revisit the combining of forward and leakage resilience, the study of which was initiated by Bellare *et al.* (CANS 2017). Bellare *et al.* combine forward security with continual leakage resilience, dubbed FS+CL. In particular, they construct a FS+CL public-key encryption (PKE) and signatures, but with various shortcomings in terms of leakage rate and assumptions. Our first result significantly improve on Bellare *et al.*'s FS+CL PKE scheme, building a FS+CL PKE from any continuous leakage-resilient binary-tree encryption scheme (in contrast Bellare *et al.* required extractable witness encryption which is a suspect assumption). Our construction preserves the leakage rate and hence yield FS+CL PKE with optimal leakage rate from standard assumption.

We next explore alternative combinations of forward security and leakage resilience. As argued by Dziembowski *et al.* (CRYPTO 2011), it is desirable to have a model allowing a deterministic key-update procedure, which FS+CL does not. We put forth a combination of forward security with *entropy bounded* leakage (FS+EBL) that allows such key updates. Then we construct FS+EBL non-interactive key exchange (NIKE) based on indistinguishability obfuscation ($i\mathcal{O}$), and DDH or LWE. Additionally, to make the public keys constant size, we rely on the Superfluous Padding Assumption (SuPA) of Brzuska and Mittelbach (Eprint 2015). Crucially, we *do not* use auxiliary information in SuPA. SuPA notwithstanding, our scheme improves on the recent bounded leakage-resilient NIKE of Li *et al.* (CRYPTO 2020) and also the FS NIKE construction of Pointcheval and Sanders (SCN 2014) from generic multilinear maps. Finally, we argue that using *computational entropy* (FS+CEBL) is more compelling in the context of deterministic updates. We pose achieving a FS+CEBL NIKE as an important open problem.

1 Introduction

1.1 Background and Contributions in Brief

Leakage Resilience. When a cryptographic algorithm (or any algorithm) is implemented and run, it must be done on some *physical* system. This introduces *side channel* attacks where the adversary obtains some leakage about

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secrets (*e.g.*, some bits of them), such as execution time, power consumption, and even sound waves [23,31,32]. The cryptographic community has responded by extending the attack model, which had previously only considered black-box access to the algorithms, so that in the extended model the adversary gets some "bounded" leakage about the secrets [2, 9, 30, 37]. Works further extended the model to consider "continual" leakage (CL) attacks [11, 15]. In this model, the life of a secret key is divided into time periods/epochs, and in time period $t+1$ one runs an update algorithm on the secret key of time period t to derive the new secret key for time period $t+1$. (The old secret key is erased.) In each time period the adversary queries for a function with bounded output length applied to the current secret key.

Forward Security. Meanwhile, *forward security* (FS) [5] is a related notion that addresses the threat of exposure of the secret key *in whole*. This can happen due to "too much" leakage in the above sense, but notably also due to separate compromise of the system via break-in or key-exfiltration, for example. Prevalence of the latter was particularly brought to light by the Snowden revelations. Forward security also employs the key update paradigm, such that the secret key is updated every time period. If a break-in happens during time period i, it is required that security (for example, unforgeability of signatures) still holds relative to keys in time periods $1 \leq i' < i$. The initial work on forward security has been extended and optimized, *e.g.* [1, 7, 8, 28, 33, 36], making a signification impact on both theory and practice.

Combining Leakage Resilience and Forward security. As they employ the same attack model with related goals, it makes sense to ask for both leakage resilience with forward security simultaneously. Indeed, one would want security to be maintained if keys are partially exposed, while retaining security as much as possible if keys are fully exposed. Surprisingly, this combination was only considered relatively recently, by Bellare *et al.* [6]. In fact, they propose a security notion that is stronger than asking both notions hold. Namely, they ask for forward security to hold *in the presence of leakage*. Their notion of leakage is continual leakage (CL). They start by examining tree-based constructs as in [5] and give such a construction of FS+CL signatures based on CL-signatures, dismissing the approach for FS+CL encryption. Then they provide a generic approach to construct FS+CL encryption and signature schemes by combining what they call a key-evolution scheme (KE) that is forward one-way under continual-leakage (FOWCL KE) with "witness primitives", namely (extractable) witness encryption [22] and witness signatures [4, 14] respectively. They thus obtain FS+CL PKE from extractable witness encryption and an alternative construction of FS+CL signatures from witness signatures, although they instantiate the construction of FOWCL KE in both cases using tree-based approach.

Main Questions. Regarding Bellare *et al.*'s FS+CL constructs, extractable witness encryption is suspicious [21]. More generally, we would like to improve the assumptions and leakage rates of the schemes. Finally, we would broaden the

set of primitives, and ways of combining forward security and continual leakage, considered. This leads to our main questions:

> Can we improve on Bellare *et al.*'s FS+CL constructions? Are there alternative models combining forward security and leakage resilience that may be fruitful?

Improved FS+CL Encryption Scheme. As mentioned above, Bellare *et al.* [6] construct FS+CL secure encryption scheme from extractable witness encryption (EWE), which is a suspect assumption [20], and their FS+CL PKE scheme also does not enjoy optimal leakage rate. We significantly improve upon their construction by carefully re-examining the tree-based construct as explained below.

In their effort to construct FS+CL PKE, Bellare *et al.* explicitly dismissed the idea of using a CL-secure binary-tree encryption (BTE) [13], because it seems the underlying hierarchical identity-based encryption scheme (HIBE) scheme must tolerate *joint* leakage on multiple keys, whereas CL-security only allows leakage on each such key *individually. We show this intuition is false*, and construct a FS+CL-secure PKE scheme from any CL-secure BTE scheme. This can in turn can be realized from any CL-secure hierarchical identity-based encryption (HIBE) scheme.[1] CL-secure HIBE is known, for example, from simple assumptions on composite-order bilinear groups [10]. We also show that our construction preserves the leakage rate of the base scheme, yielding optimal leakage rate for the appropriate choice of parameters. Hence, we obtain a FS+CL encryption scheme enjoying optimal leakage rate and from standard assumption.

Finally, somewhat interestingly, we observe that in general FS+CL does *not* actually imply CL, because the former relies on time periods (*e.g.*, when encrypting (resp. signing), the encryptor (resp. signer) needs to know the current time period).

Alternative Models: FS+(C)EBL. Dziembowski *et al.* [18] argue it is desirable to have leakage models allowing a deterministic key-update procedure, which FS+CL does not. (Although FS+CL does not imply CL, it still requires randomized update for the same reason as CL.) The reason deterministic updates are desirable is they are both convenient from a design standpoint, and moreover the fundamental reason randomized key update is required by FS+CL and CL is arguably contrived.[2] To address this need, we mesh forward security with *entropic bounded leakage* (EBL) [37] instead of CL. In other words, we look at scaling the notion of entropic bounded leakage to span multiple time periods.

To this end, we introduce a model called FS+EBL (pronounced *ee-bull*). The FS+EBL model is defined with respect to any key evolving scheme equipped with an update function. Our definition requires forward security in the presence of

[1] The only difference between HIBE and BTE is that in the latter we insist on a binary tree

[2] That reason involves the leakage function computing the update function itself, leaking a future key bit-by-bit over time.

leakage such that the current key always meets an entropy bound. In particular, we require that the secret key in each time period prior to the period of exposure retain enough residual entropy conditioned on the leakage from each of the keys. The FS+EBL model is scheme-dependent, in the sense that, the above entropic condition is required to hold with respect to a (deterministic) update function corresponding to the underlying key evolving primitive. We believe that our model is a stepping stone towards a more complete definition of adapting a entropic-bounded leakage to the continual leakage setting. We leave this for future work. We strongly believe that any ensuing model would not render our NIKE scheme insecure but would only require a new security proof for it.

Such a restriction on key entropy seems overly severe, however, and motivates additional consideration of *computational entropy*. The point is that if considering information-theoretic entropy, leakage on the current key necessarily reduces entropy of all other keys. But consider leaking (noisy) hamming weight or physical bits of the current key, or even some one-way function of the key. After an appropriate update function is applied, however, it is plausible that computational entropy of the new key is restored (implicitly, this assumes the leakage function did not compute the update function). To profit in such a case, we introduce the FS+CEBL (pronounced *see-bull*) that parallels FS+EBL but uses computational entropy.

NIKE in FS+EBL Model. Broadening our set of primitives considered, we study non-interactive key exchange (NIKE) in the FS+(C)EBL model. We give a FS+EBL-secure NIKE in the common reference string (CRS) model from indistinguishability obfuscation ($i\mathcal{O}$), either DDH or LWE, and a relaxed variant of the *Superfluous Padding Assumption* (SuPA) on $i\mathcal{O}$ introduced in [12] (more on this in the technical overview below). The scheme has optimal leakage rate and uses a common reference string, which is necessary as per [35].

We remark that, before this work, even *FS*-NIKE was not known from $i\mathcal{O}$. The recent work of Jain *et al.* [29] makes remarkable progress on constructing $i\mathcal{O}$ from standard assumptions, thus making it attractive for feasibility results like ours. Similar to the prior FS-NIKE construction from multilinear maps [38], our construction of FS+EBL NIKE supports an a-priori bounded (but an arbitrary polynomial) number of time periods. However, our construction achieves much better parameters than the construction of [38]. In particular, the size of the public parameter in [38] is $O(T)$, the secret key size is $O(\log T)$, and the public key size is constant (here T denotes the maximum number of time periods supported by the scheme). In contrast, our FS+EBL NIKE achieves constant size secret keys and public parameter, and the size of our public keys are $O(\log T)$. Hence, our construction essentially improves on the result of [38]. We view our construction of FS+EBL NIKE (even just FS NIKE) as a contribution of independent interest, even though it is predicated on a security model that still requires further research and development.

NIKE in FS+CEBL Model. A nice feature of our FS+EBL NIKE construction is that, the key update function can be instantiated by any entropic-leakage

resilient one-way function [9]. In the FS+CEBL setting, we suggest using the PRG of Zhandry [40], because it is secure for any computationally unpredictable seed. The issue is that a leakage from time period i could leak from secret key $i+1$ which is the output of the PRG. Existing results do not explore the case where the output of the PRG is also susceptible to leakage. We leave constructing FS+CEBL NIKE for future work.

1.2 Technical Overview

High-level Idea of the FS+CL PKE Construction. A binary-tree encryption (BTE) has a master public key (MPK) associated with the root node of a binary tree and all the nodes have an associated secret key. Moreover, the secret key of any node can be used to derive the secret keys for the children of that node. To encrypt a message for a particular node, one uses MPK and the identity of that node. The security notion requires the attacker to commit to a target node w^* in advance (i.e., before seeing MPK) and it gets the secret keys of all nodes except for those which lie on the path from the root to the "target" node (including both). In our model, the adversary can also leak continuously from the secrets keys of all these nodes. The goal of the adversary is then to win the indistinguishability game with respect to the target node w^*.

To construct a FS+CL PKE scheme for $T \leq 2^\ell - 1$ time period, we use a continuous leakage-resilient BTE (CLR-BTE) scheme of depth ℓ and associate the time periods with all nodes of the tree according to a *pre-order* traversal. Let w^i denote the node corresponding to time period i. The public key of the FS+CL PKE scheme consists of the root public key MPK and the secret key for time period i consists of sk_{w^i} (the secret key of w^i) and the secret keys of all right siblings of the nodes on the path from the root to w^i. At the end of time period i the secret key is updated as follows: If w^i is an internal node, then the secret keys of node w^{i+1} (the next node according to the pre-order traversal) and its sibling (i.e., the two children of w^i) are derived; otherwise the secret key for w^{i+1} is already stored as part of the secret key. In either case, sk_{w^i} is erased. The secret keys of the all the nodes corresponding to time period $i+1$ are then refreshed by running the key update algorithm of the underlying CLR-BTE scheme.

Proof Strategy. In our proof, the reduction (which is an adversary $\mathcal{A}_{\mathsf{clr\text{-}bte}}$ of the underlying CLR-BTE scheme) simply guesses the time period i^* in which the FS+CL adversary $\mathcal{A}_{\mathsf{kee}}$ will attack. This corresponds to a challenge node w^{i^*} which $\mathcal{A}_{\mathsf{clr\text{-}bte}}$ forwards to its own challenger. If the guess is incorrect, the reduction aborts outputting a random bit. $\mathcal{A}_{\mathsf{clr\text{-}bte}}$ then receives the secret keys of all the nodes that are right siblings of the nodes that lie in the path $P_{w^{i^*}}$ from the root node to w^{i^*} (the target node) and also the secret keys of both the children of w^{i^*}. Using the knowledge of these keys $\mathcal{A}_{\mathsf{clr\text{-}bte}}$ can simulate the update queries of $\mathcal{A}_{\mathsf{kee}}$. Now, let us see how to simulate the leakage queries of $\mathcal{A}_{\mathsf{kee}}$. Note that, the secret key corresponding to some time period i in the FS+CL scheme is of the form $SK_i = (sk_{w^i}, \{sk_{rs(P_{w^i})}\})$, where sk_{w^i} and $\{sk_{rs(P_{w^i})}\}$ denote the (possibly

refreshed versions of the) secret keys corresponding to the node w^i and the right siblings of all the nodes that lie on the path P_{w^i} respectively. Now, either of the following two cases arise: (i) either the node w^i lies in the path $P_{w^{i^*}}$ (the path from root to the target node w^{i^*}) or (ii) w^i does not lie in the path $P_{w^{i^*}}$. In the first case, $\mathcal{A}_{\mathsf{clr\text{-}bte}}$ already knows all the keys $\{sk_{rs(P_{w^i})}\}$ and hence it can translate the leakage function f (queried by $\mathcal{A}_{\mathsf{kee}}$) to a related leakage function f' *only* on the key sk_{w^i} (by hard-wiring the keys $\{sk_{rs(P_{w^i})}\}$ into f'). For the later case, $\mathcal{A}_{\mathsf{clr\text{-}bte}}$ knows the key sk_{w^i} and all the keys $\{sk_{rs(P_{w^i})}\}$, except exactly one key corresponding to a node w that lies in the path $P_{w^{i^*}}$. So, $\mathcal{A}_{\mathsf{clr\text{-}bte}}$ can again translate the joint leakage function f to leakage just on the secret key of node w.

To summarize, the key observation is that: in either case, the reduction knows the secret keys of all nodes except one, and hence it can simulate the joint leakage by leaking only on one node at a time. However, the adversary may also get multiple (continuous) leakages on the secret key of a node. For e.g., consider the secret key sk_1 (corresponding to the right child of the root node). The secret key sk_1 is included in each secret key sk_{0w} for any suffix w. However, note that, when the secret keys from one time period are updated to the next time period they are also refreshed by running the underlying key refresh algorithm of the CLR-BTE scheme. Hence the CLR property of the BTE scheme allows us to tolerate multiple leakages on the same node by making use of its' leakage oracle.

Constructing NIKE in the FS+EBL Model. The starting point of our NIKE construction is the recent bounded leakage-resilient NIKE construction of [35] (henceforth called the *LMQW protocol*) from indistinguishability obfuscation ($i\mathcal{O}$) and other standard assumptions (DDH/LWE) in the CRS model.

The main idea of the LMQW construction is as follows: Each user samples a random string s as its secret key and sets its public key as $x = G(s)$, where G is a function whose description is a part of the CRS and can be indistinguishably created in either *lossy* or in *injective* mode. In the real construction, the function G is set to be injective. To generate a shared key with an user j, user i inputs its own key pair (x_i, s_i) and the public key x_j of user j to an obfuscated program $\widehat{C}$ (which is also included as part of the CRS) which works as follows: The circuit C (which is obfuscated) simply checks if s_i is a valid pre-image of either x_i or x_j under G, i.e, it checks if either $x_i = G(s_i)$ or $x_j = G(s_i)$. If so, it returns $\mathsf{PRF}_K(x_i, x_j)$ (where the PRF key K is embedded inside the obfuscated program); else it outputs $\perp$.

Lifting the LMQW protocol to the FS setting. It is easy to see that the LMQW protocol is *not* forward-secure. This is because, each public key is an injective function of its corresponding secret key, and hence if a secret key s is updated to s', the public key no longer stays the same. We now describe how to modify the above construction to achieve security in the FS+EBL setting. The main idea of our construction is as follows: Similar to the LMQW protocol, the (initial) secret key of each user i in our construction is also a random string $s_i^{(1)}$. The CRS also contains the description of the obfuscated program $\widehat{C}$ and the function G (as

described above). However, the public key of each party i is now an obfuscated circuit $\widehat{C_i}$ (corresponding to a circuit C_i) which has the initial/root secret key $s_i^{(1)}$ (corresponding to base time period 1) of party i embedded in it. It takes as input a key $s_j^{(t)}$ of user j (corresponding to some time period t) and works as follows: (a) First, it updates the secret key $s_i^{(1)}$ of user i (hard-coded in it) to $s_i^{(t)}$ by running the (deterministic) NIKE update function (to be defined shortly) $t-1$ times, (b) computes $x_i^{(t)} = G(s_i^{(t)})$ and $x_j^{(t)} = G(s_j^{(t)})$, and finally (c) internally invokes the obfuscated circuit $\widehat{C}$ (included as part of CRS) on input the tuple $(s_j^{(t)}, x_i^{(t)}, x_j^{(t)})$. To generate the shared key with an user i corresponding to time period t, user j runs $\widehat{C_i}$ with input its secret key $s_j^{(t)}$ corresponding to time period t to obtain the shared key $\mathsf{PRF}_K(x_i^{(t)}, x_j^{(t)})$. It is easy to see that user i also derives the same shared key for time period t by running the program $\widehat{C_j}$ (public key of user j) on input its own $s_i^{(t)}$ corresponding to time period t. The key update function for our FS+EBL NIKE can be any entropic-leakage resilient OWF [9]. This is so that it remains hard to compute the prior key even given entropic leakage on the pre-image of the OWF.

Security Proof. The security proof of our construction follows the proof technique of the LMQW protocol with some major differences as explained below. The main idea of the proof of the LMQW protocol follows the punctured programming paradigm [39], where they puncture the PRF key K at the point (x_i, x_j) and program a random output y. However, instead of hard-coding y directly they hard-core $y \oplus s_i$ and $y \oplus s_j$, i.e, the one-time pad encryption of y under s_i and s_j respectively. This allows the obfuscated program to decrypt y given either s_i or s_j as input. At this point, they switch the function G to be in *lossy* mode and argue that the shared key y retain high min-entropy, even given the obfuscated program with hard-coded ciphertexts, the public keys and leakages on the secret keys s_i and s_j. The entropic key k is then converted into a uniformly random string by using an appropriate extractor.

However, for our construction, we *cannot* argue the last step of the above proof, i.e., the shared key y (for time period t) retains enough entropy given all the public information (CRS and public keys) and entropic leakage on the keys. This is because the public keys $\hat{C}_i$ and $\hat{C}_j$ completely determine the keys $s_i^{(t)}$ and $s_j^{(t)}$ respectively, even after switching the function G to be in lossy mode. This is because the obfuscated programs $\hat{C}_i$ and $\hat{C}_j$ contains the base secret keys $s_i^{(1)}$ and $s_j^{(1)}$ hard-coded in them, and hence, given the public keys the secret keys have no entropy left. To this end, we switch the public key $\hat{C}_i$ to an obfuscation of a functionally equivalent program that, instead of embedding the base secret key $s_i^{(1)}$ embeds all possible public keys $(x_i^{(1)}, \cdots, x_i^{(T)})$ in it, where T is the total number of time period supported by our scheme and $x_i^{(j)} = G(s_i^{(j)})$ for $j \in [T]$. Since, the function G is lossy the shared key y still retains enough entropy, even given the public key. A similar argument can be made for party j.

By setting the parameters appropriately, we can prove FS+EBL security of our NIKE construction with optimal leakage rate.

Compressing the size of the public key using the SuP assumption. Note that, in the above proof step we needed to embed T values and hence the public key of each user (which consists of the above obfuscated circuit) is of size at least T. However, this appears to be a "proof problem", and this can be formally captured via the *Superfluous Padding (SuP) Assumption* [12]. Intuitively the SuP assumption (SuPA) states that if two distributions are indistinguishable relative to an obfuscated circuit C which was padded before obfuscation, then the two distributions are also indistinguishable relative to the obfuscated circuit C without padding. Or in other words, if an obfuscation of a padded circuit hides something, then so does an obfuscation of the unpadded circuit.

Although non-standard, it is shown in [12] that SuPA holds for virtual black-box obfuscation (VBB) as evidence it holds for $i\mathcal{O}$. Unfortunately, as shown in [26], assuming $i\mathcal{O}$ and one-way functions SuPA does not hold for $i\mathcal{O}$ if the distinguisher is given *auxiliary information*. Crucially, we use a *relaxed* variant of SuPA that does not give the distinguisher access to any auxiliary information. This relaxed SuPA is enough to prove the security of our NIKE construction. We stress the impossibility result of [26] does not apply to this relaxed SuPA, and in fact, we conjecture that, in the absence of any auxiliary information SuPA does hold for $i\mathcal{O}$. In this case, the size of the public keys in our NIKE scheme is not linear in T, but is only $O(\log T)$. The latter is because the obfuscated circuit needs to know the maximum number of times it will need to update its keys.

2 Preliminaries

2.1 Notations

Let $x \in \mathcal{X}$ denote an element x in the support of $\mathcal{X}$. For a probability distribution $\mathcal{X}$, let $|\mathcal{X}|$ denote the size of the support of $\mathcal{X}$, i.e., $|\mathcal{X}| = |\{x \,|\, \mathsf{Pr}[\mathcal{X} = x] > 0\}|$. If x is a string , we denote $|x|$ as the length of x. Let $x \leftarrow \mathcal{X}$ be the process of sampling x from the distribution $\mathcal{X}$. For $n \in \mathbb{N}$, we write $[n] = \{1, 2, \cdots, n\}$. When A is an algorithm, we write $y \leftarrow A(x)$ to denote a run of A on input x and output y; if A is randomized, then y is a random variable and $A(x; r)$ denotes a run of A on input x and randomness r. An algorithm A is probabilistic polynomial-time (PPT) if A is randomized and for any input $x, r \in \{0,1\}^*$, the computation of $A(x; r)$ terminates in at most $poly(|x|)$ steps. For a set S, we let U_S denote the uniform distribution over S. For an integer $\alpha \in \mathbb{N}$, let U_α denote the uniform distribution over $\{0,1\}^\alpha$, the bit strings of length α. Throughout this paper, we denote the security parameter by κ, which is implicitly taken as input by all the algorithms. For two random variables X and Y drawn from a finite set $\mathcal{X}$, let $\delta(X, Y) = \frac{1}{2}|\sum_{x \in \mathcal{X}} \mathsf{Pr}(X = x) - \mathsf{Pr}(Y = x)|$ denote the statistical distance between them. Given a circuit D, define the computational distance δ^D between X and Y as $\delta^D(X, Y) = |\mathbb{E}[D(X)] - \mathbb{E}[D(Y)]|$.

2.2 Notions of Entropy

In this section, we recall some the definitions of information-theoretic and computational notions of entropy that are relevant to this work and also state the results related to them.

2.2.1 Unconditional (Information-theoretic) Entropy

Definition 1 (Min-entropy). *The min-entropy of a random variable X, denoted as $\mathrm{H}_\infty(\mathrm{X})$ is defined as $\mathrm{H}_\infty(\mathrm{X}) \stackrel{\text{def}}{=} -\log\big(\max_x \Pr[X = x]\big)$.*

Definition 2 (Conditional Min-entropy [17]). *The average-conditional min-entropy of a random variable X conditioned on a (possibly) correlated variable Z, denoted as $\widetilde{\mathrm{H}}_\infty(X|Z)$ is defined as*

$$\widetilde{\mathrm{H}}_\infty(X|Z) = \text{-}\log\big(\mathbb{E}_{z\leftarrow Z}\,[\max_x \Pr[X = x|Z = z]\big) = \text{-}\log\big(\mathbb{E}_{z\leftarrow Z}\,[2^{-\mathrm{H}_\infty(X|Z=z)}]\big).$$

Lemma 1 (Chain Rule for min-entropy [17]). *For any random variable X, Y and Z, if Y takes on values in $\{0,1\}^\ell$, then*

$$\widetilde{\mathrm{H}}_\infty(X|Y,Z) \geq \widetilde{\mathrm{H}}_\infty(X|Z) - \ell \quad \text{and} \quad \widetilde{\mathrm{H}}_\infty(X|Y) \geq \widetilde{\mathrm{H}}_\infty(X) - \ell.$$

One may also define a more general notion of conditional min-entropy $\widetilde{H}_\infty(X|\mathcal{E})$, where the conditioning happens over an arbitrary experiment $\mathcal{E}$, and not just a "one-time" random variable Y [3].

2.2.2 Computational Entropy a.k.a pseudo-entropy

Computational entropy or pseudo-entropy is quantified with two parameters-*quality* (i.e., how much distinguishable a random variable is from a source with true min-entropy to a size-bounded (poly-time) distinguisher)) and *quantity* (i.e., number of bits of entropy).

Definition 3 (Hill Entropy [24, 27]). *A distribution $\mathcal{X}$ has* **HILL entropy** *at least k, denoted by $\mathrm{H}^{\mathsf{HILL}}_{\epsilon,s}(X) \geq k$, if there exists a distribution $\mathcal{Y}$, where $\mathrm{H}_\infty(\mathcal{Y}) \geq k$, such that $\forall\, \mathcal{D} \in \mathcal{D}_s^{\mathsf{rand},\{0,1\}}$, $\delta^{\mathcal{D}}(\mathcal{X},\mathcal{Y}) \leq \epsilon$.*

Let (X,Y) be a pair of random variables. X has **conditional HILL entropy** *at least k conditioned on Y, denoted $\mathrm{H}^{\mathsf{HILL}}_{\epsilon,s}(X|Y)$, if there exists a joint distribution (Z,Y) such that $\widetilde{\mathrm{H}}_\infty(Z|Y) \geq k$, and $\forall\, \mathcal{D} \in \mathcal{D}_s^{\mathsf{rand},\{0,1\}}$, $\delta^{\mathcal{D}}\big((X,Y),(Z,Y)\big) \leq \epsilon$.*

2.3 Primitives required for our constructions.

In this section, we outline the primitives required for our constructions.

2.3.1 Puncturable Pseudo-Random Functions.

In this section, we define the syntax and security properties of a puncturable pseudorandom function family. We follow the definition given in [25].

A *puncturable* family of PRF $\mathsf{pPRF} : \mathcal{K} \times \mathcal{X} \to \mathcal{Y}$ is given by a triple of polynomial time algorithms $(\mathsf{pPRF.setup}, \mathsf{pPRF.puncture}, \mathsf{pPRF.eval})$ and equipped with an additional (punctured) key space $\mathcal{K}_p$ defined as follows:

- $\mathsf{pPRF.setup}(1^\kappa)$: This is a randomized algorithm that takes the security parameter κ as input and outputs a description of the key space $\mathcal{K}$, the punctured key space $\mathcal{K}_p$ and the PRF pPRF.
- $\mathsf{pPRF.puncture}(K, x)$: This is also a randomized algorithm that takes as input a (master) PRF key $K \in \mathcal{K}$, and an input $x \in \mathcal{X}$, and outputs a key $K_x \in \mathcal{K}_p$, often denoted as the punctured key (punctured at the point x). Without loss of generality, we can think of $\mathsf{pPRF.puncture}$ as a deterministic algorithm also. This is because we can de-randomize the algorithm by generating its random bits using a PRF keyed by a part of the master key K and given the point x as input.
- $\mathsf{pPRF.eval}(K_x, x')$: This is deterministic algorithm that takes as input the punctured key $K_x \in \mathcal{K}_p$, and an input $x' \in \mathcal{X}$. Let $K \in \mathcal{K}$, $x \in \mathcal{X}$ and $K_x \leftarrow \mathsf{pPRF.puncture}(K, x)$. The correctness guarantee stipulates that:

$$\mathsf{pPRF.eval}(K_x, x') = \begin{cases} \mathsf{pPRF}(K, x) & \text{if } x \neq x' \\ \perp & \text{otherwise} \end{cases}$$

Security of Puncturable PRFs: The security of puncturable PRFs is depicted by a game between a challenger and an adversary. The security game consists of the following four stages:

1. **Setup Phase:** The challenger chooses uniformly at random a (master) PRF key $K \in \mathcal{K}$ and a bit $b \in \{0, 1\}$.
2. **Evaluation Query Phase:** In this phase the adversary $\mathcal{A}$ queries a point $x \in \mathcal{X}$. The challenger sends back the evaluation $\mathsf{pPRF}(K, x)$ to $\mathcal{A}$. These queries can be made arbitrarily and adaptively by $\mathcal{A}$ polynomially many times. Let $E \subset \mathcal{X}$ be the set of evaluation queries.
3. **Challenge Phase:** In this phase, the adversary $\mathcal{A}$ chooses a challenge point $x^* \in \mathcal{X}$. The challenger computes $K_{x^*} \leftarrow \mathsf{pPRF.puncture}(K, x^*)$. If bit $b = 0$, $\mathcal{C}$ sends back $(K_{x^*}, \mathsf{pPRF}(K, x^*))$. Else, the challenger samples a uniformly random $y^* \leftarrow \mathcal{Y}$, and sends back (K_{x^*}, y^*) to $\mathcal{A}$.
4. **Guess Phase:** $\mathcal{A}$ outputs a guess b' for the bit b chosen by the challenger.

The advantage of $\mathcal{A}$ in the above game is defined by:

$$\mathsf{Adv}^{\mathsf{pPRF}}_{\mathcal{A}}(\kappa) = \Pr\left[b' = b \;\middle|\; x^* \leftarrow \mathcal{A}^{\mathsf{pPRF.eval}(\mathsf{K},\cdot)}(\kappa, K^*_x) \;\wedge\; x^* \notin E\right].$$

Definition 4. *The punctured PRF* pPRF *is said to be secure if for all PPT adversaries* $\mathcal{A}$ *participating in the above game,* $\mathsf{Adv}^{\mathsf{pPRF}}_{\mathcal{A}}(\kappa)$ *is negligible in* κ.

2.3.2 Indistinguishability Obfuscation.

A uniform PPT machine $i\mathcal{O}$ is called an indistinguishability obfuscator for a circuit class $\{\mathcal{C}_\kappa\}_{\kappa\in\mathbb{N}}$ if it satisfies the following conditions:

- (*Functionality Preserving*). For all security parameters $\kappa \in \mathbb{N}$, for all $C \in \mathcal{C}_\kappa$, for all inputs x, we have that:

$$\Pr\big[\, C'(x) = C(x) : C' \leftarrow i\mathcal{O}(\kappa, C) \,\big] = 1$$

- (*Indistinguishability of Obfuscation*). For any (not necessarily uniform) PPT adversaries $Samp$, $\mathcal{D}$, there exists a negligible function $\mathsf{negl}(\cdot)$ such that the following holds: if for all security parameters $\kappa \in \mathbb{N}$, $\Pr\big[\, \forall x, C_0(x) = C_1(x) : (C_0, C_1, \mathsf{st}) \leftarrow Samp(\kappa) \,\big] > 1 - \mathsf{negl}(\kappa)$, then we have:

$$\Big| \Pr\big[\, \mathcal{D}(\mathsf{st}, i\mathcal{O}(\kappa, C_0)) = 1 : (C_0, C_1, \mathsf{st}) \leftarrow Samp(\kappa) \,\big] - \Pr\big[\, \mathcal{D}(\mathsf{st}, i\mathcal{O}(\kappa, C_1)) = 1 : (C_0, C_1, \mathsf{st}) \leftarrow Samp(\kappa) \,\big] \Big| \leq \mathsf{negl}(\kappa).$$

  where the probability is over the coins of $\mathcal{D}$ and $i\mathcal{O}$.

We remark that the algorithms $Samp$ and $\mathcal{D}$ pass state st, which can equivalently be viewed as a single stateful algorithm $\mathcal{B} = (Samp, \mathcal{D})$.

2.3.3 Lossy Functions.

A family of (n, k, m)-lossy functions is given by a triple of algorithms $\mathsf{LF} = (\mathsf{Lossy}, \mathsf{Inj}, \mathsf{Eval})$ with the following syntax:

- $\mathsf{Inj}(1^\kappa)$ is a randomized algorithm which outputs an injective function evaluation key ek.
- $\mathsf{Lossy}(1^\kappa)$ is a randomized algorithm which outputs a lossy function evaluation key ek.
- $\mathsf{Eval}_{\mathsf{ek}}(x)$ is a deterministic algorithm, parametrized by an evaluation key ek which takes as input $x \in \{0,1\}^n$ and outputs some $y \in \{0,1\}^m$.

Security of Lossy Functions: We require a lossy function to satisfy the following properties:

- The function $\mathsf{Eval}_{\mathsf{ek}}(\cdot)$ where $\mathsf{ek} \leftarrow \mathsf{Inj}(1^\kappa)$ is injective with probability $1 - \mathsf{negl}(\kappa)$.
- The function $\mathsf{Eval}_{\mathsf{ek}}(\cdot)$ where $\mathsf{ek} \leftarrow \mathsf{Lossy}(1^\kappa)$ has image size at most 2^{n-k} with probability $1 - \mathsf{negl}(\kappa)$. In particular, we have that x has at least k bits of min-entropy given $\mathsf{Eval}_{\mathsf{ek}}(x)$.
- The evaluations keys $\mathsf{ek}_{inj} \leftarrow \mathsf{Inj}(1^\kappa)$ and $\mathsf{ek}_{lossy} \leftarrow \mathsf{Lossy}(1^\kappa)$ are computationally indistinguishable.

2.3.4 Entropic Leakage-resilient OWF.

In this section, we recall the definition of leakage-resilient one-way functions (LR-OWF) from [9]. Informally, a one-way function (OWF) $g : \{0,1\}^n \to \{0,1\}^m$ is leakage-resilient if it remains one-way, even in the presence of some leakage about pre-image. In entropy-bounded leakage model, instead of bounding the length of the output of leakage functions (as in bounded leakage model), we bound the entropy loss that happens due to seeing the output of the leakage functions. We follow the definition of [15] to consider the entropy loss over the uniform distribution as a measure of leakiness. We follow this definition since it has nice composability properties as stated below.

Definition 5. [15]. A (probabilistic) function $h : \{0,1\}^* \to \{0,1\}^*$ is ℓ-leaky, if for all $n \in \mathbb{N}$, we have $\widetilde{H}_\infty(U_n|h(U_n)) \geq n - \ell$, where U_n denote the uniform distribution over $\{0,1\}^n$.

As observed in [15], if a function is ℓ-leaky, i.e, it decreases the entropy of uniform distribution by at most ℓ bits, then it decreases the entropy of every distribution by at most ℓ bits. Moreover, this definition composes nicely in the sense that, if the adversary adaptively chooses different ℓ_i-leaky functions, it learns only $\sum_i \ell_i$ bits of information. We now define the security model for weak PRFs in this entropy-bounded leakage model.

Definition 6. (Entropic leakage-resilient one-wayness). *Let $\mathcal{A}$ be an adversary against $g : \{0,1\}^n \to \{0,1\}^m$. We define the advantage of the adversary $\mathcal{A}$ as* $\mathsf{Adv}^{\mathsf{LR\text{-}OWF}}_{\mathcal{A}}(\kappa) = \mathsf{Pr}[g(x) = y \,|\, x^* \xleftarrow{\$} \{0,1\}^n, y^* = g(x^*); x \leftarrow \mathcal{A}^{\mathcal{O}_{\mathsf{Leak}}(\cdot)}(y^*)]$. Here $\mathcal{O}_{\mathsf{Leak}}$ is an oracle that on input $h : \{0,1\}^n \to \{0,1\}^*$ returns $f(x^*)$, subject to the restriction that h is λ-entropy leaky. We say that g is λ-entropic leakage-resilient one-way function (λ-ELR-OWF) if not any PPT adversary $\mathcal{A}$ its advantage defined as above is negligible in κ.

As shown in [16], a second-preimage resistant (SPR) function with $n(\kappa)$ bits input and $m(\kappa)$ bits output is also a $\lambda(\kappa)$-entropy leaky OWF for $\lambda(\kappa) = n(\kappa) - m(\kappa) - \omega(\log \kappa)$.

2.4 The Superfluous Padding Assumption.

Following [12], we present the Superfluous Padding Assumption (SuPA). Intuitively SuPA states that if two distributions are indistinguishable relative to an obfuscated circuit C which was padded before obfuscation, then the two distributions are also indistinguishable relative to the obfuscated circuit C without padding. Or in other words, if an obfuscation of a padded circuit hides something, then so does an obfuscation of the unpadded circuit. Unfortunately, as shown in [26] assuming $i\mathcal{O}$ and one-way functions SuPA does not hold for $i\mathcal{O}$ if the distinguisher is given *arbitrary auxiliary* information. We present a relaxed version of the SuP assumption where the distinguisher is *not* given access to any

Game $\mathsf{SuP}[s]^{\mathcal{D}}_{\mathsf{Obf},\mathsf{Samp}_0,\mathsf{Samp}_1}$

$b \leftarrow \{0,1\}$
$C \leftarrow \mathsf{Samp}_b(1^\kappa)$
If $s(\kappa) > 0$, then return $\widehat{C} \leftarrow \mathsf{Obf}(\mathsf{PAD}(s(\kappa), C))$
Else, return $\widehat{C} \leftarrow \mathsf{Obf}(C)$)
$b' \leftarrow \mathcal{D}(1^\kappa, s, \widehat{C}, |C|)$
Return $(b' = b)$

Fig. 1. The SuP game parameterized by a polynomial $s(\kappa)$. According to s, the circuit C is padded (if $s = 0$ the original circuit C is used) before it is obfuscated and given to distinguisher $\mathcal{D}$, who additionally gets s as well as the size of the original circuit C.

auxiliary input and we observe that this relaxed variant of SuPA is enough to prove the security of our NIKE construction.

Following [12], we state the assumption in two steps: First, we define admissible sampler and then define the SuP assumption with respect to such an admissible sampler.

Definition 7 ((Relaxed) SuP-admissible Samplers). *Let* Obf *be an obfuscation scheme and let* $\mathsf{PAD} : \mathbb{N} \times \{0,1\}^* \to \{0,1\}^*$ *be a deterministic padding algorithm that takes as input an integer s and and a description of a circuit C and outputs a functionally equivalent circuit of size $s + |C|$. We say that a pair of PPT samplers* $(\mathsf{Samp}_0, \mathsf{Samp}_1)$ *is* SuP*-admissible for obfuscator* Obf *, if there exists a polynomial s such that for any PPT distinguisher D its advantage in the* $\mathsf{SuP}[s]$ *game (see* Figure 1*) is negligible:*

$$\mathsf{Adv}^{\mathsf{SuP}[s]}_{\mathsf{Obf},\mathsf{Samp}_0,\mathsf{Samp}_1,\mathcal{D}}(\kappa) = 2 \cdot \mathsf{Pr}\left[\mathsf{SuP}[s]^{\mathcal{D}}_{\mathsf{Obf},\mathsf{Samp}_0,\mathsf{Samp}_1}(\kappa)\right] - 1 \leq \mathsf{negl}(\kappa).$$

Definition 8 (The (Relaxed) SuP assumption). *Let* Obf *be an obfuscation scheme and let* Samp_0 *and* Samp_1 *be two* SuP-admissible *samplers. Then, the relaxed* Superfluous Padding Assumption *states that no efficient distinguisher $\mathcal{D}$ has a non-negligible advantage in the* $\mathsf{SuP}[0]$ *game without padding:*

$$\mathsf{Adv}^{\mathsf{SuP}[0]}_{\mathsf{Obf},\mathsf{Samp}_0,\mathsf{Samp}_1,\mathcal{D}}(\kappa) = 2 \cdot \mathsf{Pr}\left[\mathsf{SuP}[0]^{\mathcal{D}}_{\mathsf{Obf},\mathsf{Samp}_0,\mathsf{Samp}_1}(\kappa)\right] - 1 \leq \mathsf{negl}(\kappa).$$

3 Our Results in the FS+CL Model

3.1 Modeling Encryption Scheme in the FS+CL Model.

In this section, following [6] we recall the syntax and security definition of encryption schemes in the FS+CL Model.

Game $\text{FINDCL}^{\mathcal{A}}_{\mathsf{KEE}}(\kappa)$	$\underline{\mathsf{Up}()}$
$b \leftarrow\!\!\$ \{0,1\}$; $t \leftarrow 1$; $t^* \leftarrow \perp$	If $t < \mathsf{KEE.T}(\kappa)$ then
$(pk, sk_1) \leftarrow\!\!\$ \mathsf{KEE.Kg}(1^\kappa)$	$sk_{t+1} \leftarrow\!\!\$ \mathsf{KEE.Upd}(1^\kappa, pk, t, sk_t)$
$(i, m_0, m_1, state) \leftarrow\!\!\$ \mathcal{A}_1^{\mathsf{Up},\mathsf{Leak},\mathsf{Exp}}(1^\kappa, pk)$	$t \leftarrow t + 1$
If not $(1 \le i < t^*)$ then return false	Else return $\perp$
If $\lvert m_0 \rvert \neq \lvert m_1 \rvert$ then return false	$\underline{\mathsf{Leak}(L)}$
$(i, c) \leftarrow\!\!\$ \mathsf{KEE.Enc}(1^\kappa, pk, i, m_b)$	Return $L(sk)$
$b' \leftarrow\!\!\$ \mathcal{A}_2(1^\kappa, state, (i, c))$	$\underline{\mathsf{Exp}()}$
Return $(b' = b)$	$t^* \leftarrow t$; Return sk_t

Fig. 2. Game defining forward indistinguishability of key-evolving encryption scheme KEE under continual leakage.

Modeling Encryption in the FS+CL Model. A key-evolving encryption scheme $\mathsf{KEE} = (\mathsf{KEE.Kg}, \mathsf{KEE.Upd}, \mathsf{KEE.Enc}, \mathsf{KEE.Dec})$, where KEE.Dec is deterministic. Here, $\mathsf{KEE.Kg}(1^\kappa)$ is used to generate the initial key pair (sk_1, pk). The key update algorithm $\mathsf{KEE.Upd}(1^\kappa, pk, i, sk_i)$ is used to evolve/update the key from time period i to $i+1$, outputting sk_{i+1} in the process. KEE.Enc is used to encrypt a message m in time period i using the public key pk. KEE.Dec is used to decrypt a ciphertext c, produced in time i, with the help of secret key sk_i. The security game is presented in Figure 2. In this game, an attacker is given access to three oracles : Up (which it uses to update the key), Leak (which it uses to leak on the key with its choice of leakage function L, and a one-time access to Exp which gives the entire secret key sk_{t^*}. One additional constraint is that the attacker $\mathcal{A}$ is δ-bounded, i.e., that $\mathcal{A}$ makes at most $\delta(\kappa)$ length bounded leakage queries. The attacker provides challenge messages i, m_0, m_1 and a time period i. It receives an encryption of m_b for a randomly chosen bit b. $\mathcal{A}$ wins the game if it correctly guesses the bit b and if $i < t^*$.

3.2 Encryption scheme in the FS+CL model.

In this section we provide the details of our FS+CL encryption scheme. To this end, we first abstract out a notion of continuous leakage-resilient binary tree encryption (CLR-BTE) and use it to construct our FS+CL encryption scheme achieving optimal leakage rate, i.e., $1 - o(1)$.

3.2.1 Continuous Leakage-resilient Binary Tree Encryption.

We now introduce our notion of binary tree encryption in the continuous leakage model. Our security model of CLR-BTE scheme generalizes the definition of binary tree encryption (BTE) (proposed by Canetti et al. [13]) in the setting of continuous leakage. A BTE can be seen as a restricted version of HIBE, where

the identity tree is represented as a *binary* tree.[3] In particular, as in HIBE, a BTE is also associated with a "master" public key MPK corresponding to a tree, and each node in the tree has their respective secret keys. To encrypt a message for a node, one specifies the identity of the node and the public key MPK. The resulting ciphertext can be decrypted using the secret key of the target node.

Definition 9. *(Continuous leakage-resilient BTE).* A continuous leakage-resilient binary tree encryption scheme (CLR-BTE) consists of a tuple of the PPT algorithms $(\mathsf{Gen}, \mathsf{Der}, \mathsf{Upd}, \mathsf{Enc}, \mathsf{Dec})$ such that:

1. $\mathsf{Gen}(1^\kappa, \ell)$: The key generation algorithm Gen takes as input the security parameter κ and a value ℓ for the depth of the tree. It returns a master public key MPK and an initial (root) secret key SK_ε.
2. $\mathsf{Der}(MPK, w, Sk_w)$: The key derivation algorithm Der takes as input MPK, the identity of a node $w \in \{0,1\}^{\leq \ell}$, and its secret key SK_w. It returns secret keys SK_{w_0}, SK_{w_1} for the two children of w.
3. $\mathsf{Upd}(w, Sk_w)$: The key update algorithm Upd takes as input the secret key SK_w of a node w and outputs a re-randomized key SK'_w for the same node w, such that $|SK'_w| = |SK_w|$.
4. $\mathsf{Enc}(MPK, w, M)$: The encryption algorithm Enc takes as input MPK, the identity of a node $w \in \{0,1\}^{\leq \ell}$ and a message M to return a ciphertext C.
5. $\mathsf{Dec}(MPK, w, Sk_w, C)$: The decryption algorithm Dec takes as input MPK, the identity of a node $w \in \{0,1\}^{\leq \ell}$, its secret key SK_w, and a ciphertext C. It returns a message M or $\perp$ (to denote decryption failure).

Correctness: For all (MPK, SK_ε) output by Gen, any node $w \in \{0,1\}^{\leq \ell}$, any secret key SK_w correctly generated for this node (which can be output of Upd also), and any message M, we have

$$\mathsf{Dec}(MPK, w, SK_w, \mathsf{Enc}(MPK, w, M)) = M.$$

We now present our security model for CLR-BTE. Our model generalizes the notion of *selection-node chosen-plaintext attacks* (SN-CPA) put forward by Canetti et al. [13] to define the security of BTE. In our model, the adversary first specifies the identity of the target node[4] $w^* \in \{0,1\}^{\leq \ell}$. The adversary receives the public key MPK and the secret keys of all the nodes that do not trivially allow him/her to derive the secret key of w^*[5]. Besides, the adversary is also allowed to continuously leak from the secret keys of all the nodes that lie on the path from the root node and w^* (including both). The goal of the adversary is then to win the indistinguishability game with respect to the target node w^*.

[3] Recall that, in HIBE the tree can have arbitrary degree.

[4] Note that, this model where the adversary specifies the target node w^* ahead of time is weaker than the model where the adversary may choose the target *adaptively* (analogous to the adaptive security of HIBE schemes). However, as we will show, this model already suffices to construct of a FS+CL encryption scheme.

[5] In particular, the adversary receives the secret keys of all the nodes that are siblings of all the nodes that are on the path from the root node to the target node w^*.

Definition 10. A CLR-BTE scheme is secure against *continuous leakage selective-node, chosen-plaintext attacks* ($\lambda(\kappa)$-CLR-SN-CPA) if for all polynomially-bounded functions $\ell(\cdot)$, and leakage bound $\lambda(\kappa)$, the advantage of any PPT adversary $\mathcal{A}$ in the following game is negligible in the security parameter κ:

1. The adversary $\mathcal{A}(1^\kappa, \ell)$ outputs the name of a node $w^* \in \{0,1\}^{\leq \ell}$. We will denote the path from the root node to the target node w^* by P_{w^*}.
2. The challenger runs the algorithm $\mathsf{Gen}(1^\kappa, \ell)$ and outputs (MPK, SK_ε). In addition, it runs $\mathsf{Der}(\cdot,\cdot,\cdot)$ to generate the secret keys of all the nodes on the path P_{w^*}, and also the secret keys for the two children w_0^* and w_1^*. The adversary is given MPK and the secret keys $\{SK_w\}$ for all nodes w of the following form:
   - $w = w'\bar{b}$, where $w'b$ is a prefix of w^* and $b \in \{0,1\}$ (i.e., w is a sibling of some node in P_{w^*}).
   - $w = w_0^*$ or $w = w_1^*$ (i.e., w is a child of w^*; this is only when $|w^*| < \ell$).

   The challenger also creates a set $\mathcal{T}$ that holds tuples of all the (node) identities, secret keys and the number of leaked bits from each key so far.
3. The adversary $\mathcal{A}_{\mathsf{clr\text{-}bte}}$ may also ask leakage queries. The adversary runs for arbitrarily many leakage rounds. In each round:
   - The adversary provides the description of a probabilistic leakage function $h : \{0,1\}^* \to \{0,1\}^{\lambda(\kappa)}$, and an identity of a node w in the path P_{w^*} (that may also include both the root note and the target node w^*).[6] The challenger scans $\mathcal{T}$ to find the tuple with identity w. It should be of the form (w, SK_w, L_w). The challenger then checks if $L_w + |h(SK_w)| \leq \lambda(\kappa)$. If this is true, it responds with $h(SK_w)$ and updates $L_w = L_w + |h(SK_w)|$. If the check fails, it returns $\perp$ to the adversary.
   - At the end of each round, the challenger computes $SK'_w \leftarrow \mathsf{Upd}(w, SK_w)$ and updates $SK_w = SK'_w$.
4. The adversary $\mathcal{A}$ then sends two messages M_0 and M_1 to the challenger such that $|M_0| = |M_1|$. The challenger samples a random bit $b \xleftarrow{\$} \{0,1\}$, and computes $C^* \leftarrow \mathsf{Enc}(MPK, w^*, M_b)$. It then returns C^* to the adversary $\mathcal{A}$. The adversary is not allowed to ask any further leakage queries after receiving the challenge ciphertext C^*.[7]

At the end of this game, the adversary outputs a bit $b' \in \{0,1\}$; it succeeds if $b' = b$. The advantage of the adversary is the absolute value of the difference between its success probability and $1/2$.

[6] This is equivalent to a definition where, in each round, the adversary asks for multiple leakage functions adaptively, such that the output length of all these functions sum up to $\lambda(\kappa)$.

[7] If the adversary is allowed to ask leakage queries after receiving the challenge ciphertext, it can encode the entire decryption algorithm of C^* as a function on a secret key, and thus win the game trivially.

3.2.2 Construction of CLR-BTE scheme.

Our construction of the CLR-BTE scheme can be instantiated in a straightforward manner from the continuous leakage-resilient HIBE (CLR-HIBE) construction of Lewko et al. [34], tuned to the setting of a binary tree. The resulting CLR-BTE is *adaptively secure*, since the CLR-HIBE of [34] enjoys security against adaptive adversaries employing the dual-system encryption technique. The security of the CLR-BTE scheme can be proven under static assumptions over composite-order bilinear groups. We refer the reader to [34] for the details of the CLR-HIBE construction and its proof. As shown in [34], for appropriate choice of parameters, their CLR-HIBE scheme achieves the optimal leakage rate of $1 - o(1)$.

3.2.3 FINDCL encryption from CLR-BTE scheme.

We now show a generic construction of a FINDCL-secure encryption scheme starting from any CLR-BTE scheme. The main idea of our construction is very simple: use the Canetti-Halevi-Katz (CHK) transform [13] to the underlying CLR-BTE scheme to construct a FINDCL encryption scheme. In particular, we show the applicability of the CHK transform[8] even in the setting of continuous leakage. However, as we show later, the analysis of the CHK transform in the setting of leakage turns out to be quite tricky.

Let $(\mathsf{Gen}, \mathsf{Der}, \mathsf{Upd}, \mathsf{Enc}, \mathsf{Dec})$ be a CLR-BTE scheme. We construct our FINDCL PKE scheme $(\mathsf{KEE.Kg}, \mathsf{KEE.Upd}, \mathsf{KEE.Enc}, \mathsf{KEE.Dec})$ as shown below. The construction is identical to the CHK transform, with the underlying building blocks appropriately changed.

Some additional notation: To obtain a FINDCL-secure encryption scheme with $T = 2^\ell - 1$, time periods (labeled through 1 to T), we use a CLR-BTE of depth ℓ. We associate the time periods with all nodes of the tree according to a pre-order traversal. The node associated with time period i is denoted by w^i. In a pre-order traversal, $w^1 = \varepsilon$ (the root node), if w^i is an internal node then $w^{i+1} = w^i 0$ (i.e., left child of w^i). If w^i is a leaf node and $i < T - 1$ then $w^{i+1} = w'1$, where w' is the longest string such that $w'0$ is a prefix of w^i.

1. $\mathsf{KEE.Kg}(1^\kappa, T)$: Run $\mathsf{Gen}(1^\kappa, \ell)$, where $T \leq 2^\ell - 1$, and obtain (MPK, SK_ε). Set $pk = (MPK, T)$, and $sk_1 = SK_\varepsilon$.
2. $\mathsf{KEE.Upd}(1^\kappa, pk, i, sk_i)$: The secret key sk_i is organized as a stack of node keys, with the secret key SK'_{w^i} on top, where SK'_{w^i} is obtained by running Upd of the CLR-BTE scheme (potentially multiple times) on the key SK_{w^i}. We first pop this key off the stack. If w^i is a leaf node, the next node key on top of the stack is $SK'_{w^{i+1}}$ (a refreshed version of the key $SK_{w^{i+1}}$). If w^i is an internal node, compute $(SK_{w^i 0}, SK_{w^i 1}) \leftarrow \mathsf{Der}(pk, w^i, SK_{w^i})$

[8] The original CHK transform [13] is used to construct a forward-secure PKE scheme starting from a BTE scheme.

Then for $b \in \{0,1\}$, compute $SK'_{w^i b} \leftarrow \mathsf{Upd}(w, SK_{w^i b})$. Further, for all other node keys SK_w remaining in the stack (corresponding to sk_{i+1}), run $SK'_w \leftarrow \mathsf{Upd}(w, SK_w)$. Then push $SK'_{w^i 1}$ and then $SK'_{w^i 0}$ onto the stack. In either case, the node key SK'_{w^i} is erased.

3. $\mathsf{KEE.Enc}(pk, i, m)$: Run $\mathsf{Enc}(pk, w^i, m)$. Note that w^i is publicly computable given i and T.
4. $\mathsf{KEE.Dec}(1^\kappa, pk_i, sk_i, c_i)$: Run $\mathsf{Dec}(pk, w, SK'_{w^i}, c_i)$. Note that, SK'_{w^i} is stored as part of sk_i.

Theorem 1. *Let* $\lambda : \mathbb{N} \to [0,1]$. *Let* $\Pi = (\mathsf{Gen}, \mathsf{Der}, \mathsf{Upd}, \mathsf{Enc}, \mathsf{Dec})$ *be a* $\lambda(\kappa)$-CLR-SN-CPA *continuous leakage-resilient binary-tree encryption* (CLR-BTE) *scheme. Let* $\ell : \mathbb{N} \to \mathbb{N}$ *be a polynomial such that* $T \leq 2^\ell - 1$. *Then* $\Pi' = (\mathsf{KEE.Kg}, \mathsf{KEE.Upd}, \mathsf{KEE.Enc}, \mathsf{KEE.Dec})$ *is a* $\lambda(\kappa)$-FINDCL *secure encryption scheme supporting up to* T *time periods.*

Proof. Our proof follows the template of the CHK transformation for converting a BTE scheme to forward-secure encryption scheme, with the crucial difference in simulating the leakage queries. Assume that we have an adversary $\mathcal{A}_{\mathsf{kee}}$ with advantage $\epsilon(\kappa)$ in an $\lambda(\kappa)$-FINDCL security game of $\Pi' = (\mathsf{KEE.Kg}, \mathsf{KEE.Upd}, \mathsf{KEE.Enc}, \mathsf{KEE.Dec})$. We construct an adversary $\mathcal{A}_{\mathsf{clr\text{-}bte}}$ that obtains an advantage $\epsilon(\kappa)/T$ in the corresponding attack against the underlying the CLR-BTE scheme $\Pi = (\mathsf{Gen}, \mathsf{Der}, \mathsf{Upd}, \mathsf{Enc}, \mathsf{Dec})$. The leakage rate tolerated by Π is exactly the same as Π'. We now describe how $\mathcal{A}_{\mathsf{clr\text{-}bte}}$ simulates the environment for $\mathcal{A}_{\mathsf{kee}}$:

1. $\mathcal{A}_{\mathsf{clr\text{-}bte}}$ chooses uniformly at random a time period $i^* \in [T]$. This define the node w^{i^*} (the identity of the node corresponding to i^*). $\mathcal{A}_{\mathsf{clr\text{-}bte}}$ then forwards w^{i^*} to its challenger and obtains MPK and $\{SK_w\}$ for all the appropriate nodes w[9] from its challenger. $\mathcal{A}_{\mathsf{clr\text{-}bte}}$ then sets $pk = (MPK, T)$, and forwards the public key pk to the adversary $\mathcal{A}_{\mathsf{kee}}$.
2. When $\mathcal{A}_{\mathsf{kee}}$ decides to break into the system, it provides the time period, say j. If $j \leq i^*$, then $\mathcal{A}_{\mathsf{clr\text{-}bte}}$ outputs a random bit and halts. Otherwise, $\mathcal{A}_{\mathsf{clr\text{-}bte}}$ computes the appropriate secret key sk_j and gives it to $\mathcal{A}_{\mathsf{kee}}$. Note that, $\mathcal{A}_{\mathsf{clr\text{-}bte}}$ can efficiently compute the secret keys sk_j for any $j > i^*$ from the knowledge of $\{SK_w\}$ (the set of secret keys received in Step 1).
3. $\mathcal{A}_{\mathsf{kee}}$ may ask leakage queries on the secret key corresponding to any time period, say i. The node associated with time period i is w^i. The secret key sk_i can be seen as a stack of node keys (derived using the underlying CLR-BTE scheme) with the key SK'_{w^i} on top of the stack. The other node keys in the stack are secret keys corresponding to the right siblings of all the nodes in the path P_{w^i} from the root node to w^i. Let us denote the secret key as $sk_i = (SK'_{w^i}, \{SK\}'_{rs(P_{w^i})})$, where $\{SK\}'_{rs(P_{w^i})}$ denote the (refreshed) secret keys of the right siblings of all nodes in path from the root to the node w^i (we denote this path by P_{w^i}). Now, either one of the two cases must be true:

[9] Recall that $\mathcal{A}_{\mathsf{clr\text{-}bte}}$ receives the secret keys of all the nodes that are right siblings of the nodes that lie on the path P from the root node to w^{i^*}.

(1) $w^i \in P_{w^{i^*}}$ or (2) $w^i \notin P_{w^{i^*}}$, where $P_{w^{i^*}}$ is the path containing the nodes from the root node to the target node w^{i^*} (including both).

For the first case, $\mathcal{A}_{\mathsf{clr\text{-}bte}}$ already knows all the keys $\{SK\}_{rs(P_{w^i})}$) and it does the following:

- Receive as input the leakage function f from $\mathcal{A}_{\mathsf{kee}}$. Next, it calls Upd on all the node keys $\{SK\}_{rs(P_{w^i})}$ and receive the refreshed keys $\{SK\}'_{rs(P_{w^i})}$. It then modifies the description of the function as $h = f_{\{SK\}'_{rs(P_{w^i})}}(\cdot) = f\big(\cdot, \{SK\}'_{rs(P_{w^i})}\big)$. In other words, $\mathcal{A}_{\mathsf{clr\text{-}bte}}$ hardwires the secret keys $\{SK\}'_{rs(P_{w^i})}$ in the function f, and forwards h as the leakage function to its challenger.
- On input the answer $h(SK'_{w^i}, \{SK\}'_{rs(P_{w^i})})$ from its challenger, $\mathcal{A}_{\mathsf{clr\text{-}bte}}$ forwards this answer as the output of the leakage function f to $\mathcal{A}_{\mathsf{kee}}$.

For the second case (i.e., when $w^i \notin P_{w^{i^*}}$), there exists at most one node $w \in P_{w^{i^*}}$ whose secret key is included $\{SK\}'_{rs(P_{w^i})}$. Apart from the secret key of w, $\mathcal{A}_{\mathsf{clr\text{-}bte}}$ knows the secret key of w^i, SK_{w^i}, and the keys $\{SK\}_{rs(P_{w^i})}$). Thus, similar to above it can transform the joint leakage function f to a leakage function h on SK'_w. It then returns the result to $\mathcal{A}_{\mathsf{kee}}$.

It is clear that in both cases, $\mathcal{A}_{\mathsf{clr\text{-}bte}}$ perfectly simulates the answers to the leakage queries of the adversary $\mathcal{A}_{\mathsf{kee}}$.

4. When $\mathcal{A}_{\mathsf{kee}}$ asks an update query $\mathsf{KEE.Upd}(i)$, $\mathcal{A}_{\mathsf{blr\text{-}bte}}$ can easily compute the key for the next time period using the knowledge of the keys $\{SK_w\}$ received from its challenger in the beginning.
5. When $\mathcal{A}_{\mathsf{kee}}$ asks a challenge query with input (i, m_0, m_1), if $i \neq i^*$ then $\mathcal{A}_{\mathsf{clr\text{-}bte}}$ outputs a random bit and halts. Otherwise, it forwards the tuple (m_0, m_1) to its challenger and obtains the challenge ciphertext C^*. It then gives C^* to $\mathcal{A}_{\mathsf{kee}}$.
6. When $\mathcal{A}_{\mathsf{kee}}$ outputs b', $\mathcal{A}$ outputs b' and halts.

It is easy to see that, if $i = i^*$, the above simulation by $\mathcal{A}_{\mathsf{clr\text{-}bte}}$ is perfect. Since, $\mathcal{A}_{\mathsf{clr\text{-}bte}}$ guesses i^* with probability $1/T$, we have that $\mathcal{A}_{\mathsf{clr\text{-}bte}}$ correctly predicts the bit b with advantage $\epsilon(\kappa)/T$. □

4 Our Results in the FS+(C)EBL Model

In this section we present the FS+(C)EBL model and present a construction of NIKE in the FS+EBL model.

4.1 The FS+EBL Model

The Entropy Bounded Leakage (EBL) model was designed to capture security against adversary who leaked on the secret key. However, to make the attack non-trivial, it defines the legitimacy of the adversary. An adversary is legitimate if the secret key sk still contains enough min-entropy, parametrized by α, even after

the leakage. This is a generalization of length-bounded leakage model where a leakage function can leak at most, say, δ bits. Implicitly, the EBL model is defined in the setting of a single time period (as there is only the one secret key). The notion of length bounded leakage model was extended to the setting of multiple time periods and this was called continual leakage model. In this setting, the secret key is updated (using a randomized update function) across time periods and an adversary can leak at most δ bits in every time period. In this section, we consider *deterministic* key update functions and take the idea of entropic bounded leakage model and extend it to the setting of multiple time periods. We consider the combined problem of FS+EBL, i.e., schemes that are forward secure and which are resilient to entropic bounded leakage. Specifically, we consider the Forward Secure + Entropic Bounded Leakage Model, abbreviated as FS+EBL. It is parametrized by T and α where T is the number of time periods and α is the minimum residual entropy required. As before, one can define the legitimacy of the attacker in this model.

Definition 11 (Definition of Legitimacy - FS+EBL Model). *Let Π be any key-evolving scheme with a* deterministic *key update algorithm. Let SK_i denote the random variable produced by the key update algorithm for time period i. Then, any PPT adversary $\mathcal{A}$ making leakage queries denoted by $\mathbb{L}_i(SK_i)$ for $i = 1, \ldots, T$, is legitimate in the (T, α)-FS+EBL model if:*

$$\forall j \in [t^*], \mathrm{H}_\infty\big(SK_j \,|\, \mathbb{L}_1, \cdots, \mathbb{L}_T, R_{L_1}, \cdots, R_{L_T}\big) \geq \alpha \tag{1}$$

where R_{L_i} denote the random coins of the adversary corresponding to the leakage function L_i, $t^* \leq T$ is the time period at which $\mathcal{A}$ is given sk_{t^*} in full.

Remark 1. In the above definition, one can also have H to be computational notion of entropy such as HILL entropy or unpredictability entropy [24, 27]. In this setting the definition of legitimacy can be revised to:

Definition 12 (Definition of Legitimacy - FS+CEBL Model). *Let Π be any key-evolving scheme with a* deterministic *key update algorithm. Let SK_i denote the random variable produced by the key update algorithm for time period i. Then, any PPT adversary $\mathcal{A}$ making leakage queries denoted by $\mathbb{L}_i(SK_i)$ for $i = 1, \ldots, T$, is legitimate in the (T, α)-FS+CEBL model if:*

$$\forall j \in [t^*], \mathrm{H}^{\mathsf{HILL}}_{\epsilon,s}\big(SK_j \,|\, \mathbb{L}_1, \cdots, \mathbb{L}_T, R_{L_1}, \cdots, R_{L_T}\big) \geq \alpha \tag{2}$$

where R_{L_i} denote the random coins of the adversary corresponding to the leakage function L_i, $t^* \leq T$ is the time period at which $\mathcal{A}$ is given sk_{t^*} in full.

4.2 NIKE in FS + EBL Model.

Non-interactive key exchange (NIKE) protocols allow two (or more) parties to establish a shared key between them, without any interaction. It is assumed that the public keys of all the parties are pre-distributed and known to each other.

In this work, we consider two-party NIKE protocols and extend them to the setting of *forward-security* under *entropy-bounded* leakage model ($\mathsf{FS} + \mathsf{EBL}$). We provide the definition of NIKE in this model (we will often call such a NIKE scheme as FS-EBLR-NIKE). To bypass the black-box impossibility result of constructing leakage-resilient NIKE protocol in the plain model [35], we consider the NIKE protocols in the *common reference string* (CRS) model, where we rely on leak-free randomness to generate the CRS. Our security model for FS-EBLR-NIKE scheme can be seen as a leakage-resilient adaptation of the model of forward-secure NIKE (FS-NIKE) of Pointcheval and Sanders (dubbed as $\mathcal{PS}$ model) [38]. Hence, we call our model of NIKE as the $\mathcal{EBL}$-$\mathcal{PS}$ model.

4.3 Syntax of FS-EBLR NIKE.

A NIKE scheme NIKE in the $\mathsf{FS} + \mathsf{EBL}$ model consists of the tuple of algorithms (NIKE.Setup, NIKE.Gen, NIKE.Upd, NIKE.Key). We associate to NIKE a public parameter space $\mathcal{PP}$, public key space $\mathcal{PK}$, secret key space $\mathcal{SK}$, shared key space $\mathcal{SHK}$, and an identity space $\mathcal{IDS}$. Identities are used to track which public keys are associated with which users; we are *not* in the identity-based setting.

- $\mathsf{NIKE.Setup}(1^\kappa, (\alpha, T))$: This is a randomized algorithm that takes as input the security parameter κ (expressed in unary), parameters α and T of the (T, α)-$\mathsf{FS} + \mathsf{EBL}$ model (where α is the leakage parameter and T denotes the maximum number of time period supported by the system) and outputs public parameters $params \in \mathcal{PP}$.
- $\mathsf{NIKE.Gen}(1^\kappa, ID)$: On input an identity $ID \in \mathcal{IDS}$, the key generation outputs a public-secret key pair (pk, sk_t) for the current time period t. We assume that the secret keys implicitly contain the time periods. The current time period t is initially set to 1.
- $\mathsf{NIKE.Upd}(sk_t)$: The (deterministic) update algorithm takes as input the secret key sk_t at time period t and outputs the updated secret key sk_{t+1} for the next time period $t+1$, if $t < T$. We require that the updated key $sk_{t+1} \neq sk_t$. The key sk_t is then securely erased from memory. If $t = T$, then the secret key is erased and there is no new key.
- $\mathsf{NIKE.Key}(ID_A, pk^A, ID_B, sk_t^B)$: On input an identity $ID_A \in \mathcal{IDS}$ associated with public key pk_A, and another identity $ID_B \in \mathcal{IDS}$ with secret key sk_t^B corresponding to the current time period t, output the shared key $shk_t^{AB} \in \mathcal{SHK}$ or a failure symbol $\perp$. If $ID_A = ID_B$, the algorithm outputs $\perp$. Since the secret key sk_t^B is associated with time period t, the shared key shk_t^{AB} between the two users ID_A and ID_B also corresponds to the same time period t.

Correctness: The correctness requirement states that the shared keys computed by any two users ID_A and ID_B in the *same* time period are *identical*. In other words, for any time period $t \geq 1$, and any pair (ID_A, ID_B) of users having key pairs (pk^A, sk_t^A) and (pk^B, sk_t^B) respectively, it holds that:

$$\mathsf{NIKE.Key}(ID_A, pk^B, ID_B, sk_t^A) = \mathsf{NIKE.Key}(ID_B, pk^A, ID_A, sk_t^B).$$

Game $\underline{\mathsf{EBL\text{-}PS}^{\mathcal{A}_{\mathsf{nike}}}_{\mathsf{NIKE}}(\kappa, \alpha, T)}$

$params \leftarrow \mathsf{NIKE.Setup}(1^\kappa, \alpha, T)$; $S, C, Q \leftarrow \emptyset$ // S, C and Q maintains the list of *honest*, *corrupt* and *exposed* users respectively.

$(ID_A, ID_B, \tilde{t}) \leftarrow \mathcal{A}_1^{\mathsf{RegHon},\mathsf{RegCor},\mathsf{CorrReveal},\mathsf{Leak},\mathsf{Exp}}(params)$
$b \leftarrow\!\!\$\ \{0,1\}$
If $b = 0$ then Return $shk_{\tilde{t}}^{AB} \leftarrow \mathsf{NIKE.Key}(ID_A, pk_A, ID_B, sk_{\tilde{t}}^B)$
Else Return $shk_{\tilde{t}}^{AB} \leftarrow\!\!\$\ \mathcal{SHK}$
$b' \leftarrow\!\!\$\ \mathcal{A}_2^{\mathsf{RegHon},\mathsf{RegCor},\mathsf{CorrReveal},\mathsf{Leak},\mathsf{Exp}}(shk_{\tilde{t}}^{AB})$

If $(ID_A, -, -, \mathsf{corrupt}) \in C$ or $(ID_B, -, -, , \mathsf{corrupt}) \in C$, then return $\perp$
If $(ID_A, t^*) \in Q$ and $t^* \leq \tilde{t}$, then return $\perp$
If $(ID_B, t^*) \in Q$ and $t^* \leq \tilde{t}$, then return $\perp$
Return $(b' = b)$

Fig. 3. Game defining security of NIKE scheme NIKE in the $\mathsf{FS} + \mathsf{EBL}$ model.

4.4 Security Model for FS-EBLR NIKE.

Our security model for NIKE generalizes the model of forward-secure NIKE of [38] (often referred to as the $\mathcal{PS}$ model). We refer to our model as the $\mathcal{EBL}$-$\mathcal{PS}$ model. Security of a NIKE protocol NIKE in the $\mathcal{EBL}$-$\mathcal{PS}$ model is defined by a game $\mathsf{EBL\text{-}PS}$ between an adversary $\mathcal{A}_{\mathsf{nike}} = (\mathcal{A}_1, \mathcal{A}_2)$ and a challenger $\mathcal{C}$ (see Figure 3). Before the beginning of the game, the challenger $\mathcal{C}$ also initializes three sets S, C and Q to be empty sets. The adversary $\mathcal{A}_{\mathsf{nike}}$ can query the following oracles:

1. $\mathsf{RegHon}(ID)$: This oracle is used by $\mathcal{A}_{\mathsf{nike}}$ to register a new honest user ID at the initial time period. The challenger runs the $\mathsf{NIKE.Gen}$ algorithm with the current time period as 1, and returns the public key pk to $\mathcal{A}_{\mathsf{nike}}$. It also adds the tuple $(ID, sk_1, pk, \mathsf{honest})$ to the set S. This implicitly defines all the future keys $sk_2, \cdots, sk_T$ (since the update function is deterministic). This query may be asked at most *twice* by $\mathcal{A}_{\mathsf{nike}}$. Users registered by this query are called "honest".
2. $\mathsf{RegCor}(ID, pk)$: This oracle allows the adversary to register a new corrupted user ID with public key pk. The challenger adds the tuple $(ID, --, pk, \mathsf{corrupt})$ to the set C. We call the users registered by this query as "corrupt".
3. $\mathsf{CorrReveal}(ID_A, ID_B, t)$: $\mathcal{A}_{\mathsf{nike}}$ supplies two indices where ID_A was registered as *corrupt* and ID_B was registered as *honest*. The challenger looks up the secret key sk_1^B (corresponding to ID_B) and computes the updated key sk_t^B corresponding to time period t. Then it runs $\mathsf{NIKE.Key}(ID_A, pk^A, ID_B, sk_t^B)$ to get the shared key shk_t^{AB} for time period t and returns shk_t^{AB} to $\mathcal{A}_{\mathsf{nike}}$.
4. $\mathsf{Leak}(L, ID, t)$: The adversary $\mathcal{A}_{\mathsf{nike}}$ submits a leakage function $L : \mathcal{PP} \times \mathcal{SK} \rightarrow \{0,1\}^*$ to leak on the secret key of user ID for time period t, provided that $\mathcal{A}_{\mathsf{nike}}$ belongs to the class of legitimate adversaries (see Definition 11).

5. $\mathsf{Exp}(ID, t^*)$: This query is used by $\mathcal{A}_{\mathsf{nike}}$ to get the secret key of an honestly registered user ID corresponding to time period t^*. The challenger looks for a tuple $(ID, sk_1, pk, \mathsf{honest})$. If there is a match, it computes sk_{t^*} corresponding to t^* and returns sk_{t^*} to $\mathcal{A}_{\mathsf{nike}}$. Else, it returns $\perp$. The challenger adds (ID, t^*) to the set Q.

The formal details of our $\mathsf{EBL\text{-}PS}$ game is given in Figure 3.

Definition 13 ($\mathsf{FS} + \mathsf{EBL}$-secure NIKE). *A NIKE protocol* NIKE *is* (T, α)-forward-secure under computational-entropy-bounded leakage model ((T, α)-$\mathsf{FS} + \mathsf{EBL}$) *with respect to any legitimate adversary* $\mathcal{A}_{\mathsf{nike}}$ *playing the above* $\mathsf{EBL\text{-}PS}$ *game (see Figure 3), if the advantage defined as* $\mathsf{Adv}^{\mathsf{fs\text{-}ebl}}_{\mathcal{A}_{\mathsf{nike}}}(\kappa) = |\mathsf{Pr}[\mathsf{EBL\text{-}PS}^{\mathcal{A}_{\mathsf{nike}}}_{\mathsf{NIKE}}(\kappa, \alpha, T)) = 1] - 1/2|$ *is negligible in* κ.

In other words, the adversary $\mathcal{A}_{\mathsf{nike}}$ succeeds in the above experiment if it is able to distinguish a valid shared key between two users from a random session key. To avoid trivial win, some restrictions are enforced, namely: (i) both the targeted (or test) users needs to be *honestly registered* (ii) the adversary $\mathcal{A}_{\mathsf{nike}}$ is not allowed to obtain the secret keys corresponding to any of the test users prior to the challenge time period $\tilde{t}$, (iii) $\mathcal{A}_{\mathsf{nike}}$ is allowed to leak on the secret keys of both the target users ID_A and ID_B, as long as it satisfies the legitimacy condition (see Definition 11). We emphasize that the adversary can still obtain the secret keys of the target users ID_A and ID_B for time periods $t^* > \tilde{t}$, which models *forward security*.

Variants of NIKE. Similar to [35], we consider different variants of NIKE depending on whether the setup algorithm just outputs a uniformly random coins or sample from some structured distributions. In particular, we say a NIKE scheme is:

- a *plain* NIKE, if $\mathsf{NIKE.Setup}(1^\kappa)$ just outputs (some specified number of) uniform random coins. In particular, $\mathsf{NIKE.Setup}(1^\kappa; r) = r$.
- a NIKE in the *common reference string* model, if $\mathsf{NIKE.Setup}(1^\kappa)$ can be arbitrary (i.e., sample from an arbitrary distribution). In this case, we rely on *leak-free randomness* to run the setup algorithm.

Remark 2. We note that, in the original $\mathcal{PS}$ model of forward secure NIKE, there can be multiple honest users, and the adversary is allowed to obtain the secret keys of the honest users other than the target users (even prior to the challenge time period $\tilde{t}$). In this work, we consider a simplified version where there are only *two* honest users. The above simplified model can be shown to be polynomially equivalent to the full-fledged $\mathcal{PS}$ model by following the same reduction strategy as in [19][Theorem 8, Appendix B], where they show that the CKS-light model (with two honest users) is polynomially equivalent to the CKS-heavy model (where they can be multiple honest users). We emphasize that, in our application of constructing $\mathsf{FS} + \mathsf{EBL}$-secure PKE scheme from $\mathsf{FS\text{-}EBLR}$ NIKE, we only require the above simplified model.

4.5 Construction of NIKE scheme in the FS + EBL model

In this section, we present our construction of forward-secure NIKE protocol resilient to entropy-bounded leakage in the common reference string model.

Let $i\mathcal{O}$ be an indistinguishability obfuscator for circuits, $\mathsf{pPRF} = (\mathsf{pPRF.keygen}, \mathsf{pPRF.puncture}, \mathsf{pPRF.eval})$ be a puncturable PRF with image space $\mathcal{Y} = \{0,1\}^y$, $\mathsf{LF} = (\mathsf{Inj}, \mathsf{Lossy}, f)$ be a (κ, k, m)[10]-lossy function, and $\mathsf{LF}' = (\mathsf{Inj}', \mathsf{Lossy}', f')$ be a (κ', k', m')-lossy function, where $\kappa' \geq m$.

- $\mathsf{NIKE.Setup}(1^\kappa, T)$: Choose a random key $K \leftarrow \mathsf{pPRF.keygen}(1^\kappa)$. Sample two injective evaluation keys $\mathsf{ek} \leftarrow \mathsf{Inj}(1^\kappa)$, $\mathsf{ek}' \leftarrow \mathsf{Inj}'(1^\kappa)$. Consider the circuit $C(r, X_i, X_j)$ that has the key K hard-coded (see Figure 4) and compute $\widehat{C} = i\mathcal{O}(C)$. Set $params = (\widehat{C}, \mathsf{ek}, \mathsf{ek}')$.

Inputs: r, X_i, X_j.
Constant: $K, \mathsf{ek}, \mathsf{ek}'$
If $f_{\mathsf{ek}}(r) = X_i$ or $f_{\mathsf{ek}}(r) = X_j$, output $\mathsf{pPRF.eval}(K, (f'_{\mathsf{ek}'}(X_i), f'_{\mathsf{ek}'}(X_j))$;
Else output $\perp$.

Fig. 4. The Circuit $C(r, X_i, X_j)$

- $\mathsf{NIKE.Gen}(1^\kappa, params, ID_i)$: To compute the key pair of an user ID_i, sample $sk_1^i \xleftarrow{\$} \{0,1\}^\kappa$. Consider the circuit $C_i(sk_t, t)$ that has the keys ek, ek', the base secret key sk_1^i, and the obfuscated circuit $\widehat{C}$ (which is part of $params$) hard-coded (See Figure 5) and compute $\widehat{C_i} = i\mathcal{O}(C_i)$. Set the public key as $pk^i = \widehat{C_i}$.

Inputs: sk_t, t.
Constants: sk_1^i, ek, ek', $\hat{C}$, T.
1. Check if $t \leq T$. If not, output $\perp$.
2. Update $sk_t^i = \mathsf{NIKE.Upd}^{t-1}(sk_1^i)$.
3. Compute $X_t^i = f_{\mathsf{ek}}(sk_t^i)$ and $X_t^j = f_{\mathsf{ek}}(sk_t)$.

Output the shared key $shk_t^{ij} = \widehat{C}(sk_t, X_t^i, X_t^j)$.

Fig. 5. Circuit $C_i(sk_t, t)$

- $\mathsf{NIKE.Upd}(1^\kappa, sk_t^i)$: On input of the user $ID_i's$ secret key sk_t^i at time period t, computes sk_{t+1}^i, the secret key for the next time period $t+1$. The instantiation of the update function is mentioned below.

[10] A (κ, k, m)-lossy function maps an input from $x \in \{0,1\}^\kappa$ to an output $y \in \{0,1\}^m$. In the lossy mode, the image size of the function is at most $2^{\kappa-k}$ with high probability. The formal definition can be found in Section 2.3.3.

- $\mathsf{NIKE.Key}(ID_i, pk^i = \widehat{C_i}, ID_j, sk_t^j)$: The user ID_j runs the obfuscated circuit $\widehat{C_i} = i\mathcal{O}(C_i)$ on inputs the secret key sk_t^j corresponding to time period t to obtain the shared key shk_t^{ij} at time period t.

Note on Update Function: The update function $\mathsf{NIKE.Upd}$ is one which takes a secret key of the current period and produces a new secret key. As defined in the security model, the adversary can issue leakage queries provided the keys are α-entropic conditioned on the set of all leakage queries. It is not hard to see that the update function should necessarily satisfy the one-wayness property, essentially guaranteeing the non-invertibility of the earlier keys once the secret key is exposed. Interestingly, for the above construction, we can abstract away the update function to any *entropic leakage resilient one-way function*, i.e., $\mathsf{NIKE.Upd}(\cdot) = g(\cdot)$, where $g : \{0,1\}^\kappa \to \{0,1\}^\kappa$ be a α-entropic leakage-resilient one-way function (α-ELR-OWF). The definition of entropic leakage-resilient OWF is given in Section 2.3.4.

Correctness. It is not hard to see that both the parties ID_i and ID_j end up with the *same* shared key. For concreteness, we present the proof of correctness of the computation of the shared key shk_t^{ij} by both the parties in Appendix **??**.

Shared key computation by P_i: P_i computes the shared key as:

$$\begin{aligned}
shk_t^{ij} &= \mathsf{NIKE.Key}(ID_j, pk^j = \widehat{C_j}, ID_i, sk_t^i) \\
&= \widehat{C_j}(sk_t^i, (X_t^i, X_t^j)) \\
&= \mathsf{pPRF.eval}\big(K, f'_{\mathsf{ek}'}(X_t^i), f'_{\mathsf{ek}'}(X_t^j)\big) \\
&= \mathsf{pPRF.eval}\big(K, f'_{\mathsf{ek}'}(f_{\mathsf{ek}}(sk_t^i)), f'_{\mathsf{ek}'}(f_{\mathsf{ek}}(\mathsf{NIKE.Upd}^{t-1}(sk_1^j)))\big) \\
&= \mathsf{pPRF.eval}\big(K, f'_{\mathsf{ek}'}(f_{\mathsf{ek}}(\mathsf{NIKE.Upd}^{t-1}(sk_1^i))), f'_{\mathsf{ek}'}(f_{\mathsf{ek}}(\mathsf{NIKE.Upd}^{t-1}(sk_1^j)))\big)
\end{aligned}$$

Shared key computation by P_j: P_j computes the shared key as:

$$\begin{aligned}
shk_t^{ij'} &= \mathsf{NIKE.Key}(ID_i, pk^i = \widehat{C_i}, ID_j, sk_t^j) \\
&= \widehat{C_i}(sk_t^j, (X_t^i, X_t^j)) \\
&= \mathsf{pPRF.eval}\big(K, f'_{\mathsf{ek}'}(X_t^i), f'_{\mathsf{ek}'}(X_t^j)\big) \\
&= \mathsf{pPRF.eval}\big(K, f'_{\mathsf{ek}'}(f_{\mathsf{ek}}(\mathsf{NIKE.Upd}^{t-1}(sk_1^i))), f'_{\mathsf{ek}'}(f_{\mathsf{ek}}(sk_t^j))\big) \\
&= \mathsf{pPRF.eval}\big(K, f'_{\mathsf{ek}'}(f_{\mathsf{ek}}(\mathsf{NIKE.Upd}^{t-1}(sk_1^i))), f'_{\mathsf{ek}'}(f_{\mathsf{ek}}(\mathsf{NIKE.Upd}^{t-1}(sk_1^j)))\big)
\end{aligned}$$

Hence, we can see that shared keys computed by both P_i and P_j corresponding to time period t are *same*, i.e., $shk_t^{ij} = shk_t^{ij'}$.

Instantiations. Our FS-EBLR NIKE construction from above can be instantiated based on the recent construction of $i\mathcal{O}$ from well-founded assumptions [29]. One can construct lossy functions from DDH or LWE [35]. Besides, we need to rely on the relaxed varaint of the Superfluous Padding Assumption (SuPA). In particular we obtain FS+ECL NIKE from either DDH or LWE along with sub-exponential SXDH on asymmetric bilinear groups, sub-exponential LPN, Boolean PRGs in NC^0 and relaxed SuPA.

4.6 Security Proof of our NIKE construction in the FS + EBL model.

Theorem 2. *Let κ be the security parameter, and $T = T(\kappa)$ be an arbitrary but fixed polynomial. Assume that $i\mathcal{O}$ is an indistinguishability obfuscator for circuits, and the superfluous padding assumption holds for $i\mathcal{O}$. Let LF is an (κ, k, m)-lossy function, LF' is an (κ', k', m')-lossy function where $\kappa' \geq m$, pPRF is a family of puncturable PRFs with image size $\mathcal{Y} = \{0,1\}^y$. Then, Construction 4.5 is a (α, T)-forward-secure entropy-bounded leakage-resilient NIKE in the $\mathcal{EBL}$-$\mathcal{PS}$ model with $\alpha \geq y + rT + r' - 2\kappa$, where $r = (\kappa - k)$, $r' = (\kappa' - k')$, and T denote the total number of time periods supported by the scheme.*

Proof. We will now present the detailed proof of Theorem 2. Let $\mathsf{Ext} : \{0,1\}^n \times \{0,1\}^d \to \mathcal{Y}$ be an average-case strong randomness extractor to be determined later. We prove the security of Theorem 2 via a sequence of hybrid experiments. We will use S_i to refer to the event that $\mathcal{A}_{\mathsf{nike}}$ wins in $\mathbf{Game}_i$.

$\mathbf{Game}_0$: This corresponds to the original security game $\mathsf{EBL\text{-}PS}^{\mathcal{A}_{\mathsf{nike}}}_{\mathsf{NIKE}}(\kappa, \alpha, T)$, except that the challenger guesses that challenge time period $\tilde{t}$. In particular, the challenger does the following:

1. It chooses a time period $\tilde{t}$ as a guess for the challenge time period of the adversary $\mathcal{A}_{\mathsf{nike}}$. If $\mathcal{A}_{\mathsf{nike}}$ chooses any period other than $\tilde{t}$, the challenger aborts. Note that the guess is correct with probability $1/T$.
2. The challenger then runs $\mathsf{NIKE.Setup}$ to get $params = (\widehat{C} = i\mathcal{O}(C), \mathsf{ek}, \mathsf{ek}')$. The definition of the circuit C can be found in Figure 4.
3. It also chooses two secret keys sk_1^i and sk_1^j. It sets $pk^i = \widehat{C_i} = i\mathcal{O}(C_i)$ where C_i is defined in Figure 5.
4. It computes the challenge appropriately.
5. ll the leakage information that $\mathcal{A}_{\mathsf{nike}}$ asks can also be answered by the challenger with the knowledge of the secret keys sk_1^i and sk_1^j.

Let $\mathcal{E}$ be the event that the challenger guesses the correct challenge time period. Then,

$$\mathsf{Pr}[S_0] \leq \mathsf{Pr}[S_0|\mathcal{E}] + \mathsf{Pr}[\bar{\mathcal{E}}] = \mathsf{Adv}^{\mathsf{fs\text{-}ebl}}_{\mathcal{A}_{\mathsf{nike}}}(\kappa) + \left(1 - \frac{1}{T}\right)$$

$\mathbf{Game}_1$: This is similar to $\mathbf{Game}_0$, except that the challenger sets $pk^i = \widehat{C_i^*} = i\mathcal{O}(C_i^*)$ (resp. $pk^j = \widehat{C_j^*} = i\mathcal{O}(C_j^*)$) where C_i^* (resp. C_j^*) is a program which is closely related to the original program C_i (resp. C_j). The difference in the programs are that: Instead of embedding the base secret key sk_1^i, the challenger computes the vector of values $\vec{X}_i = (X_1^i, \ldots, X_T^i)$ (resp. $\vec{X}_j = (X_1^j, \ldots, X_T^j)$) with knowledge of sk_1^i (resp. sk_1^j). Note that, $X_t^i = f_{\mathsf{ek}}(\mathsf{NIKE.Upd}^{t-1}(sk_1^i))$. The challenger then embeds the vector $\vec{X}_i$ (resp. $\vec{X}_j$) in the program C_i^* (resp. C_j^*). The definition of C_i^* is defined in Figure 6. Note that, the two circuits C_i and C_i^* (resp. C_j and C_j^*) are functionally equivalent, and hence the distributions in $\mathbf{Game}_1$ and $\mathbf{Game}_0$ are computationally indistinguishable by security of $i\mathcal{O}$. In other words,

$$|\mathsf{Pr}[S_1] - \mathsf{Pr}[S_0]| \leq \mathsf{negl}(\kappa).$$

Constants: $\vec{X}_i = (X_1^i, \ldots, X_T^i), \mathsf{ek}, \mathsf{ek}', \widehat{C}, T$.

Inputs: sk_t, t.

1. Check if $t \leq T$. If not, output $\perp$.
2. Compute: $X_t^j = f_{\mathsf{ek}}(sk_t)$.

Output the shared key $shk_t^{ij} = \widehat{C}(sk_t, X_t^i, X_t^j)$.

Fig. 6. The Circuit $C_i^*(sk_t, t)$.

$\mathbf{Game}_2$: This is similar to $\mathbf{Game}_1$, with the exception that there is a change in the generation of *params*: Instead of computing $\widehat{C} = i\mathcal{O}(C)$, the challenger computes $\widehat{C^{(1)}} = i\mathcal{O}(C^{(1)})$ where the program $C^{(1)}$ is closely related to C. The difference in the programs are as follows: The challenger uses its guess of the challenge period $\tilde{t}$ to embed the values of $f'_{\mathsf{ek}'}(X_{\tilde{t}}^i)$ and $f'_{\mathsf{ek}'}(X_{\tilde{t}}^j)$ in the program $C^{(1)}$ (note that, the challenger can always compute these values using its knowledge of the secret keys sk_1^i, sk_1^j). Further, it also embeds the value $\mathsf{pPRF.eval}\left(K, f'_{\mathsf{ek}'}(X_{\tilde{t}}^i), f'_{\mathsf{ek}'}(X_{\tilde{t}}^j)\right)$ in $C^{(1)}$. The program is defined in Figure 7. Set $params = (\widehat{C^{(1)}}, \mathsf{ek}, \mathsf{ek}')$.

Note that the programs C and $C^{(1)}$ are functionally equivalent. This follows from the fact that the functions f and f', are injective. Therefore, we have that the distributions in $\mathbf{Game}_1$ and $\mathbf{Game}_2$ are computationally indistinguishable by security of $i\mathcal{O}$. In other words,

$$|\mathsf{Pr}[S_2] - \mathsf{Pr}[S_1]| \leq \mathsf{negl}(\kappa)$$

Constant: $K, \mathsf{ek}, \mathsf{ek}', f'_{\mathsf{ek}'}(X_{\tilde{t}}^i), f'_{\mathsf{ek}'}(X_{\tilde{t}}^j), \mathsf{pPRF.eval}\left(K, f'_{\mathsf{ek}'}(X_{\tilde{t}}^i), f'_{\mathsf{ek}'}(X_{\tilde{t}}^j)\right)$

Inputs: r, X_i, X_j.
If $f'_{\mathsf{ek}'}(X_i) = f'_{\mathsf{ek}'}(X_{\tilde{t}}^i)$ and $f'_{\mathsf{ek}'}(X_j) = f'_{\mathsf{ek}'}(X_{\tilde{t}}^j)$, and

If $f'_{\mathsf{ek}'}(f_{\mathsf{ek}}(r)) = f'_{\mathsf{ek}'}(X_{\tilde{t}}^i)$ or $f'_{\mathsf{ek}'}(f_{\mathsf{ek}}(r)) = f'_{\mathsf{ek}'}(X_{\tilde{t}}^j)$:

output $\mathsf{pPRF.eval}(K, (f'_{\mathsf{ek}'}(X_i), f'_{\mathsf{ek}'}(X_j))$.

If $f_{\mathsf{ek}}(r) = X_i$ or $f_{\mathsf{ek}}(r) = X_j$, output $\mathsf{pPRF.eval}(K, (f'_{\mathsf{ek}'}(X_i), f'_{\mathsf{ek}'}(X_j))$.
Else output $\perp$.

Fig. 7. Circuit $C^{(1)}(r, X_i, X_j)$.

$\mathbf{Game}_3$: This is similar to $\mathbf{Game}_2$ with the exception that now we will puncture the pPRF at the points $f'_{\mathsf{ek}'}(X_{\tilde{t}}^i), f'_{\mathsf{ek}'}(X_{\tilde{t}}^j)$ while generating *params*, i.e.

$$K' \leftarrow \mathsf{pPRF.puncture}(K, f'_{\mathsf{ek}'}(X_{\tilde{t}}^i), f'_{\mathsf{ek}'}(X_{\tilde{t}}^j)) \ .$$

Instead of computing $\widehat{C^{(1)}} = i\mathcal{O}(C^{(1)})$, the challenger computes $\widehat{C^{(2)}} = i\mathcal{O}(C^{(2)})$ where the program $C^{(2)}$ is defined in Figure 8. The circuit $C^{(2)}$ has hard-coded in it the values the punctured key K', $\mathsf{ek}, \mathsf{ek}', f'_{\mathsf{ek}'}(X^i_{\tilde{t}}), f'_{\mathsf{ek}'}(X^j_{\tilde{t}})$ and some *random* value $\gamma \leftarrow \mathcal{Y}$ (instead of $\mathsf{pPRF.eval}\left(K', f'_{\mathsf{ek}'}(X^i_{\tilde{t}}), f'_{\mathsf{ek}'}(X^j_{\tilde{t}})\right)$). Set $params = (\widehat{C^{(2)}}, \mathsf{ek}, \mathsf{ek}')$.

Therefore, we have that the distributions in $\mathbf{Game}_2$ and $\mathbf{Game}_1$ are computationally indistinguishable by the security of punctured PRF. In other words,

$$|\mathsf{Pr}[S_3] - \mathsf{Pr}[S_2]| \leq \mathsf{negl}(\kappa).$$

Constant: $K', \mathsf{ek}, \mathsf{ek}', f'_{\mathsf{ek}'}(X^i_{\tilde{t}}), f'_{\mathsf{ek}'}(X^j_{\tilde{t}}), \gamma$

Inputs: r, X_i, X_j.
If $f'_{\mathsf{ek}'}(X_i) = f'_{\mathsf{ek}'}(X^i_{\tilde{t}})$ and $f'_{\mathsf{ek}'}(X_j) = f'_{\mathsf{ek}'}(X^j_{\tilde{t}})$, and:

If $f'_{\mathsf{ek}'}(f_{\mathsf{ek}}(r)) = f'_{\mathsf{ek}'}(X^i_{\tilde{t}})$ or $f'_{\mathsf{ek}'}(f_{\mathsf{ek}}(r)) = f'_{\mathsf{ek}'}(X^j_{\tilde{t}})$, then output γ.

If $f_{\mathsf{ek}}(r) = X_i$ or $f_{\mathsf{ek}}(r) = X_j$, output $\mathsf{pPRF.eval}(K', (f'_{\mathsf{ek}'}(X_i), f'_{\mathsf{ek}'}(X_j))$.
Else output $\perp$.

Fig. 8. Circuit $C^{(2)}(r, X_i, X_j)$.

$\mathbf{Game}_4$: We again change how we generate the parameters $params$. The challenger first samples two seeds $s_i, s'_i \leftarrow \{0,1\}^d$.

- We first sample γ from $\mathcal{Y}$.
- We compute $z_i = \mathsf{Ext}(sk^i_{\tilde{t}}, s_i) \oplus \gamma$ and $z_j = \mathsf{Ext}(sk^j_{\tilde{t}}, s'_i) \oplus \gamma$.
- We now replace $i\mathcal{O}(C^{(2)})$ with $i\mathcal{O}(C^{(3)})$ where $C^{(3)}$ is a closely related program defined in Figure 9.

Constant: $K', \mathsf{ek}, \mathsf{ek}', f'_{\mathsf{ek}'}(X^i_{\tilde{t}}), f'_{\mathsf{ek}'}(X^j_{\tilde{t}}), z_i, z_j, s_i, s'_i$

Inputs: r, X_i, X_j.
If $f'_{\mathsf{ek}'}(X_i) = f'_{\mathsf{ek}'}(X^i_{\tilde{t}})$ and $f'_{\mathsf{ek}'}(X_j) = f'_{\mathsf{ek}'}(X^j_{\tilde{t}})$, and

If $f'_{\mathsf{ek}'}(f_{\mathsf{ek}}(r)) = f'_{\mathsf{ek}'}(X^i_{\tilde{t}})$ output $z_i \oplus \mathsf{Ext}(r, s_i)$

If $f'_{\mathsf{ek}'}(f_{\mathsf{ek}}(r)) = f'_{\mathsf{ek}'}(X^j_{\tilde{t}})$ output $z_j \oplus \mathsf{Ext}(r, s'_i)$

If $f_{\mathsf{ek}}(r) = X_i$ or $f_{\mathsf{ek}}(r) = X_j$, output $\mathsf{pPRF.eval}(K', (f'_{\mathsf{ek}'}(X_i), f'_{\mathsf{ek}'}(X_j))$.
Else output $\perp$.

Fig. 9. Circuit $C^{(3)}(r, X_i, X_j)$.

Note that the programs $C^{(2)}$ and $C^{(3)}$ are functionally equivalent. This follows from the definitions of functions f and f', i.e., they are injective, and consequently both $z_i \oplus \mathsf{Ext}(r, s_i)$ and $z_j \oplus \mathsf{Ext}(r, s'_i)$ evaluate to γ. Therefore, we have that the distributions in $\mathbf{Game}_4$ and $\mathbf{Game}_3$ are computationally indistinguishable by security of $i\mathcal{O}$. In other words,

$$|\mathsf{Pr}[S_4] - \mathsf{Pr}[S_3]| \leq \mathsf{negl}(\kappa)$$

$\mathbf{Game}_5$: This is similar to $\mathbf{Game}_4$ with the only difference being in the way we generate *params*. In this game, the challenger samples ek and ek' from Lossy instead of Inj. The keys ek and ek' are now embedded in the programs $C^{(3)}$ and C_i^* (resp. C_j^*) to generate new programs $C^{(4)}$ and C_i^{**} (resp. C_j^{**}). The resulting distribution is indistinguishable from that of $\mathbf{Game}_4$ from the security of lossy functions. In other words,

$$|\mathsf{Pr}[S_5] - \mathsf{Pr}[S_4]| \leq \mathsf{negl}(\kappa)$$

$\mathbf{Game}_6$: This is similar to $\mathbf{Game}_5$ except that the value returned as the shared key is entirely random. We will argue that $\mathbf{Game}_5$ and $\mathbf{Game}_6$ are statistically indistinguishable by relying on the security of Ext. Specifically, we prove the following claim.

Claim. The distribution of γ conditioned on $E := (pk^{(i)} = i\mathcal{O}(C_i^{**}), pk^{(j)} = i\mathcal{O}(C_j^{**}), \mathit{params}, \mathcal{L})$ is *statistically* indistinguishable from random. Here $\mathcal{L}$ denotes the view of the adversary based on the leakage queries it makes, and the responses it receives.

Proof. To prove the above claim, it is sufficient to prove that the values $z_i = \mathsf{Ext}(Y_{\tilde{t}}^i, s_i) \oplus \gamma$ and $z_j = \mathsf{Ext}(Y_{\tilde{t}}^j, s'_i) \oplus \gamma$ look uniformly random. Notice that all the elements of E can be generated from

$$\left(K', \mathsf{ek}, \mathsf{ek}', \mathcal{L}, X_1^i, \ldots, X_T^i, X_1^j, \ldots, X_T^j, T, s_i, s'_i, f'_{\mathsf{ek}'}(X_{\tilde{t}}^i), f'_{\mathsf{ek}'}(X_{\tilde{t}}^j)\right)$$

Note that, we ensured that C_i^{**}, C_j^{**} did not have the base secret keys sk_1^i and sk_1^j embedded in them. However, these do have the values of $X_1^i, \ldots, X_T^i$ and $X_1^j, \ldots, X_T^j$ which are dependent on the secret key chain. We need to estimate the loss of entropy due to this value. However, observe that we have switched the key $\mathsf{ek} \leftarrow \mathsf{Inj}$ to lossy function $\mathsf{ek} \leftarrow \mathsf{Lossy}$. By the definition of lossy function and relying on the chain rule of min-entropy, we have that:

$$\widetilde{\mathrm{H}}_\infty(Y_{\tilde{t}}^i | \mathcal{L}, X_1^i, \ldots, X_T^i, f'_{\mathsf{ek}'}(X_{\tilde{t}}^i))) \geq \alpha - (\kappa - k)T - (\kappa' - k')$$

To complete the proof, we will make use of the following Theorem.

Theorem 3 (Generalized Leftover Hash Lemma [17]). *Assume* $\{H_x : \{0,1\}^\kappa \to \{0,1\}^y\}_{x \in X}$ *be a family of universal hash functions. Then for any random variables* W *and* I*, we have:*

$$\delta\big((H_X(W), W, I), (U_y, W, I)\big) \leq \epsilon,$$

where $\epsilon \leq \frac{1}{2}\sqrt{2^{\widetilde{\mathrm{H}}_\infty(W|I)}2^y}$ *and* $\delta(X, Y)$ *denotes the statistical distance between two random variables* X *and* Y*.*

In particular, universal hash functions are average-case (κ, n, y, ϵ)*-strong extractors whenever* $y \leq n - 2\log(\frac{1}{\epsilon}) + 2$*.*

Combining the above and the fact that $y \leq \alpha - (\kappa - k)T - (\kappa' - k') + 2\kappa$, we obtain that z_i is statistically close to uniform in $\{0,1\}^y$. A similar analysis can be done for the case of z_j. As a result, the distribution of γ is statistically close to uniform in $\{0,1\}^y$. This concludes the proof of the claim. ■

Game$_7$: In the first game, i.e., **Game**$_1$, the size of the program C_i^* (resp. C_j^*) is linear in T. The reason is that we embedded the vectors $\vec{X}_i = (X_1^i, \cdots, X_T^i)$ (resp. $\vec{X}_j = (X_1^j, \cdots, X_T^j)$) in the program C_i^* (resp. C_j^*). Note that, C_i^* is a functionally equivalent padded version of the circuit C_i (used in the original construction, i.e., in **Game**$_0$) and can be written as $C_i^* = \mathsf{PAD}(s(\kappa), C_i)$ where $s = mT$. We have shown by the above sequence of hybrids that:

$$(\gamma_1, i\mathcal{O}(\mathsf{PAD}(s(\kappa), C_i)) \approx_c (\gamma_6, i\mathcal{O}(\mathsf{PAD}(s(\kappa), C_i)),$$

where γ_1 and γ_6 are the values of the shared key at the end of **Game**$_1$ and **Game**$_6$ respectively. Applying the relaxed SuP assumption for indistinguishability obfuscation and identifying Samp_0 with the sampler that runs **Game**$_1$ and, similarly, Samp_1 with the sampler that runs **Game**$_6$ we get that also

$$(\gamma_1, i\mathcal{O}(C_i) \approx_c (\gamma_6, i\mathcal{O}(C_i))$$

are computationally indistinguishable. Hence, under the SuP assumption the size of our public key is independent total number of time periods T.

Finally, note that the adversary has no advantage in **Game**$_7$. □

Parameters. We can set the value of κ appropriately, such that $\alpha = \mathsf{poly}(\kappa)$ for an arbitrary polynomial $\mathsf{poly}(\cdot)$ and any leakage rate $\rho \leq 1 - o(1)$. The size of our public parameter (*params*) and the secret keys are $O(1)$. By relying on the SuP assumption the size of our public key is $O(\log T)$.

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