# Succinct LWE Sampling, Random Polynomials, and Obfuscation

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### Abstract

We present a construction of indistinguishability obfuscation (iO) that relies on the learning with errors (LWE) assumption together with a new notion of succinctly sampling pseudorandom LWE samples. We then present a candidate LWE sampler whose security is related to the hardness of solving systems of polynomial equations. Our construction improves on the recent iO candidate of Wee and Wichs (Eurocrypt 2021) in two ways: first, we show that a much weaker and simpler notion of LWE sampling suffices for iO; and secondly, our candidate LWE sampler is secure based on a compactly specified and falsifiable assumption about random polynomials, with a simple error distribution that facilitates cryptanalysis.

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# Contents

# 1 Introduction

Indistinguishability obfuscation (iO) [BGI$^+$01, GR07] is a probabilistic polynomial-time algorithm $\mathcal{O}$ that takes as input a circuit $C$ and outputs an (obfuscated) circuit $C' = \mathcal{O}(C)$ satisfying two properties: (a) *functionality*: $C$ and $C'$ compute the same function; and (b) *security*: for any two circuits $C_1$ and $C_2$ that compute the same function (and have the same size), $\mathcal{O}(C_1)$ and $\mathcal{O}(C_2)$ are computationally indistinguishable. Since the first candidate for iO was introduced in [GGH$^+$13], a series of works have shown that iO would have a huge impact on cryptography.

In this work, we build upon the recent line of works on lattice-inspired iO candidates [Agr19, CHVW19, AP20, BDGM20b, BDGM20a, WW21, GP21] that are plausibly post-quantum secure. The dream goal here is to ultimately base iO on the hardness of the learning with errors (LWE) problem together with an assumption about simple Boolean or integer pseudorandom generators (PRGs). Such a result would, in particular, eliminate pairings from the recent break-through result basing iO on well-founded assumptions [JLS21].

## 1.1 Our Contributions

We present a candidate construction of iO that relies on LWE together with a new notion of succinctly sampling pseudorandom LWE samples. In addition, we present a candidate sampler whose security is related to the hardness of solving systems of polynomial equations. Our construction improves on the recent iO candidate of Wee and Wichs [WW21] (henceforth referred to as the WW construction) in two ways:

- First, our new notion of succinct LWE sampling simplifies and relaxes the notion of oblivious LWE sampling from WW. Instead of a simulation-based definition as in WW, we have a simple indistinguishability-based definition, where the generated LWE sample can be used to drown out the differences between certain error distributions. Furthermore, we put forth two variants of succinct LWE sampling, and provide a general amplification from a weak (falsifiable) notion that refers to a specific error distribution to a strong (non-falsifiable) notion that refers to general error distributions.

- Next, our candidate succinct LWE sampler is easy to describe and is based on random polynomials. It yields an LWE sample with a simple error distribution that facilitates cryptanalysis. This is in contrast to WW, where the LWE sampler involved complex FHE evaluation, and the resulting error distribution in the samples was dependent on the concrete implementation of the circuit being evaluated. Indeed, a recent work of [HJL21] carefully crafted circuit implementations that would render the WW candidate as well as the related candidate in [GP21] insecure (see Section 1.3 for a more detailed discussion).

## 1.2 Technical Overview

The starting point of our construction is essentially the same as that of the Wee-Wichs (WW) iO candidate, which in turn builds on [BDGM20a]. We begin by describing a notion of succinct randomized encoding (SRE), which can be seen as a relaxation of the notions of split FHE and functional encodings used in prior works. It is also very related to the notion of exponentially efficient iO (XiO) from [LPST16], and is easily seen to imply it, but we find the SRE abstraction easier to work with in the context of our work. By leveraging prior results on XiO [LPST16], our notion of SRE implies iO under the LWE assumption.

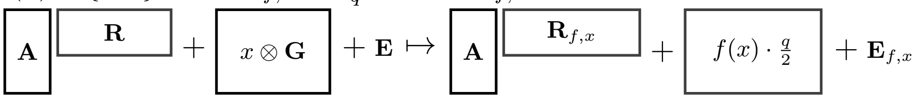
**Succinct Randomized Encodings.** A succinct randomized encoding[1] [BGL+15, LPST16] of a function $f : \{0,1\}^\ell \to \{0,1\}^N$ is an efficient probabilistic algorithm $\mathsf{Encode}$ such that:

- functionality: we can efficiently recover $f(x)$ given $f$ and $\mathsf{Encode}(f, x)$;
- security: for any $x_0, x_1$ such that $f(x_0) = f(x_1)$, we have $\mathsf{Encode}(f, x_0) \approx_c \mathsf{Encode}(f, x_1)$; and
- succinctness: $\mathsf{Encode}(f, x)$ is shorter than the output length of $f$. That is, $|\mathsf{Encode}(f, x)| = \widetilde{O}(N^\delta)$ for some constant $\delta < 1$, ignoring factors polynomial in $\ell$ and the security parameter.

Henceforth, we will focus on building SRE for circuits.

**Base Scheme.** We start with a base scheme for succinct randomized encodings implicit in WW, which is insecure, but serves as the basis of our eventual construction. The base scheme uses a variant of the homomorphic encryption/commitment schemes of [GSW13, GVW15], along with the "packing" techniques in [PVW08, MW16, BTVW17, PS19, GH19, BDGM19]. Given a commitment $\mathbf{C}$ to an input $x \in \{0,1\}^\ell$, along with a circuit $f : \{0,1\}^\ell \to \{0,1\}^N$, this scheme allows us to homomorphically compute a commitment $\mathbf{C}_f$ to the output $f(x)$. Moreover, the opening for the output commitment is shorter than the output size $N$. Concretely, we define $\mathbf{C}, \mathbf{C}_f$ as follows:

- We treat the function $f : \{0,1\}^\ell \to \{0,1\}^N$ as a function $f : \{0,1\}^\ell \to \{0,1\}^{M \times K}$, where $M$ and $K$ are parameters we shall specify shortly, such that $MK = N$.
- Given a public random matrix $\mathbf{A} \in \mathbb{Z}_q^{M \times w}$ where $M \gg w$, we define a commitment $\mathbf{C}$ to an input $x$ as
$$\mathbf{C} := \mathbf{AR} + x \otimes \mathbf{G} + \mathbf{E}$$
where $\mathbf{A} \leftarrow \mathbb{Z}_q^{M \times w}$, $\mathbf{R} \leftarrow \mathbb{Z}_q^{w \times \ell M \log q}$ are uniformly random, $\mathbf{E} \leftarrow \chi^{M \times \ell M \log q}$ has its entries chosen from an error distribution $\chi$, $\mathbf{G} \in \mathbb{Z}_q^{M \times M \log q}$ is the gadget matrix [MP12], and we treat $x$ as a row vector of length $\ell$ in $x \otimes \mathbf{G}$.
- Homomorphic evaluation of $f$ on $\mathbf{C}$ yields $\mathbf{C}_f$ satisfying
$$\mathbf{C}_f = \mathbf{AR}_{f,x} + \mathbf{E}_{f,x} + f(x) \cdot \tfrac{q}{2} \in \mathbb{Z}_q^{M \times K} \tag{1}$$
where $f(x) \in \{0,1\}^{M \times K}$, $\mathbf{R}_{f,x} \in \mathbb{Z}_q^{w \times K}$ and $\mathbf{E}_{f,x}$ has small entries.



Our base scheme[2] simply outputs
$$\mathbf{A},\ \mathbf{C} := \mathbf{AR} + x \otimes \mathbf{G} + \mathbf{E},\ \mathbf{R}_{f,x}$$

[1] Our notion of succinct randomized encodings is weaker than prior works: indeed, [BGL+15] required the encoder to run in time sublinear in $N$, whereas we allow the encoder run-time to be polynomial in $N$.

[2] In the WW terminology, this would be a candidate $K$-sim functional encoding for $f_1, \ldots, f_K : \{0,1\}^\ell \to \{0,1\}^M$.

as the encoding of $x$. Decoding computes $\mathbf{C}_f$ given $(\mathbf{C}, f)$, subtracts $\mathbf{A} \cdot \mathbf{R}_{f,x}$ to obtain $f(x) \cdot \frac{q}{2}$ plus error (following equation 1) and rounds to obtain $f(x)$.

The encoding is also succinct: The total size of the encoding (in bits) is

$$O((Mw + M^2\ell + wK) \cdot \log q).$$

Setting $M = N^{1/3}, K = N^{2/3}, w = O(\lambda)$ yields encoding size $\widetilde{O}(N^{2/3})$, where $\widetilde{O}(\cdot)$ hides polynomial factors in $\lambda, \ell$ and the depth of the circuit computing $f$.

The scheme is, however, *completely insecure* as written because, given $\mathbf{C}, \mathbf{R}_{f,x}$ and a "guess" for $x$, we can recover $\mathbf{R}$ by solving a system of linear equations, and test if our guess was correct (see WW). This allows us to easily distinguish between encodings of any $x_0$ and $x_1$.

**"Pseudorandom" LWE Sampling.** Following [WW21], we fix the insecurity of the base scheme by masking $\mathbf{R}_{f,x}$ using a "pseudorandom" LWE sample; similar ideas were used in several prior works [BDGM20a, GP21, JLS21, AR17, Agr19, JLMS19, AJL+19] with "pseudorandom" noise. That is, we generate a "pseudorandom" LWE sample $\mathbf{B}^* = \mathbf{AS}^* + \mathbf{E}^* \in \mathbb{Z}_q^{M \times K}$ and output

$$\mathsf{seed}_{\mathbf{B}^*},\ \mathbf{A},\ \mathbf{AR} + x \otimes \mathbf{G} + \mathbf{E},\ \mathbf{R}_{f,x} + \mathbf{S}^* \tag{2}$$

where $\mathsf{seed}_{\mathbf{B}^*}$ is a succinct description of $\mathbf{B}^*$, with $|\mathsf{seed}_{\mathbf{B}^*}| \leq (MK)^\delta$ for some $\delta < 1$. Correctness now relies on the fact that

$$\mathbf{A} \cdot (\mathbf{R}_{f,x} + \mathbf{S}^*) \approx \mathbf{B}^* + \mathbf{C}_f + f(x) \cdot \tfrac{q}{2}.$$

WW's security requirement for the pseudorandom LWE sample, "oblivious LWE sampling", was cumbersome to define, required a simulator, and only made sense in the common reference string model. The reliance on a simulator means the definition did not have an inherently falsifiable format that enables demonstrating insecurity by constructing an efficient attacker. Here, we reformulate a simpler and falsifiable variant that we call "succinct LWE sampling".[3]

Defining pseudorandom LWE sampling, in WW and in our work, is difficult because we want $\mathbf{B}^* = \mathbf{AS}^* + \mathbf{E}^*$ to look like a random LWE sample, but this is impossible since it is succinctly described in $\mathsf{seed}_{\mathbf{B}^*}$. Instead, we essentially want $\mathbf{E}^*$ to drown out the difference between any two sufficiently small error distributions $\mathbf{Z}_0$ and $\mathbf{Z}_1$, in the sense that $\mathsf{seed}_{\mathbf{B}^*}, \mathbf{E}^* - \mathbf{Z}_b$ hides $b$. Unfortunately, this too is impossible, since $\mathsf{seed}_{\mathbf{B}^*}$ lets us get $\mathbf{B}^* = \mathbf{AS}^* + \mathbf{E}^*$ from which we can then derive $\mathbf{AS}^* + \mathbf{Z}_b$; this allows us to distinguish between (say) $\mathbf{Z}_0 = 0$ and $\mathbf{Z}_1$ being a small Gaussian by checking rank. Our main observation is that we don't need indistinguishability to hold for worst-case distributions $\mathbf{Z}_b$, but rather only for ones where an LWE sample $\mathbf{AR} + \mathbf{Z}_b$ with the error $\mathbf{Z}_b$ and a truly random $\mathbf{R}$ would hide the bit $b$. Formally, the definition says that for any two distributions of $(\mathbf{Z}_b, \mathsf{aux}_b)$ where $\mathbf{Z}_b$ is sufficiently short:

$$\begin{aligned} &\textbf{If} \qquad (\mathsf{aux}_0, \mathbf{A}, \mathbf{AR} + \mathbf{Z}_0) \approx_c (\mathsf{aux}_1, \mathbf{A}, \mathbf{AR} + \mathbf{Z}_1), && (3) \\ &\textbf{then} \qquad (\mathsf{seed}_{\mathbf{B}^*}, \mathsf{aux}_0, \mathbf{A}, \mathbf{E}^* - \mathbf{Z}_0) \approx_c (\mathsf{seed}_{\mathbf{B}^*}, \mathsf{aux}_1, \mathbf{A}, \mathbf{E}^* - \mathbf{Z}_1). && (4) \end{aligned}$$

Note that, since $\mathsf{seed}_{\mathbf{B}^*}$ defines $\mathbf{AS}^* + \mathbf{E}^*$, giving $\mathbf{E}^* - \mathbf{Z}_b$ in (4) is equivalent to giving $\mathbf{AS}^* + \mathbf{Z}_b$, and hence we use these interchangeably in the definition.

[3] It is simpler in terms of syntax, since we do not refer to LWE trapdoors for $\mathbf{A}$, and in terms of the security requirement since we do not require a simulator, but instead have a simple indistinguishability criterion.

The above definition is not falsifiable since it quantifies over all $(\mathsf{aux}_b, \mathbf{Z}_b)$ satisfying the pre-condition (3). However, we also consider a weaker, falsifiable definition, where we fix a specific $(\mathsf{aux}_b^*, \mathbf{Z}_b^*)$ that satisfies the pre-condition (3). We then show a generic transformation that lifts any scheme realizing the weak definition into one that realizes the general definition. Specifically, in the weak definition, we fix $\mathsf{aux}_b^* = (\widehat{\mathbf{B}}, \mathbf{C})$ to consist of a commitment $\widehat{\mathbf{B}}$ to 0, along with a commitment $\mathbf{C}$ to $-b$. We then homomorphically evaluate an AND operation (multiplication) on the commitments $\widehat{\mathbf{B}}, \mathbf{C}$, which results in a commitment to 0, and we define $\mathbf{Z}_b^*$ to be the error term for this commitment. Formally,

$$\mathsf{aux}_b^* = \left(\widehat{\mathbf{B}} = \mathbf{A}\mathbf{S}_0 + \mathbf{F}, \quad \mathbf{C} = \mathbf{A}\mathbf{R} + \mathbf{E} - b\mathbf{G}\right) \quad \text{and} \quad \mathbf{Z}_b^* = \mathbf{E}\mathbf{G}^{-1}(\widehat{\mathbf{B}}) - b\mathbf{F},$$

where $\mathbf{E}$ and $\mathbf{F}$ are matrices with small entries. The transformation is inspired by a trick employed in WW to frame the security of their candidate oblivious LWE sampler construction as a falsifiable assumption. Here, we are able to abstract this trick out and formally prove that it amplifies a weak definition of security to a strong one. Therefore, we get a simple and falsifiable definition of succinct LWE sampling as our target. We refer to Section 3.3 for more details.

Our final definition introduces additional relaxations. Instead of a uniformly random matrix $\mathbf{A}$, we allow the use of matrices $\mathbf{A}^*$, which may not be uniformly random and can have some additional structure, as long as LWE still holds w.r.t. $\mathbf{A}^*$. We also allow the succinct sampler to rely on a non-succinct common reference string (CRS) of length $\mathrm{poly}(N)$. This is analogous to the reliance on a CRS in WW (as well as [BDGM20a, GP21]) and suffices for iO.

**Our Succinct Randomized Encoding.** To go from succinct LWE sampling to SRE, we essentially follow WW, and replace $\mathbf{A}$ with $\mathbf{A}^*$ in (2). The SRE consists of:

$$\mathsf{seed}_{\mathbf{B}^*}, \ \mathbf{A}^*, \quad \mathbf{A}^*\mathbf{R} + x \otimes \mathbf{G} + \mathbf{E}, \quad \mathbf{R}_{f,x} + \mathbf{S}^* \ . \tag{5}$$

Correctness and succinctness follow readily as before. To prove security, we need to argue as follows that $\mathsf{Encode}(f, x_b)$ hides $b$ as long as $f(x_0) = f(x_1)$.

- As long as $\mathbf{A}^*$ is full-rank, $(\mathbf{R}_{f,x_b} + \mathbf{S}^*)$ can be computed from $\mathbf{A}^*$ and $\mathbf{A}^* \cdot (\mathbf{R}_{f,x_b} + \mathbf{S}^*)$, so it suffices to argue that:

  $$\mathsf{seed}_{\mathbf{B}^*}, \quad \mathbf{A}^*, \quad \mathbf{A}^*\mathbf{R} + x_b \otimes \mathbf{G} + \mathbf{E}, \quad \mathbf{A}^* \cdot (\mathbf{R}_{f,x_b} + \mathbf{S}^*)$$

  hides $b$.

- Using $\mathbf{C}_f = \mathbf{A}^*\mathbf{R}_{f,x_b} + \mathbf{E}_{f,x_b} + f(x_b) \cdot \frac{q}{2}$ and deriving $\mathbf{B}^* = \mathbf{A}^*\mathbf{S}^* + \mathbf{E}^*$ from $\mathsf{seed}_{\mathbf{B}^*}$, we can write

  $$\mathbf{A}^* \cdot (\mathbf{R}_{f,x_b} + \mathbf{S}^*) = \mathbf{C}_f - f(x_b) \cdot \tfrac{q}{2} + \mathbf{B}^* - \mathbf{E}^* - \mathbf{E}_{f,x_b},$$

  so it suffices to argue that

  $$\mathsf{seed}_{\mathbf{B}^*}, \quad \mathbf{A}^*, \quad \mathbf{A}^*\mathbf{R} + x_b \otimes \mathbf{G} + \mathbf{E}, \quad \mathbf{E}^* + \mathbf{E}_{f,x_b}$$

  hides $b$.

- At this point, we will invoke security of our succinct LWE sampler with

  $$\mathsf{aux}_b = \mathbf{A}^*\mathbf{R} + x_b \otimes \mathbf{G} + \mathbf{E}, \qquad \mathbf{Z}_b = \mathbf{E}_{f,x_b}$$

  For this step, we need to show that the pre-condition (3) holds:

  $$(\mathbf{A}^*\mathbf{R} + x_0 \otimes \mathbf{G} + \mathbf{E},\ \mathbf{A}^*,\ \mathbf{A}^*\mathbf{S}' + \mathbf{E}_{f,x_0}) \approx_c (\mathbf{A}^*\mathbf{R} + x_1 \otimes \mathbf{G} + \mathbf{E},\ \mathbf{A}^*,\ \mathbf{A}^*\mathbf{S}' + \mathbf{E}_{f,x_1}).$$

  This follows from LWE w.r.t. $\mathbf{A}^*$ and the fact that $\mathbf{A}^*\mathbf{S}' + \mathbf{E}_{f,x_b} \equiv \mathbf{A}^*\mathbf{S}' + \mathbf{C}_f - f(x_b) \cdot \frac{q}{2}$, where $f(x_0) = f(x_1)$.

Note that, in the above, we only relied on the security of the LWE sampler for the special case where $\mathsf{aux}_b$ is an encryption of $x_b$ and $\mathbf{Z}_b$ is the error in the ciphertext one gets by homomorphically computing $f(x_b)$ for some function $f$ such that $f(x_0) = f(x_1)$. However, as mentioned previously, we can also rely on an even more restricted form of $(\mathsf{aux}_b, \mathbf{Z}_b)$, essentially corresponding to the extremely simple case where $f$ just computes the AND of $b$ and 0, and generically lift security to the completely general case.

**Our Candidate Succinct LWE Sampler.** We want to design a succinct LWE sampler generating $\mathbf{B}^* = \mathbf{A}^*\mathbf{S}^* + \mathbf{E}^*$. The security requirement in Equation (4) implies that $\mathbf{E}^* - \mathbf{Z}_b$ hides $b$ for any short matrices $\mathbf{Z}_0, \mathbf{Z}_1$ satisfying some additional properties which we shall ignore in the rest of this overview. In addition, we want $\mathbf{B}^*$ to admit a short description $\mathsf{seed}_{\mathbf{B}^*}$, which means that $\mathbf{E}^* \in \mathbb{Z}^{M\times K}$ should compute a "pseudorandom" noise-flooding distribution.

Following [JLMS19, AJL+19], a good candidate for $\mathbf{E}^*$ is to evaluate $MK$ random degree-$d$ polynomials in $dmk$ variables drawn from independent Gaussian distributions, where $MK \ll (dmk)^{d/2}$ to avoid linearization and potential sum-of-squares-based attacks; the ensuing distribution is plausibly indistinguishable from $MK$ independent samples from a "noise-flooding" distribution $\mathcal{D}$ for a suitable choice of parameters. Concretely, thinking of $d$ as a small constant, we sample "secret" Gaussian matrices $\mathbf{E}_1, \ldots, \mathbf{E}_d \leftarrow \chi^{m\times k}$ and public Gaussian matrices $\mathbf{P} \leftarrow \chi^{M\times m^d}$ and $\mathbf{P}' \leftarrow \chi^{k^d\times K}$ and we define

$$\mathbf{E}^* := \mathbf{P}(\mathbf{E}_1 \otimes \mathbf{E}_2 \otimes \cdots \otimes \mathbf{E}_d)\mathbf{P}' \in \mathbb{Z}^{M\times K}$$

where $\mathbf{P}, \mathbf{P}'$ are published in the CRS. In the special case of $m = M = 1$ and $\mathbf{P} = 1$, the distribution of $\mathbf{E}^* \in \mathbb{Z}^K$ corresponds roughly to the evaluation of $K$ random (i.e. Gaussian) degree-$d$ (multilinear) polynomials in $dk$ variables (where the $dk$ variables are the entries of the $\mathbf{E}_1, \ldots, \mathbf{E}_d$ and the coefficients of the polynomial are specified by $\mathbf{P}'$). In the general case, we have a collection of polynomials, where each one looks at a certain structured set of monomials. For more details, see Section 4.5.

Next, we specify $(\mathbf{B}^*, \mathbf{A}^*, \mathbf{S}^*, \mathsf{seed}_{\mathbf{B}^*})$, starting with $\mathsf{seed}_{\mathbf{B}^*}$. Following [JLMS19], we additionally sample $\mathbf{A}_i \leftarrow \mathbb{Z}_q^{m\times w}, \mathbf{S}_i \leftarrow \mathbb{Z}_q^{w\times k}$ for $i = 1, \ldots, d$ and some $w \ll m, k$, and we define:

$$\mathsf{seed}_{\mathbf{B}^*} := (\mathbf{B}_1 := \mathbf{A}_1\mathbf{S}_1 + \mathbf{E}_1\ ,\ \ldots\ ,\ \mathbf{B}_d := \mathbf{A}_d\mathbf{S}_d + \mathbf{E}_d) \in (\mathbb{Z}_q^{m\times k})^d.$$

Inspired by the homomorphic operations of the Brakerski-Vaikuntanathan FHE [BV11], we want to relate $\mathbf{E}^*$ to $\mathbf{B}_1 \otimes \cdots \otimes \mathbf{B}_d$ and from there, derive $\mathbf{B}^*, \mathbf{A}^*, \mathbf{S}^*$ such that $\mathbf{B}^* = \mathbf{A}^*\mathbf{S}^* + \mathbf{E}^*$ (we will discuss succinctness after that). We start with $d = 2$ for simplicity. By the mixed product property:

$$\mathbf{B}_1 \otimes \mathbf{B}_2 = \mathbf{A}_1\mathbf{S}_1 \otimes \mathbf{B}_2 \ + \ \mathbf{E}_1 \otimes \mathbf{A}_2\mathbf{S}_2 \ + \ \mathbf{E}_1 \otimes \mathbf{E}_2 = [\mathbf{A}_1 \otimes \mathbf{I}_m \mid \mathbf{I}_m \otimes \mathbf{A}_2]\begin{pmatrix}\mathbf{S}_1 \otimes \mathbf{B}_2 \\ \mathbf{E}_1 \otimes \mathbf{S}_2\end{pmatrix} \ + \ \mathbf{E}_1 \otimes \mathbf{E}_2.$$

We start by defining $\mathbf{B}^*$ and "pre-cursor" values $\overline{\mathbf{A}}^*, \overline{\mathbf{S}}^*$, which we will use to derive the final $\mathbf{A}^*, \mathbf{S}^*$ later, via:

$$\overbrace{\mathbf{P}\cdot(\mathbf{B}_1 \otimes \mathbf{B}_2)\cdot\mathbf{P}'}^{\mathbf{B}^*} = \overbrace{\mathbf{P}[\mathbf{A}_1 \otimes \mathbf{I}_m \mid \mathbf{I}_m \otimes \mathbf{A}_2]}^{\overline{\mathbf{A}}^*}\cdot \overbrace{\begin{pmatrix}\mathbf{S}_1 \otimes \mathbf{B}_2 \\ \mathbf{E}_1 \otimes \mathbf{S}_2\end{pmatrix}\mathbf{P}'}^{\overline{\mathbf{S}}^*} + \overbrace{\mathbf{P}(\mathbf{E}_1 \otimes \mathbf{E}_2)\mathbf{P}'}^{\mathbf{E}^*}$$

For general $d$, we have:

$$\begin{aligned}
\mathbf{B}^* &= \mathbf{P}\cdot(\mathbf{B}_1 \otimes \cdots \otimes \mathbf{B}_d)\cdot\mathbf{P}' \in \mathbb{Z}_q^{M\times K}, \quad \mathbf{E}^* = \mathbf{P}(\mathbf{E}_1 \otimes \mathbf{E}_2 \otimes \cdots \otimes \mathbf{E}_d)\mathbf{P}' \in \mathbb{Z}^{M\times K},\\
\overline{\mathbf{A}}^* &= \mathbf{P}\cdot(\mathbf{A}_1 \otimes \mathbf{I}_m \otimes \cdots \otimes \mathbf{I}_m \| \cdots\cdots \| \mathbf{I}_m \otimes \cdots \otimes \mathbf{I}_m \otimes \mathbf{A}_d) \in \mathbb{Z}_q^{M\times dwm^{d-1}},\\
\overline{\mathbf{S}}^* &= \begin{pmatrix}\mathbf{S}_1 \otimes \mathbf{B}_2 \otimes \cdots \otimes \mathbf{B}_d \\ \mathbf{E}_1 \otimes \mathbf{S}_2 \otimes \cdots \otimes \mathbf{B}_d \\ \vdots \\ \mathbf{E}_1 \otimes \mathbf{E}_2 \otimes \cdots \otimes \mathbf{S}_d\end{pmatrix}\cdot\mathbf{P}' \in \mathbb{Z}_q^{dwm^{d-1}\times K}, \text{ which we show satisfy}
\end{aligned}$$

$$\mathbf{B}^* = \overline{\mathbf{A}}^* \cdot \overline{\mathbf{S}}^* + \mathbf{E}^*.$$

Note that while the width of $\mathbf{A}$ in both the base scheme and WW is $w = \mathrm{poly}(\lambda)$, the width of $\overline{\mathbf{A}}^*$ is much larger and will in fact grow with $N$.

As mentioned above, it seems reasonable to conjecture that $\mathbf{E}^*$ on its own is pseudo-iid. However, $\overline{\mathbf{S}}^*$ is structured and does not look random on its own, which is problematic since we want $\overline{\mathbf{S}}^* + \mathbf{R}_{f,x}$ to drown out differences in the distribution of $\mathbf{R}_{f,x}$. Therefore, we will rely on a variant of Kilian randomization [Kil88] to hide the structure of $\overline{\mathbf{A}}^*, \overline{\mathbf{S}}^*$. We compute a random basis $\mathbf{A}^*$ of the column span of $\overline{\mathbf{A}}^*$ and then solve for $\mathbf{S}^*$ subject to $\mathbf{A}^*\mathbf{S}^* = \overline{\mathbf{A}}^* \cdot \overline{\mathbf{S}}^*$. This ensures that $\mathbf{A}^*, \mathbf{S}^*$ essentially do not reveal more than the product $\overline{\mathbf{A}}^*\overline{\mathbf{S}}^*$.

**Succinctness.** With the above implementation of succinct LWE sampling, from (5), the encodings of the resulting SRE have size

$$|\mathsf{Encode}(f,x)| = \widetilde{O}\left(\underbrace{M^2}_{\mathbf{A}^*\mathbf{R}+x\otimes\mathbf{G}+\mathbf{E}} + \underbrace{dmk}_{\mathsf{seed}_{\mathbf{B}^*}} + \underbrace{Mdwm^{d-1}}_{\mathbf{A}^*} + \underbrace{Kdwm^{d-1}}_{\mathbf{S}^*+\mathbf{R}_{f,x}}\right)$$

where $\widetilde{O}(\cdot)$ hides $\mathrm{poly}(\lambda, \log q, \ell)$ factors, which is in turn polynomial in $\lambda, \ell$ and circuit depth of $f$. We set

$$\begin{aligned}
w &= \mathrm{poly}(\lambda), & M &= m^{d-1/2} = N^{\frac{1}{2}-\frac{1}{4d}},\\
m &= N^{\frac{1}{2d}}, & K &= m^{d+1/2} = N^{\frac{1}{2}+\frac{1}{4d}}.\\
k &= m^5 = N^{\frac{5}{2d}}, & &
\end{aligned}$$

Then, $|\mathsf{Encode}(f,x)| = \widetilde{O}(m^{2d-1/6}) = \widetilde{O}(N^{1-\frac{1}{12d}})$, that is, our scheme achieves $(1-\frac{1}{12d})$-succinctness, which can then be lifted to iO using [AJ15, BV15, LPST16].

**Our Final Assumption: Subspace Flooding.** Combined with the transformation discussed earlier, we only need our sampler to satisfy weak security, which boils down to the following *subspace flooding* assumption: that

$$\mathbf{P}, \mathbf{P}', \mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \widehat{\mathbf{B}} = \mathbf{A}^*\mathbf{S}_0 + \mathbf{F}, \mathbf{C} = \mathbf{A}^*\mathbf{R} + \mathbf{E} - b\mathbf{G}, \mathbf{E}^* + \mathbf{E} \cdot \mathbf{G}^{-1}(\widehat{\mathbf{B}}) - b\mathbf{F} \tag{6}$$

hides $b$ where $\mathbf{P} \in \mathbb{Z}^{M \times m^d}$, $\mathbf{P}' \in \mathbb{Z}^{k^d \times K}$, $\mathbf{E} \in \mathbb{Z}^{M \times M \log q}$, and $\mathbf{F} \in \mathbb{Z}^{M \times K}$ and $\{\mathbf{E}_i\}_{i \in [d]}$ are sampled from small distributions;

$$\mathbf{E}^* = \mathbf{P}(\mathbf{E}_1 \otimes \mathbf{E}_2 \otimes \cdots \otimes \mathbf{E}_d)\mathbf{P}' \in \mathbb{Z}^{M \times K};$$

for $i = 1, \ldots, d$, $\mathbf{A}_i$ is sampled from $\mathbb{Z}_q^{m \times w}$ and $\mathbf{S}_i$ is sampled from $\mathbb{Z}_q^{w \times k}$;

$$\mathsf{seed}_{\mathbf{B}^*} = \{\mathbf{B}_i = \mathbf{A}_i\mathbf{S}_i + \mathbf{E}_i\}_{i \in [d]} \in (\mathbb{Z}_q^{m \times w})^d;$$

$\mathbf{S}_0$ is sampled from $\mathbb{Z}_q^{dwm^{d-1} \times K}$ and $\mathbf{R}$ is sampled from $\mathbb{Z}_q^{dwm^{d-1} \times M \log q}$ so $\widehat{\mathbf{B}} \in \mathbb{Z}_q^{M \times K}$ and $\mathbf{C} \in \mathbb{Z}_q^{M \times M \log q}$; and $\mathbf{A}^*$ is the result of the Kilian randomization process described above.

Note that the columns of $\mathbf{E} \cdot \mathbf{G}^{-1}(\widehat{\mathbf{B}}) \in \mathbb{Z}^{M \times K}$ live in a low-rank subspace defined by the columns of $\mathbf{E} \in \mathbb{Z}^{M \times M \log q}$ where $K \gg M \log q$ and $\mathbf{F}$ is sampled independently from a small distribution. Thus, the assumption states that $\mathbf{E}^*$ masks whether the error $\mathbf{E}\mathbf{G}^{-1}(\widehat{\mathbf{B}}) - b\mathbf{F} \in \mathbb{Z}^{M \times K}$ lives in this low-rank subspace, hence the name "subspace flooding".

A different, less syntactic, perspective on the subspace flooding assumption tells us that to protect arbitrary computations, it is sufficient to protect a single homomorphic multiplication. Indeed, consider $\mathbf{C}$ to be a GSW encryption of $-b$ and $\widehat{\mathbf{B}}$ to be a GSW encryption of 0. Their homomorphic multiplication gives us

$$\mathbf{C} \cdot \mathbf{G}^{-1}(\widehat{\mathbf{B}}) = \mathbf{A}^*(\mathbf{R}\mathbf{G}^{-1}(\widehat{\mathbf{B}}) - b\mathbf{S}_0) + (\mathbf{E} \cdot \mathbf{G}^{-1}(\widehat{\mathbf{B}}) - b\mathbf{F})$$

Subspace flooding says that adding $\mathbf{E}^*$ "protects" the error $\mathbf{E} \cdot \mathbf{G}^{-1}(\widehat{\mathbf{B}}) - b\mathbf{F}$ in the evaluated ciphertext in the sense of hiding $b$.

**Theorem 1.1** (Informal)**.** *Under the (subexponential hardness of the) learning with errors assumption and the subspace flooding assumption (Equation 6 above), there exists an indistinguishability obfuscation scheme.*

## 1.3 Discussion

**Noise Distribution in Prior Works.** The sampler in WW sampler works by homomorphically generating pseudorandom LWE samples using an encrypted (weak) pseudorandom function, such as that given by $k, u \mapsto \mathsf{round}(\langle k, u \rangle)$ for key $k$ and random input $u$. Prior works used the GSW FHE for homomorphic evaluation, but did not specify the circuit implementation for the PRF. Hopkins, Jain and Lin (HJL) [HJL21] presented attacks on these prior LWE samplers that "exploit the flexibility to choose specific implementations of circuits and LWE error distributions in the Gay-Pass and Wee-Wichs assumptions." Specifically, they showed how to introduce redundancy into the circuit used in homomorphic evaluation following the GSW FHE so that the last two bits of $\mathbf{E}^* + \mathbf{Z}_b$ leak $b$.

Note that the above attack can be circumvented by fixing some natural choice of a concrete weak PRF, such as the aforementioned, which corresponds to FHE decryption; and a circuit evaluation of it, such as [AP14], which is in fact a read-once branching program with $k$ hardwired. Unfortunately, writing down an explicit expression for the error distribution in the pseudorandom LWE sample is far from straightforward, which in turn impedes any cryptanalytic efforts. In this work, we avoid such considerations by directly considering succinct LWE samplers, as opposed to homomorphically evaluated weak PRFs.

**Relation to the "LWE with Leakage" Assumption of [JLMS19].** Our assumption basically asserts that for small $\mathbf{Z}_0, \mathbf{Z}_1$ satisfying some precondition:

$$\mathbf{A}_1, \ldots, \mathbf{A}_d, \ \ (\mathbf{B}_i := \mathbf{A}_i \mathbf{S}_i + \mathbf{E}_i)_{i \in [d]}, \ \ \mathbf{P}, \mathbf{P}', \ \ \mathbf{P}(\mathbf{E}_1 \otimes \cdots \otimes \mathbf{E}_d)\mathbf{P}' - \mathbf{Z}_b$$

hides $b$. (In fact, we do not give away $\mathbf{A}_1, \ldots, \mathbf{A}_d$, rather a random basis for the column span of $\mathbf{A}^*$. We ignore this difference for the rest of the comparison.)

The LWE with leakage assumption of [JLMS19] basically asserts that for small $\mathbf{z}_0, \mathbf{z}_1$, and $\mathbf{A}_i \in \mathbb{Z}_q^{m \times w}$, $\mathbf{s}_i \in \mathbb{Z}_q^{w \times 1}$, $\mathbf{e}_i \in \chi^{m \times 1}$:

$$\mathbf{A}_1, \ldots, \mathbf{A}_{d-2}, \ \ (\mathbf{b}_i := \mathbf{A}_i \mathbf{s}_i + \mathbf{e}_i)_{i \in [d-2]}, \ \ \mathbf{P}, \ \ \mathbf{P}(\mathbf{e}_1 \otimes \cdots \otimes \mathbf{e}_d) + \mathbf{z}_b$$

hides $b$.

The LWE with leakage assumption of [JLMS19] can be viewed as a variant of our flooding assumption. Syntactically, their definition can be recovered from ours with three modifications:

1. Set $k = 1$ as opposed to our assumption where $k \gg m$;
2. Set $\mathbf{P}$ to be very compressing, namely, the output has length $M \ll m^{d/2}$, whereas in our case $M \approx m^{d-1/2}$; and
3. Do not release $\mathbf{A}_{d-1}, \mathbf{A}_d, \mathbf{B}_{d-1}, \mathbf{B}_d$ to the distinguisher, ensuring that the only leakage about $\mathbf{e}_{d-1}, \mathbf{e}_d$ comes from $\mathbf{E}^*$.

These syntactic differences have the following consequences:

- With $k = 1$ and $M \approx m^{d-1/2}$, the assumption can indeed be broken with sum-of-squares attacks (see, e.g., [BHJ+19].) Thus, our source of security comes from the fact that $k$ is large. Semantically, this means that we take multiple, albeit correlated, instances of the [JLMS19] problem, defined by the $k^d$ columns of our matrix $\mathbf{E}_1 \otimes \cdots \otimes \mathbf{E}_d$, and output a "few", namely, $K \ll k^{d/2}$ linear combinations of them.
- An adversary in our setting can check the rank of

  $$\mathbf{P}(\mathbf{B}_1 \otimes \cdots \otimes \mathbf{B}_d)\mathbf{P}' - \mathbf{E}^* + \mathbf{Z}_b \bmod q$$

  which is something that cannot be computed in the [JLMS19] assumption since $\mathbf{B}_{d-1}, \mathbf{B}_d$ are not given to the distinguisher. This allows the latter to plausibly handle *worst-case small* $\mathbf{z}_b$, whereas we require an additional pre-condition on $\mathbf{Z}_b$.

Their final iO scheme additionally assume bilinear groups (in addition to LWE), which we do not.

**Cryptanalytic Challenges.** A central open problem from this work is to design succinct LWE samplers based on weaker assumptions and to carry out cryptanalysis of our candidate succinct LWE sampler. To facilitate the latter, we describe concrete cryptanalytic challenges in Section 4.6. Thanks to our amplification theorem, in order to base iO on our candidate LWE sampler, it suffices for security to hold for a specific pair of distributions $(\mathbf{Z}_0, \mathbf{Z}_1)$. On the other hand, the heuristic underlying our candidate sampler (related to random polynomials being indistinguishable from independent copies of a noise-flooding distribution $\mathcal{D}$) does not refer to properties of the specific distribution. For this reason, our cryptanalytic challenges also refer to more general distributions $\mathbf{Z}_0, \mathbf{Z}_1$ that may not correspond to those which are sufficient for iO.

# 2 Preliminaries

## 2.1 Notations

We will denote by $\lambda$ the security parameter. The notation $\text{negl}(\lambda)$ denotes any function $f$ such that $f(\lambda) = \lambda^{-\omega(1)}$, and $\text{poly}(\lambda)$ denotes any function $f$ such that $f(\lambda) = \mathcal{O}(\lambda^c)$ for some $c > 0$. For a probabilistic algorithm $\mathsf{alg}(\mathsf{inputs})$, we might explicitly refer to its random coins by writting $\mathsf{alg}(\mathsf{inputs}; \mathsf{coins})$. We will denote vectors by bold lower case letters (e.g. $\mathbf{a}$) and matrices by bold upper cases letters (e.g. $\mathbf{A}$). We will denote by $\mathbf{a}^\top$ and $\mathbf{A}^\top$ the transposes of $\mathbf{a}$ and $\mathbf{A}$, respectively. We will denote by $\lfloor x \rceil$ the nearest integer to $x$, rounding towards 0 for half-integers. For matrices $\mathbf{A}, \mathbf{B}$ of appropriate dimensions, we will denote by $(\mathbf{A} \| \mathbf{B})$ their horizontal concatenation and $\binom{\mathbf{A}}{\mathbf{B}}$ their vertical concatenation. For an integer $n \geq 1$, we denote by $\mathbf{I}_n$ the identity matrix of dimension $n$. For integral vectors and matrices (i.e., those over $\mathbb{Z}$), we use the notation $\|\mathbf{r}\|, \|\mathbf{R}\|$ to denote the maximum absolute value over all the entries.

For matrices $\mathbf{A}, \mathbf{B}$, we denote by $\mathbf{A} \otimes \mathbf{B}$ their tensor (or Kronecker) product. We'll use the following mixed-product property: for matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ of appropriate dimensions, we have $(\mathbf{AB}) \otimes (\mathbf{CD}) = (\mathbf{A} \otimes \mathbf{C}) \cdot (\mathbf{B} \otimes \mathbf{D})$.

For $p \in \mathbb{Q}$, we write $\mathsf{Round}_p(x) = \lfloor x \cdot 1/p \rceil$. If $\mathbf{X}$ is a matrix, $\mathsf{Round}_p(\mathbf{X})$ denotes the rounded value applied component-wise. We denote by $\lceil x \rceil$ the smallest integer larger or equal to $x$.

For a finite set $S$, $s \leftarrow S$ denotes sampling uniformly in $S$. We define the statistical distance between two random variables $X$ and $Y$ over some domain $\Omega$ as: $\mathsf{SD}(X, Y) = \frac{1}{2} \sum_{w \in \Omega} |X(w) - Y(w)|$. We say that two ensembles of random variables $X = \{X_\lambda\}$, $Y = \{Y_\lambda\}$ are *statistically indistinguishable*, denoted $X \approx_s Y$, if $\mathsf{SD}(X_\lambda, Y_\lambda) \leq \text{negl}(\lambda)$.

We say that two ensembles of random variables $X = \{X_\lambda\}$, and $Y = \{Y_\lambda\}$ are *computationally indistinguishable*, denoted $X \approx_c Y$, if, for all (non-uniform) PPT distinguishers $\mathcal{A}$, we have $|\Pr[\mathcal{A}(X_\lambda) = 1] - \Pr[\mathcal{A}(Y_\lambda) = 1]| \leq \text{negl}(\lambda)$. We also refer to sub-exponential security, meaning that there exists some $\varepsilon > 0$ such that the distinguishing advantage is at most $2^{-\lambda^\varepsilon}$.

## 2.2 Learning With Errors

**Definition 2.1** ($B$-bounded distribution)**.** *We say that a distribution $\chi$ over $\mathbb{Z}$ is $B$-bounded if*

$$\Pr[\,\chi \in [-B, B]\,] = 1.$$

We recall the definition of the (decision) *Learning with Errors* problem, introduced by Regev [Reg05].

**Definition 2.2** ((Decision) Learning with Errors ( [Reg05])). *Let $n = n(\lambda)$ and $q = q(\lambda)$ be integer parameters and $\chi = \chi(\lambda)$ be a distribution over $\mathbb{Z}$. The Learning with Errors (LWE) assumption $LWE_{n,q,\chi}$ states that for all polynomials $m = \mathrm{poly}(\lambda)$ the following distributions are computationally indistinguishable:*

$$(\mathbf{A}, \mathbf{A}\mathbf{s} + \mathbf{e}) \approx_c (\mathbf{A}, \mathbf{u})$$

*where* $\mathbf{A} \leftarrow \mathbb{Z}_q^{m\times n}, \mathbf{s} \leftarrow \mathbb{Z}_q^n, \mathbf{e} \leftarrow \chi^m, \mathbf{u} \leftarrow \mathbb{Z}_q^m$.

Just like many prior works, we rely on LWE security with the following range of parameters. We assume that for any polynomial $p = p(\lambda) = \mathrm{poly}(\lambda)$ there exists some polynomial $n = n(\lambda) = \mathrm{poly}(\lambda)$, some $q = q(\lambda) = 2^{\mathrm{poly}(\lambda)}$ and some $B = B(\lambda)$-bounded distribution $\chi = \chi(\lambda)$ such that $q/B \geq 2^p$ and the $LWE_{n,q,\chi}$ assumption holds. Throughout the paper, the *LWE assumption* without further specification refers to the above parameters. The *sub-exponentially secure LWE* assumption further assumes that $LWE_{n,q,\chi}$ with the above parameters is sub-exponentially secure, meaning that there exists some $\varepsilon > 0$ such that the distinguishing advantage of any polynomial-time distinguisher is $2^{-\lambda^\varepsilon}$.

The works of [Reg05, Pei09] showed that the (sub-exponentially secure) LWE assumption with the above parameters follows from the worst-case (sub-exponential) quantum hardness SIVP and classical hardness of GapSVP with sub-exponential approximation factors.

## 2.3 Lattice Tools

**Noise Flooding.** We will use the following fact.

**Lemma 2.3** (Flooding Lemma (e.g., [AJL+12])). *Let $B = B(\lambda), B' = B'(\lambda) \in \mathbb{Z}$ be parameters and let $U([-B, B])$ be the uniform distribution over the integer interval $[-B, B]$. Then for any $e \in [-B', B']$, the statistical distance between $U([-B, B])$ and $U([-B, B]) + e$ is $B'/B$.*

**Gadget Matrix [MP12].** For an integer $q \geq 2$, define: $\mathbf{g} = (1, 2, \cdots, 2^{\lceil \log q \rceil - 1}) \in \mathbb{Z}_q^{1\times \lceil \log q \rceil}$. The *gadget matrix* $\mathbf{G}$ is defined as $\mathbf{G} = \mathbf{g} \otimes \mathbf{I}_n \in \mathbb{Z}_q^{n\times m}$ where $n \in \mathbb{N}$ and $m = n\lceil \log q \rceil$. There exists an efficiently computable deterministic function $\mathbf{G}^{-1} : \mathbb{Z}_q^n \to \{0, 1\}^m$ such for all $\mathbf{u} \in \mathbb{Z}_q^n$ we have $\mathbf{G} \cdot \mathbf{G}^{-1}(\mathbf{u}) = \mathbf{u}$. We let $\mathbf{G}^{-1}(\$)$ denote the distribution obtained by sampling $\mathbf{u} \leftarrow \mathbb{Z}_q^n$ uniformly at random and outputting $\mathbf{t} = \mathbf{G}^{-1}(\mathbf{u})$. These extend directly to matrices: $\mathbf{G}^{-1} : \mathbb{Z}_q^{n\times k} \to \{0, 1\}^{m\times k}$ by concatenating the outputs.

## 2.4 Homomorphic Operations

In this section, we describe how to perform homomorphic operations over certain encodings of inputs. For readers familiar with lattice-based primitives, these essentially are packed versions of the GSW homomorphism.

Our operations follow readily from [WW21] (building on [GSW13, GVW15], along with the "packing" techniques in [PVW08, MW16, BTVW17, PS19, GH19, BDGM19]), who build homomorphic operations for $f : \{0, 1\}^\ell \to \{0, 1\}^M$, producing some vector $\mathbf{c}_f \in \mathbb{Z}_q^M$. We extend these operations to functions $f : \{0, 1\}^\ell \to \{0, 1\}^{M\times K}$ to produce some matrix $\mathbf{C}_f \in \mathbb{Z}_q^{M\times K}$, obtained by concatenating $K$ vectors $\mathbf{c}_{f_i}$. This yields the following.

**Definition 2.4** (Homomorphic operations)**.** *Let $M, W, q, \ell, K, t$ be parameters. We define the following efficient algorithms:*

- $\mathsf{Eval}(f : \{0,1\}^\ell \to \{0,1\}^{M\times K}, \mathbf{C} \in \mathbb{Z}_q^{M\times \ell M \log q})$*: deterministically outputs a matrix* $\mathbf{C}_f \in \mathbb{Z}_q^{M\times Q}$.
- $\mathsf{Eval}_{\mathsf{open}}(f, \mathbf{A} \in \mathbb{Z}_q^{M\times W}, x \in \{0,1\}^\ell, \mathbf{R} \in \mathbb{Z}_q^{W\times \ell M \log q}, \mathbf{E} \in \mathbb{Z}^{M\times \ell M \log q})$*: deterministically outputs two matrices* $(\mathbf{R}_{f,x} \in \mathbb{Z}_q^{W\times Q}, \mathbf{E}_{f,x} \in \mathbb{Z}^{M\times Q})$.

*These operations have the following property. For all* $f : \{0,1\}^\ell \to \{0,1\}^{M\times K}$ *of depth* $t$*,* $\mathbf{x} \in \{0,1\}^\ell$*,* $\mathbf{A} \in \mathbb{Z}_q^{M\times W}$*,* $\mathbf{R} \in \mathbb{Z}_q^{W\times \ell M \log q}$ *and* $\mathbf{E} \in \mathbb{Z}^{M\times \ell M \log q}$*, if*

$$\mathbf{C} = \mathbf{A}\mathbf{R} + x^\top \otimes \mathbf{G} + \mathbf{E} \in \mathbb{Z}_q^{M\times \ell M \log q},$$

$$\mathbf{C}_f = \mathsf{Eval}(f, \mathbf{C}),$$

$$(\mathbf{R}_{f,x}, \mathbf{E}_{f,x}) = \mathsf{Eval}_{\mathsf{open}}(f, \mathbf{A}, x, \mathbf{R}, \mathbf{E}),$$

*where we view $x$ as a row vector $x \in \{0,1\}^{1\times\ell}$, then*

$$\mathbf{C}_f = \mathbf{A}\mathbf{R}_{f,x} + q/2 \cdot f(x) + \mathbf{E}_{f,x} \in \mathbb{Z}_q^{M\times K},$$

*where $f(x) \in \{0,1\}^{M\times K}$. Furthermore $\|\mathbf{E}_{f,x}\| = \|\mathbf{E}\| \cdot M^{g(t)}$ for some efficiently computable $g$ such that $g(t) = \mathcal{O}(t)$.*

Similarly to [WW21], these algorithms extend to functions $f$ with outputs in $\mathbb{Z}_q$.

- $\mathsf{Eval}^q(f : \{0,1\}^\ell \to \mathbb{Z}_q^{M\times K}, \mathbf{C} \in \mathbb{Z}_q^{M\times \ell M \log q})$: deterministically outputs a matrix $\mathbf{C}_f \in \mathbb{Z}_q^{M\times Q}$.
- $\mathsf{Eval}^q_{\mathsf{open}}(f, \mathbf{A} \in \mathbb{Z}_q^{M\times W}, x \in \{0,1\}^\ell, \mathbf{R} \in \mathbb{Z}_q^{W\times \ell M \log q}, \mathbf{E} \in \mathbb{Z}^{M\times \ell M \log q})$: deterministically outputs two matrices $(\mathbf{R}_f \in \mathbb{Z}_q^{W\times Q}, \mathbf{E}_f \in \mathbb{Z}^{M\times Q})$.

The correctness requirement becomes:

$$\mathbf{C}_f = \mathbf{A}\mathbf{R}_{f,x} + f(x) + \mathbf{E}_{f,x} \in \mathbb{Z}_q^{M\times K},$$

where $\mathbf{C} = \mathbf{A}\mathbf{R} + x \otimes \mathbf{G} + \mathbf{E} \in \mathbb{Z}_q^{M\times \ell M \log q}$, $x$ being again seen as a row vector, $\mathbf{C}_f = \mathsf{Eval}^q(f, \mathbf{C})$ and $(\mathbf{R}_{f,x}, \mathbf{E}_{f,x}) = \mathsf{Eval}^q_{\mathsf{open}}(f, \mathbf{A}, x, \mathbf{R}, \mathbf{E})$, and $f(x) \in \mathbb{Z}_q^{M\times K}$. Again, $\|\mathbf{E}_{f,x}\| = \|\mathbf{E}\| \cdot M^{g(t)}$.

## 2.5 Succinct Randomized Encodings

Next, we define succinct randomized encodings [BGL+15, BCG+18, LPST16].

**Definition 2.5.** *A* succinct randomized encoding scheme *(SRE) for the function family $\mathcal{F}_{\ell,N,t} = \{f : \{0,1\}^\ell \to \{0,1\}^N\}$ of circuits of depth at most $t$, is a tuple of PPT algorithms* (CRSGen, Encode, Decode) *with the following syntax:*

- $\mathsf{CRSGen}(1^\lambda, \mathcal{F}_{\ell,N,t}) \to \mathsf{crs}$*: on input the security parameter and a function family, outputs* crs.
- $\mathsf{Encode}(\mathsf{crs}, f, x) \to C$*: on input* crs*, a function $f \in \mathcal{F}_{\ell,N,t}$ and $x \in \{0,1\}^\ell$, outputs an encoding $C$.*
- $\mathsf{Decode}(\mathsf{crs}, C, f) \to y$*: a deterministic algorithm which, on input* crs*, an encoding $C$, and a function $f \in \mathcal{F}_{\ell,N,t}$, outputs a value $y \in \{0,1\}^N$.*

*We require the following properties:*

**Correctness:** *For $f \in \mathcal{F}_{\ell,N,t}$ and any $x \in \{0,1\}^\ell$:*

$$\Pr\left[\mathsf{Decode}(\mathsf{crs}, \mathsf{Encode}(\mathsf{crs}, f, x), f) = f(x)\right] \geq 1 - \text{negl}(\lambda),$$

*where* $\mathsf{crs} \leftarrow \mathsf{CRSGen}(1^\lambda, \mathcal{F}_{\ell,N,t})$ *(over the randomness of* $\mathsf{CRSGen}, \mathsf{Encode}$*).*

**$\delta$-Succinctness:** *There exists a constant $\delta < 1$ such that, for all* $\mathsf{crs} \leftarrow \mathsf{CRSGen}(1^\lambda, \mathcal{F}_{\ell,N,t})$*,* $C \leftarrow \mathsf{Encode}(\mathsf{crs}, f, x)$*, we have:*

$$|C| = N^\delta \cdot \text{poly}(\lambda, \ell, t).$$

**Indistinguishability-based Security:** *For all PPT $\mathcal{A}$, all $x_0, x_1 \in \ell$, and all $f \in \mathcal{F}_{t,\ell,N}$ such that $f(x_0) = f(x_1)$, the following distributions are indistinguishable for $b = 0$ and $b = 1$:*

- $\mathcal{D}_b$*: Sample* $\mathsf{crs} \leftarrow \mathsf{CRSGen}(1^\lambda, \mathcal{F}_{t,\ell,N})$*,* $C_b \leftarrow \mathsf{Encode}(\mathsf{crs}, f, x_b)$*. Output* $(\mathsf{crs}, C_b)$*.*

**Relation to XiO.** Our notion of SRE is also very related to the notion of exponentially efficient iO (XiO) from [LPST16]. An XiO scheme obfuscates a circuit $C : \{0,1\}^{\log N} \rightarrow \{0,1\}$ with the same security guarantee as iO, but the run-time of the obfuscator can be as high as poly$(\lambda, |C|, N)$ and the only constraint that makes the problem non-trivial is that the obfuscated circuit is succinct, of size at most $N^\delta$poly$(\lambda, |C|)$ for $\delta < 1$. An SRE scheme immediately yields an XiO scheme by thinking of $f$ as the universal circuit that takes as input a circuit $x = C$ an evaluates it on all $N$ inputs in $\{0,1\}^{\log N}$. The output size of $f$ is $N$ and the depth of $f$ can be bounded by $t = \text{poly}(|C|)$, so the succinctness of the SRE yields the corresponding succinctness of the XiO. Therefore, by leveraging the prior work of [LPST16] that shows how to go from XiO (in the CRS model) to iO via LWE, we get the following theorem.

**Theorem 2.6.** [AJ15, BV15, LPST16] *Assuming sub-exponentially secure SRE exist and sub-exponentially secure LWE, there exists an iO scheme.*

# 3 Succinct LWE Sampler: Definition and Amplification

In Section 3.1, we define the notion of succinct LWE samplers. In Section 3.2, we describe a seemingly weaker notion of LWE sampler, and prove that it implies the first (and stronger) notion.

## 3.1 Definition and Discussion

**Definition 3.1** (Succinct LWE Sampler)**.** *A* succinct LWE sampler *is a tuple of PPT algorithms* $(\mathsf{SampCRSGen}, \mathsf{LWEGen}, \mathsf{Expand})$ *with the following syntax:*

- $\mathsf{SampCRSGen}(1^\lambda, 1^N, \alpha)$*: on input the security parameter $\lambda$, a size parameter $N$ and a blow-up factor $\alpha$, samples a common reference string* $\mathsf{crs}$*, which include parameters* $\mathsf{params} = (q, M, K, \overline{\chi}, \overline{B})$*.*
- $\mathsf{LWEGen}(\mathsf{crs})$*: samples* $(\mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \mathbf{S}^*)$*.*
- $\mathsf{Expand}(\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*})$ *is a deterministic algorithm that outputs a matrix* $\mathbf{B}^*$*.*

**Domains and Parameters.** *The outputs of* $\mathsf{LWEGen}$ *and* $\mathsf{Expand}$ *satisfy:*

$$\mathbf{A}^* \in \mathbb{Z}_q^{M\times W}, \quad \mathbf{S}^* \in \mathbb{Z}_q^{W\times K}, \quad \mathbf{B}^* \in \mathbb{Z}_q^{M\times K},$$

*for some integer $W$. We require that:*

- $N = MK$*;*
- $\overline{B} = \mathrm{poly}(N)$*;*
- $\overline{\chi}$ *is a $\overline{B}$-bounded noise distribution; and*
- $q \geq 8 \cdot 2^\lambda \cdot \alpha \cdot \overline{B}$*.*

**Correctness.** *We require that*

$$||\mathbf{B}^* - \mathbf{A}^*\mathbf{S}^*|| := \beta \leq q/8$$

*where* $\mathsf{crs} \leftarrow \mathsf{SampCRSGen}(1^\lambda, 1^N, \alpha)$*,* $(\mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \mathbf{S}^*) \leftarrow \mathsf{LWEGen}(\mathsf{crs})$ *and* $\mathbf{B}^* := \mathsf{Expand}(\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*})$*. Furthermore, we require that $\mathbf{A}^*$ is full-rank with overwhelming probability over the randomness of* $\mathsf{SampCRSGen}$ *and* $\mathsf{LWEGen}$*.*

**$\delta$-Succinctness.** *We require the total bit length of the output of* $\mathsf{LWEGen}$ *is small. That is,*

$$\mathsf{bitlength}(\mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \mathbf{S}^*) \leq N^\delta \cdot \mathrm{poly}(\lambda, \log q) = (MK)^\delta \cdot \mathrm{poly}(\lambda, \log q) \ ,$$

*where $\delta < 1$ is a constant. When we omit $\delta$, it means succinctness holds for some constant $\delta < 1$.*

**LWE with respect to $\mathbf{A}^*$.** *We require that*

$$(\mathsf{coins}_{\mathsf{crs}}, \mathsf{coins}_{\mathsf{seed}}, \mathbf{A}^*\mathbf{s}' + \mathbf{e}') \approx_c (\mathsf{coins}_{\mathsf{crs}}, \mathsf{coins}_{\mathsf{seed}}, \mathbf{b}),$$

*where* $\mathsf{crs} = \mathsf{SampCRSGen}(1^\lambda, 1^N, \alpha; \mathsf{coins}_{\mathsf{crs}})$*,* $(\mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \mathbf{S}^*) \leftarrow \mathsf{LWEGen}(\mathsf{crs}; \mathsf{coins}_{\mathsf{seed}})$*,* $\mathbf{s}' \leftarrow \mathbf{Z}_q^W$*, and* $\mathbf{e}' \leftarrow \overline{\chi}^M$*.*

**Security (or $\beta_0$-Flooding).** *Let $D_0, D_1$ be any two polynomial-time samplable distributions such that* $(\mathsf{aux}_b, \mathbf{Z}_b) \leftarrow D_b(\mathbf{A}^*)$ *satisfies $\mathbf{Z}_b \in \mathbb{Z}^{M\times K}$, $\|\mathbf{Z}_b\| \leq \beta_0$ where $\beta_0 \cdot 2^\lambda \leq \beta$ and*

$$(\mathsf{coins}_{\mathsf{crs}}, \mathsf{coins}_{\mathsf{seed}}, \mathbf{A}^*\mathbf{S}' + \mathbf{Z}_0, \mathsf{aux}_0) \approx_c (\mathsf{coins}_{\mathsf{crs}}, \mathsf{coins}_{\mathsf{seed}}, \mathbf{A}^*\mathbf{S}' + \mathbf{Z}_1, \mathsf{aux}_1)$$

*where* $\mathsf{crs} = \mathsf{SampCRSGen}(1^\lambda, 1^N, \alpha; \mathsf{coins}_{\mathsf{crs}})$*,* $(\mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \mathbf{S}^*) = \mathsf{LWEGen}(\mathsf{crs}; \mathsf{coins}_{\mathsf{seed}})$ *and* $\mathbf{S}' \leftarrow \mathbb{Z}_q^{W\times K}$*. Then,*

$$(\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \mathbf{A}^*\mathbf{S}^* + \mathbf{Z}_0, \mathsf{aux}_0) \approx_c (\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \mathbf{A}^*\mathbf{S}^* + \mathbf{Z}_1, \mathsf{aux}_1).$$

*We will refer to the assumption on $D_0, D_1$ as the* pre-condition *for security, and the resulting indistinguishability the* post-condition*.*
*Furthermore, as we will later describe a relaxed notion of security, we will sometimes refer to the notion above as* ***strong security*** *to avoid ambiguity.*

**Remark 3.1** (Alternate formulation)**.** Since the sampler allows us to compute $\mathsf{Expand}(\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*}) = \mathbf{B}^* = \mathbf{A}^*\mathbf{S}^* + \mathbf{E}^*$, the security post-condition can be equivalently stated as:

$$(\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \mathbf{E}^* - \mathbf{Z}_0, \mathsf{aux}_0) \approx_c (\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \mathbf{E}^* - \mathbf{Z}_1, \mathsf{aux}_1).$$

**Remark 3.2** (Implied Statements)**.** The randomness $\mathsf{coins}_{\mathsf{crs}}$ and $\mathsf{coins}_{\mathsf{seed}}$ respectively used by $\mathsf{SampCRSGen}$ and $\mathsf{LWEGen}$ allow us to compute $\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \mathbf{S}^*$. In particular, LWE with respect to $\mathbf{A}^*$ implies that

$$(\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \mathbf{S}^*, \mathbf{A}^*\mathbf{s}' + \mathbf{e}) \approx_c (\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \mathbf{S}^*, \mathbf{b}),$$

and the pre-condition on $D_0, D_1$ for security implies that

$$(\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \mathbf{S}^*, \mathsf{aux}_0, \mathbf{A}^*\mathbf{S}' + \mathbf{Z}_0) \approx_c (\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \mathbf{S}^*, \mathsf{aux}_1, \mathbf{A}^*\mathbf{S}' + \mathbf{Z}_1).$$

**Remark 3.3** (Restrictions on $\mathbf{Z}_0, \mathbf{Z}_1$)**.** We note that security (namely, the post-conditionition) cannot hold for arbitrary $\mathbf{Z}_0, \mathbf{Z}_1$, for which the pre-condition does not hold. Even if one only required that $\mathbf{Z}_0$ and $\mathbf{Z}_1$ had small entries, one can efficiently distinguish $\mathbf{Z}_0 = \mathbf{0}$ from any $\mathbf{Z}_1$ not in the column span of $\mathbf{A}^*$. In particular, the rank of $\mathbf{A}^*\mathbf{S}^* + \mathbf{Z}_b$ would leak $b$: this is because $\mathbf{A}^*\mathbf{S}^*$ is rank-deficient by succinctness. We can rule out such distinguishers simply by requiring that $\mathbf{Z}_0 - \mathbf{Z}_1$ lies in the column span of $\mathbf{A}^*$; our pre-condition is in some sense a "distributional" or "computational" relaxation of such a requirement.

**Remark 3.4** (Triviality without succinctness)**.** We remark that it is easy to build a succinct LWE sampler if there are no restrictions on the bit-length of $\mathsf{seed}_{\mathbf{B}^*}$ (looking ahead, such a sampler would not be sufficient to build iO). Indeed, without any succinctness requirement, we could set:

$$\mathsf{crs} = \emptyset, \quad \mathsf{seed}_{\mathbf{B}^*} = \mathbf{A}^*\mathbf{S}^* + \mathbf{E}^* \in \mathbb{Z}_q^{M \times K}$$

where $\mathbf{S}^*$ is random and $\mathbf{E}^*$ has small entries, but large enough to "noise-flood" $\mathbf{Z}_b$ (namely, $\beta_0/\beta = 2^{-\lambda}$).

For convenience, we consider the equivalent notion of security from Remark 3.1. We claim that this construction (unconditionally) satisfies security. To see this, first note that for all $b \in \{0, 1\}$:

$$(\mathsf{seed}_{\mathbf{B}^*},\ \mathbf{A}^*,\ \mathbf{E}^* - \mathbf{Z}_b,\ \mathsf{aux}_b) \approx_s (\mathbf{A}^*\mathbf{S}^* + (\mathbf{E}^* + \mathbf{Z}_b),\ \mathbf{A}^*,\ \mathbf{E}^*,\ \mathsf{aux}_b)$$

by noise flooding, where we use that $\mathbf{E}^*$ is sampled independently of $\mathsf{aux}_b, \mathbf{Z}_b$. The pre-condition then implies that

$$(\mathbf{A}^*, (\mathbf{A}^*\mathbf{S}^* + \mathbf{Z}_0) + \mathbf{E}^*, \mathbf{E}^*, \mathsf{aux}_0) \approx_c (\mathbf{A}^*, (\mathbf{A}^*\mathbf{S}^* + \mathbf{Z}_1) + \mathbf{E}^*, \mathbf{E}^*, \mathsf{aux}_1),$$

where we again use that $\mathbf{E}^*$ is sampled independently of $\mathsf{aux}_b, \mathbf{Z}_b, \mathbf{S}^*$, and that $\mathbf{S}^*$ is sampled uniformly at random independently of the other components (and takes the role of $\mathbf{S}'$ in the pre-condition).

**Remark 3.5** (Heuristic necessity of a CRS)**.** We heuristically show that security requires a (long) CRS if $\mathsf{seed}_{\mathbf{B}^*}$ is required to be short, namely the CRS needs to be of length $\approx N$ for any $\delta$-succinct scheme with $\delta < 1$.

Suppose for contradiction that there is such a sampler that expands some short input $(\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*})$ of length at most $N^\delta \cdot \mathrm{poly}(\lambda, \log q)$ to some $\mathsf{Expand}(\mathsf{seed}_{\mathbf{B}^*}) = \mathbf{B}^* = \mathbf{A}^*\mathbf{S}^* + \mathbf{E}^*$ of bit-length $N \log q$. Let $\mathbf{Z}_b$ be a random LWE error and let $\mathsf{aux}_b$ be an obfuscation of the following program:

$P_{b,\mathbf{A}^*,\mathbf{Z}_b}$: on input $(\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*})$ of bit-length $N^\delta \cdot \mathrm{poly}(\lambda, \log q)$, and $\widetilde{\mathbf{B}}$ of bit-length $N \log q$,

- Check that $\widetilde{\mathbf{B}} - \mathbf{Z}_b$ is in the column span of $\mathbf{A}^*$, and output $\perp$ if not.
- Compute $\mathbf{B}^* = \mathsf{Expand}(\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*}) = \mathbf{A}^*\mathbf{S}^* + \mathbf{E}^*$. Output $b$ if $\|\mathbf{B}^* - \widetilde{\mathbf{B}} + \mathbf{Z}_b\| \leq \beta$, and output $\perp$ otherwise.

Then $(\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*}, \widetilde{\mathbf{B}} = \mathbf{A}^*\mathbf{S}^* + \mathbf{Z}_0, \mathsf{aux}_0)$ is efficiently distinguishable from $(\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*}, \widetilde{\mathbf{B}} = \mathbf{A}^*\mathbf{S}^* + \mathbf{Z}_1, \mathsf{aux}_1)$, by running $\mathsf{aux}_b$ on input $((\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*}), \widetilde{\mathbf{B}})$ and using the fact that $(\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*})$ has bit-length at most $\widetilde{O}(N^\delta)$ by assumption, that $\|\mathbf{E}^*\| \leq \beta$, that $\mathbf{A}^*\mathbf{S}^*$ has low rank by succinctness, and that $\mathbf{A}^*\mathbf{S}^* + \mathbf{Z}_0 - \mathbf{Z}_1$ has high rank w.h.p.

Furthermore, suppose heuristically that $\mathsf{aux}_b$ acts like an ideal obfuscation of $P_{b,\mathbf{Z}_b}$, meaning that it does not reveal more than black-box access to the program. Then, the pre-condition would hold since given $(\mathsf{coins}_{\mathsf{crs}}, \mathsf{coins}_{\mathsf{seed}}, \mathbf{B}_b = \mathbf{A}^*\mathbf{S}' + \mathbf{Z}_b)$ and black-box access to $P_{b,\mathbf{Z}_b}$, one cannot distinguish $b = 0$ vs $b = 1$. The idea is that the only way to learn anything about $b$ is to provide a "good" input to $P_{b,\mathbf{Z}_b}$ that makes it output something other than $\perp$. Any good input must be of the form $((\mathsf{crs}', \mathsf{seed}'_{\mathbf{B}^*}), \mathbf{B}_b + \mathbf{A}^*\mathbf{S})$ for some $\mathbf{S} \in \mathbb{Z}_q^{W \times K}$. But if $\mathbf{B}_b$ was uniform, there would be no inputs of this form, where $(\mathsf{crs}', \mathsf{seed}'_{\mathbf{B}^*})$ is short, such that $\|\mathsf{Expand}(\mathsf{crs}', \mathsf{seed}'_{\mathbf{B}^*}) - \mathbf{B}_b + \mathbf{A}^*\mathbf{S}\|$ is also small, meaning that $P_{b,\mathbf{Z}_b}$ would always output $\perp$ in this case. This follows by a counting argument, where the sizes of $\mathsf{crs}'$, $\mathsf{seed}'_{\mathbf{B}^*}$ and $\mathbf{S}$ are much smaller than the size of $\mathbf{B}_b$ whenever $\delta$ is sufficiently small, and $\beta$ is relatively small compared to $q$. Therefore finding a good input to $P_{b,\mathbf{Z}_b}$ would require breaking LWE with respect to $\mathbf{A}^*$.

## 3.2 Weak Succinct LWE Samplers

We now present a weaker security notion for succinct LWE samplers. Instead of quantifying over all $(\mathbf{Z}_b, \mathsf{aux}_b)$ that satisfy the specified pre-condition as we did previously, we now fix one particular and simple choice of $(\mathbf{Z}_b, \mathsf{aux}_b)$. In particular, this makes the definition falsifiable. We then show in Theorem 3.3 that there is a generic compiler that upgrades this type of weak security to the previous definition of strong security (Definition 3.1).

**Definition 3.2.** *Weak Security (or Weak $\beta_0$-Flooding). Define $D_0, D_1$ as follows.*

$$D_b: \quad \mathsf{aux}_b = \left(\widehat{\mathbf{B}} := \mathbf{A}^*\widehat{\mathbf{S}} + \widehat{\mathbf{E}}, \quad \mathbf{C} = \mathbf{A}^*\mathbf{R} + \mathbf{E} - b \cdot \mathbf{G}\right)$$
$$\mathbf{Z}_b = \mathbf{E}\mathbf{G}^{-1}(\widehat{\mathbf{B}}) - b\widehat{\mathbf{E}},$$

*where*

- $\mathsf{SampCRSGen}$ *defines* $(q, M, K, \overline{\chi}, \overline{B}) = \mathsf{params}$*;*
- $\mathsf{LWEGen}$ *defines* $\mathbf{A}^* \in \mathbb{Z}_q^{M \times W}$*;*
- $\widehat{\mathbf{B}} \in \mathbb{Z}_q^{M \times K}$*,* $\widehat{\mathbf{S}} \leftarrow \mathbb{Z}_q^{W \times K}$*, and* $\widehat{\mathbf{E}} \leftarrow [-B_{\mathsf{flood}}, B_{\mathsf{flood}}]^{M \times K}$*, where* $B_{\mathsf{flood}} = (\beta_0 + \overline{B}) \cdot 2^\lambda$*;*
- $\mathbf{C} \in \mathbb{Z}_q^{M \times M \log q}$*,* $\mathbf{R} \leftarrow \mathbb{Z}_q^{W \times M \log q}$*, and* $\mathbf{E} \leftarrow \overline{\chi}^{M \times M \log q}$*.*

*We say that the sampler* $(\mathsf{SampCRSGen}, \mathsf{LWEGen}, \mathsf{Expand})$ *is weakly secure if*

$$(\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \mathbf{A}^*\mathbf{S}^* + \mathbf{Z}_0, \mathsf{aux}_0) \approx_c (\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \mathbf{A}^*\mathbf{S}^* + \mathbf{Z}_1, \mathsf{aux}_1).$$

**Remark 3.6** (Alternate formulation of security)**.** Similar to Remark 3.1, as the sampler allows us to compute $\mathsf{Expand}(\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*}) = \mathbf{B}^* = \mathbf{A}^*\mathbf{S}^* + \mathbf{E}^*$, weak security equivalently states that:

$$(\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \mathbf{E}^* - \mathbf{Z}_0, \mathsf{aux}_0) \approx_c (\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \mathbf{E}^* - \mathbf{Z}_1, \mathsf{aux}_1).$$

**Remark 3.7** (Pre-condition from LWE)**.** We note that the distributions $D_0, D_1$ satisfy the pre-condition for security of Definition 3.1, assuming LWE, namely:

$$(\mathsf{coins}_{\mathsf{crs}}, \mathsf{coins}_{\mathsf{seed}}, \mathbf{A}^*\mathbf{S}' + \mathbf{Z}_0, \mathsf{aux}_0) \approx_c (\mathsf{coins}_{\mathsf{crs}}, \mathsf{coins}_{\mathsf{seed}}, \mathbf{A}^*\mathbf{S}' + \mathbf{Z}_1, \mathsf{aux}_0), \tag{7}$$

where $(\mathsf{aux}_b, \mathbf{Z}_b) \leftarrow D_b$ and $\mathbf{S}' \leftarrow \mathbb{Z}_q^{W \times K}$.

This is true because one can efficiently sample $\mathbf{A}^*\mathbf{S}' + \mathbf{Z}_b$ given only $(\mathbf{A}^*, \mathsf{aux}_b)$, as follows:

- Compute $\mathbf{C}_{\widehat{\mathbf{B}}} = \mathbf{C}\mathbf{G}^{-1}(\widehat{\mathbf{B}}) \in \mathbb{Z}_q^{M \times K}$; and
- Output $\mathbf{C}_{\widehat{\mathbf{B}}} + \mathbf{A}^*\mathbf{S}$ for some random $\mathbf{S} \leftarrow \mathbb{Z}_q^{W \times K}$.

Indeed,

$$\mathbf{C}_{\widehat{\mathbf{B}}} + \mathbf{A}^*\mathbf{S} = (\mathbf{A}^*\mathbf{R} + \mathbf{E} - b\mathbf{G})\mathbf{G}^{-1}(\widehat{\mathbf{B}}) + \mathbf{A}^*\mathbf{S} = \mathbf{A}^*(\mathbf{R}\mathbf{G}^{-1}(\widehat{\mathbf{B}}) - b\widehat{\mathbf{S}} + \mathbf{S}) + (\mathbf{E}\mathbf{G}^{-1}(\widehat{\mathbf{B}}) - b\widehat{\mathbf{E}})$$

and the latter term is distributed identically to $\mathbf{A}^*\mathbf{S}' + \mathbf{Z}_b$ with a random $\mathbf{S}'$.

Therefore, to show the precondition equation (7), it suffices to prove that $(\mathsf{coins}_{\mathsf{crs}}, \mathsf{coins}_{\mathsf{seed}}, \mathsf{aux}_b)$ hides $b$. But this follows from LWE with respect to $\mathbf{A}^*$ (Definition 3.1) with noise distribution $\overline{\chi}$.

### 3.3 Amplification

We now describe a general compiler that lifts weak security (Definition 3.2) to strong security (Definition 3.1). The idea is based on a trick used in the specific oblivious LWE sampler construction of WW, but now abstracted out as a general compiler.

The compiler works as follows. We start with a weakly secure scheme and augment the seed by adding a commitment $\mathbf{C} = \mathbf{A}^*\mathbf{R} + \mathbf{E} + \mathsf{flag} \cdot \mathbf{G}$ to a "flag" bit initially set to $\mathsf{flag} = 0$.[4] We also add a long uniformly random value $\widehat{\mathbf{B}} \in \mathbb{Z}_q^{M \times K}$ to the CRS. If the original weakly secure scheme outputs some LWE sample $\mathbf{B}^* = \mathbf{A}^*\mathbf{S}^* + \mathbf{E}^*$, we define the new LWE sample to be $\mathbf{B}^* - \mathbf{C}\mathbf{G}^{-1}(\widehat{\mathbf{B}})$. In the proof, we switch $\widehat{\mathbf{B}}$ to be an LWE sample and we switch the flag bit to 1 by relying on the weakly secure LWE sampler. We can then prove strong security by "programming" the random LWE sample $\widehat{\mathbf{B}}$ in the CRS, analogously to the trivial scheme in Remark 3.4.

**Construction.** Let $(\mathsf{SampCRSGen}, \mathsf{LWEGen}, \mathsf{Expand})$ be a weakly secure succinct LWE sampler. We build one satisfying strong security as follows:

- $\overline{\mathsf{SampCRSGen}}(1^\lambda, 1^N, \alpha\,; \overline{\mathsf{coins}}_{\mathsf{crs}})$: Using randomness $\overline{\mathsf{coins}}_{\mathsf{crs}} = (\mathsf{coins}_{\mathsf{crs}}, \rho_{\mathsf{crs}})$, compute $\mathsf{crs} = \mathsf{SampCRSGen}(1^\lambda, 1^N, \min(\alpha, 2^\lambda); \mathsf{coins}_{\mathsf{crs}})$. Sample $\widehat{\mathbf{B}} \leftarrow \mathbb{Z}_q^{M \times K}$ using $\rho_{\mathsf{crs}}$. Output:

$$\overline{\mathsf{crs}} = (\mathsf{crs}, \widehat{\mathbf{B}}).$$

[4] In this section, we denote by $\mathsf{flag}$ the bit $b$ corresponding to the weak security we assume, to differentiate it from the bit $b$ corresponding to standard security which we aim to build.

- $\overline{\mathsf{LWEGen}}(\overline{\mathsf{crs}}\,; \overline{\mathsf{coins}}_{\mathsf{seed}})$ : Using randomness $\overline{\mathsf{coins}}_{\mathsf{seed}} = (\mathsf{coins}_{\mathsf{seed}}, \rho_{\mathsf{seed}})$, compute $(\mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \mathbf{S}^*) = \mathsf{LWEGen}(\mathsf{crs}; \mathsf{coins}_{\mathsf{seed}})$, and set

  $$\mathbf{C} = \mathbf{A}^*\mathbf{R} + \mathbf{E} \in \mathbb{Z}_q^{M \times M \log q},$$

  using $\rho_{\mathsf{seed}}$ to sample $\mathbf{R} \leftarrow \mathbb{Z}_q^{W \times M \log q}$ and $\mathbf{E} \leftarrow \overline{\chi}^{M \times K}$ (where $\overline{\chi}$ is defined by $\mathsf{crs}$), and output:

  $$\left(\overline{\mathsf{seed}_{\mathbf{B}^*}} = (\mathsf{seed}_{\mathbf{B}^*},\ \mathbf{C}),\ \mathbf{A}^*, \quad \overline{\mathbf{S}}^* = (\mathbf{S}^* - \mathbf{R}\mathbf{G}^{-1}(\widehat{\mathbf{B}}))\right).$$

- $\overline{\mathsf{Expand}}(\overline{\mathsf{crs}}, \overline{\mathsf{seed}_{\mathbf{B}^*}})$ : Compute $\mathbf{B}^* = \mathsf{Expand}(\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*})$ and output

  $$\overline{\mathbf{B}}^* = \mathbf{B}^* \ - \ \mathbf{C} \cdot \mathbf{G}^{-1}\left(\widehat{\mathbf{B}}\right).$$

**Theorem 3.3.** *Suppose* $(\mathsf{SampCRSGen}, \mathsf{LWEGen}, \mathsf{Expand})$ *is a weakly secure, $\delta$-succinct LWE sampler (Definition 3.2). Suppose furthermore that it satisfies $M^2 \leq N^\delta \cdot \mathrm{poly}(\lambda, \log q)$. Then, assuming LWE,* $(\overline{\mathsf{SampCRSGen}}, \overline{\mathsf{LWEGen}}, \overline{\mathsf{Expand}})$ *is a secure $\delta$-succinct LWE sampler, satisfying strong security (Definition 3.1). Moreover, with the parameters of Definition 3.2,* $(\overline{\mathsf{SampCRSGen}}, \overline{\mathsf{LWEGen}}, \overline{\mathsf{Expand}})$ *is (strongly) $\beta_0$-flooding.*

*Proof.* Correctness follows directly by setting $\overline{\mathbf{E}}^* = \mathbf{E}^* - \mathbf{E} \cdot \mathbf{G}^{-1}\left(\widehat{\mathbf{B}}\right)$. $\delta$-succinctness follows by $\delta$-succinctness of $(\mathsf{SampCRSGen}, \mathsf{LWEGen}, \mathsf{Expand})$ and as $\mathsf{bitlength}(\mathbf{C}) = M^2 \log q \leq N^\delta \cdot \mathrm{poly}(\lambda, \log q)$ by assumption. LWE with respect to $\mathbf{A}^*$ is directly inherited as $\widehat{\mathbf{B}}, \mathbf{C}$ are sampled independently from $\mathsf{coins}_{\mathsf{crs}}, \mathsf{coins}_{\mathsf{seed}}$, and thus a reduction can sample the associated randomness by itself. We will now focus on proving security. Let $D_0, D_1$ be two distributions satisfying the pre-condition. That is, for $(\overline{\mathsf{aux}}_b, \overline{\mathbf{Z}}_b) \leftarrow D_b(\mathbf{A}^*)$, we have:

$$(\overline{\mathsf{coins}}_{\mathsf{crs}}, \overline{\mathsf{coins}}_{\mathsf{seed}}, \mathbf{A}^*\mathbf{S}' + \overline{\mathbf{Z}}_0, \overline{\mathsf{aux}}_0) \approx_c (\overline{\mathsf{coins}}_{\mathsf{crs}}, \overline{\mathsf{coins}}_{\mathsf{seed}}, \mathbf{A}^*\mathbf{S}' + \overline{\mathbf{Z}}_1, , \overline{\mathsf{aux}}_1).$$

Using the equivalent notion of security from Remark 3.6, we wish to prove that:

$$(\overline{\mathsf{crs}}, \overline{\mathsf{seed}_{\mathbf{B}^*}}, \mathbf{A}^*, \overline{\mathbf{E}}^* - \overline{\mathbf{Z}}_0, \mathsf{aux}_0) \approx_c (\overline{\mathsf{crs}}, \overline{\mathsf{seed}_{\mathbf{B}^*}}, \mathbf{A}^*, \overline{\mathbf{E}}^* - \overline{\mathbf{Z}}_1, \overline{\mathsf{aux}}_1),$$

where $\overline{\mathbf{E}}^* = \mathbf{E}^* + \mathbf{E} \cdot \mathbf{G}^{-1}\left(\widehat{\mathbf{B}}\right)$. We proceed by a hybrid argument.

$H_0(b)$**.** This is the original distribution with the challenge bit $b$:

$$H_0(b) = (\overline{\mathsf{crs}} = (\mathsf{crs}, \widehat{\mathbf{B}}),\ \overline{\mathsf{seed}_{\mathbf{B}^*}} = (\mathsf{seed}_{\mathbf{B}^*}, \mathbf{C}),\ \mathbf{A}^*,\ \overline{\mathbf{E}}^* - \overline{\mathbf{Z}}_b, \overline{\mathsf{aux}}_b).$$

$H_1(b)$**.** We switch how $\widehat{\mathbf{B}}$ is computed. Instead of choosing it uniformly at random, we now set it to an LWE sample:

$$\widehat{\mathbf{B}} = \mathbf{A}^*\widehat{\mathbf{S}} + \widehat{\mathbf{E}},$$

where $\widehat{\mathbf{S}} \leftarrow \mathbb{Z}_q^{W \times K}, \widehat{\mathbf{E}} \leftarrow [-B_{\mathsf{flood}}, B_{\mathsf{flood}}]^{M \times K}$ (where we recall that $B_{\mathsf{flood}} = (\beta_0 + \overline{B}) \cdot 2^\lambda$). The resulting distribution is:

$$H_1(b) = (\overline{\mathsf{crs}} = (\mathsf{crs}, \widehat{\mathbf{B}} = \mathbf{A}^*\widehat{\mathbf{S}} + \widehat{\mathbf{E}}),\ \overline{\mathsf{seed}_{\mathbf{B}^*}} = (\mathsf{seed}_{\mathbf{B}^*}, \mathbf{C}),\ \mathbf{A}^*,\ \overline{\mathbf{E}}^* - \overline{\mathbf{Z}}_b, \overline{\mathsf{aux}}_b).$$

We claim that, assuming LWE with respect to $\mathbf{A}^*$ (Definition 3.1) with a $\overline{B}$-bounded distribution $\overline{\chi}$, we have:

$$H_0(b) \approx_c H_1(b)$$

This follows since LWE with the error distribution $\overline{\chi}$ also implies LWE with the wider error distribution $\widehat{\mathbf{E}} \leftarrow [-B_{\mathsf{flood}}, B_{\mathsf{flood}}]^{M \times K}$; in particular, $\widehat{\mathbf{E}}$ is statistically close to $\widehat{\mathbf{E}} + \mathbf{E}$ for $\mathbf{E} \leftarrow \overline{\chi}$ and hence $\mathbf{A}^*\widehat{\mathbf{S}} + \widehat{\mathbf{E}} \approx_s (\mathbf{A}^*\widehat{\mathbf{S}} + \mathbf{E}) + \widehat{\mathbf{E}} \approx_c \mathbf{U}$ where $\mathbf{U}$ is uniform. The last indistinguishability follows from LWE with respect to $\mathbf{A}^*$ for the weak scheme, which holds even given $\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \mathbf{S}^*$ by Remark 3.2; the reduction can sample all the other components given in $H_0(b), H_1(b)$ on its own.

$H_2(b)$**.** We switch how $\mathbf{C}$ is computed. We now set $\mathsf{flag} = 1$ and compute:

$$\mathbf{C} = \mathbf{A}^*\mathbf{R} - \mathsf{flag} \cdot \mathbf{G} + \mathbf{E} \in \mathbb{Z}_q^{M \times M \log q}.$$

This means that we now have

$$\overline{\mathbf{E}}^* = \mathbf{E}^* - \mathbf{E} \cdot \mathbf{G}^{-1}\left(\widehat{\mathbf{B}}\right) + \mathsf{flag} \cdot \widehat{\mathbf{E}},$$

and implicitly $\overline{\mathbf{S}}^* = \mathbf{S}^* - \mathbf{R}\mathbf{G}^{-1}(\widehat{\mathbf{B}}) + \mathsf{flag} \cdot \widehat{\mathbf{S}}$. Note that $\mathsf{flag} = 0$ corresponds to $H_1(b)$ and $\mathsf{flag} = 1$ corresponds to $H_2(b)$, where the hybrids consist of:

$$H_{1+\mathsf{flag}}(b) = (\overline{\mathsf{crs}} = (\mathsf{crs}, \widehat{\mathbf{B}}),\ \overline{\mathsf{seed}_{\mathbf{B}^*}} = (\mathsf{seed}_{\mathbf{B}^*}, \mathbf{C}),\ \mathbf{A}^*,\ \overline{\mathbf{E}}^* - \overline{\mathbf{Z}}_b, \overline{\mathsf{aux}}_b).$$

We claim that:

$$H_1(b) \approx_c H_2(b)$$

by weak security of the succinct LWE sampler $(\mathsf{SampCRSGen}, \mathsf{LWEGen}, \mathsf{Expand})$ with

$$\begin{aligned} \mathsf{aux}_{\mathsf{flag}} &= (\widehat{\mathbf{B}}, \mathbf{C} = \mathbf{A}^*\mathbf{R} - \mathsf{flag} \cdot \mathbf{G} + \mathbf{E}) \\ \mathbf{Z}_{\mathsf{flag}} &= \mathbf{E} \cdot \mathbf{G}^{-1}\left(\widehat{\mathbf{B}}\right) - \mathsf{flag} \cdot \widehat{\mathbf{E}} \end{aligned}$$

which states that:

$$(\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \underbrace{\mathbf{E}^* - \mathbf{Z}_0}_{\overline{\mathbf{E}}^*}, \underbrace{\mathsf{aux}_0}_{\widehat{\mathbf{B}},\mathbf{C}}) \approx_c (\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \underbrace{\mathbf{E}^* - \mathbf{Z}_1}_{\overline{\mathbf{E}}^*}, \underbrace{\mathsf{aux}_1}_{\widehat{\mathbf{B}},\mathbf{C}}).$$

The claim follows by having the reduction sample $(\overline{\mathsf{aux}}_b, \overline{\mathbf{Z}}_b) \leftarrow D_b(\mathbf{A}^*)$ on its own given $\mathbf{A}^*$ and $b$ to convert the above two distributions into $H_1(b)$ and $H_2(b)$ respectively.

$H_3(b)$**.** We modify how we compute $\widehat{\mathbf{B}}$ and exchange $\widehat{\mathbf{E}}$ with $\widehat{\mathbf{E}} + \overline{\mathbf{Z}}_b$. In particular, we set the distribution as:

$$H_3(b) = (\overline{\mathsf{crs}} = (\mathsf{crs}, \widehat{\mathbf{B}} = \mathbf{A}^*\widehat{\mathbf{S}} + \underbrace{\widehat{\mathbf{E}} + \overline{\mathbf{Z}}_b}_{\widehat{\mathbf{E}}_{new}}), \quad \overline{\mathsf{seed}_{\mathbf{B}^*}} = (\mathsf{seed}_{\mathbf{B}^*}, \mathbf{C}), \mathbf{A}^*, \underbrace{\overline{\mathbf{E}}^*}_{\overline{\mathbf{E}}^*_{new} - \overline{\mathbf{Z}}_b}, \overline{\mathsf{aux}}_b),$$

where $\overline{\mathbf{E}}^* = \mathbf{E}^* - \mathbf{E} \cdot \mathbf{G}^{-1}\left(\widehat{\mathbf{B}}\right) + \widehat{\mathbf{E}}$ and $\overline{\mathbf{E}}^*_{new} = \mathbf{E}^* - \mathbf{E} \cdot \mathbf{G}^{-1}\left(\widehat{\mathbf{B}}\right) + \widehat{\mathbf{E}}_{new}$. Therefore, the only difference between $H_2(b)$ vs $H_3(b)$ is the difference between using $\widehat{\mathbf{E}}$ vs $\widehat{\mathbf{E}}_{new}$ in $\widehat{\mathbf{B}}$, and the other changes then follow as a function of this change.

We argue $H_2(b) \approx_s H_3(b)$ by noise flooding (Lemma 2.3). In particular, $\widehat{\mathbf{E}}$ and $\widehat{\mathbf{E}}_{new} = \widehat{\mathbf{E}} + \overline{\mathbf{Z}}_b$ are statistically close since $\widehat{\mathbf{E}}$ is sufficiently large to flood out $\overline{\mathbf{Z}}_b$ by way we set the parameter $B_{\mathsf{flood}}$.

$H_3(0) \approx_c H_3(1)$**:** Finally, we argue that $H_3(0) \approx_c H_3(1)$ follows by the pre-condition on $D_0, D_1$. Namely, the pre-condition ensures that:

$$(\overline{\mathsf{coins}}_{\mathsf{crs}}, \overline{\mathsf{coins}}_{\mathsf{seed}}, \mathbf{A}^*\mathbf{S}' + \overline{\mathbf{Z}}_0, \overline{\mathsf{aux}}_0) \approx_c (\overline{\mathsf{coins}}_{\mathsf{crs}}, \overline{\mathsf{coins}}_{\mathsf{seed}}, \mathbf{A}^*\mathbf{S}' + \overline{\mathbf{Z}}_1, \overline{\mathsf{aux}}_1),$$

where $\overline{\mathsf{coins}}_{\mathsf{crs}} = (\mathsf{coins}_{\mathsf{crs}}, \rho_1), \overline{\mathsf{coins}}_{\mathsf{seed}} = (\mathsf{coins}_{\mathsf{seed}}, \rho_2)$ are the random coins used by $\overline{\mathsf{SampCRSGen}}$ and $\overline{\mathsf{LWEGen}}$, respectively.

The reduction computes $\mathsf{crs} = \mathsf{SampCRSGen}(1^\lambda, 1^N, \min(\alpha, 2^\lambda); \mathsf{coins}_{\mathsf{crs}})$, samples $\widehat{\mathbf{E}}$ on its own, sets $\widehat{\mathbf{B}} = (\mathbf{A}^*\mathbf{S}'+\mathbf{Z}_b)+\widehat{\mathbf{E}}$ and sets $\overline{\mathsf{crs}} = (\mathsf{crs}, \widehat{\mathbf{B}})$. It computes $(\mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \mathbf{E}^*) = \mathsf{LWEGen}(\mathsf{crs}; \mathsf{coins}_{\mathsf{seed}})$, samples $\mathbf{C}, \mathbf{E}$ on its own, and sets $\overline{\mathsf{seed}_{\mathbf{B}^*}} = (\mathsf{seed}_{\mathbf{B}^*}, \mathbf{C})$. Finally it computes $\overline{\mathbf{E}}^* = \mathbf{E}^* - \mathbf{E} \cdot \mathbf{G}^{-1}\left(\widehat{\mathbf{B}}\right) + \widehat{\mathbf{E}}$, and can therefore generate

$$H_3(b) = (\overline{\mathsf{crs}} = (\mathsf{crs},\, \widehat{\mathbf{B}} = \mathbf{A}^*\widehat{\mathbf{S}} + \widehat{\mathbf{E}} + \mathbf{Z}_b), \quad \overline{\mathsf{seed}_{\mathbf{B}^*}} = (\mathsf{seed}_{\mathbf{B}^*}, \mathbf{C}), \quad \mathbf{A}^*, \quad \overline{\mathbf{E}}^*, \quad \overline{\mathsf{aux}}_b).$$

Taken together, the above shows that $H_0(0) \approx_c H_3(0) \approx_c H_3(1) \approx_c H_0(1)$ and therefore $H_0(0) \approx_c H_0(1)$, which concludes the proof.

□

# 4 Candidate Succinct LWE Sampler

In Section 4.1, we present the template of our main candidate. In Section 4.2, we consider prove correctness and succinctness. In Section 4.3, we explain how to setup parameters, and state our conjectured security. Last, we discuss the plausibility of our conjecture in Section 4.5.

## 4.1 A Basic Framework

We describe a basic template to build succinct LWE samplers. Looking ahead, the SRE construction in Section 5 requires an additional succinctness requirement, namely, that additional encodings produced by the SRE are succinct. We make sure that our template and the parameters we propose are compatible with that constraint.

We now describe our framework. It uses a set of parameters:

$$\mathsf{parameters} := (d, m, k, w, M, K, \overline{\chi}, \chi, \beta, q)$$

which in particular includes $\mathsf{params} = (q, M, K, \overline{\chi}, \overline{B}, \chi)$ directly output by $\mathsf{SampCRSGen}$. Informally,

- the security of our sampler is related to the hardness of solving systems of random degree $d$ polynomials;
- $q$ is the underlying LWE modulus;
- $m, k, w$ define the dimensions of the ”seed” LWE samples $\mathbf{A}_i, \mathbf{S}_i, \mathbf{E}_i$, which together with $d$, determine $M, K$, which are the dimensions for “expanded” sample $\mathbf{B}^*$;
- $\chi$ is the noise distribution for $\mathbf{E}_i$; it is $B$-bounded over $\mathbb{Z}$;
- $\overline{\chi}$ is the noise distribution used for LWE w.r.t $\mathbf{A}^*$; it is $\overline{B}$-bounded over $\mathbb{Z}$;

- $D_P$ a $\sigma$-bounded distribution over $\mathbb{Z}$. We will take $D_P = \chi$ for simplicity.

We now describe our candidate $(\mathsf{SampCRSGen}, \mathsf{LWEGen}, \mathsf{Expand})$.

- $\mathsf{SampCRSGen}(1^\lambda, 1^N, \alpha)$: Derive $\mathsf{parameters} = (d, m, k, w, M, K, \overline{\chi}, \overline{B}, \chi, \beta, q)$ from $(1^\lambda, 1^N, \alpha)$ as described later in Section 4.3. Set $\mathsf{params} = (q, M, K, \overline{\chi}, \overline{B}, \chi)$.

  Sample $\mathbf{P'} \leftarrow \chi^{k^d \times K}$ and $\mathbf{P} \leftarrow \chi^{M \times m^d}$. Output
$$\mathsf{crs} = (\mathsf{params}, \mathbf{P}, \mathbf{P'}).$$

- $\mathsf{LWEGen}(\mathsf{crs})$: On input $\mathsf{crs} = (\mathsf{params}, \mathbf{P}, \mathbf{P'})$, sample, for $i \in [d]$, $\mathbf{A}_i \leftarrow \mathbb{Z}_q^{m \times w}$, $\mathbf{S}_i \leftarrow \mathbb{Z}_q^{w \times k}$, $\mathbf{E}_i \leftarrow \chi^{m \times k}$ where $\chi$ is specified in $\mathsf{params}$. Compute:
$$\mathbf{B}_i = \mathbf{A}_i \mathbf{S}_i + \mathbf{E}_i \in \mathbb{Z}_q^{m \times k}.$$

  Set:

$$\overline{\mathbf{A}}^* = \mathbf{P}\cdot( \mathbf{A}_1 \otimes \mathbf{I}_m \otimes \cdots \otimes \mathbf{I}_m \,\|\, \mathbf{I}_m \otimes \mathbf{A}_2 \otimes \mathbf{I}_m \otimes \cdots \otimes \mathbf{I}_m \,\|\, \cdots \,\|\, \mathbf{I}_m \otimes \cdots \otimes \mathbf{I}_m \otimes \mathbf{A}_d) \in \mathbb{Z}_q^{M \times dwm^{d-1}}$$

$$\overline{\mathbf{S}}^* = \begin{pmatrix} \mathbf{S}_1 \otimes \mathbf{B}_2 \otimes \cdots \otimes \mathbf{B}_d \\ \mathbf{E}_1 \otimes \mathbf{S}_2 \otimes \cdots \otimes \mathbf{B}_d \\ \vdots \\ \mathbf{E}_1 \otimes \mathbf{E}_2 \otimes \cdots \otimes \mathbf{S}_d \end{pmatrix} \cdot \mathbf{P'} \in \mathbb{Z}_q^{dwm^{d-1} \times K}.$$

  Sample a random basis $\mathbf{A}^* \in \mathbb{Z}_q^{M \times W}$ of the column space of $\overline{\mathbf{A}}^*$, and solve for $\mathbf{S}^* \in \mathbb{Z}_q^{W \times K}$ such that $\mathbf{A}^* \mathbf{S}^* = \overline{\mathbf{A}}^* \cdot \overline{\mathbf{S}}^*$. Output:
$$\mathsf{seed}_{\mathbf{B}^*} = \{\mathbf{B}_i\}_{i \in [d]}, \quad \mathbf{A}^*,\ \mathbf{S}^*.$$

- $\mathsf{Expand}(\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*})$: On input $\mathsf{crs} = (\mathsf{params}, \mathbf{P}, \mathbf{P'})$ and $\mathsf{seed}_{\mathbf{B}^*} = \{\mathbf{B}_i\}_{i \in [d]}$, output:
$$\mathbf{B}^* = \mathbf{P} \cdot (\mathbf{B}_1 \otimes \cdots \otimes \mathbf{B}_d) \cdot \mathbf{P'} \in \mathbb{Z}_q^{M \times K}.$$

### 4.2 Correctness, Succinctness, and LWE with respect to $\mathbf{A}^*$

We show that for appropriate parameters, the sampler described above is correct and succinct.

**Claim 4.0.1.** *Assume $\beta \geq B^2 (mkB)^d$. Then the sampler* $(\mathsf{SampCRSGen}, \mathsf{LWEGen}, \mathsf{Expand})$ *described above satisfies correctness (Definition 3.1).*

*Proof.* We first prove that
$$\mathbf{B}^* = \overline{\mathbf{A}}^* \cdot \overline{\mathbf{S}}^* + \mathbf{E}^*, \tag{8}$$
where $\mathbf{E}^* = \mathbf{P} \left( \bigotimes_{i=1}^d \mathbf{E}_i \right) \mathbf{P'} \in \mathbb{Z}_q^{M \times K}$.

$$\text{Let } \widehat{\mathbf{A}} := \Big( \mathbf{A}_1 \otimes \mathbf{I}_m \otimes \cdots \otimes \mathbf{I}_m \| \mathbf{I}_m \otimes \mathbf{A}_2 \otimes \mathbf{I}_m \otimes \cdots \otimes \mathbf{I}_m \,\| \\ \mathbf{I}_m \otimes \cdots \otimes \mathbf{I}_m \otimes \mathbf{A}_d \Big) \in \mathbb{Z}_q^{m^d \times dwm^{d-1}}$$

$$\widehat{\mathbf{S}} = \begin{pmatrix} \mathbf{S}_1 \otimes \mathbf{B}_2 \otimes \cdots \otimes \mathbf{B}_d \\ \mathbf{E}_1 \otimes \mathbf{S}_2 \otimes \cdots \otimes \mathbf{B}_d \\ \vdots \\ \mathbf{E}_1 \otimes \mathbf{E}_2 \otimes \cdots \otimes \mathbf{S}_d \end{pmatrix} \in \mathbb{Z}_q^{dwm^{d-1} \times k^d}.$$

In particular, $\overline{\mathbf{A}}^* = \mathbf{P}\widehat{\mathbf{A}}$ and $\overline{\mathbf{S}}^* = \widehat{\mathbf{S}}\mathbf{P}'$.

For $i \in [d]$, let $\mathbf{A}^{(i)}$ be the $i$-th block (out of $d$ blocks) of $wm^{d-1}$ columns of $\widehat{\mathbf{A}}$, namely

$$\mathbf{A}^{(i)} = \left( \bigotimes_{j=1}^{i-1} \mathbf{I}_m \right) \otimes \mathbf{A}_i \otimes \left( \bigotimes_{j=i+1}^{d} \mathbf{I}_m \right),$$

and $\mathbf{S}^{(d)}$ be the $i$-th block (out of $d$ blocks) of $wm^{d-1}$ rows of $\widehat{\mathbf{S}}$, namely

$$\mathbf{S}^{(i)} = \left( \bigotimes_{j=1}^{i-1} \mathbf{E}_j \right) \otimes \mathbf{S}_i \otimes \left( \bigotimes_{j=i+1}^{d} \mathbf{B}_j \right).$$

We have:

$$\begin{aligned} \widehat{\mathbf{A}} \cdot \widehat{\mathbf{S}} &= \sum_{i=1}^{d} \mathbf{A}^{(i)} \mathbf{S}^{(i)} \\ &= \sum_{i=1}^{d} \left( \bigotimes_{j=1}^{i-1} \mathbf{E}_j \right) \otimes (\mathbf{A}_i \mathbf{S}_i) \otimes \left( \bigotimes_{j=i+1}^{d} \mathbf{B}_i \right) \\ &= \sum_{i=1}^{d} \left( \bigotimes_{j=1}^{i-1} \mathbf{E}_j \right) \otimes (\mathbf{B}_j - \mathbf{E}_j) \otimes \left( \bigotimes_{j=i+1}^{d} \mathbf{B}_i \right) \\ &= \sum_{i=1}^{d} \left( \bigotimes_{j=1}^{i-1} \mathbf{E}_j \right) \cdot \left( \bigotimes_{j=i}^{d} \mathbf{B}_j \right) \quad - \quad \left( \bigotimes_{j=1}^{i} \mathbf{E}_j \right) \cdot \left( \bigotimes_{j=i+1}^{d} \mathbf{B}_j \right) \\ &= \bigotimes_{i=1}^{d} \mathbf{B}_i \; - \; \bigotimes_{i=1}^{d} \mathbf{E}_i, \end{aligned}$$

where the first equality is by definition of $\mathbf{A}^{(i)}, \mathbf{S}^{(i)}$, the second equality follows by the mixed product property, and the last by observing the last sum is telescoping. Multiplying on the left by $\mathbf{P}$ and on the right by $\mathbf{P}'$ gives Eq. (8).

In particular, by construction of $\mathbf{A}^*$ and $\mathbf{S}^*$, we also have:

$$\mathbf{B}^* = \mathbf{A}^* \mathbf{S}^* + \mathbf{E}^*.$$

Moreover, if $\chi$ is a $B$-bounded distribution, then $\|\mathbf{E}^*\| \leq \beta \coloneqq B^2 \cdot (mkB)^d$. Last, $\mathbf{A}^*$ is full-rank by construction.

□

**Claim 4.0.2.** *Suppose there exists $\delta < 1$ such that*

$$(dmk + MW + WK) \leq N^\delta \cdot \mathrm{poly}(\lambda, \log q),$$

*where $W$ is the width of $\mathbf{A}^*$. Then* (SampCRSGen, LWEGen, Expand) *described above is $\delta$-succinct.*

*Proof.* This follows as $\mathsf{bitlength}(\{\mathbf{B}_i\}_{i\in[d]}, \mathbf{A}^*, \mathbf{S}^*) = (dmk + MW + WK) \cdot \log q$. □

Next, we show that LWE holds with respect to $\mathbf{A}^*$ (assuming standard LWE), for our candidate sampler. We first show that it holds with respect to $\overline{\mathbf{A}}^*$.

**Lemma 4.1** (LWE with respect to $\overline{\mathbf{A}}^*$)**.** *Let $\chi(\lambda)$ be a $B(\lambda)$-bounded distribution. Let $D_P$ be a $\sigma$-bounded distribution over $\mathbb{Z}$ such that if $\mathbf{P} = D_P^{M\times m^d}(\mathsf{coins}_P)$ is sampled using randomness $\mathsf{coins}_P$, then with overwhelming probability over $\mathsf{coins}_P$, $\mathbf{P}$ is full-rank. Suppose furthermore that $M \leq m^d$.*

*Suppose LWE$_{w,q,\chi}$ holds. Let $\overline{\chi} = \mathcal{U}([-\overline{B}, \overline{B}])$ be the uniform distribution in $[-\overline{B}, \overline{B}]$, where $\overline{B} \geq \sigma m^d B \cdot 2^\lambda$. Then:*

$$\left(\mathsf{coins}_P, \mathbf{P}, \{\mathbf{A}_i\}_{i\in[d]}, \overline{\mathbf{A}}^*, \overline{\mathbf{A}}^* \cdot \mathbf{s} + \mathbf{e}\right) \approx_c \left(\mathsf{coins}_P, \mathbf{P}, \{\mathbf{A}_i\}_{i\in[d]}, \overline{\mathbf{A}}^*, \mathbf{b}\right),$$

*where $\mathbf{P} = D_P^{M\times m^d}(\mathsf{coins}_P)$, $\mathbf{b} \leftarrow \mathbb{Z}_q^M$, $\mathbf{s} \leftarrow \mathbb{Z}_q^{dwm^{d-1}}$, $\mathbf{e} \leftarrow \overline{\chi}^M$.*

*Proof.* Assuming LWE$_{w,q,\chi}$ holds, we have for all integer $L \geq 1$:

$$(\mathbf{A},\ \mathbf{A}\otimes\mathbf{I}_L,\ (\mathbf{A}\otimes\mathbf{I}_L)\cdot\overline{\mathbf{s}} + \overline{\mathbf{e}}) \approx (\mathbf{A},\ \mathbf{A}\otimes\mathbf{I}_L, \mathbf{b}), \tag{9}$$

where $\mathbf{A} \leftarrow \mathbb{Z}_q^{m\times w}$, $\overline{\mathbf{s}} \leftarrow \mathbb{Z}_q^{nL}$, $\overline{\mathbf{e}} \leftarrow \chi^{mL}$ and $\mathbf{b} \leftarrow \mathbb{Z}_q^{mL}$ (e.g. [CC17]). Therefore, writing $\overline{\mathbf{A}}^* = (\mathbf{A}_1 \otimes \mathbf{I}_{m^{d-1}} \| \widetilde{\mathbf{A}})$ for some $\widetilde{\mathbf{A}} \in \mathbb{Z}_q^{M\times(d-1)wm^{d-1}}$, and setting $L = m^{d-1}$, we have

$$\begin{aligned}
&(\mathsf{coins}_P, \mathbf{P}, \{\mathbf{A}_i\}_{i\in[d]},\ \overline{\mathbf{A}}^*,\ \overline{\mathbf{A}}^*\cdot\mathbf{s}+\mathbf{e})\\
&\qquad\equiv\ (\mathsf{coins}_P, \mathbf{P},\ \{\mathbf{A}_i\}_{i\in[d]},\quad \overline{\mathbf{A}}^*,\quad \mathbf{P}(\mathbf{A}\otimes\mathbf{I}_L)\overline{\mathbf{s}}\ +\ \mathbf{P}\widetilde{\mathbf{A}}\underline{\mathbf{s}}\ +\ \mathbf{e})\\
&\qquad\approx_s (\mathsf{coins}_P, \mathbf{P},\ \{\mathbf{A}_i\}_{i\in[d]},\quad \overline{\mathbf{A}}^*,\quad \mathbf{P}\left((\mathbf{A}\otimes\mathbf{I}_L)\overline{\mathbf{s}}+\overline{\mathbf{e}}\right) +\ \mathbf{P}\widetilde{\mathbf{A}}\underline{\mathbf{s}}\ +\ \mathbf{e})\\
&\qquad\approx_c (\mathsf{coins}_P, \mathbf{P},\ \{\mathbf{A}_i\}_{i\in[d]},\quad \overline{\mathbf{A}}^*,\quad \mathbf{P}\mathbf{b}+\mathbf{P}\widetilde{\mathbf{A}}\underline{\mathbf{s}}\ +\ \mathbf{e})\\
&\qquad\approx_s (\mathsf{coins}_P, \mathbf{P},\ \{\mathbf{A}_i\}_{i\in[d]},\quad \overline{\mathbf{A}}^*,\quad \mathbf{b}'),
\end{aligned}$$

where $\mathbf{s} = \binom{\overline{\mathbf{s}}}{\underline{\mathbf{s}}} \leftarrow \mathbb{Z}_q^{dwm^{d-1}}$, $\mathbf{e} \leftarrow [-\overline{B}, \overline{B}]^M$, $\overline{\mathbf{e}} \leftarrow \chi^{m^d}$, $\mathbf{b} \leftarrow \mathbb{Z}_q^{m^d}$ and $\mathbf{b}' \leftarrow \mathbb{Z}_q^M$. The first statistical indistinguishability follows by noise flooding (Lemma 2.3) by definition of $\overline{B}$, and the computational indistinguishability follows from Eq. (9). The last statistical indistinguishability by assumption on $\chi$, $M$ and $m^d$, which implies $(\mathsf{coins}_P, \mathbf{P}, \mathbf{P}\mathbf{b}) \approx_s (\mathsf{coins}_P, \mathbf{P}, \mathbf{b}')$ where $\mathbf{P} = D_P^{M\times m^d}(\mathsf{coins}_P), \mathbf{b} \leftarrow \mathbb{Z}_q^{m^d}$ and $\mathbf{b}' \leftarrow \mathbb{Z}_q^M$. □

**Corollary 4.2** (LWE with respect to $\mathbf{A}^*$)**.** *Let $\chi(\lambda)$ be a $B(\lambda)$-bounded distribution. Suppose furthermore that $M \leq m^d$. Then, assuming LWE$_{w,\chi,q}$, $(\mathsf{SampCRSGen}, \mathsf{LWEGen}, \mathsf{Expand})$ satisfies LWE with respect to $\mathbf{A}^*$ with noise distribution $\overline{\chi} = \mathcal{U}([-\overline{B}, \overline{B}])$ where $\overline{B} = B^2 \cdot m^d \cdot 2^\lambda$.*

*Proof.* First, by construction of $\mathbf{A}^*$, the distributions $\mathbf{A}^*\mathbf{s}'$ where $\mathbf{S} \leftarrow \mathbb{Z}_q^W$ and $\overline{\mathbf{A}}^* \cdot \mathbf{s}$ where $\mathbf{s} \leftarrow \mathbb{Z}_q^{dwm^{d-1}}$ are identically distributed, namely, they are uniformly distributed in the column-span of $\mathbf{A}^*$. In particular, by Lemma 4.1:

$$(\mathsf{coins}_P, \mathbf{P}, \{\mathbf{A}_i\}_{i\in[d]}, \mathbf{A}^*\mathbf{s}' + \mathbf{e}) \equiv (\mathsf{coins}_P, \mathbf{P}, \{\mathbf{A}_i\}_{i\in[d]}, \overline{\mathbf{A}}^* \cdot \mathbf{s} + \mathbf{e}) \approx_c (\mathbf{P}, \{\mathbf{A}_i\}_{i\in[d]}, \mathbf{b}')$$

where $\mathbf{s}' \leftarrow \mathbb{Z}_q^W$, $\mathbf{s} \leftarrow \mathbb{Z}_q^{dwm^{d-1}}$, $\mathbf{e} \leftarrow \overline{\chi}$ and $\mathbf{b}' \leftarrow \mathbb{Z}_q^M$. Observe furthermore that $D_P = \chi$ satisfies the constraints of Lemma 4.1.

The claim then follows by observing that one can generate the randomness $\mathsf{coins}_{\mathsf{crs}}, \mathsf{coins}_{\mathsf{seed}}$ used by $\mathsf{SampCRSGen}, \mathsf{LWEGen}$ given $\mathsf{coins}_P, \mathbf{A}_i$. This is because $\mathbf{A}_i$ is sampled uniformly at random in $\mathbb{Z}_q^{m\times w}$, and one can sample the additional coins used to generate $\mathbf{P}', \mathbf{S}_i, \mathbf{E}_i$ on its own.

□

### 4.3 Instantiating the Parameters

**Parameters.** We first go through our parameters, and show that they satisfy the constraints of Definition 3.1.

Our candidate is a "degree-$d$" sampler, where $d \geq 2$ is a fixed constant integer. It expands LWE samples $\mathbf{B}_i \in \mathbb{Z}_q^{m \times k}$ to a matrix $\mathbf{B}^* \in \mathbb{Z}_q^{M \times K}$, using matrices $\mathbf{P} \leftarrow \chi^{M \times m^d}$ and $\mathbf{P}' \leftarrow \chi^{k^d \times K}$.[5] This expansion has stretch $\gamma$, in the sense that $MK = (mk)^\gamma$. $w$ and $W$ are the respective widths of the underlying matrices $\mathbf{A}_i \in \mathbb{Z}_q^{m \times w}$ and $\mathbf{A}^* \in \mathbb{Z}_q^{M \times W}$. $\delta$ is the succinctness parameter of our sampler.

$\chi$ denotes a $B$-bounded distribution used to sample $\mathsf{seed}_{\mathbf{B}^*}$, namely the matrices $\{\mathbf{E}_i\}_{i \in [d]}$, and we assume that $\mathrm{LWE}_{w,q,\chi}$ holds. $\beta$ is a bound on $\|\mathbf{E}^*\|$ which depends on $B$.

$\overline{\chi}$ denotes a $\overline{B}$-bounded distribution such that LWE with respect to $\mathbf{A}^*$ holds (assuming LWE holding for some fixed parameters only dependent on the security parameter $\lambda$). $\alpha$ denotes a blow-up factor that defines the noise bound $\beta_0$ that the sampler is masking in the security property, namely $\beta_0 = \alpha \overline{B}$.

We gather the constraints on our parameters below:

- $N = MK$ //constraint of the sampler
- $(dmk + MW + WK) \leq N^\delta \cdot \mathrm{poly}(\lambda, \log q)$ for some $\delta < 1$ //$\delta$-succinctness
- $M^2 \leq N^\delta \cdot \mathrm{poly}(\lambda, \log q)$ //for SRE succinctness
- $M \leq m^d$ //LWE with respect to $\mathbf{A}^*$ (Corollary 4.2)
- $\chi$ is a $B$-bounded distribution such that $\mathrm{LWE}_{w,q,\chi}$ holds. //base LWE assumption
- $\overline{B} = B^2 m^d \cdot 2^\lambda$ //LWE with respect to $\mathbf{A}^*$ (Corollary 4.2)
- $\beta = B^2 (mkB)^d$ //bound on $\|\mathbf{E}^*\|$
- $B$ is large enough so that $\beta \geq \beta_0 \cdot 2^\lambda$ where $\beta_0 = \alpha \overline{B}$. //constraint of the sampler
- $q \geq 8\beta$. //constraint of the sampler

We additionally add the following constraints to ensure security:

- $\gamma < d/2$ //to avoid SOS attacks (Section 4.5).
- $M \leq m^d, K \leq k^d$ //to avoid rank attacks[6] (Section 4.5).

Next, we show our candidate sampler satisfies these constraints. Given the security parameter $\lambda$, fix a degree $d = \mathcal{O}(1)$, a dimension $w = w(\lambda)$, and a bound $B = B(\lambda)$. Given additional parameters $N \geq w^{6d}$ and $\alpha$ as input, our candidate sets the following parameters.

It fixes a stretch parameter $\gamma \in \left[\frac{2d}{2d-1/6}, d/2\right)$.

Set $m = N^{1/2d} \geq w^3$. It then defines the following "dimension" parameters $k, M, K$:

$$k = m^{\frac{2d}{\gamma}-1}, \quad M = m^{d-1/2}, \quad K = m^{d+1/2}$$

[5] In general, we can use a different (small) distributions $D_P$ and $D_{P'}$ for $\mathbf{P}$, $\mathbf{P}'$. We only set $D_P = D'_P = \chi$ to minimize the number of distributions and parameters.

[6] The first constraint is redundant with the constraints of Corollary 4.2.

and $wm^{d-1} \leq W = \text{rank}\left(\overline{\mathbf{A}}^*\right) \leq m^d - (m-w)^d < dwm^{d-1} = \mathsf{width}\left(\overline{\mathbf{A}}^*\right)$ by construction of $\mathbf{A}^*$.[7] Note that the second inequality is strict as $m > w$,[8] that is, $\overline{\mathbf{A}}^*$ is rank deficient.

It then defines the following "bound" parameters $\overline{B}, \beta$:

$$\overline{B} = B^2 m^d \cdot 2^\lambda, \quad \beta = B^2(mkB)^d,$$

where we assume that $\chi$ is $B$-bounded with $B \geq \frac{(\alpha \cdot 2^{2\lambda})^{1/d}}{k}$ such that $\mathrm{LWE}_{w,q,\chi}$ holds.[9]

Let $\overline{\chi} = \mathcal{U}([-\overline{B}, \overline{B}])$ be the uniform distribution over $[-\overline{B}, \overline{B}]$. It finally sets the modulus $q$ as

$$q = 8\beta.$$

We show that the setting of parameters satisfy all the constraints described above. First, by definition, $N = m^{2d} = MK$. Furthermore:

$$\begin{aligned}
\mathsf{bitlength}(\mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \mathbf{S}^*) &= dmk \log q \ + \ M \cdot W \log q \ + \ W \cdot K \log q \\
&< \left(dm^{2d/\gamma} + dwm^{2d-3/2} + dwm^{2d-1/2}\right) \cdot \log q \\
&= \left(m^{2d-\frac{1}{6}} \ + \ dm^{2d/\gamma}\right) \cdot \log q \\
&= N^\delta \cdot \mathrm{poly}(\lambda, \log q)
\end{aligned}$$

with $\delta = 1 - \frac{1}{12d} = \frac{2d-1/6}{2d}$, where we used $W < dwm^{d-1}$, $w \leq m^{1/3}$, which follows as $N \geq w^{6d}$ and $m = N^{1/2d}$, and $1/\gamma \leq \delta$.

We furthermore have $M^2 = m^{2d-1} \leq N^\delta$.

We have by construction: $\overline{B} = B^2 m^d \cdot 2^\lambda$, $\beta_0 = \alpha\overline{B}$ , $\beta = \beta_0 \cdot 2^\lambda$, $\beta \geq B^2(mkB)^d$ and $q = 8\beta$, so that the constraint $\beta \geq \beta_0 \cdot 2^\lambda$ can be rewritten as:

$$B^2(mkB)^d \geq \alpha \cdot 2^\lambda \cdot (B^2 m^d 2^\lambda),$$

which is exactly our constraint on $B$.

Last, we have $\gamma < d/2$ by definition, $M = m^{d-1/2} \leq m^d$, and $K = m^{d+1/2} \leq (m^3)^d$.

**Remark 4.1** (Length of the CRS)**.** As noted in Remark 3.5, a long CRS is required for security to hold if we allow arbitrary auxiliary information $\mathsf{aux}$. We note this is the case for the parameters of Conjecture 1. Indeed: $\mathsf{bitlength}(\mathbf{P}') = k^d K \ \log q \geq m^{4d+1/2} \ \log q \geq N \ \mathrm{poly}(\lambda, \log q) = m^{2d} \ \mathrm{poly}(\lambda, \log q)$.

**Remark 4.2** (Parameters as a function of $\gamma$.)**.** Our construction induces different parameters, according the choice of $\gamma$. The main affected parameter is $k$, which goes from $k = m^{3+o(1)}$ to $k \approx m^{2d}$. We note here that it also makes sense to use a constant $\gamma \in \left(1, \frac{2d}{2d-1/6}\right]$ for our construction. The only difference is that the succinctness of the scheme then becomes $1/\gamma$ as opposed to $1 - \mathcal{O}(1/d)$.

We gather some example parameters in the table below. In all cases, we set $d \geq 4$ be a constant, $m \geq w^3$ so that $N = m^{2d}$, $M = m^{d-1/2}$ and $K = m^{d+1/2}$. The third column represent the components that should have size bounded by $N^\delta$ to satisfy $\delta$-succinctness.

| Stretch $\gamma$ | Dimension $k$ | $M^2 + \mathsf{bitlength}(\mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \mathbf{S}^*)$ | Succinctness $\delta$ |
|---|---|---|---|
| $\gamma = d/3$ | $k = m^5$ | $\mathcal{O}(m^{2d-1/6})$ | $\delta = 1 - \frac{1}{12d}$ |
| $\gamma = \frac{2d}{2d-1/6}$ | $k = m^{2d-7/6}$ | $\mathcal{O}(m^{2d-1/6})$ | $\delta = 1 - \frac{1}{12d} = 1/\gamma$ |
| $\gamma = \frac{2d}{2d-\epsilon}$ | $k = m^{2d-\epsilon-1}$ | $\mathcal{O}(m^{2d-\epsilon})$ | $\delta = 1/\gamma$ |

Figure 1: Example parameters. In the above, we fix a constant $d \geq 4$ and $w = w(\lambda)$. The output size is $N = m^{2d}$ where $N \geq w^{6d}$.

Next, we state our main conjecture for our candidate, namely that it satisfies the weak notion of security of Definition 3.2. Looking ahead, thanks to Theorem 3.3, this suffices to imply iO.

**Conjecture 1** (Conjectured security)**.** *Let $\chi$ be a $B$-bounded distribution, and assume LWE$_{w,q,\chi}$ holds. Then* ($\mathsf{SampCRSGen}, \mathsf{LWEGen}, \mathsf{Expand}$) *with any of the parameters above satisfies weak $\beta_0$-flooding (Definition 3.2), where $\beta_0 = \alpha\overline{B}$.*

**Remark 4.3** (Security as a function of $d$)**.** Our constructions decouples the stretch $\gamma$, defined as $(\mathsf{bitlength}(\{\mathbf{B}_i\}_{i\in[d]}))^\gamma = \mathsf{bitlength}(\mathbf{B}^*)$ (up to polynomial factors in $\lambda, \log q$), from the degree $d$. In particular, for a fixed (constant) stretch $\gamma \geq \frac{2d}{2d-1/6}$, we expect Conjecture 1 to be weaker as $d$ increases.

Next, combining the above with Theorem 3.3, we describe two distributions whose indistinguishability would imply the existence of succinct LWE sampler with $\theta$-flooding (Definition 3.1) for some parameter $\theta$. Looking ahead, combined with Theorem 5.1, this suffices to imply an iO scheme.

**Conjecture 2** (Stand-alone $\theta$-flooding)**.** *Let $\beta_0 = \theta \cdot 2^\lambda$. With any of the parameters* $\mathsf{params}$ *described above, the following distributions $\Delta_b$ are indistinguishable:*

$$\Delta_b = (\mathbf{P}, \mathbf{P}', \mathsf{seed}_{\mathbf{B}^*}, \ \mathbf{A}^*, \quad \widehat{\mathbf{B}} = \mathbf{A}^*\mathbf{S}_0 + \mathbf{F}, \quad \mathbf{C} = \mathbf{A}^*\mathbf{R} + \mathbf{E} - b\mathbf{G}, \quad \mathbf{E}^* + \mathbf{E}\cdot\mathbf{G}^{-1}(\widehat{\mathbf{B}}) - b\mathbf{F})$$

*where*

$$\mathbf{P} \leftarrow \chi^{M\times m^d}, \quad \mathbf{P'} \leftarrow \chi^{k^d\times K},$$

$$\mathsf{seed}_{\mathbf{B}^*} = \{\mathbf{B}_i\}_{i\in[d]} \in (\mathbb{Z}_q^{m\times w})^d$$

$$\widehat{\mathbf{B}} \in \mathbb{Z}_q^{M\times K}, \quad \textit{where} \quad \mathbf{S}_0 \leftarrow \mathbb{Z}_q^{W\times K}, \quad \mathbf{F} \leftarrow \chi_{\mathsf{flood}}^{M\times K}$$

$$\mathbf{C} \in \mathbb{Z}_q^{M\times M\log q}, \quad \textit{where} \quad \mathbf{R} \leftarrow \mathbb{Z}_q^{W\times M\log q}, \quad \mathbf{E} \leftarrow \overline{\chi}^{M\times M\log q}$$

*where* $(\mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \mathbf{S}^*) \leftarrow \mathsf{LWEGen}(\mathsf{params}, \mathbf{P}, \mathbf{P}')$*,* $\mathbf{B}^* = \mathsf{Expand}(\mathsf{params}, \mathbf{P}, \mathbf{P}', \mathsf{seed}_{\mathbf{B}^*})$*, and* $\mathbf{E}^* = \mathbf{B}^* - \mathbf{A}^*\mathbf{S}^*$*. Furthermore, $\overline{\chi}$ is a noise distribution such that LWE with respect to $\mathbf{A}^*$ holds, and $\chi_{\mathsf{flood}}$ is a $\beta_0$-bounded distribution that floods $\theta$-bounded distributions.*

[7] We prove that rank $\left(\overline{\mathbf{A}^*}\right) \leq m^d - (m-w)^d$ in Section 4.5, paragraph Rank of $\mathbf{A}^*\mathbf{S}^*$.

[8] Writing $m = m' + w$ where $m' > 0$, the difference $(m'+w)^d - (m'^d + dw(m'+w)^{d-1})$ is the sum of monomials in $m', w$ with positive coefficients.

[9] This is without loss of generality by defining for instance $\chi' = \chi + [-B, B]$ where $B'$ is large enough to satisfy the previous constraint. A direct reduction ensures that if LWE holds with $\chi$, then it holds with $\chi'$.

### 4.4 Alternate Candidate Construction

Here we present a variant of the construction in Section 4.1. The main intuition is that this new variant sums $T$ copies of the candidate of Section 4.1, but reusing the same matrices $\mathbf{A}_i$ across all copies.

We now describe our variant ($\mathsf{SampCRSGen}, \mathsf{LWEGen}, \mathsf{Expand}$). It uses the same parameters as Section 4.1, and an additional parameter $T = T(\lambda)$.

- $\mathsf{SampCRSGen}(1^\lambda, 1^N, \alpha)$: Derive $\mathsf{parameters} = (d, m, k, w, M, K, \overline{\chi}, \overline{B}, \chi, \beta, q)$ from $(1^\lambda, 1^N, \alpha)$ as described below. Set $\mathsf{params} = (q, M, K, \overline{\chi}, \overline{B}, \chi)$.

  Sample $\mathbf{P'} \leftarrow \chi^{k^d \times K}$. Output
$$\mathsf{crs} = (\mathsf{params}, \mathbf{P'}).$$

- $\mathsf{LWEGen}(\mathsf{crs})$: On input $\mathsf{crs} = (\mathsf{params}, \mathbf{P'})$, sample $\mathbf{P} \leftarrow \chi^{M \times m^d}$. For $i \in [d]$, sample $\mathbf{A}_i \leftarrow \mathbb{Z}_q^{m \times w}$. For $i \in [d], j \in [T]$, sample $\mathbf{S}_i^{(j)} \leftarrow \mathbb{Z}_q^{w \times k}$, $\mathbf{E}_i^{(j)} \leftarrow \chi^{m \times k}$ where $\chi$ is specified in $\mathsf{params}$. Compute:
$$\mathbf{B}_i^{(j)} = \mathbf{A}_i \mathbf{S}_i^{(j)} + \mathbf{E}_i^{(j)} \in \mathbb{Z}_q^{m \times k}.$$

  Set:
$$\overline{\mathbf{A}}^* = \mathbf{P} \cdot (\, \mathbf{A}_1 \otimes \mathbf{I}_m \otimes \cdots \otimes \mathbf{I}_m \,\|\, \mathbf{I}_m \otimes \mathbf{A}_2 \otimes \mathbf{I}_m \otimes \cdots \otimes \mathbf{I}_m \,\|\, \mathbf{I}_m \otimes \cdots \otimes \mathbf{I}_m \otimes \mathbf{A}_d) \in \mathbb{Z}_q^{M \times W}$$
$$\overline{\mathbf{S}}^{*(j)} = \begin{pmatrix} \mathbf{S}_1^{(j)} \otimes \mathbf{B}_2^{(j)} \otimes \cdots \otimes \mathbf{B}_d^{(j)} \\ \mathbf{E}_1^{(j)} \otimes \mathbf{S}_2^{(j)} \otimes \cdots \otimes \mathbf{B}_d^{(j)} \\ \vdots \\ \mathbf{E}_1^{(j)} \otimes \mathbf{E}_2^{(j)} \otimes \cdots \otimes \mathbf{S}_d^{(j)} \end{pmatrix} \cdot \mathbf{P'} \in \mathbb{Z}_q^{W \times K}.$$
$$\mathbf{S}^* = \sum_{j=1}^{T} \mathbf{S}^{*(j)} \cdot \mathbf{P'}$$

  Sample a random basis $\mathbf{A}^* \in \mathbb{Z}_q^{M \times W}$ of the column space of $\overline{\mathbf{A}}^*$, and solve for $\mathbf{S}^* \in \mathbb{Z}_q^{W \times K}$ such that $\mathbf{A}^* \mathbf{S}^* = \overline{\mathbf{A}}^* \cdot \overline{\mathbf{S}}^*$. Output:
$$\mathsf{seed}_{\mathbf{B}^*} = \left(\mathbf{P},\ \{\mathbf{B}_i^{(j)}\}_{i \in [d], j \in [T]}\right), \quad \mathbf{A}^*, \quad \mathbf{S}^*.$$

- $\mathsf{Expand}(\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*})$: On input $\mathsf{crs} = (\mathbf{P'})$ and $\mathsf{seed}_{\mathbf{B}^*} = (\mathbf{P}, \{\mathbf{B}_i^{(j)}\}_{i \in [d], j \in [T]})$, output:
$$\mathbf{B}^* = \mathbf{P} \cdot \left( \sum_{j=1}^{T} \mathbf{B}_1^{(j)} \otimes \cdots \otimes \mathbf{B}_d^{(j)} \right) \cdot \mathbf{P'} \in \mathbb{Z}_q^{M \times K}.$$

Intuitively, this new candidate sums $T$ copies of the construction in Section 4.1, except it reuses the same $\mathbf{A}_i$ (and $\mathbf{P}, \mathbf{P'}$) across all copies. In particular, all of those copies generate a new $\mathbf{B}^{*(j)} = \mathbf{A}^* \mathbf{S}^{*(j)} + \mathbf{E}^{*(j)}$, for the *same* matrix $\mathbf{A}^*$; in particular the dimensions of $\mathbf{B}^*, \mathbf{A}^*$ and $\mathbf{S}^*$ are the same as in Section 4.1. In particular, correctness follows by linearity, and succinctness can be argued with a very similar argument as in Claim 4.0.2.

The parameters can be set very similarly as in Section 4.3. For concreteness, we could require $T \leq w^3$, simarly set $N = m^{2d}$ as before, and

$$k = \frac{m^{2d/\gamma - 1}}{T} \geq m^{\frac{2d}{\gamma} - 2},$$

and set all the other dimension parameters as a function of $m, d$. The bound $\beta$ on $\|\mathbf{E}^*\|$ becomes $\beta = (TmkB_{\mathsf{seed}})^d$, and the modulus remains $q = 8\beta$. Correctness, succinctness and LWE with respect to $\mathbf{A}^*$ follow with almost identical proofs as in Section 4.2.

## 4.5 Cryptanalysis

We comment here on the plausible security of our candidate from Section 4.1 instantiated with the parameters of Section 4.3.

Recall that security of a succinct LWE sampler requires

$$(\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \mathbf{E}^* - \mathbf{Z}_b, \mathsf{aux}_b)$$

to hide $b$ for appropriate $\mathsf{aux}_b$ and small $\mathbf{Z}_b$.

Ignoring the auxiliary information related to the sampler for now, the crucial requirement is that $\mathbf{E}^* - \mathbf{Z}_b$ (or, equivalently, $\mathbf{A}^*\mathbf{S}^* + \mathbf{Z}_b$) hides $b$ for sufficiently small $\mathbf{Z}_b$. As noted in the technical overview, pseudorandomness of $\mathbf{E}^*$ cannot hold given $\mathsf{seed}_{\mathbf{B}^*}$: one can compute $\mathbf{B}^* - \mathbf{E}^*$ and check that it is low rank. Nonetheless, as a sanity check, we would like to ensure that the marginal distribution of $\mathbf{E}^*$ is pseudorandom *by itself*, i.e. in the absence of $\mathsf{seed}_{\mathbf{B}^*}$. We first describe some attacks on the pseudorandomness of $\mathbf{E}^*$, and their influence on our parameters in Section 4.3.

### 4.5.1 Linearization Attacks.

A strong break for the pseudorandomness of $\mathbf{E}^*$ is to recover the initial errors $\mathbf{E}_i \in \mathbb{Z}^{m \times k}$ such that $\mathbf{P}\left(\bigotimes_{i=1}^d \mathbf{E}_i\right)\mathbf{P}' = \mathbf{E}^*$. This would be enough to break pseudorandomness: only a small fraction of small $\mathbf{E}^* \in \mathbb{Z}^{M \times K}$ have such a succinct description as long as $N = MK$ is large enough compared to $m$ and $k$ (say $MK = (mk)^\gamma$ for some constant $\gamma > 1$).

One way of recovering the $\mathbf{E}_i$'s given $\mathbf{E}^*$, $\mathbf{P}$ and $\mathbf{P}'$ is to view the equation

$$\mathbf{P}\left(\bigotimes_{i=1}^d \mathbf{E}_i\right)\mathbf{P}' = \mathbf{E}^*$$

as a set of *linear* equations with the $(mk)^d$ variables

$$X_{i_1, j_1, \cdots, i_d, j_d} = \mathbf{E}_1^{i_1, j_1} \times \cdots \times \mathbf{E}_d^{i_d, j_d}$$

where $i_1, \cdots, i_d \in [m]$ and $j_1, \cdots, j_d \in [k]$, and where $\mathbf{E}^{i,j}$ denotes the $(i, j)$th component of $\mathbf{E}$. In particular, this is solvable as long as the number of equations is no smaller than the number of variables, that is:

$$MK \geq (mk)^d.$$

Our choice of parameters reflects security against linearization attacks. We also note that the linearization attack (in contrast to the sum of squares attack) works just as well over any finite field as it does over the integers.

### 4.5.2 Low-Degree Polynomials and Sum of Squares.

The recovery attack described above can be generically improved using the more refined *sum of squares* (SOS) attacks. These ensure that pseudorandomness of $\mathbf{E}^*$ cannot hold whenever

$$MK \geq (mk)^{d/2}.$$

We refer the reader to [BHJ+19] for more details on sum of squares attacks. In our scheme, we explicitly require that the stretch of our sampler, namely $\gamma$ such that $MK = (mk)^\gamma$, is smaller than $d/2$.

**Security when $m = 1$.** When $m = 1$, $\mathbf{P}$ is a scalar that we will ignore. We are given

$$\mathbf{e}^* = \left(\bigotimes_{i=1}^{d} \mathbf{e}_i\right) \mathbf{P}'$$

which is a vector of length $K$. Since $\bigotimes_{i=1}^{d} \mathbf{e}_i$ is simply the set of all degree-$d$ multilinear monomials with a variable from each of the $\mathbf{e}_i$, this can be interpreted as evaluating $K$ degree-$d$ polynomials with Gaussian coefficients on the $dk$ variables in $\mathbf{e}_1, \ldots, \mathbf{e}_d$. Since $K \ll k^{d/2}$, neither linearization nor sum of squares seems to apply [BHJ+19].

The work of Kosov [Kos20] tells us each entry in $\mathbf{E}^*$ by itself, namely a polynomial with Gaussian coefficients evaluated on Gaussian inputs, comes from a noise-flooding distribution (for mild choices of parameters).

This analysis also points to the qualitative distinction between our assumption and the analysis above for $m = 1$. When $m = 2$, for example, we obtain $MK$ polynomials evaluated on a number of correlated random variables. That is, setting the two rows of $\mathbf{E}_i$ to be $\mathbf{e}_{i1}$ and $\mathbf{e}_{i2}$,

$$\mathbf{E}^* = \mathbf{P} \begin{bmatrix} \mathbf{e}_{11} \otimes \mathbf{e}_{21} \otimes \cdots \otimes \mathbf{e}_{d1} \\ \mathbf{e}_{12} \otimes \mathbf{e}_{21} \otimes \cdots \otimes \mathbf{e}_{d1} \\ \vdots \\ \mathbf{e}_{12} \otimes \mathbf{e}_{22} \otimes \cdots \otimes \mathbf{e}_{d2} \end{bmatrix} \mathbf{P}'$$

To the best of our knowledge, all attacks described above still fail. In fact, we don't even have an attack if $\mathbf{P} = \mathbf{I}_{2^d}$ was the identity and $M = 2^d$. However, this is certainly a cryptanalytic avenue worth pursuing in the future.

### 4.5.3 Covariance Analysis.

We now consider a class of attacks which attempt to distinguish $\mathbf{E}^*$ from $\mathbf{E}^* + \mathbf{M}$ for an arbitrary matrix $\mathbf{M}$ with small entries by computing the covariance between pairs of input entries. We rule out such an attack for $d = 2$ with the following result:

**Theorem 4.3.** *Let $\mathbf{E}^* = \mathbf{P}\left(\mathbf{E}_1 \otimes \mathbf{E}_2\right)\mathbf{P}'$ with $\mathbf{P}, \mathbf{P}', \mathbf{E}_1, \mathbf{E}_2$ drawn as in Section 4.3 where $\chi$ (the $B$-bounded distribution entries of $\mathbf{E}_1$ and $\mathbf{E}_2$ are drawn from) is $\mathcal{N}(0, \sigma^2)$. Then for any matrix $\mathbf{M}$ with entries bounded by $\beta_0$ defined as in Section 4.3 and any indices $i, j, k, l$, the following distributions are indistinguishable:*

$$\left\{\mathbb{E}\left(\mathbf{Y}_i\mathbf{Y}_j\mathbf{Y}_k\mathbf{Y}_l\right) - \mathbb{E}\left(\mathbf{Y}_i\mathbf{Y}_j\right)\mathbb{E}\left(\mathbf{Y}_k\mathbf{Y}_l\right)\right) \mid \mathbf{Y} = \mathbf{E}^* + b \cdot \mathbf{M}\right\}_{b \in \{0,1\}}.$$

*Proof.* Using Claim 4.3.1 and Claim 4.3.3, i.e. that for any $i, j, k$, $\mathbb{E}\left(\mathbf{E}_i^*\right) = \mathbb{E}\left(\mathbf{E}_i^*\mathbf{E}_j^*\mathbf{E}_k^*\right) = 0$, we have for any $i, j, k, l$ that

$$\begin{aligned}
&\mathbb{E}\left((\mathbf{E}_i^* + \mathbf{M}_i)(\mathbf{E}_j^* + \mathbf{M}_j)(\mathbf{E}_k^* + \mathbf{M}_k)(\mathbf{E}_l^* + \mathbf{M}_l)\right) - \mathbb{E}\left((\mathbf{E}_i^* + \mathbf{M}_i)(\mathbf{E}_j^* + \mathbf{M}_j)\right)\mathbb{E}\left((\mathbf{E}_k^* + \mathbf{M}_k)(\mathbf{E}_l^* + \mathbf{M}_l)\right)\\
&= \mathbb{E}\left(\mathbf{E}_i^*\mathbf{E}_j^*\mathbf{E}_k^*\mathbf{E}_l^*\right) - \mathbb{E}\left(\mathbf{E}_i^*\mathbf{E}_j^*\right)\mathbb{E}\left(\mathbf{E}_k^*\mathbf{E}_l^*\right)\\
&\quad + \mathbb{E}(\mathbf{E}_i^*\mathbf{E}_k^*)\mathbf{M}_j\mathbf{M}_l + \mathbb{E}(\mathbf{E}_i^*\mathbf{E}_l^*)\mathbf{M}_j\mathbf{M}_k + \mathbb{E}(\mathbf{E}_j^*\mathbf{E}_k^*)\mathbf{M}_i\mathbf{M}_l + \mathbb{E}(\mathbf{E}_j^*\mathbf{E}_l^*)\mathbf{M}_i\mathbf{M}_k.
\end{aligned}$$

Using Claim 4.3.2 and Claim 4.3.4, we can now simplify distribution for $b = 0$ to

$$\begin{aligned}
\mathbb{E}\left(\mathbf{E}_i^*\mathbf{E}_j^*\mathbf{E}_k^*\mathbf{E}_l^*\right) - \mathbb{E}\left(\mathbf{E}_i^*\mathbf{E}_j^*\right)\mathbb{E}\left(\mathbf{E}_k^*\mathbf{E}_l^*\right) &\geq 2 \cdot \mathbb{E}\left(\mathbf{E}_i^*\mathbf{E}_j^*\right)\mathbb{E}\left(\mathbf{E}_k^*\mathbf{E}_l^*\right)\\
&= 2\sigma^8 \cdot (\mathbf{P}\mathbf{P}^\top)_{i_1j_1}(\mathbf{P}'^\top\mathbf{P}')_{i_2j_2}(\mathbf{P}\mathbf{P}^\top)_{k_1l_1}(\mathbf{P}'^\top\mathbf{P}')_{k_2l_2}.
\end{aligned}$$

Note that all the extra $\mathbf{M}$-dependent terms in distribution 1 have coefficient $\sigma^4$, since they are multiplied by the expectation of the product of two entries of $\mathbf{E}^*$. Since the entries of $\mathbf{M}$ are bounded by $\beta_0$ and therefore will be flooded by the entries of $\mathbf{E}^*$, we can conclude that the distributions are indistinguishable.

**Claim 4.3.1.** *For any $i = (i_1, i_2)$, $\mathbb{E}\left(\mathbf{E}_i^*\right) = 0$.*

*Proof.* The entries of $\mathbf{E}_1$ and $\mathbf{E}_2$ are drawn from independent Gaussians with mean 0, so the expectation $\mathbb{E}\left(\mathbf{E}_{1,\beta_1\gamma_1}\mathbf{E}_{2,\beta_2\gamma_2}\right) = 0$, $\forall\ \beta = \beta_1 n + \beta_2, \gamma = \gamma_1 n + \gamma_2$. Then the expected value of an entry of $\mathbf{E}^*$ is by linearity of expectation

$$\mathbb{E}\left(\mathbf{E}_i^*\right) = \sum_{\beta,\gamma} \mathbf{P}_{i_1\beta}\mathbb{E}\left(\mathbf{E}_{1,\beta_1\gamma_1}\mathbf{E}_{2,\beta_2\gamma_2}\right)\mathbf{P}'_{\gamma i_2} = \sum_{\beta,\gamma}\left(\mathbf{P}_{i_1\beta} \cdot 0 \cdot \mathbf{P}'_{\gamma i_2}\right) = 0.$$

□

**Claim 4.3.2.** *For any $i = (i_1, i_2)$ and $j = (j_1, j_2)$, $\mathbb{E}\left(\mathbf{E}_i^*\mathbf{E}_j^*\right) = \sigma^4 \cdot (\mathbf{P}\mathbf{P}^\top)_{i_1j_1}(\mathbf{P}'^\top\mathbf{P}')_{i_2j_2}$.*

*Proof.* The expectation of the product of two entries of $\mathbf{E}^*$ is

$$\mathbb{E}\left(\mathbf{E}_i^*\mathbf{E}_j^*\right) = \sum_{\beta,\gamma,\beta',\gamma'} \mathbf{P}_{i_1\beta}\mathbf{P}_{j_1\beta'}\mathbf{P}'_{\gamma i_2}\mathbf{P}'_{\gamma' j_2}\mathbb{E}\left(\mathbf{E}_{1,\beta_1\gamma_1}\mathbf{E}_{2,\beta_2\gamma_2}\mathbf{E}_{1,\beta_1'\gamma_1'}\mathbf{E}_{2,\beta_2'\gamma_2'}\right) \tag{10}$$

by linearity of expectation. The entries of $\mathbf{E}_1$ and $\mathbf{E}_2$ are drawn from independent Gaussians with mean 0 and variance $\sigma^2$, so

$$\mathbb{E}\left(\mathbf{E}_{1,\beta_1\gamma_1}\mathbf{E}_{2,\beta_2\gamma_2}\mathbf{E}_{1,\beta_1'\gamma_1'}\mathbf{E}_{2,\beta_2'\gamma_2'}\right) = \begin{cases}\sigma^4 & \text{if } \beta = \beta', \gamma = \gamma'\\ 0 & \text{otherwise.}\end{cases}$$

This allows us to simplify Eq. (10) to obtain

$$\mathbb{E}\left(\mathbf{E}_i^*\mathbf{E}_j^*\right) = \sigma^4 \cdot \left(\sum_{\beta}\mathbf{P}_{i_1\beta}\mathbf{P}_{j_1\beta}\right)\left(\sum_{\gamma}\mathbf{P}'_{\gamma i_2}\mathbf{P}'_{\gamma j_2}\right) = \sigma^4 \cdot (\mathbf{P}\mathbf{P}^\top)_{i_1j_1}(\mathbf{P}'^\top\mathbf{P}')_{i_2j_2}.$$

□

**Claim 4.3.3.** *For any $i = (i_1, i_2)$, $j = (j_1, j_2)$, and $k = (k_1, k_2)$, $\mathbb{E}\left(\mathbf{E}^*_i\mathbf{E}^*_j\mathbf{E}^*_k\right) = 0$.*

*Proof.* Again using the fact that the entries of $\mathbf{E}_1$ and $\mathbf{E}_2$ are independently drawn with mean 0, it's easy to see that the expected value of the product of three entries of $\mathbf{E}^*$ is 0, since every term will include at least one entry of $\mathbf{E}_1$ and at least one entry of $\mathbf{E}_2$ that isn't squared, and thus sends the expectation to 0. □

**Claim 4.3.4.** *For any $i = (i_1, i_2)$, $j = (j_1, j_2)$, $k = (k_1, k_2)$, and $l = (l_1, l_2)$, $\mathbb{E}\left(\mathbf{E}^*_i\mathbf{E}^*_j\mathbf{E}^*_k\mathbf{E}^*_l\right) \geq 3 \cdot \mathbb{E}(\mathbf{E}^*_i\mathbf{E}^*_j)\mathbb{E}(\mathbf{E}^*_k\mathbf{E}^*_l)$.*

*Proof.* We will now use a similar technique as above to simplify the expectation of the product of four entries of $\mathbf{E}^*$, which by linearity of expectation is

$$\begin{aligned}\mathbb{E}\left(\mathbf{E}^*_i\mathbf{E}^*_j\mathbf{E}^*_k\mathbf{E}^*_l\right) = &\sum_{\beta,\gamma,\beta',\gamma',\beta'',\gamma'',\beta''',\gamma'''} \mathbf{P}_{i_1\beta}\mathbf{P}_{j_1\beta'}\mathbf{P}_{k_1\beta''}\mathbf{P}_{l_1\beta'''}\mathbf{P}'_{\gamma i_2}\mathbf{P}'_{\gamma' j_2}\mathbf{P}'_{\gamma'' k_2}\mathbf{P}'_{\gamma''' l_2} \\ &\cdot \mathbb{E}\left(\mathbf{E}_{1,\beta_1\gamma_1}\mathbf{E}_{2,\beta_2\gamma_2}\mathbf{E}_{1,\beta'_1\gamma'_1}\mathbf{E}_{2,\beta'_2\gamma'_2}\mathbf{E}_{1,\beta''_1\gamma''_1}\mathbf{E}_{2,\beta''_2\gamma''_2}\mathbf{E}_{1,\beta'''_1\gamma'''_1}\mathbf{E}_{2,\beta'''_2\gamma'''_2}\right) \quad (11)\end{aligned}$$

where we have $\beta = \beta_1 n + \beta_2, \gamma = \gamma_1 n + \gamma_2$, and so on for $\beta', \beta'', \beta''', \gamma', \gamma'', \gamma'''$. Because the entries of $\mathbf{E}_1$ and $\mathbf{E}_2$ are drawn from independent Gaussians with mean 0 and variance $\sigma^2$, we have that

$$\begin{aligned}&\mathbb{E}\left(\mathbf{E}_{1,\beta_1\gamma_1}\mathbf{E}_{2,\beta_2\gamma_2}\mathbf{E}_{1,\beta'_1\gamma'_1}\mathbf{E}_{2,\beta'_2\gamma'_2}\mathbf{E}_{1,\beta''_1\gamma''_1}\mathbf{E}_{2,\beta''_2\gamma''_2}\mathbf{E}_{1,\beta'''_1\gamma'''_1}\mathbf{E}_{2,\beta'''_2\gamma'''_2}\right) \\ &= \begin{cases} 9\sigma^8 & \text{if } \beta = \beta' = \beta'' = \beta''' \\ 3\sigma^8 & \text{if } \beta_1 = \beta'_1 = \beta''_1 = \beta'''_1, \beta_2 = \beta'_2 \neq \beta''_2 = \beta'''_2, \text{ and symmetrical cases} \\ \sigma^8 & \text{if } \beta_1 = \beta'_1 \neq \beta''_1 = \beta'''_1, \beta_2 = \beta'_2 \neq \beta''_2 = \beta'''_2, \text{ and symmetrical cases} \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

where the conditions are always symmetrical for $(\beta, \beta', \beta'', \beta''')$ and $(\gamma, \gamma', \gamma'', \gamma''')$. Let us denote

$$\begin{aligned}T^{\beta,\beta',\gamma,\gamma'}_{i,j,k,l} &= \mathbf{P}_{i_1\beta}\mathbf{P}_{j_1\beta}\mathbf{P}_{k_1\beta'}\mathbf{P}_{l_1\beta'}\mathbf{P}'_{\gamma i_2}\mathbf{P}'_{\gamma j_2}\mathbf{P}'_{\gamma' k_2}\mathbf{P}'_{\gamma' l_2} + \mathbf{P}_{i_1\beta}\mathbf{P}_{j_1\beta'}\mathbf{P}_{k_1\beta}\mathbf{P}_{l_1\beta'}\mathbf{P}'_{\gamma i_2}\mathbf{P}'_{\gamma' j_2}\mathbf{P}'_{\gamma k_2}\mathbf{P}'_{\gamma' l_2} \\ &+ \mathbf{P}_{i_1\beta}\mathbf{P}_{j_1\beta'}\mathbf{P}_{k_1\beta'}\mathbf{P}_{l_1\beta}\mathbf{P}'_{\gamma i_2}\mathbf{P}'_{\gamma' j_2}\mathbf{P}'_{\gamma' k_2}\mathbf{P}'_{\gamma l_2}.\end{aligned}$$

and for $\beta^{(0)} = \beta_1 \cdot n + \beta_2, \beta^{(1)} = \beta'_1 \cdot n + \beta_2, \beta^{(2)} = \beta_1 \cdot n + \beta'_2, \beta^{(3)} = \beta'_1 \cdot n + \beta'_2$, and symmetrically $\gamma^{(0)}, \gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)}$, let us denote

$$\begin{aligned}S^{\beta_1,\beta'_1,\gamma_1,\gamma'_1,\beta_2,\beta'_2,\gamma_2,\gamma'_2}_{i,j,k,l} &= \mathbf{P}_{i_1\beta^{(0)}}\mathbf{P}_{j_1\beta^{(1)}}\mathbf{P}_{i'_1\beta^{(2)}}\mathbf{P}_{j'_1\beta^{(3)}}\mathbf{P}'_{\gamma^{(0)}i_2}\mathbf{P}'_{\gamma^{(1)}j_2}\mathbf{P}'_{\gamma^{(2)}i'_2}\mathbf{P}'_{\gamma^{(3)}j'_2} \\ &+ \mathbf{P}_{i_1\beta^{(0)}}\mathbf{P}_{j_1\beta^{(2)}}\mathbf{P}_{i'_1\beta^{(1)}}\mathbf{P}_{j'_1\beta^{(3)}}\mathbf{P}'_{\gamma^{(0)}i_2}\mathbf{P}'_{\gamma^{(2)}j_2}\mathbf{P}'_{\gamma^{(1)}i'_2}\mathbf{P}'_{\gamma^{(3)}j'_2} \\ &+ \mathbf{P}_{i_1\beta^{(0)}}\mathbf{P}_{j_1\beta^{(1)}}\mathbf{P}_{i'_1\beta^{(3)}}\mathbf{P}_{j'_1\beta^{(2)}}\mathbf{P}'_{\gamma^{(0)}i_2}\mathbf{P}'_{\gamma^{(1)}j_2}\mathbf{P}'_{\gamma^{(3)}i'_2}\mathbf{P}'_{\gamma^{(2)}j'_2} \\ &+ \mathbf{P}_{i_1\beta^{(0)}}\mathbf{P}_{j_1\beta^{(2)}}\mathbf{P}_{i'_1\beta^{(2)}}\mathbf{P}_{j'_1\beta^{(1)}}\mathbf{P}'_{\gamma^{(0)}i_2}\mathbf{P}'_{\gamma^{(3)}j_2}\mathbf{P}'_{\gamma^{(2)}i'_2}\mathbf{P}'_{\gamma^{(1)}j'_2} \\ &+ \mathbf{P}_{i_1\beta^{(0)}}\mathbf{P}_{j_1\beta^{(2)}}\mathbf{P}_{i'_1\beta^{(3)}}\mathbf{P}_{j'_1\beta^{(1)}}\mathbf{P}'_{\gamma^{(0)}i_2}\mathbf{P}'_{\gamma^{(2)}j_2}\mathbf{P}'_{\gamma^{(3)}i'_2}\mathbf{P}'_{\gamma^{(1)}j'_2} \\ &+ \mathbf{P}_{i_1\beta^{(0)}}\mathbf{P}_{j_1\beta^{(3)}}\mathbf{P}_{i'_1\beta^{(1)}}\mathbf{P}_{j'_1\beta^{(2)}}\mathbf{P}'_{\gamma^{(0)}i_2}\mathbf{P}'_{\gamma^{(3)}j_2}\mathbf{P}'_{\gamma^{(1)}i'_2}\mathbf{P}'_{\gamma^{(2)}j'_2}\Big)\end{aligned}$$

This allows us to rewrite Eq. (11) to obtain

$$\begin{aligned}
&\mathbb{E}(\mathbf{E}^*_i\mathbf{E}^*_j\mathbf{E}^*_k\mathbf{E}^*_l) \\
&\quad = 9\sigma^8 \cdot \sum_{\beta,\gamma} \mathbf{P}_{i_1\beta}\mathbf{P}_{j_1\beta}\mathbf{P}_{k_1\beta}\mathbf{P}_{l_1\beta}\mathbf{P}'_{\gamma i_2}\mathbf{P}'_{\gamma j_2}\mathbf{P}'_{\gamma k_2}\mathbf{P}'_{\gamma l_2} \\
&\qquad + 3\sigma^8 \cdot \sum_{\beta_1,\gamma_1} \sum_{\beta_2\neq\beta'_2,\gamma_2\neq\gamma'_2} T^{\beta,\beta',\gamma,\gamma'}_{i,j,k,l} + 3\sigma^8 \cdot \sum_{\beta_1\neq\beta'_1,\gamma_1\neq\gamma'_1} \sum_{\beta_2,\gamma_2} T^{\beta,\beta',\gamma,\gamma'}_{i,j,k,l} \\
&\qquad + \sigma^8 \cdot \sum_{\beta_1\neq\beta'_1,\gamma_1\neq\gamma'_1} \sum_{\beta_2\neq\beta'_2,\gamma_2\neq\gamma'_2} T^{\beta,\beta',\gamma,\gamma'}_{i,j,k,l} + \sigma^8 \cdot \sum_{\beta_1\neq\beta'_1,\gamma_1\neq\gamma'_1} \sum_{\beta_2\neq\beta'_2,\gamma_2\neq\gamma'_2} S^{\beta_1,\beta'_1,\gamma_1,\gamma'_1,\beta_2,\beta'_2,\gamma_2,\gamma'_2}_{i,j,k,l} \\
&\quad = 3\sigma^8 \cdot \sum_{\beta,\beta',\gamma,\gamma'} T^{\beta,\beta',\gamma,\gamma'}_{i,j,k,l} \\
&\qquad - 2\sigma^8 \cdot \sum_{\beta_1\neq\beta'_1,\gamma_1\neq\gamma'_1} \sum_{\beta_2\neq\beta'_2,\gamma_2\neq\gamma'_2} T^{\beta,\beta',\gamma,\gamma'}_{i,j,k,l} + \sigma^8 \cdot \sum_{\beta_1\neq\beta'_1,\gamma_1\neq\gamma'_1} \sum_{\beta_2\neq\beta'_2,\gamma_2\neq\gamma'_2} S^{\beta_1,\beta'_1,\gamma_1,\gamma'_1,\beta_2,\beta'_2,\gamma_2,\gamma'_2}_{i,j,k,l}
\end{aligned}$$

After noting that $\mathbb{E}(\mathbf{E}^*_i\mathbf{E}^*_j)\mathbb{E}(\mathbf{E}^*_k\mathbf{E}^*_l) = \sigma^8 \cdot \sum_{\beta,\gamma,\beta',\gamma'} \mathbf{P}_{i_1\beta}\mathbf{P}_{j_1\beta}\mathbf{P}_{k_1\beta'}\mathbf{P}_{l_1\beta'}\mathbf{P}'_{\gamma i_2}\mathbf{P}'_{\gamma j_2}\mathbf{P}'_{\gamma' k_2}\mathbf{P}'_{\gamma' l_2}$, i.e. summing over only one term of $T^{\beta,\beta',\gamma,\gamma'}_{i,j,k,l}$ instead of all three, we can conclude the claim. □

□

### 4.5.4 Rank Attacks.

Towards analyzing the case of larger $m$, we attempt another class of attacks which consist of looking at the *rank* of the various matrices that arise in the assumption.

**Rank Attack on $\mathbf{E}^*$.** Note that a random (e.g. Gaussian) $\mathbf{E}^*$ would be full-rank with overwhelming probability. In particular, as

$$\mathbf{E}^* = \mathbf{P}\left(\bigotimes_{i=1}^{d} \mathbf{E}_i\right)\mathbf{P}',$$

where $\mathbf{P} \in \mathbb{Z}^{M\times m^d}$ and $\mathbf{P}' \in \mathbb{Z}^{k^d\times K}$, the rank of $\mathbf{E}^*$ is at most the rank of $\mathbf{P}, \mathbf{P}'$. In particular, $\mathbf{P}$ and $\mathbf{P}'$ need to be full-rank and compressing, meaning that $M \leq m^d$ and $K \leq k^d$, respectively. Our setting of parameters (see Section 4.3) ensure these restrictions hold.

The rank of $\bigotimes_{i=1}^d \mathbf{E}_i$ is the product of the ranks of $\mathbf{E}_i$, and is therefore, $\mathsf{min}(m^d, k^d)$ with high probability. Heuristically, then, the rank of $\mathbf{E}^*$ is exactly $\mathsf{min}(K, M)$ with high probability, as long as the Gaussians have sufficiently large width, a statement that we verified experimentally.

**Rank Attack on $\mathbf{A}^*\mathbf{S}^*$.** Note that if $\mathbf{A}^*\mathbf{S}^*$ is computationally indistinguishable from $\mathbf{A}^*\mathbf{S}'$ for a uniformly random $\mathbf{S}'$ given $\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \mathsf{aux}_b$, then the pre-condition implies the post-condition in Definition 3.1, guaranteeing security. Thus, we evaluate possible distinguishers between $\mathbf{A}^*\mathbf{S}^*$ and $\mathbf{A}^*\mathbf{S}'$.

One such class of attacks consist in comparing the rank of $\mathbf{A}^*\mathbf{S}^*$ to the rank of $\mathbf{A}^*$. We heuristically and experimentally analyzed the ranks of $\mathbf{A}^*$ and $\mathbf{A}^*\mathbf{S}^*$ to reason about these attacks.

First, note that $\overline{\mathbf{A}}^*\overline{\mathbf{S}}^* = \mathbf{A}^*\mathbf{S}^*$. Recall that the matrices $\mathbf{A}_i \in \mathbb{Z}_q^{m\times w}$ are random and therefore w.h.p. full-rank (i.e., rank $w$). Let $\mathbf{A}_i^\perp \in \mathbb{Z}_q^{(m-w)\times m}$ be a basis for the left-kernel of $\mathbf{A}_i$, that is, they are rank-$(m-w)$ matrices such that

$$\mathbf{A}_i^\perp \mathbf{A}_i = 0 \pmod q$$

We note that w.h.p. the rank of the matrix

$$(\,\mathbf{A}_1 \otimes \mathbf{I}_m \otimes \cdots \otimes \mathbf{I}_m \,\|\, \mathbf{I}_m \otimes \mathbf{A}_2 \otimes \mathbf{I}_m \otimes \cdots \otimes \mathbf{I}_m \,\|\, \cdots \,\|\, \mathbf{I}_m \otimes \cdots \otimes \mathbf{I}_m \otimes \mathbf{A}_d) \in \mathbb{Z}_q^{m^d \times dwm^{d-1}}$$

is at most $m^d - (m-w)^d \approx dwm^{d-1} - d^2w^2m^{d-2}/2$ (the approximation assumes that $m \gg w$ which is the case for us) since the row-span of $\mathbf{A}^*$ is contained in the right kernel of $(\mathbf{A}_1^\perp \otimes \cdots \otimes \mathbf{A}_d^\perp)$, and the latter has rank $m^d - (m-w)^d$. Our experiments indicate that the rank is indeed $m^d - (m-w)^d$ w.h.p. In other words, this matrix is *rank-deficient* by approximately $d^2w^2m^{d-2}/2$.

Heuristically,

$$\overline{\mathbf{A}}^* = \mathbf{P}\cdot(\,\mathbf{A}_1 \otimes \mathbf{I}_m \otimes \cdots \otimes \mathbf{I}_m \,\|\, \mathbf{I}_m \otimes \mathbf{A}_2 \otimes \mathbf{I}_m \otimes \cdots \otimes \mathbf{I}_m \,\|\, \mathbf{I}_m \otimes \cdots \otimes \mathbf{I}_m \otimes \mathbf{A}_d)$$

has the same rank since $\mathbf{P} \in \mathbb{Z}^{M\times m^d}$ is Gaussian and nearly full-rank, i.e., rank $M \approx m^{d-1/2}$. That is, w.h.p., (heuristically)

$$\mathsf{rank}(\overline{\mathbf{A}}^*) = m^d - (m-w)^d$$

Also, heuristically, $\overline{\mathbf{A}}^*\overline{\mathbf{S}}^*$ has this rank as long as $\overline{\mathbf{S}}^*$ has sufficiently many columns, i.e. as long as $K$ is large enough compared to $\mathsf{rank}(\overline{\mathbf{A}}^*)$. (Note that the entries of $\overline{\mathbf{A}}^*$ and $\overline{\mathbf{S}}^*$ are correlated.)

To test these heuristic statements, we ran experiments for $d = 3$ and a range of values of $m, k$ and $q$. We found that $\overline{\mathbf{A}}^*$ had rank $m^d - (m-w)^d$ as expected (in all the runs of our experiment, suggesting a high probability statement). We also found that when $k \geq m$ and $K$ is large enough so that $\mathbf{S}^*$ is wide, $\overline{\mathbf{A}}^*\overline{\mathbf{S}}^* = \mathbf{A}^*\mathbf{S}^*$ also had rank $m^d - (m-w)^d$ with high probability. This is the same as one would expect from $\mathbf{A}^*\mathbf{S}'$ for a random $\mathbf{S}'$, suggesting that rank attacks fail.

### 4.6 Cryptanalytic Challenges

We describe a few cryptanalytic challenges and how they relate to our candidate and our assumptions. For each of these problems, we can also consider easier challenges where (a) the challenger also gets $\mathbf{A}^*$; and (b) we replace $\mathbf{P}$ with the identity matrix.

**Pseudo-flooding in the Absence of $\mathsf{seed}_{\mathbf{B}^*}$.** Our intuition says that for any two low-norm matrices $\mathbf{Z}_0$ and $\mathbf{Z}_1$, $\mathbf{E}^* + \mathbf{Z}_b$ hides $b$. Concretely, let $\chi$ be a discrete Gaussian of sufficiently large parameter $\sigma$. A challenge is to come up with matrices $\mathbf{Z}_0$ and $\mathbf{Z}_1$ where $||\mathbf{Z}_b|| < \sigma/2^\lambda$ such that the bit $b$ can be recovered given

$$\mathbf{P}\left(\bigotimes_{i=1}^{d} \mathbf{E}_i\right)\mathbf{P}' + \mathbf{Z}_b\ .$$

We note that when $m = 1$ and $\mathbf{P} = 1$, as argued above, this seems to follow from the noise-flooding properties of random (e.g. Gaussian) polynomials [BHJ+19].

**Pseudo-flooding in the Presence of $\mathsf{seed}_{\mathbf{B}^*}$.** Our stronger notion of security (Definition 3.1) would imply that it would be hard to recover $b$ from

$$(\mathsf{seed}_{\mathbf{B}^*},\ \mathbf{A}^*\mathbf{S}^* + \mathbf{Z}_b,\ \mathbf{E}^* - \mathbf{Z}_b), \quad b \leftarrow \{0,1\}$$

for the following concrete distributions of $\mathbf{Z}_0, \mathbf{Z}_1$:

- (norm and ideal membership) $\mathbf{Z}_0$ is drawn from a Gaussian, and $\mathbf{Z}_1 = 2\mathbf{Z}_0$, and $q$ is odd. In particular, an attacker that manages to learn the parity of $\mathbf{Z}_b$ or accurately approximate the norm of $\mathbf{Z}_b$ will be able to learn $b$.
- (subspace membership) $\mathbf{Z}_b = \mathbf{E}_0\mathbf{M} + b\hat{\mathbf{E}}$ where $\|\mathbf{E}_0\| \gg \|\hat{\mathbf{E}}\|$ and $\mathbf{M}$ is a public low-norm matrix. The distribution here is closely related to that for weak flooding. Here, $\|\mathbf{Z}_0\| \approx \|\mathbf{Z}_1\|$, but an attacker that manages to learn whether $\mathbf{Z}_b$ lies in the row span of $\mathbf{M}$ will be able to learn $b$.

In both cases, an attacker could try to exploit the leakage on $b$ from $\mathbf{A}^*\mathbf{S}^* + \mathbf{Z}_b$ or from $\mathbf{E}^* - \mathbf{Z}_b$. For instance, an efficient algorithm that recovers $\mathbf{E}^*$ from $\mathsf{seed}_{\mathbf{B}^*}$ or one that recovers $b$ from $\mathbf{E}^* - \mathbf{Z}_b$ solves this problem.

**Distinguishing $\mathbf{A}^*\mathbf{S}^*$ from $\mathbf{A}^*\mathbf{S}'$.** As described above, we think the following claim is plausible:

$$\mathbf{A}^*\mathbf{S}^* \approx_c \mathbf{A}^*\mathbf{S}'$$

where $\mathbf{S}' \leftarrow \mathbb{Z}_q^{W \times K}$. As $\mathbf{A}^*\mathbf{S}^* = \overline{\mathbf{A}}^* \cdot \overline{\mathbf{S}}^*$ (where $\overline{\mathbf{A}}^*, \overline{\mathbf{S}}^*$ are defined in Section 4.1), and given that $\mathbf{A}^*$ and $\overline{\mathbf{A}}^*$ have the same column span, this is equivalent to

$$\overline{\mathbf{A}}^* \cdot \overline{\mathbf{S}}^* \approx_c \overline{\mathbf{A}}^* \cdot \mathbf{S}''$$

where $\mathbf{S}'' \leftarrow \mathbb{Z}_q^{dwm^{d-1} \times K}$, and $\overline{\mathbf{A}}^*, \overline{\mathbf{S}}^*$ have closed form expressions described in Section 4.1.

A distinguisher here does not immediately break strong or weak-flooding, but we believe it constitutes strong evidence that strong-flooding is false.

# 5 Our Succinct Randomized Encoding Construction

Let $(\mathsf{SampCRSGen}, \mathsf{LWEGen}, \mathsf{Expand})$ be a succinct LWE sampler (Definition 3.1) with parameters to be determined later.

We now describe our SRE for the family $\mathcal{F}_{\ell,N,t} = \{f : \{0,1\}^\ell \to \{0,1\}^N\}$ of depth-$t$ circuits. Let $q$ be a modulus and $\overline{\chi}$ be a $\overline{B}$-bounded distribution to be determined later.

Let $g(t) = \mathcal{O}(t)$ be the function defined in Definition 2.4.

- $\mathsf{CRSGen}(1^\lambda, \mathcal{F}_{\ell,N,t})$: Output $\mathsf{crs} \leftarrow \mathsf{SampCRSGen}(1^\lambda, 1^N, N^{g(t)})$. It in particular includes parameters $\mathsf{params} = (q, M, K, \overline{\chi}, \overline{B})$.
- $\mathsf{Encode}(\mathsf{crs}, f, x)$: Compute $(\mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \mathbf{S}^*) \leftarrow \mathsf{LWEGen}(\mathsf{crs})$, where $\mathbf{A}^* \in \mathbb{Z}_q^{M \times W}$, $\mathbf{S}^* \in \mathbb{Z}_q^{W \times K}$.

Sample $\mathbf{R} \leftarrow \{0,1\}^{W \times \ell M \log q}$, and $\mathbf{E} \leftarrow \overline{\chi}^{M \times \ell M \log q}$. Compute

$$\mathbf{C} = \mathbf{A}^* \mathbf{R} + x \otimes \mathbf{G} + \mathbf{E} \in \mathbb{Z}_q^{M \times \ell M \log q},$$

where we view $x \in \{0,1\}^{1 \times \ell}$ as a row vector, and compute $(\mathbf{R}_{f,x}, \mathbf{E}_{f,x}) = \mathsf{Eval}_{\mathsf{open}}(f, \mathbf{A}^*, x, \mathbf{R}, \mathbf{E})$. Output:

$$C = (\mathsf{seed}_{\mathbf{B}^*},\ \mathbf{C},\ \mathbf{A}^*,\ (\mathbf{R}_{f,x} + \mathbf{S}^*)).$$

- $\mathsf{Decode}(\mathsf{crs}, C, f))$: On input $C = (\mathsf{seed}_{\mathbf{B}^*}, \mathbf{C}, \mathbf{A}^*, \mathbf{V})$, compute $\mathbf{C}_f = \mathsf{Eval}(f, \mathbf{C})$, and $\mathbf{B}^* = \mathsf{Expand}(\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*})$. Output

$$f(x) = \mathsf{Round}_{q/2}\left(\mathbf{C}_f + \mathbf{B}^* - \mathbf{A}^* \cdot \mathbf{V}\right) \in \{0,1\}^{M \times K}.$$

**Theorem 5.1.** *Suppose* $(\mathsf{SampCRSGen}, \mathsf{LWEGen}, \mathsf{Expand})$ *is a succinct LWE sampler satisfying $\delta$-succinctness and $\beta_0$-flooding (Definition 3.1) with $\beta_0 = \overline{B} \cdot N^{g(t)}$. Suppose furthermore that:*

$$M^2 = N^\delta \cdot \mathrm{poly}(\lambda, \ell, t).$$

*Then* $(\mathsf{CRSGen}, \mathsf{Encode}, \mathsf{Decode})$ *is an SRE for $\mathcal{F}_{\ell,N,t}$ satisfying $\delta$-succinctness.*

Next, we show that the construction above satisfies correctness and succinctness.

**Claim 5.1.1** (Correctness)**.** *Suppose* $(\mathsf{SampCRSGen}, \mathsf{LWEGen}, \mathsf{Expand})$ *satisfy the parameters constraints and correctness Definition 3.1. Then* $(\mathsf{CRSGen}, \mathsf{Encode}, \mathsf{Decode})$ *is correct.*

*Proof.* Define $\mathbf{V} = (\mathbf{R}_{f,x} + \mathbf{S}^*)$. By Definition 2.4, we have

$$\mathbf{C}_f + \mathbf{B}^* - \mathbf{A}^* \cdot (\mathbf{R}_{f,x} + \mathbf{S}^*) = f(x) \cdot q/2 + \mathbf{E}_{f,x} + \mathbf{E}^*.$$

Let $\beta_0 = \overline{B} \cdot N^{g(t)}$. The setting of parameters $\beta$, $\overline{B}$ and $q$ from $(\mathsf{SampCRSGen}, \mathsf{LWEGen}, \mathsf{Expand})$ imply $\|\mathbf{E}\| \leq \overline{B}$ and therefore $\|\mathbf{E}_{f,x}\| \leq \overline{B} M^{g(t)} \leq N^{g(t)} = \beta_0$ by definition of $g$ (Definition 2.4), and using $M \leq N$. Furthermore $\beta \geq \beta_0 \cdot 2^\lambda$ and $q \geq 8\beta$ so that $\|\mathbf{E}_{f,x} + \mathbf{E}^*\| < q/4$, and therefore

$$\mathsf{Round}_{q/2}\left(\mathbf{C}_f + \mathbf{B}^* - \mathbf{A}^* \cdot \mathbf{V}\right)) = \mathsf{Round}_{q/2}\left(f(x) \cdot q/2 + \mathbf{E}_{f,x} + \mathbf{E}^*\right) = f(x).$$

□

**Claim 5.1.2.** *Suppose the sampler* $(\mathsf{SampCRSGen}, \mathsf{LWEGen}, \mathsf{Expand})$ *is $\delta$-succinct (Definition 3.1), and suppose that the sampler furthermore satisfies*

$$M^2 = N^\delta \cdot \mathrm{poly}(\lambda, \ell, t).$$

*Then* $(\mathsf{CRSGen}, \mathsf{Encode}, \mathsf{Decode})$ *is $\delta$-succinct.*

*Proof.* The setting of the parameters implies $\log q = \mathrm{poly}(\lambda, t)$. Then $\ell M^2 \log^2 q = N^\delta \cdot \mathrm{poly}(\lambda, \ell, t)$.

Furthermore $\mathbf{V} = (\mathbf{R}_{f,x} + \mathbf{S}^*) \in \mathbb{Z}_q^{W \times K}$ and therefore $\mathsf{bitlength}(\mathsf{seed}_{\mathbf{B}^*}, \mathbf{C}, \mathbf{A}^*, \mathbf{V}) \leq N^\delta \cdot \mathrm{poly}(\lambda, \ell, t)$ by $\delta$-succinctness of $(\mathsf{SampCRSGen}, \mathsf{LWEGen}, \mathsf{Expand})$. Therefore the SRE is $\delta$-succinct.

□

### 5.1 Security

**Claim 5.1.3** (Indistinguishability-based security.)**.** *Let $f : \{0,1\}^\ell \to \{0,1\}^N$ of depth $t$, and $x_0, x_1 \in \{0,1\}^\ell$ such that $f(x_0) = f(x_1)$. Suppose* $(\mathsf{SampCRSGen}, \mathsf{LWEGen}, \mathsf{Expand})$ *is secure (Definition 3.1), and LWE hold. Then:*

$$(\mathsf{crs}, \mathsf{Encode}(\mathsf{crs}, f, x_0)) \approx_c (\mathsf{crs}, \mathsf{Encode}(\mathsf{crs}, f, x_1)),$$

*where* $\mathsf{crs} \leftarrow \mathsf{CRSGen}(1^\lambda, \mathcal{F}_{\ell,N,t})$.

*Proof.* We prove the claim by analyzing a sequence of distributions $H_j(0), H_j(1)$, for $j = 0, 1, 2$.

$\underline{H_0(b)}$: The claim states that $H_0(0) \approx_c H_0(1)$ where

$$H_0(b) = (\mathsf{crs}, \mathsf{Encode}(\mathsf{crs}, f, x_b))) = (\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*}, \mathbf{C}_b, \mathbf{A}^*, (\mathbf{R}_{f,x_b} + \mathbf{S}^*))$$

for $b \in \{0,1\}$, where $\mathbf{C}_b = \mathbf{A}^*\mathbf{R} + x_b \otimes \mathbf{G} + \mathbf{E} \in \mathbb{Z}_q^{M \times \ell M \log q}$.

$\underline{H_1(b)}$: Next, consider

$$H_1(b) = (\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*}, \mathbf{C}_b, \mathbf{A}^*, \mathbf{F}_b = \mathbf{A}^*(\mathbf{R}_{f,x_b} + \mathbf{S}^*)),$$

where $\mathbf{C}_b$ and $\mathbf{R}_{f,x_b}$ are functions of $x_b$. We claim that

$$H_1(0) \approx_c H_1(1) \Longrightarrow H_0(0) \approx_c H_0(1)$$

Namely, if $H_0(0)$ can be efficiently distinguished from $H_0(1)$, then one can also efficiently distinguish $H_1(0)$ from $H_1(1)$. This is because $H_0(b)$ can be efficiently sampled as a randomized function of $H_1(b)$, by solving for $\mathbf{V}_b \in \mathbb{Z}_q^{M \times K}$ such that $\mathbf{A}^*\mathbf{V}_b = \mathbf{F}_b$, given $\mathbf{A}^*$. By definition of the sampler, $\mathbf{A}^* \in \mathbb{Z}_q^{M \times W}$ is full column-rank. Therefore $\mathbf{V}_b \in \mathbb{Z}_q^{W \times K}$ is the unique solution to $\mathbf{A}^*\mathbf{V}_b = \mathbf{A}^*(\mathbf{R}_{f,x_b} + \mathbf{S}^*)$, namely $\mathbf{V}_b = \mathbf{R}_{f,x_b} + \mathbf{S}^*$.

$\underline{H_2(b)}$: Finally, consider

$$H_2(b) = (\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*}, \mathbf{C}_b, \mathbf{A}^*, \mathbf{E}_{f,x_b} + \mathbf{E}^*).$$

We claim that

$$H_2(0) \approx_c H_2(1) \Longrightarrow H_1(0) \approx_c H_1(1)$$

This is because one can write $\mathbf{F}_b$ as:

$$\mathbf{F}_b = \mathbf{C}_{f,b} + \mathbf{B}^* - f(x_b) \cdot q/2 - (\mathbf{E}_{f,x_b} + \mathbf{E}^*),$$

where $\mathbf{C}_{f,b} = \mathsf{Eval}(f, \mathbf{C}_b)$, $(\mathbf{R}_{f,x_b}, \mathbf{E}_{f,x_b}) = \mathsf{Eval}_{\mathsf{open}}(f, \mathbf{A}^*, x, \mathbf{R}, \mathbf{E})$ and $\mathbf{B}^* = \mathsf{Expand}(\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*})$, by definition of $\mathsf{Eval}$ and $\mathsf{Eval}_{\mathsf{open}}$ (Definition 2.4) and correctness of the sampler (Definition 3.1).

$\underline{H_2(0) \approx_c H_2(1)}$: To prove Claim 5.1.3, it suffices to show that $H_2(0) \approx_c H_2(1)$. We claim that this follows from $\beta_0$-flooding (Definition 3.1), by setting $\mathsf{aux}_b = \mathbf{C}_b$ and $\mathbf{Z}_b = \mathbf{E}_{f,x_b}$, using the fact that $f(x_0) = f(x_1)$, and that $\|\mathbf{E}_{f,x}\| \le \overline{B}M^{g(t)} \le \overline{B}N^{g(t)} = \beta_0$ by Definition 2.4. It only remains to argue that the pre-condition of Definition 3.1 holds for $(\mathsf{aux}_b, \mathbf{Z}_b)$, namely:

$$\begin{aligned} &(\mathsf{coins}_{\mathsf{crs}}, \mathsf{coins}_{\mathsf{seed}}, \quad \mathbf{A}^*\mathbf{S}' + \mathbf{E}_{f,x_0}, \quad \mathbf{A}^*\mathbf{R} + x_0 \otimes \mathbf{G} + \mathbf{E}) \\ \approx_c\ &(\mathsf{coins}_{\mathsf{crs}}, \mathsf{coins}_{\mathsf{seed}}, \quad \mathbf{A}^*\mathbf{S}' + \mathbf{E}_{f,x_1}, \quad \mathbf{A}^*\mathbf{R} + x_1 \otimes \mathbf{G} + \mathbf{E}). \end{aligned} \tag{12}$$

where $\mathbf{S}' \leftarrow \mathbb{Z}_q^{W\times K}$, and where we view $x_0, x_1 \in \{0,1\}^{1\times\ell}$ as row vectors.

LWE wrt $\mathbf{A}^*$ implies (12). We show that LWE with respect to $\mathbf{A}^*$ (Definition 3.1) implies (12). To prove (12), it suffices to show that:

$$(\mathsf{coins}_{\mathsf{crs}}, \mathsf{coins}_{\mathsf{seed}}, \mathbf{A}^*\mathbf{R} + x_0 \otimes \mathbf{G} + \mathbf{E}) \approx_c (\mathsf{coins}_{\mathsf{crs}}, \mathsf{coins}_{\mathsf{seed}},\ \mathbf{A}^*\mathbf{R} + x_1 \otimes \mathbf{G} + \mathbf{E}).$$

This is because given $(\mathbf{A}^*, \mathbf{C}_b)$, one can efficiently sample $\mathbf{A}^*\mathbf{S}' + \mathbf{Z}_b$ by sampling $\mathbf{S} \leftarrow \mathbb{Z}_q^{W\times K}$ and computing

$$\begin{aligned}\mathsf{Eval}(f, \mathbf{C}_b) + \mathbf{A}^*\mathbf{S} - f(x_b)\cdot q/2 &= \mathbf{A}^*(\mathbf{R}_{f,x_b} + \mathbf{S}) + \mathbf{E}_{f,b}\\ &= \mathbf{A}^*\mathbf{S}' + \mathbf{Z}_b\end{aligned}$$

where $\mathbf{S}' = (\mathbf{R}_{f,x_b} + \mathbf{S})$ is uniformly random in $\mathbb{Z}_q^{W\times K}$ over the randomness of $\mathbf{S}$ alone. In particular, using again the fact that $f(x_0) = f(x_1)$, any distinguisher against the precondition implies a distinguisher against $(\mathsf{coins}_{\mathsf{crs}}, \mathsf{coins}_{\mathsf{seed}}, \mathbf{C}_b)$.

Last we have that $(\mathsf{coins}_{\mathsf{crs}}, \mathsf{coins}_{\mathsf{seed}}, \mathbf{C}_b) \approx_c (\mathsf{coins}_{\mathsf{crs}}, \mathsf{coins}_{\mathsf{seed}}, \mathbf{B})$, where $\mathbf{B} \leftarrow \mathbb{Z}_q^{M\times K}$ by LWE with respect to $\mathbf{A}^*$ (Definition 3.1), thus proving (12) and finishing the proof. □

Combining Theorem 5.1 with our candidate succinct LWE sampler (Sections 4.1 and 4.3), noting that our proposed parameters in Section 4.3 satisfy $M^2 = N^\delta \cdot \mathrm{poly}(\lambda, \ell, t)$, gives a candidate SRE. Invoking Theorem 2.6, we obtain the following.

**Corollary 5.2.** *Assuming Conjecture 1 and sub-exponential LWE, there exists an iO scheme.*

We can furthermore use Theorem 3.3 to relax the requirement on our candidate succinct LWE sampler (Section 4.1), and only rely on weak security (Definition 3.2), thus obtaining the following.

**Corollary 5.3.** *Assuming Conjecture 2 and sub-exponential LWE, there exists an iO scheme.*

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# A Relations Among Notions of LWE Sampling

## A.1 WW Oblivious LWE Sampling

We present the definition of oblivious sampling from [WW21], with a few syntactic modifications to fit our new notion. Instead of sampling the LWE matrix $\mathbf{A}$ with a trapdoor which we later use to solve our generated LWE samples, we have $\mathsf{LWEGen}$ sample the LWE matrices and output the matrix and secrets for the LWE samples that can be generated from $\mathsf{pub}$. We also allow $\mathsf{LWEGen}$ and $\mathsf{Sim}$ access to the $\mathsf{crs}$. For consistency within this paper, we will use $\mathsf{SampCRSGen}, \mathsf{LWEGen}, \mathsf{Expand}$ instead of WW's notation $(\mathsf{CRSGen}, \mathsf{Init}, \mathsf{Sample})$.

**Definition A.1** (Definition 6.1 from [WW21])**.** *An $(N, q, \hat{\chi}, B_{\mathsf{OLWE}})$-oblivious LWE sampler is a tuple of algorithms $(\mathsf{SampCRSGen}, \mathsf{LWEGen}, \mathsf{Expand}, \mathsf{Sim})$ that satisfy the following properties. We require $N = MK$. Let $\mathsf{crs} \leftarrow \mathsf{SampCRSGen}(1^\lambda, 1^N)$, $(\mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^* \in \mathbb{Z}_q^{M\times W}, \mathbf{S}^* \in \mathbb{Z}_q^{W\times K}) \leftarrow \mathsf{LWEGen}(\mathsf{crs})$, $\mathbf{B}^* \leftarrow \mathsf{Expand}(\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*})$. Then*

**Correctness.** *Let $\mathbf{E}^* = \mathbf{B}^* - \mathbf{A}^*\mathbf{S}^*$. With overwhelming probability, $\|\mathbf{E}^*\| \leq B_{\mathsf{OLWE}}$ .*

**Security.** *Let $\hat{\mathbf{S}} \leftarrow \mathbb{Z}_q^{W\times K}$, $\hat{\mathbf{E}} \leftarrow \hat{\chi}^{M\times K}$, $\hat{\mathbf{B}} := \mathbf{A}^*\hat{\mathbf{S}}+\hat{\mathbf{E}}$, $(\mathsf{crs}_{\mathsf{Sim}}, \mathsf{seed}_{\mathbf{B}^*\mathsf{Sim}}, \mathbf{S}^*_{\mathsf{Sim}}) \leftarrow \mathsf{Sim}(\mathsf{crs}, \mathbf{A}^*, \hat{\mathbf{B}})$. Then the following "real" and "simulated" distributions are indistinguishable:*

$$(\mathsf{crs}, \mathbf{A}^*, \mathsf{seed}_{\mathbf{B}^*}, \mathbf{S}^*) \approx_c (\mathsf{crs}_{\mathsf{Sim}}, \mathbf{A}^*, \mathsf{seed}_{\mathbf{B}^*\mathsf{Sim}}, \mathbf{S}^*_{\mathsf{Sim}} + \hat{\mathbf{S}}).$$

We will sketch a proof that WW's security notion implies our new security notion. Namely, given an oblivious LWE sampler $(\mathsf{SampCRSGen}, \mathsf{LWEGen}, \mathsf{Expand}, \mathsf{Sim})$ which satisfies WW security as written above, $(\mathsf{SampCRSGen}, \mathsf{LWEGen}, \mathsf{Expand})$ satisfies security from Definition 3.1.

**Claim A.1.1.** *Let $N, q, B$ be parameters and $(\mathsf{SampCRSGen}, \mathsf{LWEGen}, \mathsf{Expand}, \mathsf{Sim})$ a tuple of algorithms such that for $\mathsf{crs} \leftarrow \mathsf{SampCRSGen}(1^\lambda, 1^N; \mathsf{coins}_{\mathsf{crs}})$, $(\mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \mathbf{S}^*) \leftarrow \mathsf{LWEGen}(\mathsf{crs}; \mathsf{coins}_{\mathsf{seed}})$, $\hat{\mathbf{S}} \leftarrow \mathbb{Z}_q^{W\times K}$, $\hat{\mathbf{E}} \in [-B, B]^{M\times K}$, $\hat{\mathbf{B}} := \mathbf{A}^*\hat{\mathbf{S}} + \hat{\mathbf{E}}$, $(\mathsf{crs}_{\mathsf{Sim}}, \mathsf{seed}_{\mathbf{B}^*\mathsf{Sim}}, \mathbf{S}^*_{\mathsf{Sim}}) \leftarrow \mathsf{Sim}(\mathsf{crs}, \mathbf{A}^*, \hat{\mathbf{B}})$, we have that*

$$(\mathsf{crs}, \mathbf{A}^*, \mathsf{seed}_{\mathbf{B}^*}, \mathbf{S}^*) \approx_c (\mathsf{crs}_{\mathsf{Sim}}, \mathbf{A}^*, \mathsf{seed}_{\mathbf{B}^*\mathsf{Sim}}, \mathbf{S}^*_{\mathsf{Sim}} + \hat{\mathbf{S}}). \tag{13}$$

*Then for any two polynomial-time sampleable distributions $D_0, D_1$ such that $(\mathsf{aux}_b, \mathbf{Z}_b) \leftarrow D_b(\mathbf{A}^*)$ satisfies $\|\mathbf{Z}_b\| \leq B_0$ and for $\mathbf{S}' \leftarrow \mathbb{Z}_q^{W\times K}$,*

$$(\mathsf{coins}_{\mathsf{crs}}, \mathsf{coins}_{\mathsf{seed}}, \mathsf{aux}_0, \mathbf{A}^*\mathbf{S}' + \mathbf{Z}_0) \approx_c (\mathsf{coins}_{\mathsf{crs}}, \mathsf{coins}_{\mathsf{seed}}, \mathsf{aux}_1, \mathbf{A}^*\mathbf{S}' + \mathbf{Z}_1), \tag{14}$$

*we have that*

$$(\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \mathbf{A}^*\mathbf{S}^* + \mathbf{Z}_0, \mathsf{aux}_0) \approx_c (\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \mathbf{A}^*\mathbf{S}^* + \mathbf{Z}_1, \mathsf{aux}_1).$$

*Proof.* (Sketch.) We proceed by a hybrid argument.
$\underline{\mathcal{H}_0(b)}$ is the real distribution: $(\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \mathbf{A}^*\mathbf{S}^* + \mathbf{Z}_b, \mathsf{aux}_b)$.
$\underline{\mathcal{H}_1(\mathrm{b})}$ is the simulated distribution: $(\mathsf{crs}_{\mathsf{Sim}}, \mathsf{seed}_{\mathbf{B}^*\mathsf{Sim}}, \mathbf{A}^*, \mathbf{A}^*(\mathbf{S}^*_{\mathsf{Sim}} + \mathbf{S}') + \mathbf{Z}_b, \mathsf{aux}_b)$

- $\mathcal{H}_0(b) \approx_c \mathcal{H}_1(b)$ by reduction to security of the WW simulator (Eq. (13)): Given as challenge $(\mathsf{crs}, \mathbf{A}^*, \mathsf{seed}_{\mathbf{B}^*}, \mathbf{S})$, the reduction itself samples $(\mathsf{aux}_0, \mathbf{Z}_0) \leftarrow D_0(\mathbf{A}^*)$ and then outputs $(\mathsf{crs}, \mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \mathbf{A}^*\mathbf{S} + \mathbf{Z}_0, \mathsf{aux}_0)$.

- $\mathcal{H}_1(0) \approx_c \mathcal{H}_1(1)$ by reduction to the precondition (Eq. (14)): Given as challenge $(\mathsf{coins}_{\mathsf{crs}}, \mathsf{coins}_{\mathsf{seed}}, \mathsf{aux}_b, \mathbf{A}^*\mathbf{S}' + \mathbf{Z}_b)$, reduction samples $\mathsf{crs} \leftarrow \mathsf{SampCRSGen}(1^\lambda, 1^N; \mathsf{coins}_{\mathsf{crs}})$, $(\mathsf{seed}_{\mathbf{B}^*}, \mathbf{A}^*, \mathbf{S}^*) \leftarrow \mathsf{LWEGen}(\mathsf{crs}; \mathsf{coins}_{\mathsf{seed}})$, $(\mathsf{crs}_{\mathsf{Sim}}, \mathsf{seed}_{\mathbf{B}^*\mathsf{Sim}}, \mathbf{S}^*_{\mathsf{Sim}}) \leftarrow \mathsf{Sim}(\mathsf{crs}, \mathbf{A}^*, \mathbf{A}^*\mathbf{S}' + \mathbf{Z}_b)$ and outputs $(\mathsf{crs}_{\mathsf{Sim}}, \mathsf{seed}_{\mathbf{B}^*\mathsf{Sim}}, \mathbf{A}^*, \mathbf{A}^*\mathbf{S}' + \mathbf{Z}_b + \mathbf{A}^*\mathbf{S}^*_{\mathsf{Sim}}, \mathsf{aux}_b)$.

□