# Full Quantum Equivalence of Group Action DLog and CDH, and More

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**Abstract**

Cryptographic group actions are a relaxation of standard cryptographic groups that have less structure. This lack of structure allows them to be plausibly quantum resistant despite Shor's algorithm, while still having a number of applications. The most famous example of group actions are built from isogenies on elliptic curves.

Our main result is that CDH for abelian group actions is quantumly *equivalent* to discrete log. Galbraith et al. (Mathematical Cryptology) previously showed *perfectly* solving CDH to be equivalent to discrete log quantumly; our result works for any non-negligible advantage. We also explore several other questions about group action and isogeny protocols.

## 1 Introduction

*Proving the equivalence of breaking the Diffe-Hellman protocol and computing discrete-log is one of the oldest problems in public key cryptography.*

Boneh and Lipton [BL96]

Diffie-Hellman key agreement [DH76] is one of the most important protocols in cryptography. Given a generator $g$ of a cyclic group of order $p$, Alice and Bob choose random $a \leftarrow \mathsf{Z}_p$ and $b \leftarrow \mathsf{Z}_p$, respectively, and exchange the values $g^a$ and $g^b$. Their shared key is then $g^{ab} = (g^a)^b = (g^b)^a$.

One way to break Diffie-Hellman is to compute discrete logarithms (DLog): extract $a$ from $(g, g^a)$ and then compute $g^{ab} = (g^b)^a$ from Alice's message. Fortunately, computing discrete logs appears hard, and after decades of cryptanalytic effort the best classical algorithms on certain groups—multiplicative groups of finite fields and elliptic curves—have sub-exponential or exponential complexity.

The security of Diffie-Hellman key exchange, however, is potentially easier than solving DLog. Indeed, computing the shared key is equivalent to solving the computational Diffie-Hellman problem (CDH): computing $g^{ab}$ from $(g, g^a, g^b)$. While CDH is clearly no harder than DLog, it is not a priori obvious that the converse should hold. After all, CDH and DLog are very different problems: CDH is in essence computing multiplication $a, b \mapsto a \times b$ homomorphically on the encoded values $g^a, g^b$, whereas DLog is inverting the encoding. The good news is that there has been classical progress towards proving such an equivalence [den90, Mau94, MW96, BL96]. However, the *polynomial-time* equivalence of DLog and CDH in general groups without any auxiliary information still remains an important fundamental open question. As such, the hardness of CDH must simply be assumed in Diffie-Hellman key exchange, requiring a potentially much stronger assumption than the hardness of DLog.

**Quantum Diffie-Hellman.** Shor [Sho94] shows that DLog is easy on a quantum computer, meaning the Diffie-Hellman protocol is no longer secure. Numerous proposals have been made for replacement "post-quantum" cryptosystems. One interesting example preserving the spirit of the original Diffie-Hellman protocol is due to Couveignes [Cou06] and Rostovtsev and Stolbunov [RS06]. They propose to replace the group in Diffie-Hellman with a group *action*. Very roughly, the group action allows for a similar operation as discrete exponentiation as in Diffie-Hellman, but does *not* have an analagous operation for multiplying two group elements, as is needed by Shor's attack.

In more detail, a group action consists of a group $G$ and a set $X$, together with an action $\star : G \times X \to X$ such that for any $a, b \in G$ and $x \in X$, it holds that $(ab) \star x = a \star (b \star x)$. In this setting, DLog is the task of recovering $a$ from $(x, a \star x)$, and CDH is the task of computing $(ab) \star x$ from $(x, a \star x, b \star x)$. If we consider *abelian* and *regular*[1] group actions, we can translate Diffie-Hellman key exchange from groups to group actions by viewing $\mathsf{Z}_p$ as the group acting on the set $\langle g \rangle$ through discrete exponentiation: $a \star x = x^a$. DLog and CDH on the group immediately correspond to DLog and CDH on the group action. However, other group actions that do not correspond to plain groups are possible. The most notable example is isogenies over elliptic curves [CLM+18], one of the leading candidates for post-quantum public key cryptography proposed by Couveignes, Rostovtsev, and Stolbunov[2]. As we discuss in Section 8, other plausibly post-quantum proposals can sometimes also be phrased as group actions.

As in the classical case, the DLog-CDH equivalence is an important fundamental question in the quantum world. It may even be *more* important than the classical equivalence today, as the post-quantum hardness of group actions has so far seen a much smaller cryptanalytic effort than the classical hardness of groups, and therefore our confidence in the post-quantum CDH assumption on group actions is much weaker. An equivalence to DLog would therefore be an important step toward improving this confidence. In ordinary groups, the post-quantum equivalence is trivial: they are both easy. In group actions, however, it is less clear: group actions have less exploitable structure for proving such an equivalence, but quantum algorithms are more powerful and can potentially be used to facilitate a reduction.

In a short paper, Galbraith *et al.* [GPSV18] give a promising first step toward proving an equivalence: they show that any *perfect* algorithm for solving CDH in abelian group actions can be converted into a DLog algorithm. The core idea is that a perfect, efficient CDH algorithm essentially turns the set of a group action into a plain group, with $x_1 \times x_2 = \mathsf{CDH}(x_1, x_2)$. One can then apply Shor's DLog algorithm to the derived group. The main difficulty is that solving DLog in the derived group is not exactly identical to DLog in the original group action. Galbraith *et al.* essentially show how to translate one DLog to the other to complete the reduction.

Unfortunately, if the CDH algorithm has even relatively minor correctness error (even, say, 10%), the above algorithm does not work. On the other hand, for cryptographic applications, we want to justify that no efficient algorithm can solve CDH with any *non-negligible* success probability. It could therefore be, for example, that CDH can be broken—and hence also group action key agreement—with probability 0.9, but that DLog is still hard. In plain groups, one can amplify success probability using standard random self-reductions for CDH. However, as pointed out by Galbraith et al., the limited structure of group actions prevents such random self-reductions. They therefore leave the full quantum equivalence of DLog and CDH for group actions as an important open question.

[1] A regular group action is a group action that, for every $x_1, x_2 \in X$, there exists a *unique* element $g \in G$ such that $x_1 = g \star x_2$.

[2] A few very recent works [CD22, MM22, Rob22] break a certain isogeny-based protocol called SIDH. SIDH, however, is just one of a number of isogeny protocols, and in particular it is *not* a group action. For a slightly more in depth discussion about different isogeny protocols, see Section 2.5.

### 1.1 This Work: Full Quantum Equivalence of DLog and CDH

In this work, we resolve the open question above, showing that DLog and CDH are quantumly equivalent for abelian group actions (Section 3). Since the most commonly used group actions in cryptography (from isogenies) are abelian, our results here have wide applicability and can be used directly on isogeny-based cryptosystems such as CSI-FiSh [BKV19][3].

As a secondary result, we also show that the same cannot hold generically for Decisional Diffie-Hellman (DDH), which is equivalent to asking that the shared key not only cannot be predicted by the adversary, but that it is indistinguishable from a random string. In other words, there is no black box quantum equivalence between DLog (or even CDH) and DDH (Section 4). We also formally specify a generic model for group actions (Section 5), explore relaxations of group actions relevant to certain isogeny protocols (Section 9), and discuss the relationship between group actions and the dihedral hidden subgroup problem (Section 10).

**Our reduction (Section 3).** Our DLog-CDH equivalence will use Galbraith et al. to reduce the problem of proving equivalence to that of boosting the success probability of a CDH algorithm. However, this comes with many challenges, which we now explore. Consider a deterministic algorithm $A$ such that:

$$\Pr_{a,b\leftarrow G}[A(x, a\star x, b\star x) = (ab)\star x] = p \qquad \Pr_{a,b\leftarrow G}[A(x, a\star x, b\star x) = (uab)\star x] = 1-p$$

for some constant $p \in [0,1]$ and fixed known group element $u \in G \setminus \{1\}$. This would certainly be a valid CDH adversary with success probability $p$.

**Remark 1.** *Throughout, we will consider $x$ as being fixed; this is usually how CDH is modeled, and typically makes hardness results for CDH more challenging. It is also possible to consider a variant where $x$ is chosen randomly and $A$ works for a random $x$. [BMZ19] explore the fixed vs random question for plain groups.*

In the plain group setting, the equivalent setup would be that $A$ on input $(g, g^a, g^b)$, outputs $g^{ab}$ with probability $p$ and $g^{uab}$ with probability $1-p$. An easy random self-reduction for this $A$ would be to run $h \leftarrow A(g, (g^a)\times g^c, (g^b)\times g^d)$ for random choices of $c, d$. Each trial will run $A$ on random independent inputs, so we know that $h = g^{(a+c)(b+d)}$ with probability $p$, and $h = g^{u(a+c)(b+d)}$ with probability $1-p$. We can then compute $h' = h\times(g^a)^{-d}(g^b)^{-c}g^{-cd}$. If $h = g^{(a+c)(b+d)}$, then $h' = g^{ab}$. Meanwhile, if $h = g^{u(a+c)(b+d)}$, then $h' = g^{(u-1)(a+c)(b+d)+ab}$, which is a uniformly random element. Therefore, by repeating this process many times on independent $c, d$, a $p$ fraction of the elements will be identical to $g^{ab}$, and the rest will be uniformly random. Taking a majority therefore gives $g^{ab}$ with overwhelming probability. An important feature of this self-reduction is that when $A$ is correct, the self-reduction gives the correct answer, and when $A$ is incorrect, the self-reduction gives a uniformly random answer. The self-reduction can be strengthened to handle arbitrary $A$, thus giving a generic way to boost success probability.

Unfortunately, the above re-randomization is not possible with group actions, since there is no multiplication analog for set elements. Given $(x, a\star x, b\star x)$, one could try choosing a random $c, d$ and running $(cd)^{-1}\star A(x, c\star(a\star x), d\star(b\star x))$. The result will be $(cd)^{-1}\star[(ac)(bd)]\star x = (ab)\star x$ with probability $p$ and $(uab)\star x$ with probability $1-p$. This allows us to obtain many samples of each. But unlike the plain group self-reduction, now when $A$ is incorrect we do not output a uniformly random answer, but instead output a fixed incorrect answer $(uab)\star x$. This means we

[3] We note that our result does not directly apply to *restricted* effective group actions (REGAs) like CSIDH [CLM+18] and explain this in more detail later.

cannot in general take a majority since if $p < 1/2$ this would actually give the incorrect answer. In this case, if we *knew* that $p < 1/2$, we would know to actually take the minority element as output. This would require making non-black box use of $A$, which is non-standard but acceptable. However, if $p = 1/2$, then the majority or minority element is just a random sample between $(ab) \star x$ and $(uab) \star x$. In this case, even knowing $p$ is not enough to identify the correct answer.

We will now show how to resolve the reduction for this particular class of adversaries. To do so, we consider two cases: $u^2 = 1$, or not. The exponent 2 in $u^2 = 1$ is a result of our algorithm $A$ outputting a random choice amongst two elements, and in more general settings we could consider higher, but still polynomial, exponents. Note that group actions are defined and plausibly hard for non-cyclic or non-prime order groups, so it is reasonable to consider group orders that have small factors. For isogenies, the group order is indeed smooth.

If $u^2 = 1$ and $p = 1/2$, we are basically stuck: $A$ is simply outputting a random sample in the orbit of $(ab) \star x$ under action by $u$. Nothing we can do will amplify the success probability. Instead, we observe that $A$ can be viewed as essentially solving CDH—with perfect probability!—in the subgroup $G/\langle u \rangle$. We then apply Galbraith et al. to this subgroup to solve DLog relative to $G/\langle u \rangle$. We can then solve for the full DLog in $G$ by brute forcing the $\langle u \rangle$ component. This works regardless of $p$, but requires $u$ to generate a small group.

If $u^2 \neq 1$ and/or if $p \neq 1/2$, another approach will work. Here, we can first run our re-randomized $A$ several times on $(x, a \star x, b \star x)$ to obtain $y_0 = (ab) \star x$ and $y_1 = (uab) \star x$, but we do not yet know which is which. But in this case, we can use the fact that $A$ is *not* generating uniform outputs in the orbit of $(ab) \star x$ to distinguish the two cases. Concretely, we run the re-randomized $A$ several times on $(x, x, y_0)$ and $(x, x, y_1)$. Since $x = 1 \star x$, we know that $(x, x, y_0)$ will output $y_0$ with probability $p$ and $u \star y_0 = y_1$ with probability $1 - p$. This distribution of outputs exactly matches the distribution from our original set of trials on $(x, a \star x, b \star x)$. Meanwhile, $(x, x, y_1)$ will output $y_1$ and $u \star y_1 = (u^2 ab) \star x$ with probabilities $p$ and $1 - p$. This distribution will be different than that from our original set of trials. Thus by comparing the distributions generated from $(x, x, y_0)$ and $(x, x, y_1)$ with the distribution generated from $(x, a \star x, b \star x)$, we can identify which of $y_0, y_1$ are the correct CDH output.

Our result generalizes the approach above to work with arbitrary adversaries $A$, and to work without needing any side-information (like the probability $p$) about the distribution of outputs of $A$. Essentially, we show that there is always a polynomial-sized subgroup $H$ of $G$ such that we can amplify $A$ to have near-perfect success probability on $G/H$. We then apply Galbraith et al. to the subgroup, and then brute-force the quotient group.

There are a number of challenges to getting this sketch to work. One issue is to actually identify the subgroup of $G$. Suppose $G$ has order $n = 2 \times 3 \times 5 \times ....$ Then the number of subgroups of polynomial-size will be $\lambda^{O \log(\lambda)}$; if $G$ is non-cyclic, the number of small subgroups can even be exponential. So we cannot simply guess the subgroup, and must instead compute it.

Another issue is *thresholding*: we need to make decisions about whether various distributions of elements are close or far. These decisions are made by sampling a number of samples from the distributions, and comparing frequencies. But we can only obtain frequency estimates with inverse-polynomial error. For whatever criteria we use to distinguish distributions, if two distributions are close but not too close, the noise in our estimates will cause the criteria to output just a random bit. The question is then: if the various decisions underlying our algorithm may have random answers, how can we guarantee consistent outputs, as required to achieve a high success probability?

The randomness from thresholding seems impossible to fully overcome. However, we show via careful arguments that the randomness can all be contained within the choice of the subgroup $H$. Once this subgroup is fixed, we show that we can set our decision-making criteria such that we always make consistent decisions, resulting in consistent CDH solutions.

We note that our main proof assumes the group action is regular, meaning for a fixed $x$, $a \star x$ is a bijection. This is the most relevant setting to isogeny-based group actions. Nevertheless, we explain in Section 3.1 how to extend to arbitrary abelian group actions.

**Impossibility of Extending to DDH (Section 4).** Given the above, one may hope to actually prove that DLog implies DDH, namely that $(ab) \star x$ is indistinguishable from $c \star x$ for a random $c$, given $x, a \star x, b \star x$.

Unfortunately, we refute this possibility, at least in the composite-order setting that is most relevant to post-quantum cryptosystems. The idea is simple: we start with any group action $\star : G \times X \to X$ where CDH—and maybe even DDH—is hard. We then define a slightly larger group and set $G' = G \times \mathsf{Z}_p$ and $X' = X \times \mathsf{Z}_p$, for some polynomially bounded $p$. We expand $\star$ to an action of $G'$ on $X'$ by defining $(a, u) \star (x, y) = (a \star x, u + y)$. DLog and CDH easily hold for the expanded group action, but DDH is trivially false just by looking at the $\mathsf{Z}_p$ component, which has no hardness. We note that if $G$ is cyclic, we can make $G'$ cyclic as well by choosing $p$ to be relatively prime to the order of $G$.

**Generic Group Actions (Section 5).** Next, we propose a generic group action model, analogous to the generic group model of [Sho97]. In this model, the set elements $X$ are just random strings, and the action of $G$ on $X$ is provided by an oracle which can be queried by the adversary. This model is implicit in much of the prior work on group actions, but we are not aware of it being formally written down. We also note that the model trivially extends to the quantum setting, where classical queries are replaced by quantum queries.

**On REGAs (Section 9).** Many isogeny protocols cannot be phrased as clean group actions. Essentially, in some isogeny-based protocols (such as CSIDH [CLM+18]) there is a set of generators $g_1, \ldots, g_\ell \in G$, and it is only known how to efficiently compute the actions of the $g_i$ or $g_i^{-1}$; one can then compute the action of any $g \in G$ provided one has a representation of $g = \prod_{i=1}^{\ell} g_i^{\alpha_i}$ for polynomially-sized $\alpha_i$. In general, finding such a representation is believed to be hard. This setting is referred to as a *Restricted Effective Group Action (REGA)*.

Our reduction (as with Galbraith et al.) does not apply to REGAs, since applying Shor's algorithm requires the ability to compute the action of arbitrary group elements $g$. Formalizing some discussion from Galbraith *et al.*, we show that the reduction works for REGAs if a problem similar to the 1D Short Integer Solution (1D-SIS) problem is easy which we call REGA-SIS.[4] In the case that $G = \mathbb{Z}_p$–which we can assume since we are focused on abelian groups–the problem becomes essentially the one-dimensional version of the inhomogeneous SIS (ISIS) problem [BGLS19]: given a target integer $t \in \mathbb{Z}_p$ and a vector of integers $\mathbf{s} \in \mathbb{Z}_p^\ell$ defined by the REGA description, the problem is to find a vector of integers $\mathbf{v} \in [-\beta, \beta]^\ell$ such that $t = \mathbf{s} \cdot \mathbf{v}$. The only difference between REGA-SIS and what a natural definition of "1D-ISIS" would be is that the given vector of integers $\mathbf{s}$ is defined by the REGA rather than sampled randomly.

Essentially, we show that such a REGA-SIS oracle is enough to compute a representation of $g$ in terms of the $g_i$, which converts the REGA into a standard group action. This shows that in a world where REGA-SIS is easy, our equivalence between DLog and CDH also holds for REGAs. It turns out that the hardness of REGA-SIS is, in fact, inherent in solving DLog on REGAs: we also show that any algorithm which solves DLog on REGAs can be used to solve this REGA-SIS

[4] We defer a formal definition of this problem to the body of the paper. It is shown in [BLP+13] that 1D-SIS, for certain paramater settings, is equivalent to the "standard" LWE problem.

problem. This result is quite interesting since it implies DLog on REGAs is at least as hard as a (not necessarily randomized, and thus maybe not hard) version of a hard lattice problem.

If we could somehow strengthen this to show that a CDH solver on REGAs must also solve REGA-SIS, then we would obtain a full quantum equivalence between DLog and CDH for REGAs. We do not know how to prove such a result, but we give some evidence that *generic* adversaries for CDH on REGAs may have to solve REGA-SIS or, for certain groups, 1D-SIS itself. More precisely, we show a reduction that *generic* adversaries for CDH on REGAs that make *classical* group and group action "queries" can solve REGA-SIS.[5] We leave formally proving this equivalence as an interesting and practically important open problem.

**The Dihedral Hidden Subgroup Problem (Section 10).** Childs et al. [CJS14] apply the Dihedral Hidden Subgroup Problem (DHSP) algorithm of [Kup05] to compute isogenies between elliptic curves. This is a special case of the folklore result that any algorithm for DHSP yields an algorithm for DLog on regular, abelian group actions. We prove this folklore theorem.

The DHSP is the main approach for cryptanalyzing regular, abelian group actions, and no known better general algorithm is known. However, we point out that the two are *not* trivially equivalent: group actions have significant extra structure that could potentially be used for attacks that is not exploited by the reduction to DHSP. We are not aware of this observation being explicitly mentioned previously.

We next conjecture that, nevertheless, DHSP and regular, abelian group actions are *generically* equivalent, meaning any generic algorithm for solving these group actions can be used to solve DHSP generically. We offer some evidence of this conjecture, but leave proving or disproving it as a fascinating open question.

## 2 Preliminaries

In this section we discuss background material that is used in the rest of the paper. We expect that experienced readers can skip this section.

### 2.1 Min-entropy and Leftover Hash Lemma

Let $Z$ be a discrete random variable $Z$ with sample space $\Omega$. Its *min-entropy* is

$$H_\infty(Z) = \min_{\omega \in \Omega} \{-\log \Pr[Z = \omega]\}.$$

For two random variables $Y$ and $Z$, we use $H_\infty(Z|Y)$ to denote the min-entropy of $Z$ conditioned on $Y$. We will use the following lemma, which is a simplified version of the leftover hash lemma [ILL89].

**Lemma 1.** *Let $\{H_s : \mathcal{Z} \to Y\}_{s \in \mathcal{S}}$ be a family of pairwise independent hash functions, and $Z$ and $S$ be discrete random variables over $\mathcal{Z}$ and $\mathcal{S}$, respectively. If $H_\infty(Z) > \log|Y| + 2\log(\varepsilon^{-1})$ we have $\Delta[(S, H_S(Z)), (S, U)] \leq \varepsilon$, where $\Delta$ denotes statistical distance and $U$ denotes the uniform distribution over $Y$.*

We will also use the following corollary of the leftover hash lemma.

[5] The adversary could be quantum but is restricted to classical queries to the group and group action oracles.

**Lemma 2.** *Let $G$ be an (additive) finite abelian group such that $|G| = \lambda^{\omega(1)}$. Let $n \in Z$ such that $n > \log|G| + \omega(\log(\lambda))$. If $\mathbf{g} \leftarrow G^n$ and $\mathbf{s} \leftarrow \{0,1\}^n$, then*

$$\left(\mathbf{g}, \sum_{i=1}^{n} s_i \cdot g_i\right) \overset{s}{\approx} (\mathbf{g}, u),$$

*where $u \leftarrow G$ is a uniformly chosen element from $G$.*

### 2.2 1D-SIS Problem

The 1D-SIS problem dates to the original work of Ajtai [Ajt96] and has been used in many cryptographic applications [BV15, BKM17]. These cases use special moduli, but the case for general moduli follows from [BLP+13], where it is shown that the 1D-SIS problem with certain parameters but no special restrictions on the modulus is as hard as standard polynomial modulus LWE.

**Definition 3.** *Let $m$, $\beta$, and $q$ be positive integers. In the $1D\text{-}SIS_{m,q,\beta}$ problem, an adversary is given a random vector $\mathbf{v} \leftarrow Z_q^m$ and asked to provide a vector $\mathbf{u} \in Z_q^m$ such that $||\mathbf{u}|| < \beta$. We say that an adversary efficiently solves the $1d\text{-}SIS_{m,q,\beta}$ problem if it can provide such a vector in PPT time.*

### 2.3 Cryptographic Group Actions

Here we define cryptographic group actions following Alamati *et al.* [ADMP20], which are based on those of Brassard and Yung [BY91] and Couveignes [Cou06].

**Definition 4.** (Group Action) *A group $G$ is said to* act on *a set $X$ if there is a map $\star : G \times X \to X$ that satisfies the following two properties:*

1. *Identity: If $e$ is the identity of $G$, then $\forall x \in X$, we have $e \star x = x$.*
2. *Compatibility: For any $g, h \in G$ and any $x \in X$, we have $(gh) \star x = g \star (h \star x)$.*

We may use the abbreviated notation $(G, X, \star)$ to denote a group action. We extensively consider group actions that are *regular*:

**Definition 5.** *A group action $(G, X, \star)$ is said to be* regular *if, for every $x_1, x_2 \in X$, there exists a* unique *$g \in G$ such that $x_2 = g \star x_1$.*

We emphasize that most results in group action-based cryptography have focused on regular actions. As emphasized by [ADMP20], if a group action is regular, then for any $x \in X$, the map $f_x : g \mapsto g \star x$ defines a bijection between $G$ and $X$; in particular, if $G$ (or $X$) is finite, then we must have $|G| = |X|$.

In this paper, unless we specify otherwise, we will work with *effective* group actions (EGAs). An effective group action $(G, X, \star)$ is, informally speaking, a group action where all of the (well-defined) group operations and group action operations are efficiently computable, there are efficient ways to sample random group elements, and set elements have unique representation. Since the focus of this paper is on abelian group actions in a quantum world, we note that we can efficiently map any abelian group to $\mathsf{Z}_p$ for some integer $p$ (see Appendix A and our discussion on KEGAs), and all of the less obvious properties needed for EGAs follow automatically. However, the definition of an EGA itself is a little bit tedious (and quite formal so as to properly model isogeny-based constructions in a classical world) so we defer it to Appendix A.

## 2.4 Computational Problems

We next define problems related to group action security that are more semantically similar to typical group-based problems than those that are traditionally used in isogeny litaterature. We define the formal definitions that are typically used in isogenies (based on [ADMP20] in Appendix A and compare them to our (intuitively simpler) notions of security here. We emphasize that we are defining *problems* here and not *assumptions* because these are easier to use in reductions.

**Definition 6.** (Group Action Discrete Logarithm) *Given a group action $(G, X, \star)$ and distributions $(\mathcal{D}_X, \mathcal{D}_G)$, the group action discrete logarithm problem is defined as follows: sample $g \leftarrow \mathcal{D}_G$ and $x \leftarrow \mathcal{D}_X$, compute $y = g \star x$, and create the tuple $T = (x, y)$. We say that an adversary solves the group action discrete log problem if, given $T$ and a description of the group action and sampling algorithms, the adversary outputs $g$.*

**Definition 7.** (Group Action Computational Diffie-Hellman (CDH)) *Given a group action $(G, X, \star)$ and distributions $(\mathcal{D}_X, \mathcal{D}_G)$, the group action CDH problem is defined as follows: sample $g \leftarrow \mathcal{D}_G$ and $x, x' \leftarrow \mathcal{D}_X$, compute $y = g \star x$, and create the tuple $T = (x, y, x')$. We say that an adversary solves the group action CDH problem if, given $T$ and a description of the group action and sampling algorithms, the adversary outputs $y' = g \star x'$.*

**Definition 8.** (Group Action Decisional Diffie-Hellman (DDH)) *Given a group action $(G, X, \star)$ and distributions $(\mathcal{D}_X, \mathcal{D}_G)$, the group action DDH problem is defined as follows: sample $g_1, g_2 \leftarrow \mathcal{D}_G$ and $x, z' \leftarrow \mathcal{D}_X$, compute $y_1 = g_1 \star x$, $y_2 = g_2 \star x$, and $z = g_1 g_2 \star x$.*

*The group action DDH problem is to distinguish whether a tuple is of the form $(x, y_1, y_2, z)$ or $(x, y_1, y_2, z')$.*

**Remark 2.** *The above definitions allow for different distributions $\mathcal{D}_X$ on $X$. In particular, $\mathcal{D}_X$ could be uniform over $X$, or it could be a singleton distribution that places all its weight on a single fixed $x$. Whether $x$ is fixed or uniform potentially changes the the nature of these problems (see [BMZ19] for an exploration in the group-based setting). Looking ahead, our reduction between DLog and CDH will preserve $x$, and therefore it works no matter how $x$ is modeled.*

## 2.5 Instantiations of Cryptographic Group Actions

We next discuss various instantiations of cryptographic group actions and where they fall into our definitions. We start by discussing isogenies. For more details, we refer the reader to [ADMP20], which has an extensive discussion on the classification of various isogeny protocols into group action definitions.

### 2.5.1 Isogenies that are EGAs.

CSI-FiSh [BKV19] and its derivatives/applications [DM20a] have EGA functionality and are conjectured to even have weak pseudorandomness. However, there have recently been some subexponential attacks on CSI-FiSh [Pei20, BS20] and current cryptosystems built from CSI-FiSh are not particularly efficient. In fact, there are not efficient algorithms to (asymptotically) generate parameter sets for CSI-FiSh. However, if a powerful quantum computer were available, then efficient (quantum) computation of the class group structure could be used to generate arbitrary parameter sets for CSI-FiSh and improve efficiency.

### 2.5.2 Isogenies that are *restricted* EGAs (REGAs).

Recall that, in a *REGA*, there is a set of generators $g_1, \ldots, g_\ell \in G$, and it is only known how to efficiently compute the actions of the $g_i$ or $g_i^{-1}$; one can then compute the action of any $g \in G$ provided one has a representation of $g = \prod_{i=1}^{\ell} g_i^{\alpha_i}$ for polynomial $\alpha_i$. We define REGAs formally in Appendix A. Many of the most commonly used isogeny protocols are based on CSIDH [CLM+18], which is a REGA. These include things like the signature scheme SeaSign [DG19] or OT protocols [LGdSG21].

### 2.5.3 Isogenies that are not GAs.

There are many isogeny-based schemes that cannot be modeled as group actions. Examples include SIDH [DJP14] and the recently proposed OSIDH [CK20, Onu21, DDF21]. Most isogeny-based protocols that are not group actions are typically used for key exchange or other very simple cryptographic applications.

**Remark 3.** *A few very recent works [CD22, MM22, Rob22] break SIDH by showing how to solve the discrete log problem. However, the attack crucially exploits certain extra points that are made public in SIDH, and these points are precisely one of the reasons that SIDH is not a group action. In particular, the the attack does not seem to apply to CSI-FISH or CSIDH, the main instantiations of EGAs and REGAs, respectively.*

### 2.5.4 Non-Isogeny Group Actions.

Currently all instantiations of *abelian* candidate cryptographic group actions that are thought to be secure are isogeny-based [DDF21]. There have been a number of attempts to build key exchange and other basic primitives from nonabelian groups that amount to group actions or have hardness assumptions that can be modeled in some way as group actions [KLC+00, Sti05, SU05a], but the proposed instantiations of these schemes have been completely cryptanalyzed [Shp08, BKT18].

We note that these candidate cryptosystems typically propose an abstract scheme and then attempt to instantiate it with a group. We note that it is not usually the case that the abstract schemes themselves are broken: the cryptanalysis typically works directly on the instantiations, so it is possible that some of these protocols could be implemented securely with different choices of groups.

There have also been some candidate nonabelian cryptographic group actions proposed [JQSY19]. While these are not known to be insecure, they have far fewer applications than abelian group actions.

## 3 Reducing DLog to CDH Quantumly

Let $(G, X, \star)$ be a regular abelian group action. In Section 3.1 we explain how to extend our reduction to non-regular abelian actions. Let $x \in X$ be a fixed set element.

**Theorem 9.** *If DLog is post-quantum hard in $(G, X, \star)$, then so is CDH. More precisely, there exists an oracle algorithm $R^{\mathcal{A},(G,X,\star)}(\mu, y)$ that runs in time poly$(1/\mu, \log|G|)$ and makes poly$(1/\mu, \log|G|)$ total queries to a supposed CDH adversary $\mathcal{A}$ and group action $(G, X, \star)$, such that the following holds. If* $\Pr_{a,b\leftarrow G}[\mathcal{A}(a \star x, b \star x) = (ab) \star x] \geq \mu$, *then for any $a \in G$,* $\Pr[R^{\mathcal{A},(G,X,\star)}(\mu, a \star x) = a] \geq 0.99$.

We note that the above means that $R$ is very slightly non-black box, in that its running time and number of calls to $\mathcal{A}$ depend on the success probability $\mu$ of $\mathcal{A}$. We note that any amplification of success probability (say, from $\mu$ to 0.99) will always come with such a dependence on $\mu$. In our case, amplification is critical to our algorithm, and the dependence on $\mu$ would persist even if we only wanted $R^{\mathcal{A}}$ to have very small success probability. The remainder of this section is devoted to proving Theorem 9.

Define $\mathsf{CDH}$ to be the function which correctly solves CDH relative to $x$: $\mathsf{CDH}(a \star x, b \star x) = (ab) \star x$. We will also allow $\mathsf{CDH}$ to take as input a vector of elements, behaving as $\mathsf{CDH}(a_1 \star x, \cdots, a_n \star x) = (a_1 \cdots a_n) \star x$. Furthermore, we will allow $\mathsf{CDH}$ to take as input distribution(s) over the set $X$; in this case, $\mathsf{CDH}$ will also output a distribution.

Let $a, b \in G$ be group elements, and let $y = a \star x$ and $z = b \star x$. Suppose $\mathcal{A}$ is an efficient (quantum) algorithm such that

$$q := \Pr[\mathcal{A}(y, z) = \mathsf{CDH}(y, z)]$$

is a non-negligible function in the security parameter, where $a$ and $b$ are random elements in $G$, and the probability is over the randomness of $a$ and $b$ and $\mathcal{A}$.

Our goal is to turn $\mathcal{A}$ into a quantum algorithm for discrete logarithms. As a first step, we introduce a random self-reduction for CDH. In the case of groups (as opposed to group actions), a more powerful random self-reduction allows for amplifying the success probability on any input. The result would be an algorithm for CDH with overwhelming success probability. In our case, due to the restricted nature of group actions, we can only perform a more limited self-reduction. Nevertheless, this self-reduction has useful properties.

**The Basic Random Self-reduction.** The random self-reduced version of $\mathcal{A}$, denoted $\mathcal{A}_0$, works as follows:

- On input $y = a \star x, z = b \star x$, choose random $a', b' \in G$.
- Let $y' = a' \star y, z' = b' \star z$.
- Run $w' \leftarrow \mathcal{A}(y', z')$.
- Output $w = (a'b')^{-1} \star w'$.

Note that each run of $\mathcal{A}_0$ runs $\mathcal{A}$ exactly once, and uses a constant number of group action operations. This reduction is correct since, if $\mathcal{A}$ is correct, then we output

$$w = \left(a'b'\right)^{-1} \mathsf{CDH}\left(\left(a'a\right) \star x, \left(b'b\right) \star x\right) = \left(a'b'\right)^{-1} \left(aa'bb'\right) \star x = (ab) \star x$$

which is the correct output for $\mathsf{CDH}$. Moreover, the set elements $y', z'$ are uniformly distributed over the possible set elements.

Let $\mathcal{D}$ be the distribution $\mathcal{A}_0(x, x)$. That is, we are feeding the "dummy" distribution to our random self-reduction. While we know what the answer should be ($x = \mathsf{CDH}(x, x)$), we use this distribution to learn more about $\mathcal{A}$'s behavior.

**Lemma 10.** $\Pr[x \leftarrow \mathcal{D}] = q$.

*Proof.* Recall that $\mathcal{D}$ is the distribution $\mathcal{A}_0(x, x)$. $\mathcal{A}_0$ on input $(x, x)$ calls $\mathcal{A}(a' \star x, b' \star x)$ for random $a', b' \in G$. With probability $q$, $\mathcal{A}(a' \star x, b' \star x)$ returns $(a'b') \star x$, and in this case we have $w = x$ as desired. □

We next generalize our notation. For any $y, z \in X$ where $y = a \star x$ and $z = b \star x$ for some $a, b \in G$, let $D_{y,z}$ be the distribution of outputs of $\mathcal{A}_0(y, z)$.

**Lemma 11.** *For every $y, z \in X$ such that there exist $a, b \in G$ where $y = a \star x$ and $z = b \star x$, $\mathcal{D}_{y,z} = \mathsf{CDH}(y, z, \mathcal{D})$, where $\mathsf{CDH}(\cdot,\cdot,\cdot)$ is the 3-way CDH function. In other words, $\mathcal{A}_0(a \star x, b \star x)$ is identically distributed to $(ab) \star \mathcal{A}_0(x, x)$.*

*Proof.* Fix $a, b \in G$. Consider the probability that $\mathcal{A}_0(a \star x, b \star x)$ outputs $w$:

$$\begin{aligned}
\Pr[\mathcal{A}_0(a \star x, b \star x) = w] &= \Pr_{a',b' \in G}[\left(a'b'\right)^{-1} \star \mathcal{A}((aa') \star x, (bb') \star x) = w] \\
&= \Pr_{a',b' \in G}[\mathcal{A}((aa') \star x, (bb') \star x) = (a'b') \star w] \\
&= \Pr_{a'',b'' \in G}[\mathcal{A}(a'' \star x, b'' \star x) = (a''b'' \left(ab\right)^{-1}) \star w] \\
&= \Pr[\mathcal{A}_0(x, x) = (ab)^{-1} \star w]
\end{aligned}$$

Thus, $\mathcal{A}_0(a \star x, b \star x)$ is just the distribution $\mathcal{A}_0(x, x)$, but shifted by $ab$. □

Using this "shift invariance," we can define $\mathcal{D}_w := \mathcal{D}_{w,x} = \mathcal{D}_{x,w} = \mathcal{D}_{y,z}$, if $\mathsf{CDH}(y, z) = w$. Lemma 11 shows that $\mathcal{D}_{y,z}$ outputs $\mathsf{CDH}(y, z)$ with probability $q$. Thus, by running $\mathcal{A}_0$ many times, the right answer is almost certainly amongst the list of outputs. However, to amplify the success probability, we would need to know which of the list of outputs is the correct answer; we cannot determine this yet.

In the following, we will take steps to remedy this issue. Throughout this section, it is instructive to keep the following examples in mind:

1. Let $g \in G \setminus \{1\}$. $\mathcal{A}(Y, Z)$ outputs $\mathsf{CDH}(y, z)$ with probability $1/3$, and $g \star \mathsf{CDH}(Y, Z)$ with probability $2/3$. Notice that in this case, $\mathcal{A}_0$ has the same distribution of outputs as $\mathcal{A}$. Also notice that taking the majority element will give the wrong answer. Thus, we cannot immediately decide which of the outputs of $\mathcal{A}_0$ is the right answer just by looking at the frequencies.

2. Let $\mathcal{H}$ be a subgroup of $G$ of size $1/q$. Then consider the case where $\mathcal{A}(y, z)$ outputs $c \star \mathsf{CDH}(y, z)$, where $c \leftarrow \mathcal{H}$ is chosen uniformly. Note that $\mathcal{A}$ is still correct with probability $q$ in this case, since $c = 1_{\mathcal{H}}$ with probability $q$. Similar to Example 1, there is no way to identify the correct output just by looking at frequencies.

3. Suppose $\mathcal{H} = \mathsf{Z}_2^{\log \lambda}$, which we can decompose as a chain of subgroups $\mathcal{H}_i = \mathsf{Z}_2^i$ with $\mathcal{H}_{i-1} \subseteq \mathcal{H}_i$. $\mathcal{A}$ outputs $c \star \mathsf{CDH}(y, z)$, where $c \in \mathcal{H}$. However, $c$ is not uniform. Instead, $i \in [0, \log \lambda]$ is chosen according to some probability distribution, and then $c$ is chosen uniformly from $\mathcal{H}_i$.

4. Suppose $\mathcal{H} = \mathsf{Z}_2^{\log \lambda}$. Again, $\mathcal{A}$ outputs $c \star \mathsf{CDH}(y, z)$, where $c \in \mathcal{H}$ but not uniform. Here, $c$ occurs with probability $1 - \alpha|c|_1$, where $|c|_1$ denotes the Hamming weight of $c$.

**Example 1.** It turns out Example 1 can be handled using the shifting property from Lemma 11. Suppose we are given a $\mathsf{CDH}$ challenge parameterized by $(y = a \star x, z = b \star x)$. Basically, after repeating many runs of $\mathcal{A}_0(y, z)$, we obtain two elements: $w_0 = (ab) \star x$ and $w_1 = (gab) \star x$. In theory, in this example we could exploit the fact that we know the probabilities with which $\mathcal{A}$ outputs the correct set element and the "$g$-multiplied" set element, but let's assume that we do not know this. What can we do?

Suppose we feed these outputs back into $\mathcal{A}_0$, running $\mathcal{A}_0(w_0, x)$ and $\mathcal{A}_0(w_1, x)$ several times each. Each of these two runs will output two distinct elements. Since $w_0 = (ab) \star x$, Lemma 11 shows that $\mathcal{A}_0(w_0, x) = \mathcal{D}_{w_0} = \mathcal{D}_{y,z} = \mathcal{A}_0(y, z)$ as distributions. Likewise, since $w_1 = (gab) \star x$, we have $\mathcal{A}_0(w_1, x) = \mathcal{A}_0(g \star y, z)$.

Therefore, because $\mathcal{A}_0(w_0, x)$ is distributed the same as $\mathcal{A}_0(y, z)$ and $\mathcal{A}_0(w_1, x)$ is not, we can effectively distinguish $w_0$ from $w_1$ and find the correct $\mathsf{CDH}$ output.

**Example 2.** On the other hand, Example 2 is much harder to handle. Mimicking the above, we first run $\mathcal{A}_0$ several times, obtaining the list of values $c \star \mathsf{CDH}(y, z)$ as $c$ ranges over $\mathcal{H}$, but we don't know $c$. We can then try, for each $c \star \mathsf{CDH}(y, z)$, running $\mathcal{A}_0(c \star \mathsf{CDH}(y, z), x)$ several times, to obtain tuples of elements. However, this will not give us any useful information: each tuple will be exactly the same list as in the original run of $\mathcal{A}_0$, namely the entire set $\mathcal{H} \star \mathsf{CDH}(y, z)$. The problem is that the output distribution of $\mathcal{A}_0$ is invariant under action by $\mathcal{H}$.

Looking ahead, we cannot improve the CDH algorithm for this example. However, this particular example gives a *perfect* CDH oracle relative to the group $G/\mathcal{H}$ acting on $X/\mathcal{H} := \{\mathcal{H} \star w : w \in X\}$. We will use such an algorithm to solve discrete log in $G/\mathcal{H}$. We can then solve discrete logarithms in $\mathcal{H}$ by brute force, and then piece the two results together to solve discrete logarithms in $G$.

**Examples 3 and 4.** In general, however, we may not get a perfect CDH oracle for $\mathcal{H}$, and are not even obviously guaranteed that the outputs lie in a small subgroup. In Example 3, consider the distribution over $i$ such that larger subgroups are very unlikely, but not *too* unlikely. For any fixed number of queries, it could be that, with probability $1/2$, all results end up in $\mathcal{H}_i$, but with probability $1/2$ some of the results will end up in $\mathcal{H}_{i+1}$. It might, a priori, not even be possible to identify when you have all the elements from a subgroup, since "chaining" calls to $\mathcal{A}_0$ as we have done above might move us outside a subgroup. So it is unclear if there is a way to always output a consistent complete subgroup, so as to get a near-perfect $\mathsf{CDH}$ solver relative to $G$ mod this subgroup.

Next, we will gradually improve our $\mathsf{CDH}$ solver to resolve these difficulties.

**Restricting to a small subgroup.** We show how to discard some wrong outputs of $\mathcal{A}_0$ so that the remaining outputs lie in a reasonably-small subgroup of $G$, while still guaranteeing that we keep $\mathsf{CDH}(y, z)$.

We first give some notation. For any two distributions $\mathcal{D}_0, \mathcal{D}_1$ over $X$, let $\|\mathcal{D}_0 - \mathcal{D}_1\|_\infty = \max_{w \in X} |\Pr[w \leftarrow \mathcal{D}_0] - \Pr[w \leftarrow \mathcal{D}_1]|$. For a distribution $\mathcal{D}$ over $X$, consider sampling $T$ elements $w_1, \ldots, w_T$ from $\mathcal{D}$. This vector of $w_i$ gives rise to an "empirical" distribution $\tilde{\mathcal{D}}$, where the probability of any $w$ is just the relative frequency of $w$ amongst the $w_i$. Note that even though $\tilde{\mathcal{D}}$ has a domain of exponential size, we can represent it by the list $w_1, \ldots, w_T$, which has size $T$. Also note that there are two distributions here: the empirical distribution $\tilde{\mathcal{D}}$ itself, and the distribution over empirical distributions. We denote the latter as $\tilde{\mathcal{D}} \leftarrow \mathcal{D}^T$.
We are now ready to give our next algorithm, $\mathcal{A}_1(y, z)$:

- Let $T = \lambda/\delta^2$ for some parameter $\delta \in (0, 1)$.
- Run $\tilde{\mathcal{D}}^* \leftarrow \mathcal{A}_0(y, z)^T$
- For each $w$ in the support of $\tilde{\mathcal{D}}^*$, run $\tilde{\mathcal{D}}_w \leftarrow \mathcal{A}_0(w, x)^T$.
- Output $L$, the set of $w$ in the support of $\tilde{\mathcal{D}}^*$ such that $\|\tilde{\mathcal{D}}_w - \tilde{\mathcal{D}}^*\|_\infty \leq \delta/2$.

We will think of $\lambda$ being $\mathrm{poly}(\log q)$, so that $2^{-\Omega(\lambda)}$ is negligible in $1/q$. Note that $\mathcal{A}_1$ makes at most $T^2 + T = O(\lambda^2/\delta^4)$ evaluations of $\mathcal{A}_1$, and hence $T^2 + T$ evaluations of $\mathcal{A}_0$ and $O(T^2 + T)$ group action operations. In order to analyze the algorithm $\mathcal{A}_1$, we need to give some basic results. First we recall the Dvoretzky-Kiefer-Wolfowitz inequality:

**Lemma 12** ([Mas90]). *For any $\zeta > 0$ and distribution $\mathcal{D}$, except with probability $2e^{-2\zeta^2 T}$, $\|\tilde{\mathcal{D}} - \mathcal{D}\|_\infty \le \zeta$, where $\tilde{\mathcal{D}} \leftarrow \mathcal{D}^T$.*

In other words, the empirical distribution converges to the underlying distribution $\mathcal{D}$ as the number of samples $T$ grows large.

Now consider the distribution $\mathcal{D} = \mathcal{A}_0(x, x)$ from before, and the derived distributions $\mathcal{D}_w = \mathsf{CDH}(w, \mathcal{D})$. Let $d_w = \|\mathcal{D}_w - \mathcal{D}\|_\infty$.

**Lemma 13.** *$\forall y, z \in X$, $\|\mathcal{D}_{\mathsf{CDH}(y,z)} - \mathcal{D}_y\|_\infty = d_z$ and $d_{\mathsf{CDH}(y,z)} \le d_y + d_z$.*

*Proof.* For the equality, note that $\|\mathcal{D}_{\mathsf{CDH}(y,z)} - \mathcal{D}_y\|_\infty = \|\mathsf{CDH}(y, \mathcal{D}_z) - \mathsf{CDH}(y, \mathcal{D})\|_\infty$. Since $\mathsf{CDH}(y, \cdot)$ simply permutes the elements of $X$—more precisely, it maps $v \in X$ to $a \star v$ where $y = a \star x$—it does not affect the distance between distributions, and therefore $|\mathsf{CDH}(y, \mathcal{D}_z) - \mathsf{CDH}(y, \mathcal{D})| = |\mathcal{D}_z - \mathcal{D}| = d_z$. For the inequality, we have $d_{\mathsf{CDH}(y,z)} = |\mathcal{D}_{\mathsf{CDH}(y,z)} - \mathcal{D}|_\infty \le |\mathcal{D}_{\mathsf{CDH}(y,z)} - \mathcal{D}_y|_\infty + |\mathcal{D}_y - \mathcal{D}|_\infty = d_z + d_y$, where we used the equality in the second to last step. □

Now we prove the following general result about abelian groups. Fix an abelian group $\mathcal{H}$ and a set of generators $\mathbf{a} = (a_1, \ldots, a_n)$. For any vector $\mathbf{e} \in \mathbb{N}^n$ of non-negative integers, define $\mathbf{a}^\mathbf{e} := \prod_{i=1}^n a_i^{e_i}$. Let $\|\mathbf{e}\|_1 := \sum_{i=1}^n |e_i|$. Then for any $r \in \mathcal{H}$, we define $\|r\| := \min_{\mathbf{e} \in \mathbb{N}^n : r = \mathbf{a}^\mathbf{e}} \|\mathbf{e}\|_1$.

**Lemma 14.** *If $U = \{r \in \mathcal{H} : \|r\| \le ns\}$ has size at most $s$, then $U = \mathcal{H}$.*

In other words, if the subset of $\mathcal{H}$ with small $\|\cdot\|$ is not too big, then in fact all of $\mathcal{H}$ has small $\|\cdot\|$.

*Proof.* Clearly $U \subseteq H$. In the other direction, consider a single $a_i$. Since $U$ has size at most $s$, then so does the set $\{a_i^{e_i} : 0 \le e_i \le s\} \subseteq U$. As there are $s+1$ different possibilities for $e_i$, there must be $e_i' < e_i$ such that $a_i^{e_i} = a_i^{e_i'}$. Then $a_i^{e_i' - e_i} = 1$, and $0 < e_i' - e_i \le s$. For any $r \in \mathcal{H}$, write $r = \mathbf{a}^\mathbf{e}$. Since $a_i$ has order at most $s$, we can reduce each $e_i$ to an integer smaller than $s$ without changing $r$. After such a reduction, $\|\mathbf{e}\|_1 \le ns$, and so $r \in U$. Hence $H \subseteq U$. □

Let $L_\delta \subset G$ be the set of all $a \in G$ such that $d_{a \star x} \le \delta$, and $\mathcal{H}_\delta$ be the subgroup of $G$ generated by $L_\delta$. We have the following:

**Lemma 15.** *Let $\epsilon \in (0, 1]$ be a real number. Then if $\delta \le \epsilon q^4/8$, $|\mathcal{H}_\delta| \le q^{-1} + \epsilon$.*

Note that $\epsilon$ is necessary: $\mathcal{D}$ may output $g \star x$ for a $g$ in a subgroup $\mathcal{H}$ of size $n$, with $q^{-1}$ negligibly smaller than $n$. Suppose $\Pr[x \leftarrow D] = q$ and $\Pr[g \star x \leftarrow D]$ is slightly less than $q$ for all other $g$. Then $\mathcal{H}_\delta = \mathcal{H}$ for any non-negligible $\delta$.

*Proof.* We first prove that $|L_\delta| \le q^{-1} + \epsilon$. Note that $d_{1_G \star x} = d_x = 0$ and so $1_G \in L_\delta$. From Lemma 10, $\Pr[x \leftarrow \mathcal{D}] = q$. Therefore, for any $a \in L_\delta$,

$$\Pr[a^{-1} \star x \leftarrow \mathcal{D}] = \Pr[x \leftarrow \mathcal{D}_{a \star x}] \ge \Pr[x \leftarrow \mathcal{D}] - \delta = q - \delta,$$

where the inequality follows since $d_{a \star x} \le \delta$ for $a \in L_\delta$. Then

$$\begin{aligned} 1 &= \sum_{a \in G} \Pr[a^{-1} \star x \leftarrow \mathcal{D}] \ge \sum_{a \in L_\delta} \Pr[a^{-1} \star x \leftarrow \mathcal{D}] \\ &= \Pr[1 \star x \leftarrow \mathcal{D}] + \sum_{a \in L_\delta \setminus \{1\}} \Pr[a^{-1} \star x \leftarrow \mathcal{D}] \ge q + (|L_\delta| - 1)(q - \delta) \end{aligned}$$

Solving for $|L_\delta|$ gives $|L_\delta| \leq (1-\delta)/(q-\delta)$. Setting the right hand side to be $\leq q^{-1}+\epsilon$ gives the desired bound whenever $\delta \leq \epsilon q^2/(1-q+q\epsilon)$. Note that $(1-q+q\epsilon) \leq 1$. Therefore, $\delta \leq \epsilon q^4/8$ is only a stronger bound on $\delta$.

We now bound $|\mathcal{H}_\delta|$ by applying Lemma 14 to $\mathcal{H} = \mathcal{H}_\delta$ and $\mathbf{a} = L_\delta$ and $s = 1/q + \epsilon$. Consider some $r = \mathbf{a}^{\mathbf{e}}$ in $\mathcal{H}_\delta$. Then by iteratively applying Lemma 13,

$$d_{r\star x} = d_{\mathsf{CDH}(\underbrace{a_1\star x,\cdots,a_1\star x}_{e_1},\underbrace{a_2\star x,\cdots,a_2\star x}_{e_2},a_3\star x,\cdots)} \leq \sum_i e_i d_{a_i\star x} \leq \sum_i e_i\delta = |\mathbf{e}|_1\delta$$

By minimizing over all $\mathbf{e}$, we have that $d_{r\star x} \leq \|r\|\delta$. For $U$ as in Lemma 14, this means that $\Pr[r^{-1}\star x \leftarrow \mathcal{D}] = \Pr[x \leftarrow \mathcal{D}_{r\star x}] \geq q - \|r\|\delta \geq q - ns\delta$. Since the probabilities of each outcome sum to at most 1, we therefore have that $|U| \leq (q-ns\delta)^{-1}$. In order to satisfy the conditions of Lemma 14, we therefore need $1/(q-ns\delta) \leq s$, which is equivalent to $1 \leq s(q-ns\delta)$. Since $n = |L_\delta| \leq 1/q+\epsilon$, we have that this inequality is satisfied whenever $\delta \leq \epsilon q^4/(1+\epsilon q)^3$. As $1+\epsilon q \leq 2$, our bound of $\delta \leq \epsilon q^4/8$ is only a stronger bound, showing that $\mathcal{H}_\delta = L_\delta$. Our prior bound on $|L_\delta|$ thus proves Lemma 15. □

We are finally ready to analyze the algorithm $\mathcal{A}_1$. Let $\mathcal{D}'$ be the distribution $\mathcal{A}_1(x,x)$, and $\mathcal{D}'_{y,z}$ be the distribution $\mathcal{A}_1(y,z)$. The next lemma follows immediately from Lemma 11:

**Lemma 16.** *For every $y,z \in X$ where $y = a\star x$ and $z = b\star x$ for some $a,b \in G$, $\mathcal{D}'_{y,z} = \mathsf{CDH}(y,z,\mathcal{D}')$.*

Thus, we define $\mathcal{D}'_w := \mathcal{D}'_{w,1} = \mathcal{D}'_{1,w} = \mathcal{D}'_{y,z}$, if $\mathsf{CDH}(y,z) = w$. We now prove:

**Lemma 17.** *Except with probability $2(T+1)e^{-\delta^2 T/8} + (1-q)^T \leq 2^{-\Omega(\lambda)}$ over $L \leftarrow \mathcal{A}_1(x,x)$, we have that $x \in L \subseteq \mathcal{H}_\delta \star x$.*

*Proof.* Suppose we set $\zeta = \frac{\delta}{4}$. By Lemma 12, we have that $|\tilde{\mathcal{D}}^\star - \mathcal{D}_{\mathsf{CDH}(y,z)}|_\infty \leq \delta/4$ and for each $w$ in the support of $\tilde{\mathcal{D}}^\star$, $|\tilde{\mathcal{D}}_w - \mathcal{D}_w| \leq \delta/4$, each individually except with probability at most $2e^{-\delta^2 T/8}$. We also have that with probability $1-(1-q)^T$, $x$ will be amongst the $T$ samples of $\mathcal{A}_1(x,x)$. By a union bound, all of these happen simultaneously, except with probability $2(T+1)e^{-\delta^2 T/8} + (1-q)^T$.

If all of these happen, then $|\tilde{\mathcal{D}}_x - \tilde{\mathcal{D}}^\star| \leq |\tilde{\mathcal{D}}_x - \mathcal{D}| + |\tilde{\mathcal{D}}^\star - \mathcal{D}| \leq 2\delta/4 = \delta/2$. Thus $x \in L$ assuming the above hold. On the other hand, for any $w \in L$, $d_w = |\mathcal{D}_w - \mathcal{D}| \leq |\tilde{\mathcal{D}}_w - \mathcal{D}_w| + |\tilde{\mathcal{D}}_w - \tilde{\mathcal{D}}^\star| + |\tilde{\mathcal{D}}^\star - \mathcal{D}| \leq \delta$. Hence $w \in L_\delta \star x$ by the definition of the set $L_\delta$, which immediately implies that each $w \in \mathcal{H}_\delta \star x$. □

As a consequence, we have that $\mathcal{D}'$ has negligible support outside of $\mathcal{H}_\delta \star x$. Note that $\mathcal{D}'$ may not be random in $\mathcal{H}_\delta \star x$, as the list $L$ may not include all of $\mathcal{H}_\delta \star x$, and $L$ itself may be randomized. Indeed, in Example 4, $\alpha$ may be such that $\mathcal{D}$ and $\mathcal{D}_{c\star w}$ are sufficiently close for $c$ with small Hamming weight, but $\mathcal{D}_{c\star w}$ is far for $c$ with large Hamming weight. Some $c$ may even be right on the cusp, being included in $L$ with constant probability. The result is that the output may not be a whole subgroup and may have entropy.

We note that by setting $\epsilon$ a constant and $\delta = \epsilon q^4/8 = O(q^4)$, we have that $\mathcal{A}_1$ runs in time $O(\lambda q^{-8}) = \tilde{O}(q^{-8})$ and makes $\tilde{O}(q^{-8})$ total queries to $\mathcal{A}$ and the group action operations.

**Filling an entire subgroup.** $\mathcal{A}_1$ outputs a subset of $\mathcal{H}_\delta \star \mathsf{CDH}(y,z)$, and the subset must include $\mathsf{CDH}(y,z)$. We will now devise a new algorithm $\mathcal{A}_2$ which outputs $\mathcal{H}\star\mathsf{CDH}(y,z)$, where $\mathcal{H}$ is a (potentially unknown) subgroup of $\mathcal{H}_\delta$. We split $\mathcal{A}_2(y,z)$ into two phases, $\mathcal{A}_2^0()$, which outputs the set $\mathcal{H}\star x$, and then $\mathcal{A}_2^1(y,z,\mathcal{H}\star x)$, which outputs the set $\mathcal{H}\star\mathsf{CDH}(y,z)$. We first give $\mathcal{A}_2^0()$:

- Initialize list $L = \{x\}$. Let $s = q^{-1} + \epsilon$ be an upper bound on the size of $\mathcal{H}_\delta$.
- Let $T = s\lambda/\tau$, for a parameter $\tau \in (0, 1)$ to be chosen later.
- Repeat the following at least $T$ times:
  - For each pair $(w, w') \in L^2$, run $L_{w,w'} \leftarrow \mathcal{A}_1(w, w')$
  - Let $L' = \cup_{w,w'} L_{w,w'}$
  - If $|L'| = |L|$ and the number of iterations so far is $\geq T$, terminate and output $L$. Otherwise (if the number of iterations is $< T$ or $|L'| \neq |L|$), replace $L$ with $L'$, and continue.

We now analyze the algorithm $L \leftarrow \mathcal{A}_2^0()$.

**Lemma 18.** *Except with negligible probability $2^{-\Omega(\lambda)}$, all of the following hold:*

- *$L = \mathcal{H} \star x$ for some (potentially unknown) subgroup $\mathcal{H} \subseteq \mathcal{H}_\delta$.*
- *$\mathcal{A}_2^0()$ will terminate in at most $T + s$ steps.*
- *For the resulting $\mathcal{H}$, $\Pr[M \notin \mathcal{H} : M \leftarrow D'] < \tau$.*

*Proof.* Combining Lemmas 16 and 17, we know that except with probability $2^{-\Omega(\lambda)}$, $L_{w,w'}$ will be a list containing $\mathsf{CDH}(w, w')$. Throughout the rest of the proof of Lemma 18, we will therefore assume $\mathsf{CDH}(w, w') \in L_{w,w'}$ for all iterations and for all $w, w'$.

We first argue that $L \subseteq L'$ in every iteration, except with probability $2^{-\Omega(\lambda)}$. In particular, since $L$ is set to $L'$ at the end of each iteration, this means that $L$ is never decreasing in size, and once an element is added to $L$ it will remain for the rest of the algorithm. Indeed, $L$ initially contains $x$. By induction, assume $L$ contains $x$ for the first $i$ iterations, and consider computing $L'$ in this iteration. $L'$ is set to $L' = \cup_{w,w'} L_{w,w'}$ where $L_{w,w'} \leftarrow \mathcal{A}_1(w, w')$ as $w, w'$ range over $L$. In particular, since $x \in L$, $L'$ will contain $L_{w,x} \leftarrow \mathcal{A}_1(w, x)$ for every $w \in L$. Since we assume $L_{w,x}$ contains $\mathsf{CDH}(w, x) = w$, every $w \in L$ will be included in $L'$.

Therefore, if $|L'| = |L|$, it must mean that $L' = L$. Additionally, once we terminate, we know that $\mathsf{CDH}(w, w') \in L' = L$ for every $w, w' \in L$, meaning $L$ is closed under CDH/multiplication once we terminate. Hence, $L$ forms $\mathcal{H} \star x$ for some subgroup $\mathcal{H}$. By Lemma 17, our algorithm maintains the invariant that $L \subseteq \mathcal{H}_\delta$ at all times, and hence $\mathcal{H} \subseteq \mathcal{H}_\delta$.

Now consider any $w \in \mathcal{H}_\delta$ such that $\Pr[w \in M : M \leftarrow \mathcal{D}'] \geq \tau/s$. Then after $T$ iterations, the probability $w$ never gets added to $L$ is $(1 - \tau/s)^T = (1 - \tau/s)^{s\lambda/\tau} \approx e^{-\lambda}$. Union bounding over at most $s$ such $w$, we see that all such $w$ get added to $L$, except with probability at most $2^{-\Omega(\lambda)}$. In this case, a union bound over the $w$ such that $\Pr[w \in M : M \leftarrow \mathcal{D}'] < \tau/s$, of which there are at most $s$, shows that the probability of sampling any value not in $\mathcal{H}$ is less than $\tau$. □

We now give the algorithm $\mathcal{A}_2^1(y, z, L)$:

- Initialize $\mathcal{M}$ to be an empty list of unordered sets.
- Repeat the following $\lambda$ times:
  - Run $M \leftarrow \mathcal{A}_1(y, z)$.
  - For each $w \in M, w' \in L$, run $M_{w,w'} \leftarrow \mathcal{A}_1(w, w')$.
  - Let $M = \cup_{w,w'} M_{w,w'}$. Add $M$ to $\mathcal{M}$ (keeping duplicates).

- Let $M^*$ be the most common element in $\mathcal{M}$.

We now analyze the algorithm $\mathcal{A}_2^1(y, z, L)$.

**Lemma 19.** *If $\tau \leq 1/4(s^2+1)$, then except with probability $2^{-\Omega(\lambda)}$, $L = \mathcal{H} \star x$ for some subgroup $\mathcal{H} \subseteq \mathcal{H}_\delta$, and $M^* = \mathsf{CDH}(y, z, \mathcal{H} \star x)$.*

*Proof.* Define $w^* = \mathsf{CDH}(y, z)$. We assume the bullets of Lemma 18 hold, which Lemma 18 shows hold except with probability $2^{-\Omega(\lambda)}$. Therefore, $L = \mathcal{H} \star x$ for some subgroup $\mathcal{H} \subseteq \mathcal{H}_\delta$. It remains to show that $M^* = \mathsf{CDH}(y, z, \mathcal{H} \star x) = \mathcal{H} \star w^*$. By union-bounding over the $s^2+1$ runs of $\mathcal{A}_1$ in each iteration and invoking the last bullet of Lemma 18, the following holds: for each iteration, except with probability at most $\tau \times (s^2+1) \leq 1/4$, we have that

- $M_{w,w'} \subseteq \mathcal{H} \star w^*$ for each $w \in M, w' \in L$, and therefore in particular $M \subseteq \mathcal{H} \star w^*$.
- $w^* \in M$.

Provided $M \subseteq \mathcal{H} \star w^*$, except with probability $2^{-\Omega(\lambda)}$, we have $\mathsf{CDH}(w, w') \in M_{w,w'}$, and so $\mathcal{H} \star w^* = \mathsf{CDH}(w^\star, \mathcal{H} \star x) \subseteq M$. Therefore, $M = \mathcal{H} \star w^*$ with probability at least $3/4 - 2^{-\Omega(\lambda)} \geq 2/3$. Since each iteration samples independently the distribution over $M$, by simple concentration bounds $\mathcal{H} \star w^\star$ will be the majority element of $\mathcal{M}$, except with probability $2^{-\Omega(\lambda)}$. □

Note that $\mathcal{A}_2^0$ runs $\mathcal{A}_1$ for $(T+s)|L|^2 = O(|L|^2\lambda/q^3) = \tilde{O}(q^{-5})$ times, giving $\tilde{O}(q^{-13})$ total queries to $\mathcal{A}$ and the group action operation. Meanwhile, $\mathcal{A}_2^1$ runs $\mathcal{A}_1$ for $\lambda|L|^2$ times, giving $\tilde{O}(q^{-10})$ queries to $\mathcal{A}$ and the group operation.
From this point on, we fix a single $L \leftarrow \mathcal{A}_2^0()$ once and for all.

**Removing Superfluous Information.** We will next want to run quantum period-finding algorithms which make queries to $\mathcal{A}_2^1$ on superpositions of inputs. These algorithms, however, assume $\mathcal{A}_2^1$ is a function. Unfortunately, our algorithm generates significant side information, namely all the intermediate computations used to arrive at the final answer. Fortunately, since our algorithm outputs a single answer with overwhelming probability, we can use the standard trick of purifying the execution of $\mathcal{A}_2^1$ and then un-computing all the intermediate values. The result is that $\mathcal{A}_2^1$ is negligibly close to behaving as the function mapping $(y, z) \mapsto \mathcal{H} \star \mathsf{CDH}(y, z)$. From now on, we will therefore assume that $\mathcal{A}_2^1$ is such a function.

**Computing $\mathcal{H}$.** Given algorithm $\mathcal{A}_2^1$, we can compute the subgroup $\mathcal{H}$ using quantum period-finding [BL95]. Concretely, the function $a \mapsto \mathcal{A}_2^1(a \star x, x, L)$ will output $(a\mathcal{H}) \star x$, which is periodic with set of periods $\mathcal{H}$. Therefore, applying quantum period finding to the procedure $a \mapsto \mathcal{A}_2^1(a \star x, x, L)$ will recover $\mathcal{H}$. This will make $O(\log |G|)$ calls to $\mathcal{A}_2^1(a \star x, x, L)$.

**Solving DLog in $G/\mathcal{H}$.** Notice that $\mathcal{A}_2^1$ is a (near) perfect CDH-solver, just in the group action corresponding to $G/\mathcal{H}$. Concretely, the group $G/\mathcal{H}$ acts on the set $X/\mathcal{H} := \{\mathcal{H} \star y : y \in X\}$ in the obvious way; the distinguished element of $X/\mathcal{H}$ is $\mathcal{H} \star x$. Our algorithm $\mathcal{A}_2^1$ gives a perfect CDH algorithm for this group action: we compute $\mathsf{CDH}(\mathcal{H} \star y, \mathcal{H} \star z)$ as $\mathcal{A}_2^1(y', z')$ for an arbitrary $y' \in \mathcal{H} \star y, z' \in \mathcal{H} \star z$.

We apply Galbraith et al. [GPSV18] to our CDH adversary for $(G/\mathcal{H}, X/\mathcal{H})$ to obtain a DLog adversary $\mathcal{B}(g\mathcal{H} \star x)$ which computes $g\mathcal{H}$. For completeness, we sketch the idea: Let $\mathbf{a}$ be a set of generators for $G/\mathcal{H}$. Since $G$ is abelian, we can write any $g$ as $\mathbf{a}^{\mathbf{v}}$ for some vector $\mathbf{v} \in \mathsf{Z}_{n_1} \times \cdots \times \mathsf{Z}_{n_k}$ where $n_i$ is the period of $a_i$. We assume the $n_i$ are fully reduced, so that the choice of $\mathbf{v}$ is unique.

Shor's algorithm is used in this step, and we note that Shor's algorithm will not necessarily work if $G$ is not abelian and our group action is not regular, which is why we need this restriction.

The $\mathsf{CDH}$ oracle allows, given $h \star (\mathcal{H} \star x)$, to compute $h^y \star (\mathcal{H} \star x)$ in $O(\log y)$ steps using repeated squaring. Given a DLog instance $g \star (\mathcal{H} \star x) = \mathbf{a}^{\mathbf{v}} \star (\mathcal{H} \star x)$, we define the function $(\mathbf{x}, y) \mapsto \mathbf{a}^{\mathbf{x}+y\mathbf{v}} \star (\mathcal{H} \star x)$, which can be computed using the CDH oracle. Then this function is periodic with period $(\mathbf{v}, -1)$. Running quantum period-finding therefore gives $\mathbf{v}$, which can be used to compute $h$.

**Solving DLog in $G$.** We now have an algorithm which solves, with overwhelming probability, DLog in $G/\mathcal{H}$. We now turn this into a full DLog adversary, which works as follows:

- Given $y = c \star x$, first apply the DLog adversary for $G/\mathcal{H}$, which outputs $c\mathcal{H}$.
- For each $a \in c\mathcal{H}$ (which is polynomial sized), test if $y = a \star x$. We output the unique such $a$.

Overall, assuming $q$ is small relative to $\log |G|$, the running time of the algorithm is dominated by the cost of running $\mathcal{A}_2^0$, namely $\tilde{O}(q^{-13})$ total calls to $\mathcal{A}$ and the group action operations.

**Remark 4.** *The dependence on $q$ in our reduction is not ideal. The cost of our attack, however, is dominated by the cost of determining the subgroup. Typically, however, we expect the possible small-order subgroups to be known, and for there to only be a very limited number of options. In this case, we expect the complexity of our attack could be drastically improved.*

### 3.1 Extending to Non-Regular Group Actions

The above assumed a regular group action, which captures all the cryptographic abelian group actions currently known. Here, we briefly sketch how to extend to an arbitrary abelian group action. The idea is that, within any ablelian group action, we can pull out a regular group action, and then apply the reduction above.

Concretely, we first consider restricting $(G, X, \star)$ to the orbit of $x$ under $G$, namely $G \star x$. Let $S \subseteq G$ the the set of $a$ that "stabilizes" $x$, namely $a \star x = x$. Then $S$ is a subgroup. Moreover, for any $y \in G \star x$, the set of $a$ that stabilize $y$ is also exactly $S$.

The first step is to compute the (representation of the) subgroup $S$. Let $f : G \to X$ be defined as $f(a) = a \star x$. Then $f$ is an instance of the abelian hidden subgroup problem with hidden subgroup exactly $S$. Therefore, we can find $S$ using Shor's quantum algorithm.

Then we can define the new group action $(G/S, G \star x, \star)$, which is a regular abelian group action. CDH in this group action is identical to CDH in the original group action, in that a CDH adversary for one is also a CDH adversary for the other. We can also solve DLog in $(G, X, \star)$ by solving DLog in $(G/S, G \star x, \star)$, and then lifting $a \in G/S$ to $a' = (a, g) \in G$ for an arbitrary $g \in S$.

The main challenge is that our CDH adversary $\mathcal{A}$ may not always output elements in $G \star x$, and it may be infeasible to tell when it outputs an element in $G \star x$ versus a different orbit. Nevertheless, the same reduction as used above applies, and the analysis can be extended straightforwardly but tediously to handle the fact that $\mathcal{A}$ may output elements in different orbits. The rough idea is that $L$ outputted by $\mathcal{A}_1$ may no longer be a subset of $\mathcal{H}_\delta \star x$, as it may have pieces from elements from different orbits. But $L \cap G \star x$ is still a subset of $\mathcal{H}_\delta \star x$, and similar statements hold for $\mathcal{A}_2^0, \mathcal{A}_2^1$ as well. This is enough to ensure that we obtain a near-perfect CDH algorithm on $(G/S)/\mathcal{H}$.

## 4 On the DDH and CDH (In)equivalence

A natural question to ask is whether we can show that the group action variants of CDH and DDH are equivalent. In traditional groups, there are a number of ways to argue that CDH and DDH are not equivalent, including by positing the existence of bilinear maps [BF01].

We show that for general group actions, the problems are also *not* equivalent. We do this by providing examples of group actions where "CDH" is hard and "DDH" is easy. In particular, we show that any group action where the group can be written as a non-trivial product group has the potential to be "CDH" hard but not "DDH" hard. This mirrors what we know classically and in the plain group setting, since there we can have groups that are CDH hard but not DDH hard. We state this formally in the following lemma.

**Lemma 20.** *Let $(G, X, \star)$ be an effective group action as defined in definition 42 such that no efficient adversary can solve the group action CDH problem (as defined in definition 7) over it. Then there exists a group action $(G', X', \bigstar)$ where no efficient adversary can solve the CDH problem, but there exists a PPT algorithm for solving the group action DDH problem (as defined in definition 8).*

*Proof.* Consider some extra group $\tilde{G}$. We can define a "group action" $\tilde{G} \times \tilde{G} \to \tilde{G}$ where the group action operation is simply group multiplication in $\tilde{G}$. Discrete log is trivial on this group since group inversion is efficient.

From our secure group action $(G, X, \star)$ and our insecure "group action," we construct another group action $(G', X', \bigstar)$ which we define as follows:

$$G' = G \times \tilde{G}$$

$$X' = X \times \tilde{G}$$

$$\bigstar : \left\{G \times \tilde{G}\right\} \times \left\{X \times \tilde{G}\right\} \to \left\{X \times \tilde{G}\right\}$$

For some $g \in G$, $x \in X$, $\tilde{g}_1, \tilde{g}_2 \in \tilde{G}$, we define the action as follows:

$$\{g, \tilde{g}_1\} \bigstar \{x, \tilde{g}_2\} = \{g \star x, \tilde{g}_1 \tilde{g}_2\}$$

Note that this definition meets all of the requirements of the group action. $G \times \tilde{G}$ is a (product) group, and all of the group action axioms hold.

We can immediately build a PPT distinguisher: given a DDH tuple $(x_1' = (x, \tilde{g}_1), g' \star x_1' = (g \star x_1, \tilde{g}\tilde{g}_1), x_2' = (x, \tilde{g}_2), g' \star x_2' = (g \star x_2, \tilde{g}\tilde{g}_2))$, we can perform the following check:

$$(\tilde{g}\tilde{g}_2)^{-1} (\tilde{g}\tilde{g}_1) = \tilde{g}_2^{-1}\tilde{g}_1$$

This immediately breaks the pseudorandomness of the group action, meaning that the group action DDH problem is not hard over $\left(\tilde{G}, \tilde{X}, \bigstar\right)$. However, any adversary that breaks the group action CDH problem on $\left(\tilde{G}, \tilde{X}, \bigstar\right)$ also breaks it on $(G, X, \star)$, which contradicts our assumption that the CDH problem is hard on this group action. □

In the above example, we used a product group. A nice question is as follows: what happens if we assume that the group must be, say, prime-order cyclic? This case is much harder to show interesting results since we don't have efficiently computable bilinear pairings as in the standard group setting.

## 5 A Generic Group Action Framework

In this section, we define a generic group action framework. We create two models: one for classical queries, and one which allows quantum queries. Our framework is based on the generic group framework of Shoup [Sho97]. We borrow from Shoup's description in our own explanation below.

Let $G$ be a group of order $n$, let $X$ be a set that is representable by bit strings of length $m$, and let $(G, X, \star)$ be a group action. We define additional sets $S_G$ and $S_X$ such that they have cardinality of at least $n$ and $2^m$, respectively. We define *encoding functions* of $\sigma_G$ and $\sigma_X$ on $S_G$ and $S_X$, respectively, to be injective maps of the form $\sigma_G : G \to S_G$ and $\sigma_X : X \to S_X$.

A generic algorithm $\mathcal{A}$ for $(G, X, \star)$ on $(S_G, S_X)$ is a probabilistic algorithm that behaves in the following way. It takes as input two *encoding lists* $(\sigma_G(g_1), ..., \sigma_G(g_k))$ and $(\sigma_X(x_1), ..., \sigma_X(x_{k'}))$ where each $g_i \in G$ and $x_i \in X$ and where $\sigma_G$ and $\sigma_X$ are encoding functions of $G$ on $S_G$ and $X$ on $S_X$, respectively. As the algorithm executes, it may consult two oracles, $\mathcal{O}_G$ and $\mathcal{O}_X$.

The oracle $\mathcal{O}_G$ takes as input two strings $y, z$ representing group elements and a sign "+" or "−", computes $\sigma_G\left(\sigma_G^{-1}(y) \pm \sigma_G^{-1}(z)\right)$. The oracle $\mathcal{O}_X$ takes as input a string $y$ representing a group element and string $z$ representing a set element, and computes $\sigma_X\left(\sigma_G^{-1}(y) \star \sigma_X^{-1}(z)\right)$. As is typical in the literature, we can force all queries to be on either the initial encoding lists or the results of previous queries by making the string length $m$ very long. We typically measure the running time of the algorithm by the number of oracle queries.

We can also extend the generic group action model to the quantum setting, where we allow *quantum* queries to the oracles. We model quantum queries in the usual way: $\mathcal{O}_G \sum_{y,z,\pm,w} \alpha_{y,z,\pm,w}|y, z, \pm, w\rangle = \sum_{y,z,\pm,w} \alpha_{y,z,\pm,w}|y, z, \pm, w \oplus \mathcal{O}_G(y, z, \pm)\rangle$ and $\mathcal{O}_X \sum_{y,z,w} \alpha_{y,z,w}|y, z, w\rangle = \sum_{y,z,w} \alpha_{y,z,w}|y, z, w \oplus \mathcal{O}_X(y, z)\rangle$.

## 6 On REGAs

Our reductions showing the equivalence of group action DLog and CDH unfortunately only hold for EGAs and not for REGAs. In their work showing an equivalence for a perfect oracle [GPSV18], Galbraith *et al.* suggest that applying the BKZ algorithm [SE94] or other lattice reduction techniques can be used to complete the reduction. In this section, we formalize this idea with a number of resuts on the relationship between REGAs and lattices, and, in particular, focus on the 1D-SIS problem, which is a lattice problem that is equivalent to the standard form of LWE modulo PPT reductions. Due to space constraints, we only state the relevant lemmas in this section and defer proofs to the appendix. We present the full, unabridged version of this section as well as the formal definitions related to REGAs in full in the appendix.[6]

In this section, we will rely on the fact that, using a generalization of Shor's algorithm [CM01], we can (quantumly) efficiently compute the isomorphism between any abelian group $G$ and a product group over groups of the integers

$$G \cong \mathbb{Z}_1 \times ... \times \mathbb{Z}_m.$$

We additionally note that most of our results here only hold for *regular* group actions. We do not consider this a major drawback since all popular REGAs (e.g. CSIDH and its derivatives) are regular REGAs.

[6] While this results in some duplication, we find that it makes for much easier reading.

**A "1D-SIS Oracle" Completes the DLog/CDH Reduction for REGAs.** We begin by formalizing the argument from Galbraith *et al.* [GPSV18] that efficient lattice reductions could be used to show the discrete log/CDH equivalence of REGAs. While doing this in full would involve completely replicating our earlier proof, we simply point out at which stages using a REGA makes a difference and how we can handle these points.

We first need to ensure that we can randomly sample elements from a REGA. We define a notion of "sampleable REGA" capturing this:

**Definition 21.** ***Sampleable REGA****: Let $(G, X, \star)$ be a REGA with group element vector $\mathbf{g} = (\mathbf{g}_1, ..., \mathbf{g}_m)$ for some $m$. We say that such a REGA is sampleable if there exists an efficient way to sample a vector $\mathbf{b} \in \{-\gamma, \gamma\}^m$ for some polynomial $\gamma$ such that the vector $\mathbf{r} = \sum_{i=1}^{m} \mathbf{b}_i \mathbf{g}_i$ is distributed statistically close to uniform over $G$.*

This requirement essentially just requires that some form of the leftover hash lemma applies over the group with the action-computable elements as the "base." We note that many cryptosystems build on REGAs (i.e. those using CSIDH) implicitly make this assumption. We need this to rule out cases where the elements of $\mathbf{g}$ are too clustered: for instance, if $G$ is $\mathsf{Z}_p$ and all of the $\mathbf{g}_i$ are small integers, we will not be able to effectively compute the group action on randomly distributed group elements. Next, we define a specialized problem we call "REGA-SIS." Note that this is not a standard problem because, among other things, the $\mathbf{g}_i$ distribution comes from the definition of the REGA.

**Definition 22.** ***REGA-SIS****: Let $(G, X, \star)$ be a REGA with group element vector $\mathbf{g} = (\mathbf{g}_1, ..., \mathbf{g}_m)$ for some $m$. We define $SIS_{REGA,\beta}$ in the following way: given a random element $h \leftarrow G$, the problem is to find some vector $\mathbf{u} \in [-\beta, \beta]^m$ such that $h = \sum_{i=1}^{m} \mathbf{u}_i \mathbf{g}_i$.*

This problem is parameterized by the REGA and, in particular, by both the group and the computable elements. Furthermore, for $G = \mathsf{Z}_q$ and when each coefficient of $\mathbf{g}$ is distributed uniformly at random, REGA-SIS is exactly the 1D-inhomogeneous SIS (1D-ISIS) problem (which is reducible to standard 1D-SIS with a slight loss in parameters, and 1D-SIS itself is again reducible to and from standard LWE, for appropriate parameter settings). So this problem can be viewed as a slightly unnatural generalization of SIS. We can now state our core lemma on REGAs.

**Lemma 23.** *Consider any efficiently sampleable REGA as defined in definition 32. Then any adversary that can solve the CDH problem on the REGA with advantage $\epsilon_1$ and the $SIS_{REGA,\beta}$ problem for the same REGA and some polynomial $\beta$ with advantage $\epsilon_2$ can be used to solve the discrete log problem on the same REGA with advantage $\epsilon_1\epsilon_2$.*

**Discrete Log on REGAs and 1D-SIS.** Recall from Definition 43 that a REGA is a group action $(G, X, \star)$ where the action is only computable on a set of group elements defined by a vector $\mathbf{g} = (g_1, \ldots, g_n)$. Suppose that $G$ is an abelian group. We claim that if these group elements are distributed randomly, then any adversary that can solve discrete log on the REGA can be used to solve the 1D-SIS problem for certain parameter settings (which are all reducible to some form of standard LWE). The analysis of most practical REGAs (e.g. CSIDH) assume follow this convention, so this is not an unreasonable assumption to make. We formalize this with the following lemma.

**Lemma 24.** *Let $q$ and $m$ be integers such that $m \geq 3 \log q$. Let $\mathcal{A}$ be an adversary that can solve the group action DLog problem on regular REGAs of the form $(Z_q, X, \star)$ where the vector of group elements $\mathbf{g} = (g_1, ..., g_m)$ is $m$ elements long and distributed uniformly at random with advantage $\epsilon$. Then $\mathcal{A}$ can be used to solve the $1D\text{-}SIS_{m,q,\beta}$ problem for some polynomial $\beta$ with advantage $\epsilon$.*

**CDH on REGAs.** We above showed that an adversary that can solve discrete log on a REGA can solve a variant of the SIS problem, and that any adversary that can solve this SIS variant can also be used to complete the CDH/DLog reduction. Can we tie all of this together to get an unconditional CDH to DLog reduction to work for REGAs?

We give some mild evidence in this direction. We can show that any *generic* adversary that makes only *classical* queries to a generic group action oracle (that may still be able to perform quantum computations) can be used to solve the REGA-SIS problem we defined above in Definition 33. We can then use this to complete the CDH to DLog reduction for generic, classically-querying adversaries. Of course, classically we can prove CDH and DLog are unconditionally hard (this follows from the unconditional hardness of these problems in plain groups), and therefore equivalent. But phrasing the equivalence as a reduction suggests a possible starting point for a quantum equivalence

**Lemma 25.** *Consider some regular, abelian, and efficiently sampleable REGA* $(G, X, \star)$ *with computable elements* $\mathbf{g} = (\mathbf{g}_1, ..., \mathbf{g}_m)$*. Suppose there exists a* generic *adversary* making only classical group and group action queries *that can solve the GA-CDH problem on this REGA with advantage* $\epsilon$*. Then there exists an adversary that can solve REGA-SIS for some polynomial parameter* $\beta$ *with advantage* $\epsilon/2$*.*

**Discussion.** We have shown three core results on REGAs (stated informally): an adversary for our REGA-SIS problem would complete our CDH/DLog reduction for REGAs, an adversary for DLog on REGAs solves this REGA-SIS problem, and a generic adversary that only makes classical queries that can solve CDH on REGAs can be used to solve REGA-SIS as well. All together, these seemingly tightly bind CDH and DLog on a REGA to a SIS-like problem that appears to be vulnerable to lattice-based cryptanalysis [GPSV18]. We therefore provide some evidence for a quantum DLog-CDH equivalence on REGAs.

# 7 Hidden Subgroup Problems and GAs

In this section, we discuss some similarities between different kinds of hidden subgroup problems (HSPs) and solving group actions. We particularly focus on the *generalized* dihedral group. We note that, among other things, formalizing a connection between group actions and these kinds of problem would allow us to potentially tie two of the most popular forms of post-quantum cryptosystems (lattices and isogenies) together. Once again, due to space constraints, we just state lemmas here and defer the full presentation to the appendix.

**The Generalized Dihedral Hidden Subgroup Problem.** We begin by defining the *generalized* dihedral group.

**Definition 26.** ***Generalized Dihedral Group****: Let* $A$ *be an abelian group. The* generalized dihedral group *on* $A$*, denoted* $D_A$*, is the group defined by* $\mathbb{Z}_2 \ltimes A$*.*

When $A \cong \mathbb{Z}_n$, we get back the standard notion of the dihedral group on $2n$ elements. The dihedral group has a number of nice geometric explanations and properties, but we defer those to others [KLG06]. We next define the general dihedral hidden subgroup problem. However, rather than defining this problem in its traditional sense, we will use an equivalent formulation known as the *abelian hidden shift problem*. These problems are well known to be equivalent [CVD05].

**Definition 27.** ***Abelian Hidden Shift Problem (equivalent to GDHSP)****: Consider some functions* $f, g$ *such that, for some* $c \in A$ *and for all* $b \in \mathbb{Z}_n$*,* $f(b) = g(b + c)$*. We also require*

*that each of the $||A||$ output values of $f$ and $g$ are also distinct. We say that an algorithm solves the abelian hidden shift problem if, given descriptions of $f$ and $g$, it outputs $c$ (which reveals the subgroup in the generalized dihedral hidden subgroup version of the problem).*

The dihedral hidden subgroup problem has strong connections to lattice problems [Reg02], in that if an efficient algorithm for the DHS problem that uses a special type of "coset sampling" exists, then an efficient algorithm for the LWE problem exists as well. The best known algorithms for solving the DHS problem are subexponential and based on Kuperberg's algorithm [Kup05, Reg04, Kup13].

**An Algorithm for the AHSP Breaks Regular, Abelian Group Actions.** We first show a relatively straightforward result: any algorithm that can solve the abelian hidden shift problem can be used to solve DLog on a regular, abelian group action. This is essentially already folklore since there have been many instances (starting with [CJS14]) using Kuperberg's algorithm or related principles to build attacks against isogenies that can be modelled as EGAs.

**Lemma 28.** *Let $(G, X, \star)$ denote a regular, abelian group action. Suppose there exists a PPT algorithm $\mathcal{A}$ for solving the abelian hidden shift problem on $A$ with probability $\epsilon$. Then there exists for solving the GA-DLog problem on $(G, X, \star)$ with probability $\epsilon$.*

**Using Group Action Algorithms to Solve the AHSP.** What about the other direction? Can we show that an adversary that can break DLog on a group action can solve the AHSP? Unfortunately, this seems difficult: because the AHSP is described so generally–the functions $f$ and $g$ can be anything as long as the functions are injective–so it seems difficult or impossible to prove this for any non-generic algorithm.

But what about generic algorithms? Could we prove that the AHSP is equivalent to generically solving DLog over group actions? This seems like it might be plausible. The most interesting result would show equivalence in a generic group action model with quantum queries. While this may be attainable, unfortunately we do not know how to achieve this result. However, we can show that an adversary that can generically solve group action DLog with classical queries can be used to solve the AHSP, which is seemingly a step in the right direction. We formalize this result below.

**Lemma 29.** *Let $(G, X, \star)$ be an abelian, regular group action (EGA). Suppose there exists a generic adversary $\mathcal{A}$ that breaks the group action DLog problem (as defined in definition 6) with advantage $\epsilon$ on this group action. Then there exists an algorithm that solves that AHSP on $G$ with advantage $\epsilon$.*

**Discussion.** Unfortunately, it seems difficult to show a full quantum equivalence between the generalized dihedral hidden subgroup problem and solving DLog on a generic group action. The challenge comes from the fact that it is difficult quantumly to "remember" an adversary's query for later use in the simulation. One possible direction is to use compressed oracles [Zha19], which offer some ability to record quantum queries. However, it appears challenging to adapt the compressed oracle framework to highly structured oracles such as generic group actions. Nevertheless, we close this section with the following conjecture, which we think is very interesting future work:

**Conjecture 30.** *The generalized dihedral hidden subgroup problem on an abelian group $A$ is equivalent to the group action discrete logarithm problem on a regular, abelian group action $(A, X, \star)$ in a quantum generic model.*

# 8 Group Actions Beyond Isogenies?

The main candidate group actions are those from isogenies. However, just as cryptography from groups transitioned from focusing primarily on multiplicative groups of finite fields to considering additional groups such as elliptic curves, we anticipate cryptographic group actions to be broadly more applicable. In such a case, our reduction will be quite useful, as it means that quantum cryptanalytic effort can focus on increasing confidence in DLog, which would immediately translate to confidence in CDH.

Next, we discuss how group actions naturally arise implicitly in several other proposals, and what implications (if any) our reduction has in these settings.

## 8.1 Stickel's Key Exchange Protocol

In [Sti05], Stickel proposed a new paradigm for key exchange. We can define Stickel's protocol in a general form as follows:

**Definition 31.** ***Stickel's Key Exchange Protocol****: let $G$ be a nonabelian group, and let $g, h \in G$ be public elements with order $N$ and $M$, respectively. Let $u \in G$ be an additional element. The key exchange proceeds as follows:*

- *Alice picks $a_1 \in \mathbb{Z}_N$ and $a_2 \in \mathbb{Z}_M$ uniformly at random, computes $s_a = g^{a_1} u h^{a_2}$ and sends $s_a$ to Bob.*
- *Bob picks $b_1 \in \mathbb{Z}_N$ and $b_2 \in \mathbb{Z}_M$ uniformly at random, computes $s_b = g^{b_1} u h^{b_2}$ and sends $s_b$ to Alice.*
- *Alice receives $s_b$ and computes $K = g^{a_1} s_b h^{a_2} = g^{a_1+b_1} u h^{a_2+b_2}$.*
- *Bob receives $s_a$ and computes $K = g^{b_1} s_a h^{b_2} = g^{a_1+b_1} u h^{a_2+b_2}$.*

*Both parties use $K$ as their shared key.*

While it was not observed in the original line of work, we note that Stickel's protocol can, in fact, be viewed as an abelian group action. Let the set $X$ be defined to be all possible group elements of the form $g^i u h^j$ for $i \in [0, N-1]$ and $j \in [0, M-1]$. Let $G'$ be defined as the group $G' = \mathbb{Z}_N \times \mathbb{Z}_M$. We can define the operation $\star : G' \times X \to X$ in the following way:

$$(a, b) \star g^x u h^y = g^{a+x} u h^{b+y}$$

Then $(G', X, \star)$ is an abelian group action, even though $G$ itself is not abelian. Moreover, Stickel's protocol can be seen as an instantiation of the most primitive form of key exchange from an abelian group action: to complete the key exchange, Alice and Bob sample some $g'_a, g'_b \in G'$, compute $g'_a \star x$ and $g'_b \star x$ for "base set element" $x$ (in this case, the set element corresponding to the identity element in the group), and then compute the final key as $g'_a \star (g'_b \star x) = g'_b \star (g'_a \star x)$.

**Instantiation.** Stickel instantiates the above protocol with matrix groups, where $g, h, u$ are matrices. Unfortunately, Shpilrain [Shp08] used a clever re-linearization trick to compute the shared key for this instantiation.

**Implications of our Reduction.** When viewing Stickel's protocol as a group action, Shpilrain's attack exactly solves the CDH problem. However, the attack does not explicitly recover the user's secret keys. Our reduction shows how to use Shpilrain's attack to do exactly this. The result is that we obtain a quantum algorithm which computes $a_1, a_2$ from $g^{a_1}uh^{a_2}$, where $g, h, u$ are matrices.

Though Stickel's protocol is insecure for matrix groups, it may be secure in other non-abelian groups. Our reduction proves the security of the key agreement protocol based on the hardness of computing Alice or Bob's individual keys.

### 8.2 Other Protocols: Non-abelian Group Actions

Some non-commutative key exchange protocols are based (implicitly or explicitly) on the *decomposition search problem* (DSP), formally defined by [SU05b]: given two subgroups $A, B \subseteq G$ of a non-abelian group $G$ and elements $u, w \in G$, find $a \in A, b \in B$ such that $aub = w$. Examples of such protocols include [KLC+00, SU05b, SU05a]. The precise instantiations of these protocols are considered insecure [BKT14], but the generic frameworks may still be useful. All these protocols correspond to a group action: $A \times B$ acts on $G$ by $(a, b) * u = aub$. Then DSP is exactly DLog in this group action. However, because $A, B$ are in general non-abelian, the group action is non-abelian.

In these protocols, Alice and Bob send $aub$ and $a'ub'$ respectively. The decomposition search problem/DLog corresponds to computing their individual secrets. The actual security of the protocols are not known to be reducible to this problem, analogous to standard groups and isogenies. Unfortunately our reduction says nothing about this setting since the group action is non-abelian.

## 9 On REGAs

Our reductions showing the equivalence of group action DLog and CDH unfortunately only hold for EGAs and not for REGAs. In their work showing an equivalence for a perfect oracle [GPSV18], Galbraith *et al.* suggest that applying the BKZ algorithm [SE94] or other lattice reduction techniques can be used to complete the reduction. In this section, we formalize this idea with a number of resuts on the relationship between REGAs and lattices, and, in particular, focus on the 1D-SIS problem, which is a lattice problem that is equivalent to the standard form of LWE modulo PPT reductions.

In this section, we will rely on the fact that, using a generalization of Shor's algorithm [CM01], we can (quantumly) efficiently compute the isomorphism between any abelian group $G$ and a product group over groups of the integers

$$G \cong \mathbb{Z}_1 \times ... \times \mathbb{Z}_m.$$

We additionally note that most of our results here only hold for *regular* group actions. We do not consider this a major drawback since all popular REGAs (e.g. CSIDH and its derivatives) are regular REGAs.

### 9.1 A "1D-SIS Oracle" Completes the DLog/CDH Reduction for REGAs

We begin by formalizing the argument from Galbraith *et al.* [GPSV18] that efficient lattice reductions could be used to show the discrete log/CDH equivalence of REGAs. While doing this in full would involve completely replicating our earlier proof, we simply point out at which stages using a REGA makes a difference and how we can handle these points.

We first need to ensure that we can randomly sample elements from a REGA. We define a notion of "sampleable REGA" below that enables us to sample random elements.

**Definition 32.** ***Sampleable REGA****: Let $(G, X, \star)$ be a REGA with group element vector $\mathbf{g} = (\mathbf{g}_1, ..., \mathbf{g}_m)$ for some $m$. We say that such a REGA is sampleable if there exists an efficient way to sample a vector $\mathbf{b} \in \{-\gamma, \gamma\}^m$ for some polynomial $\gamma$ such that the vector $\mathbf{r} = \sum_{i=1}^{m} \mathbf{b}_i \mathbf{g}_i$ is distributed statistically close to uniform over $G$.*

This requirement essentially just requires that some form of the leftover hash lemma applies over the group with the action-computable elements as the "base." We note that many cryptosystems build on REGAs (i.e. those using CSIDH) implicitly make this assumption. Next, we define a specialized problem we call "REGA-SIS." Note that this is not a standard problem since the elements are not parameterized distributionally.

**Definition 33.** ***REGA-SIS****: Let $(G, X, \star)$ be a REGA with group element vector $\mathbf{g} = (\mathbf{g}_1, ..., \mathbf{g}_m)$ for some $m$. We define $SIS_{REGA,\beta}$ in the following way: given a random element $h \leftarrow G$, the problem is to find some vector $\mathbf{u} \in [-\beta, \beta]^m$ such that $h = \sum_{i=1}^{m} \mathbf{u}_i \mathbf{g}_i$.*

This problem is parameterized by the REGA and, in particular, by both the group and the computable elements. Furthermore, for $G = \mathsf{Z}_q$ and when each coefficient of $\mathbf{g}$ is distributed uniformly at random, REGA-SIS is exactly the 1D-inhomogeneous SIS (1D-ISIS) problem (which is reducible to standard 1D-SIS with a slight loss in parameters, and 1D-SIS itself is again reducible to and from standard LWE, for appropriate parameter settings). So this problem can be viewed as a slightly unnatural generalization of SIS.

We can now state our core lemma on REGAs.

**Lemma 34.** *Consider any efficiently sampleable REGA as defined in definition 32. Then any adversary that can solve the CDH problem on the REGA with advantage $\epsilon_1$ and the $SIS_{REGA,\beta}$ problem for the same REGA and some polynomial $\beta$ with advantage $\epsilon_2$ can be used to solve the discrete log problem on the same REGA with advantage $\epsilon_1\epsilon_2$.*

*Proof.* Rather than recreate our entire previous proof, we mention parts of the reduction that would be affected by the use of a REGA and point out how the reduction can be modified to still work. There are two points we need to address where we use properties of an EGA that are not also properties of a REGA:

**Sampling random elements.** We need to ensure that we can sample (statistically close to) uniform random elements in our reduction. Because we are using a sampleable REGA, we know that this can be done efficiently.

**Solving the final Dlog.** Like the original work of Galbraith *et al.* [GPSV18], assuming we have efficient sampleability, our algorithm will work for REGAs until the very last stage. At this point, we will be given some $h \in G$ such that $h$ is the correct discrete logarithm. However, we will not be given a description of how to "reach" $h$ using the vector of group elements $\mathbf{g}$, and thus we will have not solved Dlog over the REGA; it is required that we output an efficiently computable Dlog solution in order to solve the Dlog problem.

To rectify this, suppose we are given some $h \in G$ and wish to write it in the form $h = \sum_{i=1}^{m} \mathbf{u}_i \mathbf{g}_i$ for some $\mathbf{u} \in \mathbb{Z}_q^m$ where each $\mathbf{u}_i \in [-\beta', \beta']$.[7] We first use the REGA sampling algorithm to generate a random $h' \in G$ for which we know the decomposition in terms of $\mathbf{g}$, and then send $hh'$ to the $SIS_{REGA,\beta}$ adversary. The adversary responds with some $\mathbf{u}' \in \mathbb{Z}^m$ with each $||\mathbf{u}_i|| \leq \beta$. We can then use the decomposition from the sampling algorithm and $\mathbf{u}'$ to determine an appropriate solution for $\mathbf{u}$ with coefficients bounded by $\beta + \beta'$, giving us the desired solution. □

[7] We intentionally leave $\beta$ and $\beta'$ here "polynomial" rather than quantify them due so we do not have to quantify the effectiveness of the sampling algorithm.

## 9.2 Discrete Log on REGAs and 1D-SIS

Recall from definition 43 that a REGA is a group action $(G, X, \star)$ where the action is only computable on a set of group elements defined by a vector $\mathbf{g} = (g_1, \ldots, g_n)$. Suppose that $G$ is an abelian group with order $q$. We claim that if these group elements are distributed randomly when we map the group to $\mathsf{Z}_q$, then any adversary that can solve discrete log on the REGA can be used to solve the 1D-SIS problem for certain parameter settings (which can all be reducible to some form of standard LWE). We note that most practical REGAs (e.g. CSIDH) assume that these elements are distributed randomly for their applications without proof, so this is not an unreasonable assumption to make.

We formalize this below in the following lemma.

**Lemma 35.** *Let $q$ and $m$ be integers such that $m \geq 3 \log q$. Let $\mathcal{A}$ be an adversary that can solve the group action DLog problem on regular REGAs of the form $(Z_q, X, \star)$ where the vector of group elements $\mathbf{g} = (g_1, ..., g_m)$ is $m$ elements long and distributed uniformly at random with advantage $\epsilon$. Then $\mathcal{A}$ can be used to solve the $1D\text{-}SIS_{m,q,\beta}$ problem for some polynomial $\beta$ with advantage $\epsilon$.*

*Proof.* Suppose we are given a 1D-SIS challenge instance consisting of uniformly random vector $\mathbf{v} \in \mathbb{Z}_q^m$. We must use $\mathcal{A}$ to compute some vector $\mathbf{u} \in Z^m$ such that $\mathbf{u} \cdot \mathbf{v} = 0 \mod q$ and $||\mathbf{u}|| \leq \beta$. We set $G = \mathbb{Z}_q$ and $\mathbf{g} = (\mathbf{v}_1, ..., \mathbf{v}_m)$.

To create a DLog challenge, we sample a random binary string $\mathbf{b} \in \{0,1\}^m$ and compute $h = \sum_{i=1}^m \mathbf{b}_i * \mathbf{v}_i$. By lemma 2 (a corollary of the leftover hash lemma), we know that $h$ is distributed uniformly at random in $G$. Note that we can also efficiently compute $h \star x$ for any set element $x \in X$ because we can decompose $h$ using the elements of $\mathbf{g}$.

So, suppose we sample some $x \leftarrow X$ uniformly at random, compute $y = h \star x$, and then send the tuple $((G, X, \star, \mathbf{g}), x, y)$ to the adversary $\mathcal{A}$. In order to solve discrete log on the REGA, $\mathcal{A}$ must produce some vector $\mathbf{u}' \in \mathbb{Z}^m$ such that for $h' = \sum_{i=1}^m \mathbf{u}_i * \mathbf{g}_i$, $h' \star x = h \star x$. Since we are assuming our group action is regular, we have $h = h'$ by definition, and thus $(\mathbf{u}' - \mathbf{u}) \cdot \mathbf{v} = 0 \mod q$. Since $m \geq 3 \log q$, we can again apply lemma 2 to argue that there will be many binary subset sums evaluating to $h$, and thus the adversary is unlikely to pick $\mathbf{u} = \mathbf{u}'$. Moreover, since $\mathcal{A}$ is polynomially bounded, it must be only able to compute the group action a polynomial number of times, and thus $\mathbf{u}'$ must have polynomially bounded coefficients.[8] Therefore $\mathbf{u} - \mathbf{u}'$ is a solution to the SIS problem for polynomial $\beta$, which completes the reduction. Note that our reduction is direct, so the advantage $\epsilon$ is preserved. □

## 9.3 CDH on REGAs

We have previously showed that an adversary that can solve discrete log on a REGA can solve a variant of the SIS problem, and that any adversary that can solve this SIS variant can also be used to complete the CDH/DLog reduction. Can we tie all of this together and somehow get a CDH to DLog reduction to work for REGAs? This seems difficult because, unlike a DLog adversary that must give us the raw "multipliers for the group elements," a CDH adversary only has to provide set elements. On the other hand, if we force an adversary to be generic–and thus we can learn the queries and "multipliers" through queries to the generic oracle–could we possibly show a reduction?

Unfortunately, even in the case of a generic algorithm, the answer is currently no for us, but we can prove a sort of consolation prize. Namely, suppose we assume a *classical* generic group action (we aren't allowed to make superposition queries). Then we can show that any *generic* adversary

[8] If $\mathcal{A}$ could somehow compute the group action more efficiently–i.e. compute "powers" of elements of $\mathbf{g}$ in some way more efficient than brute force–then the computational model would be outside the scope of a REGA.

that makes classical queries to the group action oracle (that may still be able to perform quantum computations) can be used to solve the REGA-SIS problem we defined above in definition 33. We can then use this to complete the CDH to DLog reduction for generic, classically-querying adversaries. Of course, classically we can prove CDH and DLog are unconditionally hard (this follows from the unconditional hardness of these problems in plain generic groups), and therefore equivalent. But phrasing the equivalence as a reduction suggests a possible starting point for a quantum equivalence.

**Lemma 36.** *Consider some regular, abelian, and efficiently sampleable REGA* $(G, X, \star)$ *with computable elements* $\mathbf{g} = (\mathbf{g}_1, ..., \mathbf{g}_m)$*. Suppose there exists a* generic *adversary* making only classical group and group action queries *that can solve the GA-CDH problem on this REGA with advantage* $\epsilon$*. Then there exists an adversary that can solve REGA-SIS for some polynomial parameter* $\beta$ *with advantage* $\epsilon/2$*.*

*Proof.* In this proof, we will need to simulate a generic group action oracle (with classical queries). We begin by explaining this. We will assume the existence of two random oracles $RO_1, RO_2 : G \to \{0,1\}^\ell$ such that $2^\ell > ||X||^2$. The reason for the length is to ensure that, with all but negligible probability, there are no collisions in the random oracle.[9] We will simulate terms in a way that should be very familiar for a reader who is familiar with generic group models.

**Generic Simulation.** In order to simulate the group, we will maintain a list of queries of the form $(g_i, \tilde{g}_i)$ for $\tilde{g}_i \in \{0,1\}^\ell$. Initially, we will add $(g_I, RO_1(g_I))$ to the table, and $RO_1(g_I)$ will represent the identity element. Then, for each $\mathbf{g}_i \in \mathbf{g}$, we add the tuples $(\mathbf{g}_i, RO_1(\mathbf{g}_i))$ and $\left(\mathbf{g}_i^{-1}, RO_1\left(\mathbf{g}_i^{-1}\right)\right)$ to the table as well, giving access to the computable elements.

When an adversary makes a query of the form $\left(\tilde{h}, \tilde{h}', +\right)$ or $\left(\tilde{h}, \tilde{h}', -\right)$, we will look up $x$ in our table and find the entries $\left(h, \tilde{h}\right)$ and $\left(h', \tilde{h}'\right)$. We know that such an entry must exist because our algorithm is generic. Then we add the appropriately signed entry $(h \pm h', RO_1(h \pm h'))$ to our table and return $RO_1(h \pm h')$.

In order to simulate the set elements and group action computation queries, we will maintain a list of queries of the form $(g_i, \tilde{x}_i)$. Initially, we will add $(g_I, RO_2(g_I))$ to the table, and $RO_2(g_I)$ will be our "base point." When an adversary makes a query of the form $\left(\tilde{h}, \tilde{x}\right)$ we will first look up $\tilde{h}$ in our table of group elements and recover the tuple $\left(h, \tilde{h}\right)$. Then we will look up $\tilde{x}$ in our set table and recover the tuple $(g, \tilde{x})$. We will then add $(hg, RO_1(hg))$ to our group table and $(hg, RO_2(hg))$ to our set table, and return $RO_2(hg)$. Since we are using a REGA, our oracle should reject any group action computation queries that do not involve one of the elements $\mathbf{g}_i$.

**CDH Proof.** Now, suppose we are given a REGA-SIS instance consisting of a vector $\mathbf{g} \in G^m$ and a target element $g^* \in G$. We set up our simulation as discussed before using a description of $G$ and $\mathbf{g}$. We use the efficient sampleability of the REGA to generate some vector $\mathbf{b} \in \mathbb{Z}^m$ and some element $h = \sum_{i=1}^m \mathbf{b}_i \mathbf{g}_i$ which is distributed uniformly at random. We then sample some random $r \leftarrow G$ in a similar way to $h$ such that we know the decomposition in terms of $\mathbf{g}$.

Define the group elements $y_0 = h$, $y_1 = h + g^* + r$, and note that the tuple $(x_0, y_0 \star x_0, y_1 \star x_0)$ is distributed uniformly at random from the set elements (as required by the statement of the CDH adversary). Moreover, note that $y_0 + y_1 = g^* + r$. Finally, note that we do not know the

[9] Note that the set size and group order are the same in a regular group action.

decomposition of $g^*$ into proper sums of $\mathbf{g}_i$ terms, but we get around this because the random oracle hides whether or not we have knowledge of the decomposition or not.

Suppose we give a generic adversary that only makes classical queries the tuple $(x_0, y_0 \star x_0, y_1 \star x_0)$ and access to the generic group action oracle. The adversary must output $(y_0 + y_1) \star x_0 = (g^* + r) \star x_0$. Since set elements cannot be combined in any way, to do this, the adversary must star wtih one set element in the tuple $(x_0, y_0 \star x_0, y_1 \star x_0)$ and use group action queries to reach the desired output. A (efficiently determinable) subset of the adversary's queries $q_1, ..., q_Q$ must take one of the following forms:

$$\sum_{i=1}^{Q} q_i = g^* + r \text{ or } \sum_{i=1}^{Q} q_i = g^* + r - h \text{ or } \sum_{i=1}^{Q} q_i = h$$

where the exact form depends on the set element from the tuple the adversary decides to use to generate the final element. Since each of the $q_i$ terms must be $\pm\mathbf{g}_i$ for some $\mathbf{g}_i$, we know that the first two forms can be used to immediately generate REGA-SIS solutions for polynomial $\beta$ since we have the knowledge of the decomposition of $r$. So assuming the adversary uses $x_0$ or $x_1$ to generate its challenge query, it solves REGA-SIS.

Unfortunately, the adversary could choose to use $x_2$. However, this has an easy fix: we can just flip the description of $x_1$ and $x_2$ randomly and then feed this modified tuple to the adversary. Since both $x_1$ and $x_2$ are distributed uniformly at random, the adversary cannot determine whether we have performed the switch or not, and thus only reduces the overall advantage by a factor of 2, giving us our final result. □

### 9.4 Discussion

We have shown three core results on REGAs (stated informally): an adversary for our REGA-SIS problem enables us to complete our CDH/DLog reduction for REGAs, an adversary for DLog on REGAs can be used to solve this REGA-SIS problem, and a generic adversary that only makes classical queries that can solve CDH on REGAs can be used to solve REGA-SIS as well. All together, these seemingly tightly bind CDH and DLog on a REGA to a SIS-like problem that appears to be vulnerable to lattice-based cryptanalysis [GPSV18]. We therefore provide some evidence for a quantum DLog-CDH equivalence on REGAs.

## 10 Hidden Subgroup Problems and GAs

In this section, we discuss some similarities between different kinds of hidden subgroup problems (HSPs) and solving group actions. We particularly focus on the *generalized* dihedral group. We note that, among other things, formalizing a connection between group actions and these kinds of problem would allow us to potentially tie two of the most popular forms of post-quantum cryptosystems (lattices and isogenies) together.

### 10.1 The Generalized Dihedral Hidden Subgroup Problem

We begin by defining the *generalized* dihedral group.

**Definition 37.** ***Generalized Dihedral Group****: Let $A$ be an abelian group. The* generalized dihedral group *on $A$, which we denote $D_A$ is the group defined by $\mathbb{Z}_2 \ltimes A$.*

When $A \cong \mathbb{Z}_n$, we get back the standard notion of the dihedral group on $2n$ elements. The dihedral group has a number of nice geometric explanations and properties, but we defer those to others here [KLG06]. We next define the general dihedral hidden subgroup problem. However, rather than defining this problem in its traditional sense, we will use an equivalent formulation known as the *abelian hidden shift problem*. These problems are well known to be equivalent [CVD05].

**Definition 38.** ***Abelian Hidden Shift Problem (equivalent to GDHSP)****: Consider some functions $f, g$ such that, for some $c \in A$ and for all $b \in \mathbb{Z}_n$, $f(b) = g(b + c)$. We also require that each of the $||A||$ output values of $f$ and $g$ are also distinct. We say that an algorithm solves the abelian hidden shift problem if, given descriptions of $f$ and $g$, it outputs $c$ (which reveals the subgroup in the generalized dihedral hidden subgroup version of the problem).*

The dihedral hidden subgroup problem has strong connections to lattice problems [Reg02], in that if an efficient algorithm for the DHS problem that uses a special type of "coset sampling" exists, then an efficient algorithm for the LWE problem exists as well. The best known algorithms for solving the DHS problem are subexponential and based on Kuperberg's algorithm [Kup05, Reg04, Kup13].

### 10.2 An Algorithm for the AHSP Breaks Many Group Actions

We first show a relatively straightforward result: any algorithm that can solve the dihedral hidden shift problem can be used to solve DLog on a regular, abelian group action. This is essentially already folklore since there have been many instances (starting with [CJS14]) using Kuperberg's algorithm or related principles to build attacks against isogenies that can be modelled as EGAs.

**Lemma 39.** *Let $(G, X, \star)$ denote a regular, abelian group action.. Suppose there exists a PPT algorithm $\mathcal{A}$ for solving the abelian hidden shift problem on $A$ with probability $\epsilon$. Then there exists an algorithm for solving the GA-DLog problem on $(G, X, \star)$ with probability $\epsilon$.*

*Proof.* Suppose we are given a discrete log challenge of the form $(x, g \star x)$ for appropriately sampled $g \in G$, $x \in X$. Suppose we define our function $f : G \to X$ as $f(h) = h \star x$, and we define our shifted function $g : G \to X$ as $f(h) = h \star (g \star x)$. Given access to a discrete log challenge and a group action description, a challenger can immediately provide oracle access to appropriately defined hidden shift functions on $A$. Since the action is regular and abelian, we meet all of the criteria of the AHSP (i.e. the fact that the outputs must be unique), meaning the reduction is immediate. □

### 10.3 Using Group Action Algorithms to Solve the AHSP

What about the other direction? Can we show that an adversary that can break DLog on a group action can solve the AHSP? Unfortunately, this seems difficult: because the AHSP is described so generally–the functions $f$ and $g$ can be anything as long as the functions are injective–so it seems difficult or impossible to prove this for any non-generic algorithm.

But what about generic algorithms? Could we prove that the AHSP is equivalent to generically solving DLog over group actions? This seems like it might be plausible. The most interesting result would show equivalence in a generic group action model with quantum queries. While this may be attainable, unfortunately we do not know how to achieve this result. However, we can show that an adversary that can generically solve group action DLog with classical queries can be used to solve the AHSP, which is seemingly a step in the right direction. We formalize this result below.

**Lemma 40.** *Let $(G, X, \star)$ be an abelian, regular group action (EGA). Suppose there exists a generic adversary $\mathcal{A}$ that breaks the group action DLog problem (as defined in definition 6) with advantage $\epsilon$ on this group action. Then there exists an algorithm that solves that AHSP on $G$ with advantage $\epsilon$.*

*Proof.* As in lemma 36, we will need to simulate a generic group action oracle (with classical queries). Unlike before, though, we will need to use a more complicated strategy to handle both functions $f$ and $g$ from the AHSP. We will assume the existence of two random oracles $RO_1, RO_2 : G \to \{0,1\}^\ell$ such that $2^\ell > ||X||^2$. The reason for the length is to ensure that, with all but negligible probability, there are no collisions in the random oracle.[10] We will simulate terms in a way that should be very familiar for a reader who is familiar with generic group models.

**Generic Simulation.** In order to simulate the group, we will maintain a list of queries of the form $(g_i, \tilde{g}_i)$ for $\tilde{g}_i \in \{0,1\}^\ell$. Initially, we will add $(g_I, RO_1(g_I))$ to the table, and $RO_1(g_I)$ will represent the identity element. Then, for each $\mathbf{g}_i \in \mathbf{g}$, we add the tuples $(\mathbf{g}_i, RO_1(\mathbf{g}_i))$ and $\left(\mathbf{g}_i^{-1}, RO_1\left(\mathbf{g}_i^{-1}\right)\right)$ to the table as well, giving access to the computable elements.

When an adversary makes a query of the form $\left(\tilde{h}, \tilde{h}', +\right)$ or $\left(\tilde{h}, \tilde{h}', -\right)$, we will look up $x$ in our table and find the entries $\left(h, \tilde{h}\right)$ and $\left(h', \tilde{h}'\right)$. We know that such an entry must exist because our algorithm is generic. Then we add the appropriately signed entry $(h \pm h', RO_1(h \pm h'))$ to our table and return $RO_1(h \pm h')$.

Unlike before, we need a more complicated strategy to simulate the set elements. Instead of having one list of set elements, we will have two: one for the function $f$ and one for the function $g$. Both of these lists will contain queries of the form $(g_i, \tilde{x}_i)$.

Initially, we will add $(g_I, RO_2(f(g_I)))$ to the $f$-list, and $RO(f(g_I))$ will be our "base point." We will add $(g_I, RO_2(g(g_I)))$ to the $g$-list, and $RO_2(g(g_I))$ will be our "DLog challenge point."

When an adversary makes a query of the form $\left(\tilde{h}, \tilde{x}\right)$ we will first look up $\tilde{h}$ in our table of group elements and recover the tuple $\left(h, \tilde{h}\right)$. Then we will look up $\tilde{x}$ and see which list it is in and recover the tuple $(g, \tilde{x})$.

If it is in the $f$ list, then we will add $(hg, RO_1(hg))$ to our group table and $(hg, RO_2(f(hg)))$ to our set table for $f$, and return $RO_2(f(hg))$. If it is in the $g$ list, then we will still add $(hg, RO_1(hg))$ to our group table but then add $(hg, RO_2(g(hg)))$ to our set table for $g$, and return $RO_2(g(hg))$.

**Actual Reduction.** With the simulation above, our reduction becomes very straightforward. Given oracles $f$ and $g$ to the AHSP, we simulate a generic group action (with classical queries) in the above way. The solution to the AHSP will be exactly the solution to the discrete log problem defined by the "base point" and the "DLog challenge point." So any adversary that can solve the DLog problem generically with classical queries can be used to solve the AHSP problem. □

### 10.4 Discussion

Unfortunately, it seems difficult to show a full quantum equivalence between the generalized dihedral hidden subgroup problem and solving DLog on a generic group action. The challenge comes from the fact that it is difficult quantumly to "remember" an adversary's query for later use in the simulation. One possible direction is to use compressed oracles [Zha19], which offer some ability to

[10] Note that the set size and group order are the same in a regular group action.

record quantum queries. However, it appears challenging to adapt the compressed oracle framework to highly structured oracles such as generic group actions.

Nevertheless, we close this section with the following conjecture, which we think is very interesting future work:

**Conjecture 41.** *The generalized dihedral hidden subgroup problem on an abelian group $A$ is equivalent to the group action discrete logarithm problem on a regular, abelian group action $(G, X, \star)$ in a quantum generic model.*

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# A Formal Group Action Definitions

In this section we provide formal definitions of all of our primitives. We include full, formal versions of the traditional definitions that have been used in the isogeny literature. Our definitions here generally mirror those of [ADMP20].

## A.1 Effective Group Action

Our first formal definition is that of an effective group action. We define an *effective group action* (EGA) as follows:

**Definition 42.** (Effective Group Action) *A group action* $(G, X, \star)$ *is* effective *if the following properties are satisfied:*

1. *The group* $G$ *is finite and there exist efficient (PPT) algorithms for:*
   (a) *Membership testing, i.e., to decide if a given bit string represents a valid group element in* $G$*.*
   (b) *Equality testing, i.e., to decide if two bit strings represent the same group element in* $G$*.*
   (c) *Sampling, i.e., to sample an element* $g$ *from a distribution* $\mathcal{D}_G$ *on* $G$*. In this paper, We consider distributions that are statistically close to uniform.*
   (d) *Operation, i.e., to compute* $gh$ *for any* $g, h \in G$*.*
   (e) *Inversion, i.e., to compute* $g^{-1}$ *for any* $g \in G$*.*
2. *The set* $X$ *is finite and there exist efficient algorithms for:*
   (a) *Membership testing, i.e., to decide if a bit string represents a valid set element.*
   (b) *Unique representation, i.e., given any arbitrary set element* $x \in X$*, compute a string* $\hat{x}$ *that canonically represents* $x$*.*
3. *There exists a distinguished element* $x_0 \in X$*, called the* origin*, such that its bit-string representation is known.*
4. *There exists an efficient algorithm that given (some bit-string representations of) any* $g \in G$ *and any* $x \in X$*, outputs* $g \star x$*.*

## A.2 Restricted Effective Group Action

Some isogeny-based protocols (e.g. CSIDH [CLM+18]) are structured in a way that almost (but not quite) resembles a group action. To model this kind of protoocol, we use he definition of *Restricted Effective Group Action* (REGA) from [ADMP20]. A REGA is a weakening of EGA, where we can only evaluate the action of a generating set of small cardinality.

**Definition 43.** (Restricted Effective Group Action) *Let $(G, X, \star)$ be a group action and let $\mathbf{g} = (g_1, \ldots, g_n)$ be a (not necessarily minimal) generating set for $G$. The action is said to be* $\mathbf{g}$-restricted effective*, if the following properties are satisfied:*

- *$G$ is finite and $n = poly(\log(|G|))$.*
- *The set $X$ is finite and there exist efficient algorithms for:*
  1. *Membership testing, i.e., to decide if a bit string represents a valid set element.*
  2. *Unique representation, i.e., to compute a string $\hat{x}$ that canonically represents any given set element $x \in X$.*
- *There exists a distinguished element $x_0 \in X$, called the* origin*, such that its bit-string representation is known.*
- *There exists an efficient algorithm that given any $i \in [n]$ and any bit string representation of $x \in X$, outputs $g_i \star x$ and $g_i^{-1} \star x$.*

Although a REGA is limited to evaluations of the form $g_i \star x$ and $g_i^{-1} \star x$, this is actually enough to evaluate the action of many, and potentially all elements of $G$ without even needing axioms on the effectivity of $G$. It is typically assumed (but not always proven) in a REGA that taking a random subset product of all of the elements $g_i$ results in a group element that is statistically close to uniform. We defer commentary on this assumption, as well as more discussion on modelling isogeny-based protocols as REGAs, to [ADMP20].

## A.3 Known-order Effective Group Actions

In this paper, we will typically assume that all parties have access to a quantum computer. For abelian group actions, this means that we can use Shor's algorithm and its generalization [Sho94, CM01] to decompose the structure of the group. More generally, and again following the definition of [ADMP20], for a strengthening of EGA, we may assume that the group structure of $G$ is known. The most important case is where $G$ is abelian, which, by abuse of language, we call *known-order* EGA.

By "known order" we mean that a minimal list of generators $\mathbf{g} = (g_1, \ldots, g_n)$ together with their orders $(m_1, \ldots, m_n)$ is known, which in turn is equivalent to a decomposition

$$G \simeq \mathsf{Z}/m_1\mathsf{Z} \oplus \cdots \oplus \mathsf{Z}/m_n\mathsf{Z}.$$

An important special case is when $G$ is cyclic, i.e., $G = \langle g \rangle \simeq \mathsf{Z}/m\mathsf{Z}$.

Denote by $\mathcal{L}$ the lattice $m_1\mathsf{Z} \oplus \cdots \oplus m_n\mathsf{Z}$, the map

$$\phi : \mathsf{Z}^n/\mathcal{L} \to G$$
$$(a_1, \ldots, a_n) \mapsto \prod_{i=1}^{n} g_i^{a_i}$$

is an effective isomorphism, its inverse being a generalized discrete logarithm. If $(G, X, \star)$ is an EGA, then it is immediate to verify that $(\mathbb{Z}^n/\mathcal{L}, X, \star)$ is an EGA through $\phi$. We may just use $\mathbb{Z}^n/\mathcal{L}$ as the standard representation for $G$.

**Definition 44.** (Known-order Effective Group Action (KEGA).) *A* Known-order Effective Group Action *is an EGA $(\mathcal{Z}^n/\mathcal{L}, X, \star)$ where the lattice $\mathcal{L}$ is given by the tuple $(m_1, \ldots, m_n)$.*

We note that for an abelian group action an EGA and a KEGA are quantumly equivalent. However, a REGA and a KEGA are not necessarily quantum equivalent, even if the group action is abelian. We discuss this relationship more in the body of the paper, and [ADMP20] also includes a substantial discussion on the topic. In the quantum world, KEGA is a more appropriate tool to design protocols, owing to its simplicity. There are in fact some protocols that explicitly require the KEGA setting, e.g. [DM20b].

### A.4 Formal Definitions of Group Action Security

Here we present the formal definitions of group action securtiy from [ADMP20] and discuss their relationship with our group-inspired security definitions. These definitions are, to our knowledge, the most popular current definitions for security notions of isogeny-based group actions.

**Definition 45.** (One-Way Group Action) *A group action $(G, X, \star)$ is $(\mathcal{D}_X, \mathcal{D}_G)$-one-way if the family of efficiently computable functions $\{f_x : G \to X\}_{x \in X}$ is $(\mathcal{D}_X, \mathcal{D}_G)$-one-way, where $f_x : g \mapsto g \star x$, and $\mathcal{D}_X$, $\mathcal{D}_G$ are distributions on $X$, $G$ respectively.*

Note that any adversary that this definition is exactly equivalent to the definition of group action discrete logarithm, as we defined in definition 6.

**Definition 46.** (Weak Unpredictable Group Action) *A group action $(G, X, \star)$ is $(\mathcal{D}_G, \mathcal{D}_X)$-weakly unpredictable if the family of efficiently computable permutations $\{\pi_g : X \to X\}_{g \in G}$ is $(\mathcal{D}_G, \mathcal{D}_X)$-weakly unpredictable, where $\pi_g$ is defined as $\pi_g : x \mapsto g \star x$ and $\mathcal{D}_X$, $\mathcal{D}_G$ are distributions on $X$, $G$ respectively.*

Note that any adversary that can win the group action CDH game (as defined in definition 7) can be immediately used to win the weak unpredictable group action game. On the other hand, if $\mathcal{D}_G$ and $\mathcal{D}_X$ are uniformly distributed and the group action is regular, then an adversary that can break the weak unpredictability of a group action can be used to solve the group action CDH problem. However, if the group is not regular, the problems may not be equivalent: given just a CDH instance, it may be impossible to simulate the correct distribution of "fresh" outputs $(x_i, g \star x_i)$ for some $x_i \leftarrow \mathcal{D}_X$ when the only information provided about $g$ is $y = g \star x$. However, we note that in practice these distributions are typically uniform (or assumed to be uniform) and popular group actions are almost always regular, and in these cases the CDH problem exactly models key exchange[11] which is the focus of most applications of group actions.

**Definition 47.** (Weak Pseudorandom Group Action) *A group action $(G, X, \star)$ is $(\mathcal{D}_G, \mathcal{D}_X)$-weakly pseudorandom if the family of efficiently computable permutations $\{\pi_g : X \to X\}_{g \in G}$ is $(\mathcal{D}_G, \mathcal{D}_X)$-weakly pseudorandom, where $\pi_g : x \mapsto g \star x$, and $\mathcal{D}_X$, $\mathcal{D}_G$ are distributions on $X$, $G$ respectively.*

In each of the definitions above, if $\mathcal{D}_G$ is a probability distribution on $G$ and $\mathcal{D}_X$ is the distribution induced on $X$ by taking $g \leftarrow \mathcal{D}_G$ and outputting $g \star x_0$, then we simply write $\mathcal{D}_G$-OW group action for $(\mathcal{D}_X, \mathcal{D}_G)$-OW group action, and similarly for weak unpredictable/pseudorandom group actions. If both distributions are uniform (or statistically close to uniform), we omit them.

[11] The weak unpredictable group action definition is more useful than the CDH problem for constructing more complicated primitives; see [ADMP20].