Fast and Frobenius: Rational Isogeny Evaluation over Finite Fields

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Abstract. Consider the problem of efficiently evaluating isogenies $\phi : \mathcal{E} \to \mathcal{E}/H$ of elliptic curves over a finite field $\mathbb{F}_q$, where the kernel $H = \langle G \rangle$ is a cyclic group of odd (prime) order: given $\mathcal{E}$, G, and a point (or several points) P on $\mathcal{E}$, we want to compute $\phi(P)$. This problem is at the heart of efficient implementations of group-action- and isogeny-based post-quantum cryptosystems such as CSIDH. Algorithms based on Vélu's formulæ give an efficient solution when the kernel generator G is defined over $\mathbb{F}_q$, but for general isogenies G is only defined over some extension $\mathbb{F}_{q^k}$, even though $\langle G \rangle$ as a whole (and thus ϕ) is defined over the base field $\mathbb{F}_q$; and the performance of Vélu-style algorithms degrades rapidly as k grows. In this article we revisit isogeny evaluation with a special focus on the case where $1 \leq k \leq 12$. We improve Vélu-style evaluation for many cases where $k = 1$ using special addition chains, and combine this with the action of Galois to give greater improvements when $k > 1$.

1 Introduction

Faced with the rising threat of quantum computing, demand for quantum-secure, or post-quantum, cryptographic protocols is increasing. Isogenies have emerged as a useful candidate for post-quantum cryptography thanks to their generally small key sizes, and the possibility of implementing post-quantum group actions which offer many simple post-quantum analogues of classical discrete-log-based algorithms (see e.g. [26]).

A major drawback of isogeny-based cryptosystems is their relatively slow performance compared with many other post-quantum systems. In this paper, we improve evaluation times for isogenies of many prime degrees $\ell > 3$ given a generator of the kernel; these computations are the fundamental building blocks of most isogeny-based cryptosystems. Specifically, we propose simple alternative

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differential addition chains to enumerate points of (subsets of) the kernel more efficiently. This speeds up many ℓ-isogeny computations over the base field by a factor depending on ℓ, and also permits a full additional factor-of-k speedup for ℓ-isogenies over $\mathbb{F}_q$ whose kernel generators are defined over an extension $\mathbb{F}_{q^k}$.

Our techniques have constructive and destructive applications. First, accelerating basic isogeny computations can speed up isogeny-based cryptosystems. The methods in §4 apply for many $\ell > 3$, so they would naturally improve the performance of commutative isogeny-based schemes such as CSIDH [5], and CSI-FiSh [4] and its derivatives (such as [12] and [14]), which require computing many ℓ-isogenies for various primes ℓ. They may also improve the performance of other schemes like SQISign [16], which computes many ℓ-isogenies in its signing process. (We discuss applications further in §6.)

In §5 we focus on rational isogenies with irrational kernels; our methods there could further accelerate the improvements of [13] for Couveignes–Rostovtsev–Stolbunov key exchange (CRS) and related protocols of Stolbunov [11,24,27,28]. This is a small step forward on the road to making CRS a practical "ordinary" fallback for CSIDH in the event of new attacks making specific use of the full supersingular isogeny graph (continuing the approach of [6], for example).

Our results also have applications in cryptanalysis: the best classical and quantum attacks on commutative isogeny-based schemes involve computing massive numbers of group actions, each comprised of a large number of ℓ-isogenies (see e.g. [3] and [8]). Any algorithm that reduces the number of basic operations per ℓ-isogeny will improve the effectiveness of these attacks.

Proof-of-concept implementations of our algorithms in SageMath are available at `https://github.com/vgilchri/k-velu`. The scripts include operation-counting code to verify the counts claimed in this article.

Disclaimer. In this paper, we quantify potential speedups by counting finite field operations. Real-world speed increases depend on many additional variables including parameter sizes; the application context; implementation choices; specificities of the runtime platform (including architecture, vectorization, and hardware acceleration); and the availability of optimized low-level arithmetic.

2 Background

We work over (extensions of) the base field $\mathbb{F}_q$, where q is a power of a prime $p > 3$. The symbol ℓ always denotes a prime $\neq p$. In our applications, $3 < \ell \ll p$.

Elliptic curves. For simplicity, elliptic curves are supposed to be in a general Weierstrass form $\mathcal{E} : y^2 = f(x)$. Our algorithms focus on *Montgomery models*

$$\mathcal{E} : By^2 = x(x^2 + Ax + 1) \qquad \text{where} \qquad B(A^2 - 4) \neq 0\,.$$

but our results extend easily to other models such as twisted Edwards and short Weierstrass models. The multiplication-by-m map is denoted by $[m]$. The q-power Frobenius endomorphism is $\pi : (x, y) \mapsto (x^q, y^q)$.

Field operations. While the curve $\mathcal{E}$ will always be defined over $\mathbb{F}_q$, we will often work with points defined over $\mathbb{F}_{q^k}$ for $k \geq 1$. We write $\mathbf{M}$, $\mathbf{S}$, and $\mathbf{a}$ for the cost of multiplication, squaring, and adding (respectively) in $\mathbb{F}_{q^k}$. We write $\mathbf{C}$ for the cost of multiplying an element of $\mathbb{F}_{q^k}$ by an element of $\mathbb{F}_q$ (typically a curve constant, or an evaluation-point coordinate). Note that $\mathbf{C} \approx (1/k)\mathbf{M}$ (when k is not too large). Later, we will write $\mathbf{F}$ for the cost of evaluating the Frobenius map on $\mathbb{F}_{q^k}$; see §5.1 for discussion on this.

x-only arithmetic. Montgomery models are designed to optimize x-only arithmetic (see [20] and [10]). The `xADD` operation is

$$\texttt{xADD} : (x(P), x(Q), x(P-Q)) \longmapsto x(P+Q)\,;$$

it can be computed at a cost of $4\mathbf{M} + 2\mathbf{S} + 6\mathbf{a}$ using the formulæ

$$\begin{cases} X_+ = Z_- \left[(X_P - Z_P)(X_Q + Z_Q) + (X_P + Z_P)(X_Q - Z_Q)\right]^2 \,, \\ Z_+ = X_- \left[(X_P - Z_P)(X_Q + Z_Q) - (X_P + Z_P)(X_Q - Z_Q)\right]^2 \end{cases} \tag{1}$$

where $(X_P : Z_P)$, $(X_Q : Z_Q)$, $(X_+ : Z_+)$, and $(X_- : Z_-)$ are the x-coordinates $x(P)$, $x(Q)$, $x(P+Q)$, and $x(P-Q)$, respectively (so $x(P) = \frac{X_P}{Z_P}$, and so on).

The `xDBL` operation is

$$\texttt{xDBL} : x(P) \longmapsto x([2]P)\,;$$

it can be computed at a cost of $2\mathbf{M} + 2\mathbf{S} + \mathbf{C} + 4\mathbf{a}$ using the formulæ

$$\begin{cases} X_{[2]P} = (X_P + Z_P)^2 (X_P - Z_P)^2 \,, \\ Z_{[2]P} = (4X_P Z_P)((X_P - Z_P)^2 + ((A+2)/4)(4X_P Z_P)) \,. \end{cases} \tag{2}$$

Isogenies. Let $\mathcal{E}_1, \mathcal{E}_2$ be elliptic curves over a finite field $\mathbb{F}_q$. An isogeny $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ is a non-constant morphism mapping the identity point of $\mathcal{E}_1$ to the identity point of $\mathcal{E}_2$. Such a morphism is automatically a homomorphism. For more details see [25, Chapter 3, §4]. The kernel of ϕ is a finite subgroup of $\mathcal{E}_1$, and vice versa: every finite subgroup $\mathcal{G}$ of $\mathcal{E}_1$ determines a separable *quotient isogeny* $\mathcal{E}_1 \to \mathcal{E}_1/\mathcal{G}$. The *kernel polynomial* of ϕ is

$$D(X) := \prod_{P \in S} (X - x(P))$$

where $S \subset \mathcal{G}$ is any subset satisfying

$$S \cap -S = \emptyset \qquad \text{and} \qquad S \cup -S = \mathcal{G} \setminus \{0\}\,. \tag{3}$$

Every separable isogeny $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ defined over $\mathbb{F}_q$ can be represented by a rational map in the form

$$\phi : (x, y) \longmapsto \big(\phi_x(x), \phi_y(x, y)\big) \tag{4}$$

with

$$\phi_x(x) = \frac{N(x)}{D(x)^2} \qquad \text{and} \qquad \phi_y(x, y) = c \cdot y \frac{d\phi_x}{dx}(x) \tag{5}$$

where D is the kernel polynomial of ϕ, N is a polynomial derived from D, and c is a normalizing constant in $\mathbb{F}_q$.

Vélu's formulæ. Given a curve $\mathcal{E}$ and a finite subgroup $\mathcal{G} \subset \mathcal{E}$, Vélu [29] gives explicit formulæ for the rational functions that define a separable isogeny $\phi : \mathcal{E} \to \mathcal{E}' := \mathcal{E}/\mathcal{G}$ with kernel $\mathcal{G}$, as well as the resulting codomain curve $\mathcal{E}'$. Although the quotient curve $\mathcal{E}'$ and the isogeny ϕ are defined up to isomorphism, Vélu's formulæ construct the unique *normalized* isogeny (i.e. with $c = 1$ in (5)). See Kohel's thesis [18, §2.4] for a modern treatment of Vélu's results.

3 Evaluating isogenies

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{F}_q$, and let $\langle G \rangle$ be a subgroup of prime order ℓ (where ℓ is not equal to the field characteristic p). We suppose $\langle G \rangle$ is defined over $\mathbb{F}_q$; then, the quotient isogeny $\phi : \mathcal{E} \to \mathcal{E}/\langle G \rangle$ is also defined over $\mathbb{F}_q$.

When we say $\langle G \rangle$ is defined over $\mathbb{F}_q$, this means $\langle G \rangle$ is *Galois stable*: that is, $\pi(\langle G \rangle) = \langle G \rangle$ (where π is the q-power Frobenius endomorphism). We will mostly be concerned with algorithms taking $x(G)$ as an input, so it is worth noting that

$$x(G) \in \mathbb{F}_{q^{k'}} \qquad \text{where} \qquad k' := \begin{cases} k & \text{if } k \text{ is odd}\,, \\ k/2 & \text{if } k \text{ is even}\,. \end{cases}$$

The set of projective x-coordinates of the nonzero kernel points is

$$\mathcal{X}_G := \big\{ (X_P : Z_P) = x(P) : P \in \langle G \rangle \setminus \{0\} \big\} \subset \mathbb{P}^1(\mathbb{F}_{q^{k'}})\,;$$

each X_P/Z_P corresponds to a root of the kernel polynomial $D(X)$, and vice versa. If $\#\langle G \rangle$ is an odd prime ℓ, then $\#\mathcal{X}_G = (\ell - 1)/2$.

3.1 The isogeny evaluation problem

We want to evaluate the isogeny $\phi : \mathcal{E} \to \mathcal{E}/\langle G \rangle$. More precisely, we want efficient solutions to the problem of Definition 1:

Definition 1 (Isogeny Evaluation). *Given an elliptic curve $\mathcal{E}$ over $\mathbb{F}_q$, a list of points $(P_1, \ldots, P_n)$ in $\mathcal{E}(\mathbb{F}_q)$, and a finite subgroup $\mathcal{G}$ of $\mathcal{E}$ corresponding to the separable isogeny $\phi : \mathcal{E} \to \mathcal{E}/\mathcal{G}$, compute $(\phi(P_1), \ldots, \phi(P_n))$.*

In most cryptographic applications n is relatively small, especially compared to ℓ. We do *not* assume the codomain curve $\mathcal{E}/\mathcal{G}$ is known; if required, an equation for $\mathcal{E}/\mathcal{G}$ can be interpolated through the image of well-chosen evaluation points.

For each separable isogeny ϕ of degree d defined over $\mathbb{F}_q$, there exists a sequence of primes $(\ell_1, \ldots, \ell_n)$ and a sequence of isogenies $(\phi_1, \ldots, \phi_n)$, all defined over $\mathbb{F}_q$, such that $\phi_n \circ \cdots \circ \phi_1$ and

- $\phi_i = [\ell_i]$ (the non-cyclic case) or
- ϕ_i has cyclic kernel of order ℓ_i.

The kernel of ϕ_1 is $\ker\phi \cap \mathcal{E}[\ell_1]$, and so on. The maps $[\ell_i]$ can be computed in $O(\log \ell_i)$ $\mathbb{F}_q$-operations, so we reduce quickly to the case where ϕ has prime degree ℓ, assuming the factorization of d is known (as it is in our applications).

In general, the isogeny evaluation problem can be reduced to evaluating the map $\alpha \mapsto D(\alpha)$, where D is the kernel polynomial and α is in $\mathbb{F}_q$ or some $\mathbb{F}_q$-algebra (see e.g. [2, §4]). The polynomial D need not be explicitly computed.

3.2 The Costello-Hisil algorithm

The Costello-Hisil algorithm [9], generalized in [23], is the state-of-the-art for evaluating isogenies of Montgomery models. This algorithm is a variation of Vélu's formulæ working entirely on the level of x-coordinates, using the fact that for an ℓ-isogeny ϕ with kernel $\langle G\rangle$, the rational map on x-coordinates is

$$\phi_x(x) = x\cdot\left(\prod_{i=1}^{(\ell-1)/2}\left(\frac{x\cdot x([i]G)-1}{x-x([i]G)}\right)\right)^2. \tag{6}$$

Moving to projective coordinates $(U:V)$ such that $x = U/V$ and using the fact that $\mathcal{X}_G = \{(x([i]G):1) : 1 \le i \le (\ell-1)/2\}$, Eq. (6) becomes

$$\phi_x\big((U:V)\big) = (U':V'), \quad \begin{cases} U' = U\big[\prod_{(X_Q:Z_Q)\in\mathcal{X}_G}(UX_Q - VZ_Q)\big]^2, \\ V' = V\big[\prod_{(X_Q:Z_Q)\in\mathcal{X}_G}(UZ_Q - VX_Q)\big]^2. \end{cases} \tag{7}$$

Algorithm 1 (from [9]) and Algorithm 2 (our space-efficient variant) compute ϕ_x at a series of input points using an efficient evaluation of the expressions in (7). For the moment, we assume that we have subroutines

- `KernelPoints` (see §4): given $(X_G : Z_G)$, returns $\mathcal{X}_G$ as a list.
- `KernelRange` (see §4): a *generator* coroutine which, given $(X_G : Z_G)$, constructs and yields the elements of $\mathcal{X}_G$ to the caller one by one.
- `CrissCross` [9, Algorithm 1]: takes $(\alpha,\beta,\gamma,\delta)$ in $\mathbb{F}_{q^k}^4$ and returns $(\alpha\delta+\beta\gamma, \alpha\delta-\beta\gamma)$ in $\mathbb{F}_{q^k}^2$ at a cost of $2\mathbf{M}+2\mathbf{a}$.

4 Accelerating Vélu: faster iteration over the kernel

Let $\mathcal{E}/\mathbb{F}_q$ be an elliptic curve, and let G be a point of prime order ℓ in $\mathcal{E}$. For simplicity, in this section we will assume that G is defined over $\mathbb{F}_q$, but all of the results here apply when G is defined over an extension $\mathbb{F}_{q^k}$: in that case $\mathbf{M}$, $\mathbf{S}$, and $\mathbf{a}$ represent operations in the extension field $\mathbb{F}_{q^k}$, while $\mathbf{C}$ represents multiplication of an element of $\mathbb{F}_{q^k}$ by a curve constant of the subfield $\mathbb{F}_q$ (which is roughly k times cheaper than $\mathbf{M}$). We return to the case where $k > 1$ in §5.

Algorithm 1: Combines Algorithms 3 and 4 from [9] to evaluate an ℓ-isogeny of Montgomery models at a list of input points. The total cost is $2n\ell\mathbf{M} + 2n\mathbf{S} + ((n+1)(\ell+1) - 2)\mathbf{a}$, *plus* the cost of `KernelPoints`.

Input: The x-coordinate $(X_G : Z_G)$ of a generator G of the kernel of an ℓ-isogeny ϕ, and a list of evaluation points $((U_i : V_i) : 1 \le i \le n)$
Output: The list of images $((U_i' : V_i') = \phi_x((U_i : V_i)) : 1 \le i \le n)$

$((X_1, Z_1), \ldots, (X_{(l-1)/2}, Z_{(l-1)/2})) \leftarrow$ `KernelPoints`$((X_G : Z_G))$ `// See §4`
for $1 \le i \le (\ell-1)/2$ **do**
  $(\hat{X}_i, \hat{Z}_i) \leftarrow (X_i + Z_i, X_i - Z_i)$ `// 2a`
for $i = 1$ ***to*** n **do**
  $(\hat{U}_i, \hat{V}_i) \leftarrow (U_i + V_i, U_i - V_i)$ `// 2a`
  $(U_i', V_i') \leftarrow (1, 1)$
  for $j = 1$ ***to*** $(\ell-1)/2$ **do**
    $(t_0, t_1) \leftarrow$ `CrissCross`$(\hat{X}_j, \hat{Z}_j, \hat{U}_i, \hat{V}_i)$ `// 2M + 2a`
    $(U_i', V_i') \leftarrow (t_0 \cdot U_i', t_1 \cdot V_i')$ `// 2M`
  $(U_i', V_i') \leftarrow (U_i \cdot (U_i')^2, V_i \cdot (V_i')^2)$ `// 2M + 2S`
return $((U_1', V_1'), \ldots, (U_n', V_n'))$

Algorithm 2: A generator-based version of Algorithm 1, with much lower space requirements when $\ell \gg n$. The total cost is $2n\ell\mathbf{M} + 2n\mathbf{S} + (2n + (\ell-1)(n+1))\mathbf{a}$, *plus* the cost of a full run of `KernelRange`.

Input: The x-coordinate $(X_G : Z_G)$ of a generator G of the kernel of an ℓ-isogeny ϕ, and a list of evaluation points $((U_i : V_i) : 1 \le i \le n)$
Output: The list of images $((U_i' : V_i') = \phi_x((U_i : V_i)) : 1 \le i \le n)$

for $1 \le i \le n$ **do**
  $(\hat{U}_i, \hat{V}_i) \leftarrow (U_i + V_i, U_i - V_i)$ `// 2a`
  $(U_i', V_i') \leftarrow (1, 1)$
for $(X : Z)$ ***in*** `KernelRange`$((X_G : Z_G))$ **do** `// See §4`
  $(\hat{X}, \hat{Z}) \leftarrow (X + Z, X - Z)$ `// 2a`
  for $1 \le i \le n$ **do**
    $(t_0, t_1) \leftarrow$ `CrissCross`$(\hat{X}, \hat{Z}, \hat{U}_i, \hat{V}_i)$ `// 2M + 2a`
    $(U_i', V_i') \leftarrow (t_0 \cdot U_i', t_1 \cdot V_i')$ `// 2M`
for $1 \le i \le n$ **do**
  $(U_i', V_i') \leftarrow (U_i \cdot (U_i')^2, V_i \cdot (V_i')^2)$ `// 2M + 2S`
return $((U_1', V_1'), \ldots, (U_n', V_n'))$

4.1 Kernel point enumeration and differential addition chains

We now turn to the problem of enumerating the set $\mathcal{X}_G$. This process, which we call *kernel point enumeration*, could involve constructing the entire set (as in `KernelPoints`) or constructing its elements one by one (for `KernelRange`).

For $\ell = 2$ and 3, there is nothing to be done because $\mathcal{X}_G = \{(X_G : Z_G)\}$; so from now on we consider the case $\ell > 3$.

We allow ourselves two curve operations for kernel point enumeration: `xADD` and `xDBL`. In §5, where G is assumed to be defined over a nontrivial extension of the base field, we will also allow the Frobenius endomorphism.

Every algorithm constructing a sequence of elements of $\mathcal{X}_G$ using a series of `xADD` and `xDBL` instructions corresponds to a *modular differential addition chain*.

Definition 2. *A **Modular Differential Addition Chain (MDAC)** for a set $S \subset \mathbb{Z}/\ell\mathbb{Z}$ is a sequence of integers $(c_0, c_1, c_2, \ldots, c_n)$ such that*

1. *every element of S is represented by some $c_i \pmod{\ell}$,*
2. *$c_0 = 0$ and $c_1 = 1$, and*
3. *for each $1 < i \leq n$ there exist $0 \leq j(i), k(i), d(i) < i$ such that $c_i \equiv c_{j(i)} + c_{k(i)} \pmod{\ell}$ and $c_{j(i)} - c_{k(i)} \equiv c_{d(i)} \pmod{\ell}$.*

Algorithms to enumerate $\mathcal{X}_G$ using `xADD` and `xDBL` correspond to MDACs $(c_0, \ldots, c_n)$ for $\{1, \ldots, (\ell-1)/2\}$: the algorithm starts with $x([c_0]G) = x(0) = (1 : 0)$ and $x([c_1]G) = x(G) = (X_G : Z_G)$, then computes each $x([c_i]G)$ using

$$x([c_i]G) = \begin{cases} \texttt{xADD}(x([c_{j(i)}]G), x([c_{k(i)}]G), x([c_{d(i)}]G)) & \text{if } d(i) \neq 0\,, \\ \texttt{xDBL}([c_{j(i)}]G) & \text{if } d(i) = 0\,. \end{cases}$$

4.2 Additive kernel point enumeration

The classic approach is to compute $\mathcal{X}_G$ using repeated `xADD`s. Algorithm 3 is Costello and Hisil's `KernelPoints` [9, Algorithm 2], corresponding to the MDAC $(0, 1, 2, 3, \ldots, (\ell-1)/2)$ computed by repeatedly adding 1 (except for 2 which is computed by doubling 1). The simplicity of this MDAC means that Algorithm 3 converts to a `KernelRange` with a small internal state: to generate the next $(X_{i+1} : Z_{i+1})$, we only need the values of $(X_i : Z_i)$, $(X_{i-1} : Z_{i-1})$, and $(X_1 : Z_1)$.

4.3 Replacing `xADD`s with `xDBL`s

Comparing x-only operations on Montgomery curves, replacing an `xADD` with an `xDBL` trades $2\mathbf{M}$ and $2\mathbf{a}$ for $1\mathbf{C}$. We would therefore like to replace as many `xADD`s as possible in our kernel enumeration with `xDBL`s.

As a first attempt, we can replace Line 4 of Algorithm 3 with

$$(X_i : Z_i) \leftarrow \begin{cases} \texttt{xDBL}((X_{i/2} : Z_{i/2})) & \text{if } i \text{ is even}, \\ \texttt{xADD}\big((X_{i-1} : Z_{i-1}), (X_G : Z_G), (X_{i-2}, Z_{i-2})\big) & \text{if } i \text{ is odd}. \end{cases}$$

But applying this trick systematically requires storing many more intermediate values, reducing the efficiency of `KernelRange`. It also only replaces half of the `xADD`s with `xDBL`s, and it turns out that we can generally do much better.

Algorithm 3: Basic kernel point enumeration by repeated addition.
Uses exactly 1 `xDBL` and $(\ell-5)/2$ `xADD` operations (for prime $\ell > 3$).

Input: The x-coordinate $(X_G : Z_G) = x(G)$ of a point G of order ℓ in $\mathcal{E}(\mathbb{F}_q)$
Output: $\mathcal{X}_G$ as a list

$(X_1 : Z_1) \leftarrow (X_G : Z_G)$
$(X_2 : Z_2) \leftarrow \texttt{xDBL}\big((X_G : Z_G)\big)$
for $i = 3$ ***to*** $(\ell-1)/2$ **do** `// Invariant:` $(X_i : Z_i) = x([i]G)$
  $(X_i : Z_i) \leftarrow \texttt{xADD}\big((X_{i-1} : Z_{i-1}), (X_G : Z_G), (X_{i-2}, Z_{i-2})\big)$
return $\big((X_1 : Z_1), \ldots, (X_{(\ell-1)/2} : Z_{(\ell-1)/2})\big)$

4.4 Multiplicative kernel point enumeration

We can do better for a large class of ℓ by considering the quotient

$$M_\ell := \big(\mathbb{Z}/\ell\mathbb{Z}\big)^\times / \langle \pm 1 \rangle \,.$$

(note: M_ℓ is a quotient of the *multiplicative* group.) For convenience, we write

$$m_\ell := \#M_\ell = (\ell-1)/2 \,.$$

We can now reframe the problem of enumerating $\mathcal{X}_G$ as the problem of enumerating a complete set of representatives for M_ℓ. The MDAC of Algorithm 3 computes the set of representatives $\{1, 2, \ldots, m_\ell\}$, but for the purposes of enumerating $\mathcal{X}_G$, *any* set of representatives will do. Example 1 is particularly useful.

Example 1. Suppose 2 generates M_ℓ. This is the case if 2 is a primitive element modulo ℓ—that is, if 2 has order $(\ell-1)$ modulo ℓ—but also if 2 has order $(\ell-1)/2$ modulo ℓ and $\ell \equiv 3 \pmod 4$. In this case

$$M_\ell = \{2^i \bmod \ell : 0 \le i < m_\ell\} \,,$$

so $(0, 1, 2, 4, 8, \ldots, 2^{m_\ell})$ is an MDAC for M_ℓ using *only* doubling, and *no* differential additions. The corresponding `KernelPoints` replaces *all* of the `xADD`s in Algorithm 3 with cheaper `xDBL`s, trading $(\ell-5)\mathbf{M} + (\ell-5)\mathbf{a}$ for $(\ell-5)/2$ $\mathbf{C}$. The corresponding `KernelRange` is particularly simple: each element depends only on its predecessor, so the state consists of a single $(X_i : Z_i)$.

How often does this trick apply? The quantitative form of Artin's primitive root conjecture (see [30]) says that $M_\ell = \langle 2 \rangle$ for a little over half of all ℓ. Experimentally, 5609420 of the first 10^7 odd primes ℓ satisfy $M_\ell = \langle 2 \rangle$.

One might generalize Example 1 to other generators of M_ℓ: for example, if $M_\ell = \langle 3 \rangle$, then we could find an MDAC for $\{3^i \bmod \ell : 0 \le i < (\ell-1)/2\}$. But this is counterproductive: x-only tripling is *slower* than differential addition.

4.5 Stepping through cosets

What can we do when $M_\ell \neq \langle 2 \rangle$? A productive generalization is to let

$$A_\ell := \langle 2 \rangle \subseteq M_\ell \quad \text{and} \quad a_\ell := \#A_\ell \,,$$

and to try to compute a convenient decomposition of M_ℓ into cosets of A_ℓ. Within each coset, we can compute elements using repeated `xDBL`s as in Example 1; then, it remains to step from one coset into another using differential additions.

This can be done in a particularly simple way for the primes ℓ such that

$$M_\ell = \langle 2, 3 \rangle \,, \qquad \text{so} \qquad M_\ell = \bigsqcup_{i=0}^{m_\ell/a_\ell - 1} 3^i A_\ell \,. \tag{$*$}$$

We can move from the i-th to the $(i+1)$-th coset using the elementary relations

$$\begin{cases} c \cdot 2^{j+1} + c \cdot 2^j = 3c \cdot 2^j \\ c \cdot 2^{j+1} - c \cdot 2^j = c \cdot 2^j \end{cases} \qquad \text{for all integers } c \text{ and } j \geq 0. \tag{8}$$

In particular, having enumerated $3^i A_\ell$ by repeated doubling, we can compute an element of $3^{i+1} A_\ell$ by applying a differential addition to any two consecutive elements of $3^i A_\ell$ (and the difference is the first of them). Algorithm 4 minimizes storage overhead by using the last two elements of the previous coset to generate the first element of the next one. The `KernelRange` of Algorithm 4 therefore has an internal state of only two x-coordinates—so not only is it faster than the `KernelRange` of Algorithm 3, but it also has a smaller memory footprint.

Algorithm 4: Kernel enumeration for $\ell > 3$ satisfying $(*)$. Cost: $(1 - 1/a_\ell) \cdot m_\ell$ `xDBL`s and $m_\ell/a_\ell - 1$ `xADD`s.

Input: Projective x-coordinate $(X_G : Z_G)$ of the generator G of a cyclic subgroup of order ℓ in $\mathcal{E}(\mathbb{F}_q)$, where ℓ satisfies $(*)$.
Output: $\mathcal{X}_G$ as a list

$(a, b) \leftarrow (a_\ell, m_\ell/a_\ell)$
for $i = 0$ ***to*** $b - 1$ **do** `// Invariant:` $(X_{ai+j} : Z_{ai+j}) = x([3^i 2^{i(a-2)+(j-1)}]G)$
  if $i = 0$ **then**
    $(X_1 : Z_1) \leftarrow (X_G : Z_G)$
  else `// Compute new coset representative`
    $(X_{ai+1} : Z_{ai+1}) \leftarrow \texttt{xADD}\big((X_{ai} : Z_{ai}), (X_{ai-1} : Z_{ai-1}), (X_{ai-1} : Z_{ai-1})\big)$
  for $j = 2$ ***to*** a **do** `// Exhaust coset by doubling`
    $(X_{ai+j} : Z_{ai+j}) \leftarrow \texttt{xDBL}\big((X_{ai+j-1} : Z_{ai+j-1})\big)$
return $\big((X_1 : Z_1), \ldots, (X_{(\ell-1)/2} : Z_{(\ell-1)/2})\big)$

Algorithm 4 performs better the closer a_ℓ is to m_ℓ. In the best case, when $A_\ell = M_\ell$, it uses $m_\ell - 1$ `xDBL`s and no `xADD`s at all. The worst case is when the order of 2 in M_ℓ is as small as possible: that is, $\ell = 2^k - 1$. In this case $a_\ell = k$, and compared with Algorithm 3 we still reduce the `xADD`s by a factor of k.

4.6 The remaining primes

While 1878 of the 2261 odd primes $\ell \leq 20000$ satisfy $(*)$, there are still 383 primes that do not. We can, to some extent, adapt Algorithm 4 to handle these primes, but on a case-by-case basis and with somewhat less satisfactory results.

For example, the CSIDH-512 parameter set specifies 74 isogeny-degree primes

$$\ell = 3, 5, 7, 11, 13, \ldots, 367, 373, \text{ and } 587\,.$$

All but seven of these ℓ satisfy $(*)$: the exceptions are $\ell = 73$, 97, 193, 241, 313, and 337. Table 1 lists a candidate decomposition of M_ℓ for each of these ℓ. In each case, we need to produce an element of either $5A_\ell$ or $7A_\ell$. This can certainly be done using previously-computed elements, but this requires a larger internal state and a more complicated execution pattern, depending on ℓ.

Table 1. Primes ℓ in the CSIDH-512 parameter set that do not satisfy $(*)$.

Prime ℓ	a_ℓ	$[M_\ell : \langle 2, 3\rangle]$	Coset decomposition of M_ℓ	Notes
73	9	2	$M_{73} = A_{73} \sqcup 3A_{73} \sqcup 5A_{73} \sqcup 5 \cdot 3A_{73}$	
97	24	2	$M_{97} = A_{97} \sqcup 5A_{97}$	3 is in A_{97}
193	48	2	$M_{193} = A_{193} \sqcup 5A_{193}$	3 is in A_{193}
241	12	2	$M_{241} = \left(\bigsqcup_{i=0}^{4} 3^i A_{241}\right) \sqcup \left(\bigsqcup_{i=0}^{4} 7 \cdot 3^i A_{241}\right)$	
307	51	3	$M_{307} = A_{307} \sqcup 5A_{307} \sqcup 7A_{307}$	3 is in A_{313}
313	78	2	$M_{313} = A_{313} \sqcup 5A_{313}$	3 is in A_{193}
337	21	2	$M_{337} = \left(\bigsqcup_{i=0}^{3} 3^i A_{337}\right) \sqcup \left(\bigsqcup_{i=0}^{3} 5 \cdot 3^i A_{337}\right)$	

Example 2. Consider $\ell = 97$. Now $3 \equiv 2^{19} \pmod{97}$, so 3 is in A_{97}, and in fact $M_{97} = A_{97} \sqcup 5A_{97}$. To adapt Algorithm 4 to this case, we can still enumerate A_{97} using repeated doubling. Then, we must construct an element of $5A_{97}$ from elements of A_{97}, using a differential addition like $5 \cdot 2^i = 2^{i+2} + 2^i$ (difference $3 \cdot 2^i$) or $5 \cdot 2^i = 2^{i+1} + 3 \cdot 2^i$ (difference 2^i). Each involves near powers of 2 (modulo 97), but also $3 \cdot 2^i \equiv 2^{i+19} \pmod{97}$, which must be stored while enumerating A_{97}. This gives an algorithm using one `xADD` and 48 `xDBL`s, just like Algorithm 4, but with a slightly larger state and a more complicated execution pattern specific to $\ell = 97$. Alternatively, after enumerating A_{97}, we could redundantly recompute $3 = 1 + 2$ (difference 1) to get 5 as $1 + 4$ (difference 3) or $2 + 3$ (difference 1).

Ultimately, there does not seem to be a "one size fits all" generalization of Algorithm 4 for enumerating $\mathcal{X}_G$ without a more complicated state or redundant recomputations. We can get reasonable results for many ℓ not satisfying $(*)$ by finding a good MDAC for $M_\ell / \langle 2, 3\rangle$ and then using Algorithm 4 to exhaust the coset containing each representative but the savings are generally not optimal.

4.7 (In)Compatibility with Vélusqrt

It is natural to ask whether these techniques can be used to further accelerate the Vélusqrt algorithm [2], which evaluates isogenies of large prime degree ℓ in $\widetilde{O}(\sqrt{\ell})$ time (with $O(\sqrt{\ell})$ space). Vélusqrt never explicitly computes all of $\mathcal{X}_G$. Instead, it relies on the existence of a decomposition

$$S := \{1, 3, 5, \ldots, \ell - 2\} = (I + J) \sqcup (I - J) \sqcup K \tag{9}$$

where I, J, and K are sets of integers of size $O(\sqrt{\ell})$ such that the maps $(i, j) \to i + j$ and $(i, j) \to i - j$ are injective with disjoint images. In [2], these sets are

$$\begin{aligned} I &:= \{2b(2i + 1) : 0 \le i < b'\} && (\text{“giant steps”}), \\ J &:= \{2j + 1 : 0 \le j < b\} && (\text{“baby steps”}), \\ K &:= \{4bb' + 1, \ldots, \ell - 4, \ell - 2\} && (\text{“the rest”}), \end{aligned}$$

where $b := \lfloor \sqrt{\ell - 1}/2 \rfloor$ and $b' := \lfloor (\ell - 1)/4b \rfloor$.

The key thing to note here is that this decomposition is essentially additive, and the elements of I, J, and K are in arithmetic progression. Algorithm 4, however, is essentially multiplicative: it works with subsets in geometric progression. We cannot exclude the existence of subsets I, J, and K of size $O(\sqrt{\ell})$ satisfying (9) and which are amenable to enumeration by a variant of Algorithm 4 for some ℓ, but it seems difficult to construct nontrivial and useful examples.

5 Irrational kernel points: exploiting Frobenius

Now suppose G is defined over a nontrivial extension $\mathbb{F}_{q^k}$ of $\mathbb{F}_q$, but $\langle G \rangle$ is defined over the subfield $\mathbb{F}_q$: that is, it is Galois-stable. In particular, the q-power Frobenius endomorphism π of $\mathcal{E}$, which maps points in $\mathcal{E}(\mathbb{F}_{q^k})$ to their conjugates under $\mathrm{Gal}(\mathbb{F}_{q^k}/\mathbb{F}_q)$, maps $\langle G \rangle$ into $\langle G \rangle$, and hence restricts to an endomorphism of $\langle G \rangle$. But since the endomorphisms of $\langle G \rangle$ are $\mathbb{Z}/\ell\mathbb{Z}$, and Frobenius has no kernel (so π is not 0 on $\langle G \rangle$), it must act as multiplication by an eigenvalue $\lambda \neq 0$ on $\langle G \rangle$. The precise value of λ is not important here, but we will use the fact that λ has order k in $(\mathbb{Z}/\ell\mathbb{Z})^\times$ and order k' in $(\mathbb{Z}/\ell\mathbb{Z})^\times/\langle \pm 1 \rangle$.

Now let

$$F_\ell := \langle \lambda \rangle \subseteq M_\ell \qquad \text{and} \qquad c_F := [M_\ell : F_\ell] = m_\ell/k' \,.$$

Let R_0 be a set of representatives for M_ℓ/F_ℓ; set $S_0 := \{[r]G : r \in R_0\}$, and note

$$\#S_0 = (\ell - 1)/k' \,.$$

5.1 The cost of Frobenius

We want to use the Galois action to replace many **M** and **S** with a few **F**. For this to be worthwhile, **F** must be cheap: and it is, even if this is not obvious

given the definition of the Frobenius map on $\mathbb{F}_{q^k}$ as q-th powering. Indeed, we do not compute Frobenius by powering. Instead, we use the fact that Frobenius is $\mathbb{F}_q$-linear, acting as a $k \times k$ matrix (with entries in $\mathbb{F}_q$) on the coefficient vectors of elements in $\mathbb{F}_{q^k}$. The form of this matrix, and the cost of applying it, depends on the basis of $\mathbb{F}_{q^k}/\mathbb{F}_q$. For example:

1. If $k = 2$ and $\mathbb{F}_{q^2} = \mathbb{F}_q(\sqrt{\Delta})$, then Frobenius simply negates $\sqrt{\Delta}$ and the matrix is $\mathrm{diag}(1, -1)$, so $\mathbf{F} \approx 0$.
2. If $\mathbb{F}_{q^k}/\mathbb{F}_q$ is represented with a normal basis, then the matrix represents a cyclic permutation, and again $\mathbf{F} \approx 0$.

Even in the worst case where the basis of $\mathbb{F}_{q^k}/\mathbb{F}_q$ has no special Galois structure, $\mathbf{F}$ is just the cost of multiplying a k-vector by a $k \times k$ matrix over $\mathbb{F}_q$: that is, k^2 multiplications and $k^2 - k$ additions. This is close to the cost of one "schoolbook" $\mathbb{F}_{q^k}$-multiplication; so when $k \leq 12$, we have $\mathbf{F} \approx \mathbf{M}$.

5.2 Galois orbits

Each point $P \in \mathcal{E}(\mathbb{F}_{q^k})$ is contained in a *Galois orbit* containing all the conjugates of P. The kernel subgroup $\langle G \rangle$ breaks up (as a set) into *Galois orbits*: if we write

$$\mathcal{O}_P := \{P, \pi(P), \ldots, \pi^{k-1}(P)\} \qquad \text{for } P \in \mathcal{E}(\mathbb{F}_{q^k}),$$

then

$$\langle G \rangle = \{0\} \sqcup \begin{cases} \bigsqcup_{P \in S_0} \mathcal{O}_P & \text{if } k \text{ is even,} \\ \left(\bigsqcup_{P \in S_0} \mathcal{O}_P\right) \sqcup \left(\bigsqcup_{P \in S_0} \mathcal{O}_{-P}\right) & \text{if } k \text{ is odd.} \end{cases} \tag{10}$$

To get a picture of where we are going, recall from §3 that in general, isogeny evaluation can be reduced to evaluations of the kernel polynomial

$$D(X) := \prod_{P \in S} (X - x(P)),$$

where $S \subset \langle G \rangle$ is any subset such that $S \cap -S = \emptyset$ and $S \cup -S = \langle G \rangle \setminus \{0\}$. The decomposition of (10) can be seen in the factorization of $D(X)$ over $\mathbb{F}_{q^k}$:

$$D(X) = \prod_{P \in S} (X - x(P)) = \prod_{P \in S_0} \prod_{i=0}^{k'-1} (X - x(\pi^i(P))) = \prod_{P \in S_0} \prod_{i=0}^{k'-1} (X - x(P)^{q^i}),$$

and the factors corresponding to each P in S_0 are the irreducible factors of D over $\mathbb{F}_q$. Transposing the order of the products, if we let

$$D_0(X) := \prod_{P \in S_0} (X - x(P))$$

then for α in the base field $\mathbb{F}_q$, we can compute $D(\alpha)$ using

$$D(\alpha) = \mathrm{Norm}(D_0(\alpha)) \quad \text{for all } \alpha \in \mathbb{F}_q$$

where

$$\mathrm{Norm}(x) := \prod_{i=0}^{k'-1} x^{q^i} = x(x(\cdots(x(x)^q)^q\cdots)^q)^q\,,$$

which can be computed for the cost of $(k-1)\mathbf{F}+(k-1)\mathbf{M}$ (some multiplications can be saved with more storage, but for small k this may not be worthwhile).

Similarly, we can rewrite the rational map ϕ_x from (6) as

$$\phi_x(x) = x\cdot\Big[\prod_{P\in S}\Big(\frac{x\cdot x(P)-1}{x-x(P)}\Big)\Big]^2 = x\cdot\Big[\prod_{P\in S_0}\prod_{i=0}^{k'-1}\Big(\frac{x\cdot x(P)^{q^i}-1}{x-x(P)^{q^i}}\Big)\Big]^2.$$

Evaluating ϕ_x at α in $\mathbb{F}_q$, rearranging the products gives

$$\phi_x(\alpha) = \alpha\cdot\Big[\prod_{P\in S_0}\prod_{i=0}^{k'-1}\Big(\frac{\alpha\cdot x(P)^{q^i}-1}{\alpha-x(P)^{q^i}}\Big)\Big]^2 = \alpha\cdot\mathrm{Norm}(\overline{\phi}_x(\alpha))^2\,,$$

where

$$\overline{\phi}_x(X) := \prod_{P\in S_0}\frac{X\cdot x(P)-1}{X-x(P)}\,.$$

Projectively, from (7) we get $\phi_x : (U:V)\mapsto(U':V')$ where

$$\begin{cases} U' = U\cdot\big[\prod_{i=0}^{k'-1}\prod_{P\in S_0}(UX_P^{q^i}-Z_P^{q^i}V)\big]^2,\\ V' = V\cdot\big[\prod_{i=0}^{k'-1}\prod_{P\in S_0}(UZ_P^{q^i}-X_P^{q^i}V)\big]^2,\end{cases}$$

so if we set

$$F(U,V) := \prod_{P\in S_0}(U\cdot X_P - Z_P\cdot V)\quad\text{and}\quad G(U,V) := \prod_{P\in S_0}(U\cdot Z_P - X_P\cdot V)\,,$$

then for α and β in $\mathbb{F}_q$ we get

$$\phi_x((\alpha:\beta)) = (\alpha':\beta') := \big(\alpha\cdot\mathrm{Norm}(F(\alpha,\beta))^2 : \beta\cdot\mathrm{Norm}(G(\alpha,\beta))^2\big)\,.$$

5.3 Enumerating representatives for the Galois orbits.

We now need to enumerate a set S_0 of representatives for the Galois orbits modulo ±1 or, equivalently, a set of representatives R_0 for the cosets of F_ℓ in M_ℓ. Given an MDAC driving enumeration of the coset representatives, there are obvious adaptations of Algorithms 1 and 2 to this extension field case. Rather than iterating over all of the kernel x-coordinates, we just iterate over a subset representing the cosets of $\mathbb{F}_\ell$, and then compose with the norm.

Concretely, in Algorithm 2, we should

1. Replace `KernelRange` in Line 4 with a generator driven by an efficient MDAC for M_ℓ/F_ℓ;
2. Replace Line 10 with $(U_i', V_i') \leftarrow (U_i\cdot\mathrm{Norm}(U_i')^2, V_i\cdot\mathrm{Norm}(V_i')^2)$.

First, we can consider Algorithm 3: that is, enumerating M_ℓ/F_ℓ by repeated addition. Unfortunately, we do not have a nice bound on the length of this MDAC: the coset representatives may not be conveniently distributed over M_ℓ, so we could end up computing a lot of redundant points.

Example 3. Consider the "naive" S_0 comprised of the minimal elements (up to negation) in each Galois orbit. We computed the percentage of primes $3 \leq \ell < 10^4$ where an optimal MDAC (without redundant values) exists for this S_0:

k	1	2	3	4	5	6	7	8	9	10	11	12
%	100	100	100	100	84	86	76	67	60	56	45	42

For an example of what can go wrong, take $(\ell, k) = (89, 11)$. In this case, we get $R_0 = \{1, 3, 5, 13\}$; the shortest MDAC is $(0, 1, 2, 3, 5, 8, 13)$, which enumerates the kernel using one `xDBL` operation and six `xADD` operations, but requires the computation of two intermediate points not used in the final result.

But when we say that the coset representatives are not conveniently distributed over M_ℓ, we mean convenient with respect to addition. If we look at M_ℓ multiplicatively, then the path to efficient MDACs is clearer.

If $M_\ell = \langle 2, \lambda \rangle$ then we can take $R_0 = \{2^i : 0 \leq i < c_F\}$, which brings us to the 2-powering MDAC of Example 1—except that we stop after $c_F - 1$ `xDBL`s. We thus reduce the number of `xDBL`s by a factor of $\approx k'$, at the expense of two norm computations. This MDAC actually applies to more primes ℓ here than in §4, because we no longer need 2 to generate all of M_ℓ; we have λ to help. (In fact, the suitability of this MDAC depends not only on ℓ, but also on k.)

We can go further if we assume

$$M_\ell = \langle 2, 3, \lambda \rangle \,. \tag{$**$}$$

To simplify notation, we define

$$a_{\ell,k} := [\langle 2, \lambda \rangle : F_\ell] \,, \qquad b_{\ell,k} := [\langle 2, 3, \lambda \rangle : \langle 2, \lambda \rangle] = c_F / a_{\ell,k} \,.$$

Algorithm 5 is a truncated version of Algorithm 4 for computing S_0 instead of $\mathcal{X}_G$ when $(**)$ holds. Algorithm 6 uses Algorithm 5 to evaluate an ℓ-isogeny over $\mathbb{F}_q$ with kernel $\langle G \rangle$ at n points of $\mathcal{E}(\mathbb{F}_q)$, where $x(G)$ is in $\mathbb{F}_{q^{k'}}$ with $k' > 1$.

Table 2 compares the total costs of Algorithms 6 and 5 with Algorithms 1 and 3. In both algorithms, we can take advantage of the fact that many of the multiplications have one operand in the smaller field $\mathbb{F}_q$: notably, the multiplications involving coordinates of the evaluation points. In the context of isogeny-based cryptography (where curve constants look like random elements of $\mathbb{F}_q$), this means that in Algorithm 1, we can replace the $2\mathbf{M} + 2\mathbf{a}$ in Line 8 and the $2\mathbf{M} + 2\mathbf{S}$ in Line 10 with $2\mathbf{C} + 2\mathbf{a}$ and $2\mathbf{C} + 2\mathbf{S}$, respectively. Table 3 gives examples of the resulting costs for various (ℓ, k) with a single evaluation point.

Algorithm 5: Compute S_0 when $(**)$ holds. Cost: $b_{\ell,k} - 1$ `xADD`s and $(c_F - b_{\ell,k})$ `xDBL`s, or $(2c_F + 2b_{\ell,k} + 4)\mathbf{M} + (2c_F - 2)\mathbf{S} + (c_F - b_{\ell,k})\mathbf{C} + (4c_F + 4b_{\ell,k} - 6)\mathbf{a}$

Input: Projective x-coordinate $(X_G : Z_G)$ of the generator G of a cyclic subgroup of order ℓ in $\mathcal{E}(\mathbb{F}_{q^k})$, where ℓ satisfies $M_\ell = \langle 2, 3, \lambda \rangle$.
Output: S_0 as a list

Function `SZeroPoints`$((X_G : Z_G))$
  $(a, b) \leftarrow (a_{\ell,k}, b_{\ell,k})$
  for $i = 0$ ***to*** $b - 1$ **do** `// Invariant:` $\mathsf{x}_{ai+j} = x([3^i 2^{i(a-2)+(j-1)}]G)$
    if $i = 0$ **then**
      $\mathsf{x}_1 \leftarrow (X_G : Z_G)$
    else `// Compute new coset representative`
      $\mathsf{x}_{ai+1} \leftarrow \mathtt{xADD}(\mathsf{x}_{ai}, \mathsf{x}_{ai-1}, \mathsf{x}_{ai-1})$
    for $j = 2$ ***to*** a **do** `// Exhaust coset by doubling`
      $\mathsf{x}_{ai+j} \leftarrow \mathtt{xDBL}(\mathsf{x}_{ai+j-1})$
  return $(\mathsf{x}_1, \ldots, \mathsf{x}_{c_F})$

Algorithm 6: Isogeny evaluation using `SZeroPoints` and Frobenius. Cost: $2(c_F + k - 1)n\mathbf{M} + 2n\mathbf{S} + 2(cF + 1)n\mathbf{C} + 2c_F(n + 1)\mathbf{a} + 2(k - 1)n\mathbf{F}$ *plus* the cost of `SZeroPoints`.

Input: The x-coordinate $(X_G : Z_G)$ of a generator G of the kernel of an ℓ-isogeny ϕ, and a list of evaluation points $((U_i : V_i) : 1 \le i \le n)$
Output: The list of images $((U_i' : V_i') = \phi_x((U_i : V_i)) : 1 \le i \le n)$

$((X_1, Z_1), \ldots, (X_{c_F}, Z_{c_F})) \leftarrow$ `SZeroPoints`$((X_G : Z_G))$ `// Algorithm 5`
for $1 \le i \le c_F$ **do**
  $(\hat{X}_i, \hat{Z}_i) \leftarrow (X_i + Z_i, X_i - Z_i)$ `// 2a`
for $i = 1$ ***to*** n **do**
  $(\hat{U}_i, \hat{V}_i) \leftarrow (U_i + V_i, U_i - V_i)$ `// 2a`
  $(U_i', V_i') \leftarrow (1, 1)$
  for $j = 1$ ***to*** c_F **do**
    $(t_0, t_1) \leftarrow$ `CrissCross`$(\hat{X}_j, \hat{Z}_j, \hat{U}_i, \hat{V}_i)$ `// 2C + 2a`
    $(U_i', V_i') \leftarrow (t_0 \cdot U_i', t_1 \cdot V_i')$ `// 2M`
  $(U_i', V_i') \leftarrow (\mathtt{Norm}(U_i'), \mathtt{Norm}(V_i'))$ `// 2(k' - 1)M + 2(k' - 1)F`
  $(U_i', V_i') \leftarrow (U_i \cdot (U_i')^2, V_i \cdot (V_i')^2)$ `// 2C + 2S`
return $((U_1', V_1'), \ldots, (U_n', V_n'))$

Table 2. ℓ-isogeny evaluation comparison for kernels $\langle G \rangle$ defined over $\mathbb{F}_q$ but with $x(G) \in \mathbb{F}_{q^{k'}}$. Here, **C** denotes multiplications of elements of $\mathbb{F}_{q^{k'}}$ by elements of $\mathbb{F}_q$.

	Costello–Hisil (Algorithms 1 and 3)	This work (Algorithm 6)
M	$(\ell-1)n + 2\ell - 8$	$2(c_F + k' - 1)n + 2c_F + 2b_{\ell,k} + 4$
S	$2n + \ell - 3$	$2n + 2c_F - 2$
C	$(\ell+1)n + 1$	$2(c_F + 1)n + c_F - b_{\ell,k}$
a	$(n+1)(\ell+1) + 3\ell + 17$	$2c_F(n+1) + 4c_F + 4b_{\ell,k} - 6$
F	0	$2(k'-1)n$

Table 3. Examples of costs for evaluating an ℓ-isogeny at a single point over $\mathbb{F}_q$, with $x(G) \in \mathbb{F}_{q^{k'}}$, using Costello–Hisil (Algorithm 1 with 3, in white) and Algorithm 6 (in gray). For these k, it is reasonable to use the approximation $\mathbf{F} \approx \mathbf{M}$ (see §5.1).

$\ell = 13$						$\ell = 19$						$\ell = 23$					
k'	**M**	**S**	**C**	**a**	**F**	k'	**M**	**S**	**C**	**a**	**F**	k'	**M**	**S**	**C**	**a**	**F**
any	30	12	15	54	0	any	48	18	21	84	0	any	60	22	25	104	0
1	22	12	19	46	0	1	34	18	28	70	0	1	42	22	34	86	0
3	10	4	7	14	4	3	14	6	10	22	4	11	22	2	4	6	20
						9	18	2	4	6	16						

6 Applications to key exchange

Our algorithms could be applied in any cryptosystem involving isogenies of prime degree $\ell > 3$. We focus on key exchanges like CSIDH [5] here, but similar remarks apply for other schemes such as SQISign [16,17], SeaSign [15], and CSI-FiSh [4].

6.1 CSIDH and constant-time considerations

CSIDH is a post-quantum non-interactive key exchange based on the action of the class group of the imaginary quadratic order $\mathbb{Z}[\sqrt{-p}]$ on the set of supersingular elliptic curves $\mathcal{E}/\mathbb{F}_p$ with $\mathrm{End}_{\mathbb{F}_p}(\mathcal{E}) \cong \mathbb{Z}[\sqrt{-p}]$. The action is computed via compositions of ℓ_i-isogenies for a range of small primes $(\ell_1, \ldots, \ell_m)$.

CSIDH works over prime fields $\mathbb{F}_p$, so the methods of §5 do not apply; but Algorithm 4 may speed up implementations at least for the ℓ_i satisfying $(*)$. (We saw in §4.6 that 67 of the 74 primes ℓ_i in the CSIDH-512 parameter set met $(*)$).

The true speedup depends on two factors. The first is the number of evaluation points. Costello and Hisil evaluate at a 2-torsion point other than $(0,0)$ in order to interpolate the image curve. The constant-time CSIDH of [19] evaluates at one more point (from which subsequent kernels are derived)—that is, $n = 2$; [21] uses $n = 3$; [7] discusses $n > 3$. For large n, the cost of Algorithm 1 overwhelms kernel enumeration, but our results may still make a simple and interesting improvement when n is relatively small.

The second factor is the organisation of primes into batches for constant-time CSIDH implementations. CTIDH [1] hides the degree ℓ using the so-called

matryoshka property: ℓ_i-isogeny evaluation is a sub-computation of ℓ_j-isogeny computation whenever $\ell_i < \ell_j$ using Algorithms 1 and 3. Organising primes into similar-sized batches, we can add dummy operations to disguise smaller-degree isogenies as isogenies of the largest degree in their batch.

Our Algorithm 4 has a limited matryoshka property: ℓ_i-isogenies are sub-computations of ℓ_j-isogenies if $a_{\ell_i} \leq a_{\ell_k}$ and $m_{\ell_i}/a_{\ell_i} \leq m_{\ell_j}/a_{\ell,j}$. For constant-time implementations, it would make more sense to make all primes in a batch satisfying $(*)$ a sub-computation of an algorithm using the maximum a_ℓ and maximum m_ℓ/a_ℓ over ℓ in the batch. Redistributing batches is a delicate matter with an important impact on efficiency; therefore, while our work improves the running time for a fixed ℓ, its impact on batched computations remains uncertain, and ultimately depends on specific parameter choices.

6.2 CRS key exchange

The historical predecessors of CSIDH, due to Couveignes [11] and Rostovtsev and Stolbunov [24,27,28], are collectively known as CRS. Here the class group of an quadratic imaginary order $\mathcal{O}$ acts on an isogeny (sub)class of elliptic curves $\mathcal{E}$ with $\mathrm{End}(\mathcal{E}) \cong \mathcal{O}$. CRS performance was greatly improved in [13] using Vélu-style isogeny evaluation, but this requires finding ordinary isogeny classes over $\mathbb{F}_p$ with rational ℓ_i-torsion points over $\mathbb{F}_{q^{k_i}}$ with k_i as small as possible for as many ℓ_i as possible.

One such isogeny class over a 512-bit prime field is proposed in [13, §4]. The curves ℓ-isogenies with kernel generators over $\mathbb{F}_p$ for $\ell = 3$, 5, 7, 11, 13, 17, 103, 523, and 821, and over $\mathbb{F}_{p^k}$ for $\ell = 19$, 29, 31, 37, 61, 71, 547, 661, 881, 1013, 1181, 1321, and 1693. These "irrational" ℓ are an interesting basis of comparison for our algorithms: Table 4 shows that there are substantial savings to be had.

References

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Table 4. Costello–Hisil (Algorithms 1 and 3, in white) vs. Algorithm 6 (in gray) for the CRS parameters with $k > 1$ proposed in [13]. We omit $(\ell, k) = (1321, 5)$, since in this case $M_\ell \neq \langle 2, 3, \lambda \rangle$. Here **M**, **S**, **a**, and **F** refer to operations on elements of $\mathbb{F}_{q^{k'}}$, while **C** denotes multiplications of elements of $\mathbb{F}_{q^{k'}}$ by elements of $\mathbb{F}_q$.

k	ℓ	$a_{\ell,k}$	$b_{\ell,k}$	**M**	**S**	**C**	**a**	**F**
3	19	3	1	18n + 30	2n + 16	20n + 1	20n + 64	0
				10n + 4	2n + 4	8n + 2	8n + 14	4n
	661	110	1	660n + 1314	2n + 658	662n + 1	662n + 2632	0
				224n + 218	2n + 218	222n + 109	222n + 656	4n
4	1013	23	11	1012n + 2018	2n + 1010	1014n + 1	1014n + 4040	0
				48n + 524	2n + 504	48n + 242	48n + 1074	2n
	1181	59	5	1180n + 2354	2n + 1178	1182n + 1	1182n + 4712	0
				120n + 596	2n + 588	120n + 290	120n + 1302	2n
5	31	1	3	30n + 54	2n + 28	32n + 1	32n + 112	0
				10n + 8	2n + 4	4n	4n + 14	8n
	61	6	1	60n + 114	2n + 58	62n + 1	62n + 232	0
				20n + 10	2n + 10	14n + 5	14n + 32	8n
7	29	2	1	28n + 50	2n + 26	30n + 1	30n + 104	0
				16n + 2	2n + 2	6n + 1	6n + 8	12n
	71	5	1	70n + 134	2n + 68	72n + 1	72n + 272	0
				22n + 8	2n + 8	12n + 4	12n + 26	12n
	547	39	1	546n + 1086	2n + 544	548n + 1	548n + 2176	0
				90n + 76	2n + 76	80n + 38	80n + 230	12n
8	881	55	2	880n + 1754	2n + 878	882n + 1	882n + 3512	0
				116n + 220	2n + 218	112n + 108	112n + 548	6n
9	37	2	1	36n + 66	2n + 34	38n + 1	38n + 136	0
				20n + 2	2n + 2	6n + 1	6n + 8	16n
	1693	94	1	1692n + 3378	2n + 1690	1694n + 1	1694n + 6760	0
				204n + 186	2n + 186	190n + 93	190n + 560	16n

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A Computing a kernel generator

One task that poses a challenge is to find a point $G \in \mathcal{E}(\mathbb{F}_{q^k})$. In this section, we will illustrate an efficient method for computing a point with the necessary properties for use in the isogeny evaluation.

A.1 The subgroup H_k.

To compute a rational isogeny, our first step will be to sample a random point $P \in \mathcal{E}(\mathbb{F}_{q^k})$ of order ℓ. For this, letting $N_k := \#\mathcal{E}(\mathbb{F}_{q^k})$, one could sample a random point P, and compute $P_\ell = [N_k/\ell]P$. Then P_ℓ is either 0 or a point of order ℓ. If the order of P_ℓ is not ℓ, one tries again with a new choice of P.

Remark 1. In the special case that ℓ^2 divides $\#\mathcal{E}(\mathbb{F}_{q^k})$, we instead choose $N_k = exp(\mathbb{F}_{q^k})$, the exponent of the group order. We do this to avoid having a cofactor, N_k/ℓ, that "kills" certain torsion points.

In our context, we are assured that the P_ℓ we are looking for is *not* in $\mathcal{E}(\mathbb{F}_q)$, or indeed in any $\mathcal{E}(\mathbb{F}_{q^i})$, for any proper divisor i of k. We can therefore save some effort by sampling P_ℓ from the genuinely "new" subgroup of $\mathcal{E}(\mathbb{F}_{q^k})$.

Recall that $\mathcal{E}(\mathbb{F}_{q^i}) = \ker(\pi^i - [1])$ for each $i > 0$. For each $k > 0$, then we define an endomorphism

$$\eta_k := \Phi_k(\pi) \in \mathrm{End}(\mathcal{E})$$

where $\Phi_k(X)$ is the k-th cyclotomic polynomial (that is, the minimal polynomial over $\mathbb{Z}$ of the primitive k-th roots of unity in $\overline{\mathbb{F}}$). The subgroup

$$H_k := \ker(\eta_k) \subset \mathcal{E}(\mathbb{F}_{q^k})$$

satisfies

$$\mathcal{E}(\mathbb{F}_{q^k}) = H_k \oplus \sum_{i|k, i\neq k} \mathcal{E}(\mathbb{F}_{q^i}).$$

The key fact is that in our situation, $\mathcal{E}[\ell](\mathbb{F}_{q^k}) \subset H_k$.

Generating elements of $\mathcal{E}[\ell](\mathbb{F}_{q^k})$. We always have $\Phi_k(X) \mid X^k - 1$, so for each $k > 0$ there is an endomorphism

$$\delta_k := (\pi^k - [1])/\eta_k \in \mathrm{End}(\mathcal{E})\,,$$

and $\delta_k(\mathcal{E}(\mathbb{F}_{q^k})) \subset H_k$. We can therefore sample a point P_ℓ in $\mathcal{E}[\ell](\mathbb{F}_{q^k})$ by computing

$$P_\ell = [h_k/\ell]\delta_k(P) \qquad \text{where} \qquad h_k := \#H_k$$

and P is randomly sampled from $\mathcal{E}(\mathbb{F}_{q^k})$.

Table 5 lists the first few values of h_k and δ_k. We see that evaluating δ_k amounts to a few Frobenius operations (which are almost free, depending on the field representation) and a few applications of the group law, so this approach

saves us a factor of at least $1/k$ in the loop length of the scalar multiplication (compared wih computing P_ℓ as $[N_k/\ell]P$), but for highly composite k we save much more.

The value $\varphi(k)$ of the Euler totient function plays an important role. We have $h_k = q^{\varphi(k)} + o(q^{\varphi(k)})$, so computing $[h_k/\ell]$ instead of $[N_k/\ell]$ allows us to reduce the loop length of basic scalar multiplication algorithms from $k \log_2 q$ to $\varphi(k) \log_2 q$, which is particularly advantageous when k is highly composite.

The action of Frobenius on H_k. The Frobenius endomorphism π commutes with η_k, and therefore restricts to an endomorphism of H_k. If $G \subset H_k$ is a subgroup of prime order ℓ and fixed by π, then π will act on G as multiplication by an integer eigenvalue λ (defined modulo ℓ). Since $\eta_k = \Phi_k(\pi) = 0$ on H_k by definition, we know that λ is a k-th root of unity in $\mathbb{Z}/\ell\mathbb{Z}$.

Scalar multiplication with Frobenius. Now, $H_k \cong \mathbb{Z}/d_k\mathbb{Z} \times \mathbb{Z}/e_k\mathbb{Z}$, where $d_k \mid e_k$ and (by the rationality of the Weil pairing) $d_k \mid q^k - 1$. Typically, d_k is very small compared with e_k. If $\ell \nmid d_k$, then we can replace H_k with the cyclic subgroup $H'_k := [d_k]H_k$, and h_k with $h'_k := e_k/d_k$. Now, π induces an endomorphism of H'_k, and therefore acts as multiplication by an eigenvalue λ defined modulo h'_k.

We want to compute $[c_k]P$ for P in H'_k, where $c_k := h'_k/\ell$. Since $\Phi_k(\pi) = 0$, the eigenvalue λ is a root of Φ_k (i.e., a primitive k-th root of unity) modulo h'_k. We can compute $a_0, \ldots, a_{k-1}$ such that

$$c_k \equiv \sum_{i=0}^{k-1} a_i \lambda^i \pmod{h'_k},$$

with each coefficient $a_i \approx (h'_k)^{1/\varphi(k)}$ in $O(q)$, and then

$$[c_k]P = \sum_{i=0}^{k-1} [a_i]\pi^i(P).$$

If we precompute the various sums of conjugates of P,then we can compute $[c_k]$ using a multiscalar multiplication algorithm with a loop of length only $\log_2 q$. This might be particularly interesting in the cases where $\varphi(k) = 2$ (which corresponds to GLV multiplication) or 4.

Example 4. Consider $k = 3$: we have $\mathcal{E}(\mathbb{F}_{q^3}) \cong \mathcal{E}(\mathbb{F}_q) \oplus H_3$, and $\#H_3 = N_3/N_1$.

We first note that π^3 fixes the points in $\mathcal{E}(\mathbb{F}_{q^3})$, so $\pi^3 - [1] = [0]$ on $\mathcal{E}(\mathbb{F}_{q^3})$. By similar logic, the regular Frobenius map will fix the points in $\mathcal{E}(\mathbb{F}_q)$, meaning $\pi - [1] = [0]$ holds only for points contained entirely in the $\mathcal{E}(\mathbb{F}_q)$ portion of $\mathcal{E}(\mathbb{F}_{q^3})$. Therefore, by computing $P_H = (\pi - 1)P$, we are "killing" the $\mathcal{E}(\mathbb{F}_q)$ part of P, leaving only the part lying in the subgroup H_3. This computation is easy enough to do, and so now we need only compute $P_\ell = [N_3/N_1/\ell]P_H$, thereby saving us about a third of the multiplications.

Table 5. The first few values of h_k and δ_k.

k	h_k	δ_k	$\varphi(k)$
1	1	$[1]$	
2	$N_2/N_1 = q + O(\sqrt{q})$	$\pi - [1]$	1
3	$N_3/N_1 = q^2 + O(q^{3/2})$	$\pi - [1]$	2
4	$N_4/N_2 = q^2 + O(q)$	$\pi^2 - [1]$	2
5	$N_5/N_1 = q^4 + O(q^{7/2})$	$\pi - [1]$	4
6	$(N_6N_1)/(N_2N_3) = q^2 + O(q^{3/2})$	$(\pi + [1])(\pi^3 - [1])$	2
7	$N_7/N_1 = q^6 + O(q^{7/2})$	$\pi - [1]$	6
8	$N_8/N_4 = q^4 + O(q^2)$	$\pi^4 - [1]$	4
9	$N_9/N_3 = q^6 + O(q^{4/2})$	$\pi^3 - [1]$	6
10	$(N_{10}N_1)/(N_2N_5) = q^4 + O(q^{7/2})$	$(\pi + [1])(\pi^5 - [1])$	4
11	$N_{11}/N_1 = q^{10} + O(q^{19/2})$	$\pi - [1]$	10
12	$(N_{12}N_2)/(N_4N_6) = q^4 + O(q^3)$	$(\pi^2 + [1])(\pi^6 - [1])$	4

The "twist trick". When k is even, if we use x-only scalar multiplication, then the following lemma allows us to work over $\mathbb{F}_{q^{k/2}}$ instead of $\mathbb{F}_{q^k}$. In the case $k = 2$, this is known as the "twist trick".

Lemma 1. *If k is even, then every point P in H_k has $x(P)$ in $\mathbb{F}_{q^{k/2}}$.*

Proof. If k is even, then η_k divides $\pi^{k/2}+1$, so $\pi^{k/2}$ acts as -1 on $H_k = \ker(\eta_k)$: that is, if P is in H_k, then $\pi^{k/2}(P) = -P$, so $x(P)$ is in $\mathbb{F}_{q^{k/2}}$.

Example 5. Consider $k = 6$. We take a random point R in $E(\mathbb{F}_{q^6})$, and compute $R' := \pi^3(R) - R$, then $P := \pi(R') + R'$; now $P = \delta_6(R)$ is in H_6, and $x(P)$ is in $\mathbb{F}_{q^3}$. We have $h_6 = N_1^2 - (q+1)N_1 + q^2 - q + 1 \approx q^2$, and we need to compute $x([c_{\ell,6}]P)$ where $c_{\ell,6} := h_6/\ell$. Since $x(P)$ is in $\mathbb{F}_{q^3}$, we can do this using x-only arithmetic and the Montgomery ladder working entirely over $\mathbb{F}_{q^3}$.

The improvements outlined in this section are summarized in Algorithm 7.

Algorithm 7: Computation of Kernel Generator.

Input: $\mathcal{E}$ an elliptic curve defined over $\mathbb{F}_q$, ℓ an integer, k such that $\mathbb{F}_{q^k}$ contains an ℓ-torsion point.
Output: P, a point on $\mathcal{E}(\mathbb{F}_{q^k})$ of order ℓ.

$P_\ell \leftarrow (0:1:0)$
repeat
  $P \leftarrow$ `RandomPoint`$(\mathcal{E}(\mathbb{F}_{q^k}))$
  $c \leftarrow h_k/\ell$ `// `h_k` taken from Table 5.`
  $P' \leftarrow \delta_k(P)$ `// `δ_k` taken from Table 5.`
  $P_\ell \leftarrow [c]P'$
until $P_\ell = (0:1:0)$
return P_ℓ