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A Case Study of the Diversity and Optimization of Mathematical Definitions and Symbolization

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Abstract. Taking the idea of "reflective and critical optimization thinking", this article first examines "the outlook on development of big mathematics" in International Mathematics Day in 2003 from the perspective of the history of mathematics. Then according to the requirements of mathematical core literacy and mathematical beauty, a deep analysis is conducted on the limitations of the teaching methods in the Turing mathematics and statistics series "Terence Tao Analysis". The case study shows that (1) the sets, relations, functions and operations present a concentric circle relationship of layer-by-layer proper subsets; (2) rejecting mathematical notation damages the beauty of simplicity and learning efficiency; and (3) forcing the first mapping function to be a surjection unnecessarily limits the wider range of applications that composite functions can have, and it lacks the unified beauty of mathematical definitions. Finally, itemized strategies and suggestions for improvement and optimization are proposed.

1 INTRODUCTION

On February 24, 2023, during a meeting with Mr. Shing-Tung Yau, the first Chinese in the world to win the Fields Award, the highest honor in international mathematics community Premier Li Keqiang [1] of the State Council pointed out that scientific and technological innovation and development require important support from basic sciences, and mathematics is the foundation of basic sciences, which can be said to be the crown of natural sciences. Mr. Yau's mathematical world outlook [2] includes but is not limited to "the beauty and truth of mathematics will lead the encyclopedia and drive science to the forefront of the world." As academician Li Daqian pointed out, "High technology is essentially a mathematical technology."

March 14, 2023 is the fourth "International Mathematics Day". Zhang Jiping [3], a Boya lecture professor at Peking University, was invited to give a speech on the topic of "the development view of big mathematics". The famous mathematician Poincare [2,4] summarized mathematical beauty as the beauty of symmetry in graphical structures, the beauty of order in logical reasoning, the beauty of simplicity in mathematical formulas, the beauty of singularity in methods and strategies, and the unified beauty in mathematical definitions. This paper first focuses on examining the keynote speech from the perspective of the history of mathematics. Then, based on the optimization ideas called for in the era of innovation driven technology, such as "dare to be the first, adhere to integrity and innovation", "reflective and critical thinking", and "wise people (including masters and mathematical geniuses) must have one mistake after a thousand considerations", and according to the requirements of mathematical core literacy and mathematical beauty, in-depth analysis of the limitations and shortcomings of the teaching methods of the popular textbook "Terence Tao Analysis"[5, 6] is conducted, and advice and suggestions are provided.

2 ANALYSIS OF THE KEYNOTE SPEECH FROM THE HISTORY OF MATHEMATICS

Based on the professional qualities and professional habits of mathematics teachers with rigorous logic, after reviewing the full text [3] we find that the following two claims are contrary to the history of mathematics.

Claim 1: "There is no distinction between old and new mathematics."

Analysis: It ignores the truth of the history of mathematics at all times and in all countries (e.g., in the first half of the 19th century, mathematical genius Galois creatively invented and applied the theory of group theory to study the solutions of algebraic higher order equations, which had a decisive impact on the development of algebra and was the watershed between ancient algebra and modern

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algebra), and it also violates the "functional view of movement and change" advocated by the mathematical community and dialectical materialism epistemology (people's understanding of the world has the characteristics of scientific nature, repetition, and ascendancy, as well as the process of infinite deepening)[7].

Claim 2: "Over 150 years since Galois was killed at the age of 21, the finite fields he created had no practical use except to arouse people's pure mathematical interest." The result of our calculation shows that this claim is not true. In fact, based on a conservative estimate, 1832+151=1983 (years). On the other hand, as early as in 1976, American scholars Diffie and Hellman [8] first proposed a public-key cryptography algorithm in their famous paper "New Directions in Cryptography" to solve key distribution and management issues, the security of which is based on the difficulty of solving discrete logarithm problems over finite field Z_p (p is a large prime). Now this key exchange technology is applied in many commercial products. Our overall sense of this speech is exactly what Lao Tzu said: "Great things in the world must be done in detail. "

3 A CASE STUDY OF THE DIVERSITY AND OPTIMIZATION OF MATHEMATICAL DEFINITIONNS

3.1 Proper inclusions of sets, relations, functions and operations

Definition 1[9-11]: An n-ary operation on a set S is a function $\star : S \times S \times \cdots \times S \to S$, where the domain is the product of n factors.

Definition 2[5, 6]: Informally, a function f: $X \to Y$ from one set X to another

set Y is an operation which assigns to each element (or "input") x in X, a single element (or "output") $f(x)$ in Y.

We point out that although the author claims that Definition 2 is described “informally" with the right to use metaphors such as "input... output ...", there is a serious mistake in the logic of Claim 3: "A function is an operation".

Analysis: To visually represent the proper inclusions of sets, relationships, functions and operations, we draw the following Venn diagram (Fig. 1) based on references [9-12]. We determine that the function $f = \{(0,1)\}$ from domain $\{0\}$ to range $\{1\}$ is not any n-ary operation (nor is it any scalar multiplication in a vector space), for $\{0\} \neq \{1\}$. This counterexample indicates that the innermost two layers in Fig. 1 are a proper inclusion relation, and therefore Claim 3 is logically false. Conversely, the proposition "An operation is a special kind of function." is logically true.

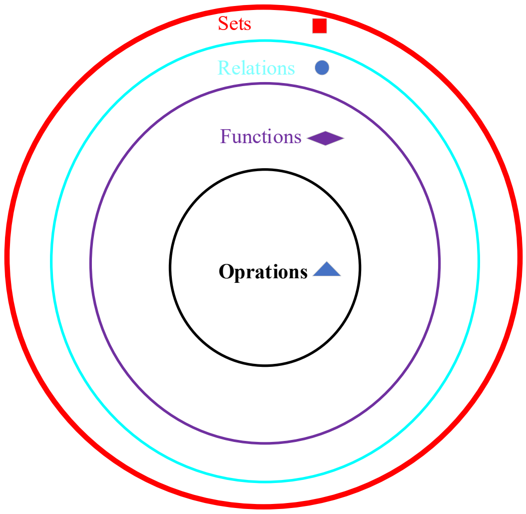

Fig. 1. Proper inclusions of sets, relations, functions and operations

3.2 Functions are essentially "sets" from modern mathematics

Claim 4[5, 6]: "Strictly speaking, a function is not a set (S_1), nor is a set a function (S_2)."

From the perspective of mathematical history[11, 12], since the time when ordered pairs were set theorized (i.e., $(a,b) = \{\{a\},\{a,b\}\}$ or $(a,b) = \{a,\{a,b\}\}$), a function has strictly been defined as a binary relation with a unique functional value for each argument, and the latter has further been defined as any subset f of the Cartesian product of two sets A (domain) and B (co-domain). Thus S_1 is inconsistent with the modern mathematical viewpoint of keeping pace with the times and the objective reality. Additionally, as a counterexample to S_2 the above set $f = \{(0,1)\}$ is indeed a function from $\{0\}$ to $\{1\}$. Secondly, we point out that the above counterexample is by no means accidental: as long as $(0,1)$ is replaced by any element (a,b) of the double Cartesian product R^2 of the set of real numbers, uncountable (infinite) counterexamples can be constructed.

It also deserves attention that literature [13] highlights an exclusive view on the traditional definition of a function: “One should avoid using the commonly used function view as a rule for assigning values to arguments, because it incorrectly shows that an implementable rule always exists.” We understand their purport: "logically, a diagram does not represent any rigorous proof." In the era of big data and artificial intelligence, to implement the educational concept of "computational thinking", even the concept of a function in "Discrete Mathematics"[9], a national planning textbook for new engineering subjects, has been defined as "a binary relation with a unique image" according to the language of set theory. The first author of this article has employed the above textbook since 2008, and years of practical experience [9,11,12] shows that as long as we step by step and follow up with examples, it is feasible for most students to accept this modern definition, which does not constitute a teaching difficulty in itself. Therefore, we sincerely advise that the author had better not mention Claim 4 in the new version of [5,6] anymore in the future.

3.3 Rejecting mathematical logic symbols inevitably pay significant costs in time and space

An excellent mathematician knows how to "learn from masters (Abel)": mathematics is signs plus logic（Russell）； the simplicity of mathematical symbols helps improve the efficiency of thinking (Kline). However, in quite a few cases, Mr. Tao prefers to use "heavy and long" text strings rather than using standard mathematical logic symbols, such as “$=, \geq, \leq, \Rightarrow, \forall, \exists$”. The consequence of this teaching method is to damage the simplicity beauty and learning efficiency; because a little makes a lot, one cannot despise it. Here is just one example. Definition 5.5.5 (Least upper bound) [5, 6]: Let E be a subset of R , and M be a real number. We say that M is a *least upper bound* for E iff (a) M is an upper bound for E , and (b) any other upper bound M' for E must be larger than or equal to M .

First, we observe the phenomenon of "loose logic" in the above definition: of course, "other" implies "not equal to". Next, we propose a concise version of the definition as follows. Let $E \subset \mathrm{R},\ M \subset \mathrm{R}$. M is called a *least upper bound* for E if (a) M is an upper bound for E , and (b) for E any upper bound $M' \geq M$.

3.4 The prerequisite for composite functions should not be too narrow—learn mainstream standards from masters

Definition 3.3.10 (Composition) [5, 6]: Let f: $X \rightarrow Y$ and g: $Y \rightarrow Z$ be two functions such that the range of f is the same set as the domain of g (hereinafter, S denotes this condition). We then define the composition $g \circ f : X \rightarrow Z$ of the two functions g and f to be the function defined by the formula $(g \circ f)(x) := g(f(x))$.

We propose an example to show that the condition S will largely limit the research scope of composite functions. Let

$$X = \{1,\ 2\} \xrightarrow{f} Y = \{3,\ 4\} \xrightarrow{g} Z = \{5\}$$

where $f = \{(1,3),(2,3)\},\ g = \{(3,5),(4,5)\}$. On the one hand, by the composite definition (without S) in [9-14], we have the composite function $g \circ f = \{(1,\ 5),(2,\ 5)\}$. On the other hand, because $\mathrm{Ran}\ f = \{3\} \neq Y = \mathrm{Dom}\ g$, we see by definition 3.3.10 that $g \circ f$ is not defined (meaningless). This example indicates that the limitation of definition 3.3.10 stems from the inability to conduct relevant research due to the excessively high requirement for “f is surjective” and misses the unified beauty in mathematical definitions. Advice: remove S. Finally, it is worth noting that in Proposition 3.1.28[5, 6] (Sets form a Boolean algebra), neither $A \cap A = A$ nor $A \cup A = A$ can be called “Identity”, which must be rewritten as “Idempotence” [9, 15].

4 CONCLUSIONS

The case study in this article shows that (1) the history of mathematics is a necessary quality for cultivating people to form a correct view of mathematics; (2) an operation is a function, but the converse is not true; (3) just like "symmetric cryptosystems" and "public-key cryptosystems" each have their own advantages in information security and computational efficiency, the traditional "rule" definition of functions and the "set" definition of modern mathematics should also respect each other, and take their own needs based on different application objects; (4) making full use of the symbols of mathematical logic, we propose a concise definition version of a least upper bound; and (5) forcing the first mapping function (metaphorically referred to as "vanguard") to be a surjection unnecessarily limits the wider range of applications that composite functions can have, and it lacks the unified beauty of mathematical definitions.

Our perception is as follows: In the face of science, everyone is equal. Teaching is endless, achieving the utmost in detail. On the basis of learning from excellent mathematical culture at home and abroad, we innovate upholding fundamental principles, make efforts to improve the quality of three-dimensional high-quality teaching materials for a mathematically powerful country, and pay attention to balancing contemporaneity, scientificity, readability and mathematical beauty. The major prerequisite for truly obtaining high-quality undergraduate textbooks and teaching methods is a work attitude of "persistent focus and attention to details", and the key lies in the responsibility of "striving for excellence and fulfilling one's duties to serve the country".

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