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Теоретико-числові функції для цілих ґаусових чисел

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Number-theoretic functions for Gaussian integers

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Класичні теоретико-числові функції: кількість дільників $\tau(n)$, сума дільників $\sigma(n)$ і добуток дільників $\pi(n)$ числа n — узагальнюються на кільце $\mathbb{Z}[i]$ цілих ґаусових чисел. Для обчислення відповідних функцій $\tau^*(\alpha)$, $\sigma_m^*(\alpha)$ і $\pi^*(\alpha)$ одержано явні формули, що використовують канонічний розклад числа α. Досліджено ряд властивостей цих функцій, зокрема, оцінки згори для функцій $\tau^*(\alpha)$ і $\sigma_m^*(\alpha)$ та властивості, пов'язані із подільністю значень цих функцій на ті чи інші числа. Досліджуються також властивості сум добутків степенів дільників числа $\alpha \in \mathbb{Z}[i]$.

Ключові слова: цілі ґаусові числа, дільник, кількість дільників, сума дільників, добуток дільників.

The classical number-theoretic functions—a number of divisors $\tau(n)$, sum of the divisors $\sigma(n)$ and product of the divisors $\pi(n)$ of a positive integer n—were generalized to the ring $\mathbb{Z}[i]$ of Gaussian integers. For the evaluation of the corresponding functions $\tau^*(\alpha), \sigma_m^*(\alpha)$ and $\pi^*(\alpha)$, obtained were the explicit formulae that use the canonical representation of α. A number of properties of these functions were studied, in particular, estimates from above for the functions $\tau^*(\alpha)$ and $\sigma_m^*(\alpha)$ and the properties connected with divisibility of their values by certain numbers. Researched are also sums of products of powers of the divisors for $\alpha \in \mathbb{Z}[i]$.

Key Words: Gaussian integers, divisor, number of divisors, sum of the divisors, product of the divisors.

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Introduction

Gaussian integers, i.e. complex numbers of the form $a + bi$, where $a, b \in \mathbb{Z}$, were introduced by Gauss in the late 1820s in his studies of the law of biquadratic reciprocity [1]. Then he explored a number of their properties: relation of divisibility, unit divisors, primality criteria for Gaussian integers, residue classes, multiplicative orders, primitive roots, and proved an analogue of the fundamental theorem of arithmetic [2]. After Gauss, these numbers were used in studying of multiple number-theoretic issues (particularly effectively for researching solvability of certain classes of diophantine equations), as well as in other mathematical branches, for instance, in the theory of elliptic functions. However, there was in fact no systematic research of the ring $\mathbb{Z}[i]$, especially since in the middle of the XIX century the much more general theory of algebraic numbers began to develop. In particular, for $\mathbb{Z}[i]$ there was no study of the properties of the natural number-theoretic functions—the number, sum and product of the divisors—that play a great role in classical number theory. In this paper, we partially complete this gap.

The ring $\mathbb{Z}[i]$ of Gaussian integers

The ring $\mathbb{Z}[i]$ of Gaussian integers is a subset

$$\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$$

of the field $\mathbb{C}$ of complex numbers. Further we present (without proof) only a few properties of this ring that we need (one may have a more detailed view of these properties in [3], [4], [5]).

The invertible elements of the ring $\mathbb{Z}[i]$ are called units. There are exactly four of them: $1, -1, i, -i$. Gaussian integers α and β are called associates (which is denoted by $\alpha \sim \beta$) if $\beta = \gamma\alpha$, where γ is a unit. Note that associates have the same divisors.

For a Gaussian integer $\alpha = a+bi$, the number $N(\alpha) = a^2 + b^2$ is called the norm of α. The complex conjugate $a - bi$ of α is denoted by $\overline{\alpha}$.

They say that a Gaussian integer α is divisible by a non-zero Gaussian integer β (or, equivalently, β is said to divide α, which is denoted by $\beta \mid \alpha$) if there exists a Gaussian integer γ such that $\alpha = \beta\gamma$. In this case, β is called a divisor of α.

Proposition 1. *A Gaussian integer β divides a Gaussian integer α if and only if $\overline{\beta}$ divides $\overline{\alpha}$.*

The divisors of α of the form $\pm 1, \pm i, \pm\alpha, \pm i\alpha$ are called trivial. All the other divisors are called non-trivial. Gaussian integers ± 1 and $\pm i$ have exactly 4 divisors each. All the other non-zero Gaussian integers have at least 8 divisors.

A common divisor of two Gaussian integers α and β that is divisible by all their common divisors is called a greatest common divisor. Every two Gaussian integers α and β have exactly 4 greatest common divisors, and these divisors are associates. A non-zero Gaussian integer is said to be prime if all of its divisors are units or its associates.

Two Gaussian integers are said to be relatively prime if their greatest common divisors are units.

Theorem 1. *Every Gaussian integer α with $N(\alpha) > 1$ can be factorized into primes in a unique way up to the order of the factors and their associativity.*

Proposition 2. *A Gaussian integer $a + bi$ is divisible by $1 + i$ if and only if the numbers a and b are of the same parity.*

Theorem 2. *Every Gaussian prime ρ satisfies one of the following conditions*:
1) *ρ is an associate of* $1 + i$;
2) *ρ is an associate of a prime positive integer* $p \equiv 3 \pmod 4$;
3) *the norm $N(\rho)$ is a prime positive integer* $p \equiv 1 \pmod 4$.

Proposition 3. *Two non-zero Gaussian integers α and $\overline{\alpha}$ are associates if and only if one of the following conditions is satisfied:*

$$\operatorname{Re}\alpha = 0; \tag{1}$$

$$\operatorname{Im}\alpha = 0; \tag{2}$$

$$\operatorname{Re}\alpha = \operatorname{Im}\alpha; \tag{3}$$

$$\operatorname{Re}\alpha = -\operatorname{Im}\alpha. \tag{4}$$

Proposition 4. *A Gaussian integer $\alpha \neq 0$ is an associate of its conjugate if and only if there exists a positive integer m such that $\alpha^m \in \mathbb{Z}$.*

The number of divisors of a Gaussian integer

Consider $\alpha \in \mathbb{Z}[i], N(\alpha) > 1$. By theorem 1, there exists a representation $\alpha = \mu\rho_1^{a_1}\rho_2^{a_2}\cdots\rho_k^{a_k}$, where μ is a unit and $\rho_1, \rho_2, \ldots, \rho_k$ are non-associate Gaussian primes. We call this representation canonical.

By $\tau^*(\alpha) = \sum\limits_{\delta\mid\alpha} 1$ we denote the number of divisors of $\alpha \neq 0$.

Theorem 3. *If $\alpha = \mu\rho_1^{a_1}\rho_2^{a_2}\cdots\rho_k^{a_k}$ is the canonical representation of α, then*

$$\tau^*(\alpha) = 4\prod_{j=1}^{k}(a_j + 1). \tag{5}$$

Доведення. Taking into account the uniqueness of the prime representation (theorem 1), one can rewrite every divisor δ of the number α uniquely as follows: $\delta = \nu\rho_1^{c_1}\rho_2^{c_2}\cdots\rho_k^{c_k}$, where $\nu \in \{\pm 1; \pm i\}$ and $0 \leqslant c_j \leqslant a_j$ for all j. □

Proposition 5 (properties of the τ^* function)**.**
(a) *$\tau^*(\alpha)$ is divisible by* 4;
(b) *$\tau^*(\alpha) = 4$ if and only if α is a unit*;
(c) *$\tau^*(\alpha) = 8$ if and only if α is a Gaussian prime*;
(d) *$\tau^*(\alpha) = 12$ if and only if α is an associate of a square of a Gaussian prime*;
(e) *if α is a Gaussian prime and $c \in \mathbb{N}, \mu \sim 1$, then $\tau^*(\mu\alpha^c) = 4(c + 1)$*;
(f) *for every positive integer c that is not equal to* 1, *there exist infinitely many numbers α, such that $\tau^*(\alpha) = 4c$*;
(g) *$\tau^*(\overline{\alpha}) = \tau^*(\alpha)$.*

Доведення. Propositions (a)-(f) follow from the definition of the τ^* function and theorem 3. Proposition (g) follows from proposition 1. □

Proposition 6. *$\tau^*(\alpha)$ is not divisible by* 8 *if and only if α is an associate of a square of a non-zero Gaussian integer.*

Доведення. Necessity. Suppose $\alpha = \mu\rho_1^{a_1}\rho_2^{a_2}\cdots\rho_k^{a_k}$. If $8 \nmid \tau^*(\alpha)$, then in (5), all the factors $a_1 + 1, a_2 + 1, \ldots, a_k + 1$ are odd. Hence, $a_1, a_2, \ldots, a_k$ are even, and α is an associate of $\left(\rho_1^{a_1/2}\rho_2^{a_2/2}\cdots\rho_k^{a_k/2}\right)^2$.

Sufficiency follows immediately from equality (5). □

Proposition 7. *For every Gaussian integer α that is divisible by $1+i$, the following inequality holds:*

$$\tau^*(\alpha) \leqslant 6|\alpha|$$

with equality if and only if $\alpha \sim 2$.

Lemma 1. *For all positive integers n, the inequality $\frac{3(\sqrt{2})^n}{2} \geqslant n+1$ holds with equality if and only if $n = 2$.*

Доведення. For $n \leqslant 3$ the statement of the lemma can be easily checked. The next part of the proof is done using induction. Since $(n+1)\cdot\sqrt{2} > n+2$ for all $n \geqslant 3$, we have

$$\frac{3(\sqrt{2})^{n+1}}{2} = \frac{3(\sqrt{2})^n}{2}\cdot\sqrt{2} > > (n+1)\cdot\sqrt{2} > n+2. \qquad \square$$

Lemma 2. *For all non-negative integers n, the inequality $(\sqrt{5})^c \geqslant n+1$ holds with equality if and only if $n = 0$.*

Доведення. For $n = 0$, the statement of the lemma is true. The next part of the proof is done using induction:

$$(\sqrt{5})^{n+1} = (\sqrt{5})^n\cdot\sqrt{5} > > (n+1)\cdot\sqrt{5} > n+2. \qquad \square$$

Proof of proposition 7. Let α be divisible by $1+i$. Then the canonical representation of α is of the form $\mu(1+i)^a\rho_1^{a_1}\rho_2^{a_2}\cdots\rho_k^{a_k}$, where $a > 0$.

If ρ is a Gaussian prime that is not an associate of $1+i$, then theorem 2 implies that $N(\rho) \geqslant 5$. For this reason, $|\rho_j| \geqslant \sqrt{5}$ for all $j = \overline{1,k}$. Applying lemmas 1 and 2 we obtain:

$$6|\alpha| = 6|1+i|^a|\rho_1^{a_1}||\rho_2^{a_2}|\cdots|\rho_k^{a_k}| \geqslant \geqslant 4\cdot\frac{3(\sqrt{2})^a}{2}\cdot(\sqrt{5})^{a_1}(\sqrt{5})^{a_2}\cdots(\sqrt{5})^{a_k} \geqslant \geqslant 4(a+1)(a_1+1)(a_2+1)\cdots(a_k+1) = \tau^*(\alpha).$$

In the foregoing inequalities, equality holds if and only if $a = 2$ and $k = 0$, i.e. if $\alpha = \mu(1+i)^2 \sim \sim 2$. $\square$

Proposition 8. *For every Gaussian integer α that is not divisible by $1+i$, the inequality*

$$\tau^*(\alpha) \leqslant 4|\alpha|$$

holds with equality if and only if $\alpha \sim 1$.

The proof of this proposition is analogous to the previous one.

Theorem 4. *For any non-zero Gaussian integers α and β the inequality*

$$\tau^*(\alpha\beta) \leqslant \frac{1}{4}\tau^*(\alpha)\tau^*(\beta)$$

holds with equality if and only if α and β are relatively prime.

Доведення. The statement of the theorem is true if at least one of the numbers α and β is a unit. Now suppose none of α and β is a unit and

$$\alpha = \mu\rho_1^{a_1}\rho_2^{a_2}\cdot\ldots\cdot\rho_k^{a_k}, \quad \beta = \nu\rho_1^{b_1}\rho_2^{b_2}\cdot\ldots\cdot\rho_k^{b_k}$$

are their canonical representations. Then $\alpha\beta = = \vartheta\rho_1^{a_1+b_1}\rho_2^{a_2+b_2}\cdot\ldots\cdot\rho_k^{a_k+b_k}$, where $\vartheta = \mu\nu \sim 1$.

Since the exponents are non-negative, we have $(a_j+1)(b_j+1) = a_jb_j + a_j + b_j + 1 \geqslant a_j + b_j + 1$ with equality if and only if $a_j = 0$ or $b_j = 0$, which implies:

$$\tau^*(\alpha\beta) = \tau^*\left(\rho_1^{a_1+b_1}\rho_2^{a_2+b_2}\cdot\ldots\cdot\rho_k^{a_k+b_k}\right) = = 4\prod_{j=1}^{k}(a_j+b_j+1) \leqslant 4\prod_{j=1}^{k}(a_j+1)(b_j+1) = = \frac{1}{4}\cdot 4\prod_{j=1}^{k}(a_j+1)\cdot 4\prod_{t=1}^{k}(b_t+1) = \frac{1}{4}\tau^*(\alpha)\tau^*(\beta).$$

Equality holds if and only if for all j, either $a_j = 0$ or $b_j = 0$, i.e. if α are β relatively prime. $\square$

Corollary 1. *Two non-zero Gaussian integers α and β are relatively prime if and only if $\tau^*(\alpha\beta) = = \frac{1}{4}\tau^*(\alpha)\tau^*(\beta)$.*

Corollary 2. *If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are relatively prime Gaussian integers, then $\tau^*(\alpha_1\alpha_2\cdots\alpha_n) = = \frac{1}{4^{n-1}}\tau^*(\alpha_1)\tau^*(\alpha_2)\cdots\tau^*(\alpha_n)$.*

Corollary 3. *For every positive integer m, the inequality*

$$\tau^*(\alpha^m) \leqslant \frac{1}{4^{m-1}}(\tau^*(\alpha))^m$$

holds with equality if and only if $\alpha \sim 1$ or $m = 1$.

The sum of the divisors of a Gaussian integer

For a Gaussian integer $\alpha \neq 0$ and a positive integer m by $\sigma^*_m(\alpha)$ we denote the sum $\sigma^*_m(\alpha) = \sum\limits_{\delta|\alpha} \delta^m$ of the m-th powers of the divisors of α.

Proposition 9. *If m is not divisible by 4, then $\sigma^*_m(\alpha) = 0$.*

Доведення. If β is a divisor of α, so are $-\beta$, $i\beta$ and $-i\beta$. Hence, the set of divisors of α breaks down into quadruples: $(\delta_1, -\delta_1, i\delta_1, -i\delta_1)$, $(\delta_2, -\delta_2, i\delta_2, -i\delta_2)$, $\ldots$, $(\delta_k, -\delta_k, i\delta_k, -i\delta_k)$, where $k = \frac{1}{4}\tau^*(\alpha)$. If m is not divisible by 4, then the sum of the m-th powers of each quadruple is equal to 0. □

Theorem 5. *Suppose $\alpha \not\sim 1$. If the canonical representation of α is of the form $\alpha = \mu\rho_1^{a_1}\rho_2^{a_2}\cdots\rho_k^{a_k}$, then for every positive integer m,*

$$\sigma^*_{4m}(\alpha) = 4\prod_{j=1}^{k} \frac{\rho_j^{4m(a_j+1)} - 1}{\rho_j^{4m} - 1}. \qquad (6)$$

Доведення. Every divisor of α is of the form $\nu\rho_1^{c_1}\rho_2^{c_2}\cdots\rho_k^{c_k}$, where $\nu \in \{\pm 1; \pm i\}$ and $0 \leqslant c_j \leqslant a_j$ for all j. Note that $\nu^{4m} = 1$.

The right-hand side of the equality (6) can be rewritten as follows:

$$(1+1+1+1)\prod_{j=1}^{k}(1 + \rho_j^{4m} + \rho_j^{8m} + \ldots + \rho_j^{4a_j m}).$$

After opening the brackets, we obtain the sum of all numbers of the form

$$\nu^{4m}\rho_1^{4mc_1}\rho_2^{4mc_2}\cdots\rho_k^{4mc_k},$$

where $0 \leqslant c_j \leqslant a_j$ for all j, i.e. the sum of the m-th powers of the divisors of α. □

Note that (6) implies that if $\alpha \sim \beta$, then $\sigma^*_{4m}(\alpha) = \sigma^*_{4m}(\beta)$.

Proposition 10 (properties of the σ^*_{4m} function)**.**

(*a*) *$\sigma^*_{4m}(\alpha)$ is divisible by* 4;
(*b*) *$\sigma^*_{4m}(\alpha) = 4$ if and only if $\alpha \sim 1$;*
(*c*) *if α is a Gaussian prime, then $\sigma^*_{4m}(\alpha) = 4(\alpha^{4m} + 1)$;*
(*d*) *if α is a Gaussian prime, then $\sigma^*_{4m}((\mu\alpha)^c) = \sigma^*_{4m}(\alpha^c)$ for any $c \in \mathbb{N}$ and $\mu \sim 1$;*
(*e*) *$\sigma^*_{4m}(\overline{\alpha}) = \overline{\sigma^*_{4m}(\alpha)}$;*
(*f*) *if α and $\overline{\alpha}$ are associates, then $\sigma^*_{4m}(\alpha) \in \mathbb{Z}$.*

Доведення. Propositions (*a*)-(*d*) from the definition of the σ^*_{4m} function and theorem 5.

(*e*) This follows from proposition 1.

(*f*) Applying propositions 10.*e* and 10.*d*, we obtain:

$$\overline{\sigma^*_{4m}(\alpha)} = \sigma^*_{4m}(\overline{\alpha}) = \sigma^*_{4m}(\mu\alpha) = \sigma^*_{4m}(\alpha).$$

Hence, $\sigma^*_{4m}(\alpha) \in \mathbb{Z}$. □

Proposition 11. *$\sigma^*_{4m}(\alpha)$ is not divisible by $4(1+i)$ if and only if either α or $(1+i)\alpha$ is an associate of a square of a Gaussian integer.*

Proof. It can be easily seen that for $\alpha \sim 1$, this proposition holds. Now suppose $\alpha \nsim 1$ and $\alpha = \mu\rho_1^{a_1}\rho_2^{a_2}\cdots\rho_k^{a_k}$ is the canonical representation of α.

From formula (6), we see that the number $\sigma^*_{4m}(\alpha)$ is not divisible by $4(1+i)$ if and only if for all j, the sum $1 + \rho_j^{4m} + \rho_j^{8m} + \cdots + \rho_j^{4a_j m}$ is not divisible by $1+i$. If $\rho_j \sim 1+i$, then this sum is not divisible by $1+i$. Hence, the divisibility of the number $\sigma^*_{4m}(\alpha)$ by $4(1+i)$ is influenced only by the exponents of those primes from the canonical representation of α that are not associates of $1+i$.

If $\rho_j \nsim 1+i$, then $(1+i) \nmid \rho_j^{4m}$. Suppose $\rho_j^{4m} = s_j + t_j i$, where $s_j, t_j \in \mathbb{Z}$. Then by proposition 2, the numbers s_j and t_j do not have the same parity; hence, the numbers $s_j + 1$ and t_j are of the same parity. Thus, by proposition 2, $(1+i)|(\rho_j^{4m} + 1)$. Notice that

$$1 + \rho_j^{4m} + \cdots + \rho_j^{4a_j m} = \begin{cases} (1+\rho_j^{4m})(1 + \rho_j^{8m} + \cdots + \rho_j^{4(a_j-2)m}) + \rho_j^{4a_j m} & \text{if } 2 \mid a_j; \\ (1+\rho_j^{4m})(1 + \rho_j^{8m} + \cdots + \rho_j^{4(a_j-1)m}) & \text{if } 2 \nmid a_j. \end{cases}$$

From this we obtain that the sum $1 + \rho_j^{4m} + \rho_j^{8m} + \cdots + \rho_j^{4a_j m}$ is not divisible by $1+i$ if and only if the number a_j is even. Hence, $4(1+i) \nmid \sigma^*_{4m}(\alpha)$ if and only if in the canonical representation of α, all the primes that are not associates of $1+i$ have even exponents, i.e. if and only if there exists β such that $\mu\rho_1^{a_1}\rho_2^{a_2}\cdots\rho_k^{a_k} \sim \beta^2$, or, equivalently, $\alpha \sim (1+i)\beta^2$. □

Proposition 12. *$\sigma^*_{4m}(\alpha)$ is divisible by 8 if and only if neither α nor $(1+i)\alpha$ is an associate of a square of a non-zero Gaussian integer.*

Доведення. Necessity follows from the previous proposition and the fact that $4(1+i)|8$.

Sufficiency. Suppose neither α nor $(1+i)\alpha$ is an associate of a square of a non-zero Gaussian integer and the canonical representation of α is of the form $\mu\rho_1^{a_1}\rho_2^{a_2}\cdots\rho_k^{a_k}$. Then there exists a number l such that $\rho_l \nsim 1+i$ and the exponent a_l is odd. The odd parity of a_l implies that for any positive integer m the following equality holds:

$$1+\rho_l^{4m}+\rho_l^{8m}+\cdots+\rho_l^{4a_lm}= \\ =(1+\rho_l^{4m})(1+\rho_l^{8m}+\rho_l^{16m}+\cdots+\rho_l^{4(a_l-1)m}).$$

Hence, $(1+\rho_l^{4m}) \mid (1+\rho_l^{4m}+\rho_l^{8m}+\cdots+\rho_l^{4a_lm})$.

Suppose $\rho_l^m = s+ti$, where $s,t\in\mathbb{Z}$. Since $(1+i)\nmid\rho_l^m$, by proposition 2, the numbers s and t do not have the same parity. Therefore, on the right-hand side of the equality

$$1+\rho_l^{4m}=1+(s+ti)^4= \\ =(1+s^4+t^4)-6s^2t^2+(4s^3t-4st^3)i,$$

each of the numbers $1+s^4+t^4$ and $4s^3t-4st^3$ is divisible by 2. Thus, $2|(1+\rho_l^{4m})$, which implies that $2|(1+\rho_l^{4m}+\rho_l^{8m}+\cdots+\rho_l^{4a_lm})$.

Therefore, on the right-hand side of (6), aside from 4, there is one more factor that is divisible by 2. Hence, $\sigma^*_{4m}(\alpha)$ is divisible by 8. □

Proposition 13. *For all non-zero Gaussian integers α and positive integers m, the following inequality holds:*

$$|\sigma^*_{4m}(\alpha)| \leqslant |\alpha|^{4m}(\tau^*(\alpha)-4)+4. \qquad (7)$$

In (7), equality holds only for $\alpha\sim 1$ or $\alpha\sim p$, where p is a prime positive integer of the form $4k+3$; and when m is even, it also holds for $\alpha\sim(1+i)$.

Доведення. If $\alpha\sim 1$, then for all positive integers m, we have:

$$|\sigma^*_{4m}(\alpha)| = 4 = |\alpha|^{4m}(\tau^*(\alpha)-4)+4. \qquad (8)$$

Now suppose $\alpha\nsim 1$. Then the number α has exactly four divisors whose absolute values are equal to 1, and the absolute values of each of the rest (there are $\tau^*(\alpha)-4$ of them) do not exceed $|\alpha|$. Hence,

$$|\sigma^*_{4m}(\alpha)| = \Big|\sum_{\delta|\alpha}\delta^{4m}\Big| \leqslant \\ \leqslant \sum_{\delta|\alpha}|\delta|^{4m} \leqslant |\alpha|^{4m}(\tau^*(\alpha)-4)+4. \qquad (9)$$

To prove the second part of the proposition, we only need to consider the case when $\alpha\nsim 1$. Note that for each divisor δ of a Gaussian integer α, the inequality $|\delta|\leqslant|\alpha|$ holds with equality if and only if $\delta\sim\alpha$. Hence, in inequality (9), equality holds if and only if the divisors of α are the units and the associates of α only, i.e. if α is prime.

Therefore, it only remains to check for what primes α equality (8) holds. Since α is a Gaussian prime, then (8) implies that

$$|\sigma^*_{4m}(\alpha)| = |4\alpha^{4m}+4| = 4|\alpha^{4m}|+4.$$

This and the triangle inequality for complex numbers imply that $\alpha^{4m}\in\mathbb{Z}$. By theorem 2, this is possible only if $\alpha\sim p$, where p is a prime positive integer of the form $4k+3$, or if $\alpha\sim 1+i$.

It can be easily checked that in the first case, the equality in (7) holds always, and in the second case, $\alpha^{4m}=(-4)^m=(-1)^m4^m$. Hence, $|\sigma^*_{4m}(\alpha)| = |(-1)^m4^{m+1}+4|$, while $|\alpha|^{4m}(\tau^*(\alpha)-4)+4=4^{m+1}+4$. Therefore, the equality in (7) holds only if m is even. □

Theorem 6. *For any relatively prime Gaussian integers α and β,*

$$\sigma^*_{4m}(\alpha\beta)=\frac{1}{4}\sigma^*_{4m}(\alpha)\sigma^*_{4m}(\beta).$$

Доведення. The statement of the theorem is true if at least one of the numbers α and β is a unit.

Now suppose $\alpha=\mu\rho_1^{a_1}\rho_2^{a_2}\cdots\rho_k^{a_k}$ and $\beta= =\nu\xi_1^{b_1}\xi_2^{b_2}\cdots\xi_l^{b_l}$ are the canonical representations α of β. Since α and β are relatively prime, all the prime numbers $\rho_1,\rho_2,\ldots,\rho_k$ and $\xi_1,\xi_2,\ldots,\xi_l$ are non-associate. Hence, every divisor of the number $\alpha\beta$ is of the form $\vartheta\rho_1^{a_1}\rho_2^{a_2}\cdots\rho_k^{a_k}\cdot\xi_1^{b_1}\xi_2^{b_2}\cdots\xi_l^{b_l}$, where $\vartheta\sim 1$. Then we have:

$$\sigma^*_{4m}(\alpha\beta)=4\prod_{j=1}^{k}\frac{\rho_j^{4m(a_j+1)}-1}{\rho_j^{4m}-1}\cdot\prod_{t=1}^{l}\frac{\xi_t^{4m(b_t+1)}-1}{\xi_t^{4m}-1}= \\ =\frac{1}{4}\sigma^*_{4m}(\alpha)\sigma^*_{4m}(\beta). \qquad □$$

Corollary 4. *If $\alpha_1,\alpha_2,\ldots,\alpha_n$ are n relatively prime Gaussian integers, then $\sigma^*_{4m}(\alpha_1\alpha_2\cdots\alpha_n)= =\frac{1}{4^{n-1}}\sigma^*_{4m}(\alpha_1)\sigma^*_{4m}(\alpha_2)\cdots\sigma^*_{4m}(\alpha_n)$.*

The product of the divisors of a Gaussian integer

For a Gaussian integer $\alpha\neq 0$, by $\pi^*(\alpha)$ we denote the product $\pi^*(\alpha)=\prod_{\delta|\alpha}\delta$ of its divisors. Note that if α and β are associates, then $\pi^*(\alpha)=\pi^*(\beta)$.

Theorem 7.

$$\pi^*(\alpha) = \begin{cases} \alpha^{\frac{1}{2}\tau^*(\alpha)} & \textit{if } \alpha \textit{ is not a square of a Gaussian integer;} \\ -\alpha^{\frac{1}{2}\tau^*(\alpha)} & \textit{if } \alpha \textit{ is a square of a Gaussian integer.} \end{cases}$$

Доведення. If β is a divisor of α, so is α/β. Let $\beta_1, \beta_2, \ldots, \beta_{\tau^*(\alpha)}$ be all the divisors of α. Then we have:

$$\pi^*(\alpha) = \beta_1\beta_2\cdots\beta_{\tau^*(\alpha)} = \frac{\alpha}{\beta_1}\cdot\frac{\alpha}{\beta_2}\cdots\frac{\alpha}{\beta_{\tau^*(\alpha)}}.$$

Hence,

$$(\pi^*(\alpha))^2 = \beta_1\beta_2\cdots\beta_{\tau^*(\alpha)}\cdot\frac{\alpha}{\beta_1}\cdot\frac{\alpha}{\beta_2}\cdots\frac{\alpha}{\beta_{\tau^*(\alpha)}} = \\ = \underbrace{\alpha\cdot\alpha\cdots\alpha}_{\tau^*(\alpha)} = \alpha^{\tau^*(\alpha)}.$$

By theorem 3, the number $\tau^*(\alpha)$ is always even. Therefore,

$$\pi^*(\alpha) = \alpha^{\frac{1}{2}\tau^*(\alpha)} \text{ or } \pi^*(\alpha) = -\alpha^{\frac{1}{2}\tau^*(\alpha)}. \quad (10)$$

The set of divisors of α naturally breaks down into quadruples:
$(\delta_1, -\delta_1, i\delta_1, -i\delta_1), (\delta_2, -\delta_2, i\delta_2, -i\delta_2), \ldots,$
$(\delta_k, -\delta_k, i\delta_k, -i\delta_k)$, where $k = \frac{1}{4}\tau^*(\alpha)$. Suppose $\delta_1\delta_2\cdots\delta_{\frac{1}{4}\tau^*(\alpha)} = \gamma$. Then

$$\pi^*(\alpha) = \prod_{j=1}^{\frac{1}{4}\tau^*(\alpha)} ((\delta_j)\cdot(-\delta_j)\cdot(i\delta_j)\cdot(-i\delta_j)) = \\ = (-1)^{\frac{1}{4}\tau^*(\alpha)}\gamma^4. \quad (11)$$

Now we consider 3 cases:

1) α is not an associate of a square of a Gaussian integer. Then, by proposition 6, $8|\tau^*(\alpha)$, so the number $\frac{1}{4}\tau^*(\alpha)$ is even. Hence, (11) implies that $\pi^*(\alpha) = \gamma^4$, and from (10), we obtain either $\gamma^4 = \alpha^{\frac{1}{2}\tau^*(\alpha)}$ or $\gamma^4 = -\alpha^{\frac{1}{2}\tau^*(\alpha)}$. The second equality is impossible, since -1 is not a fourth power of a Gaussian integer.

Thus, in this case, $\pi^*(\alpha) = \alpha^{\frac{1}{2}\tau^*(\alpha)}$.

2) $\alpha = i\omega^2$ (that is, α is an associate of a square, but is not itself a square). Propositions 5.a and 6 imply that $4|\tau^*(\alpha)$, but $8\nmid\tau^*(\alpha)$. Hence, the number $\frac{1}{4}\tau^*(\alpha)$ is odd and (11) implies that $\pi^*(\alpha) = -\gamma^4$. Besides, $\alpha^{\frac{1}{2}\tau^*(\alpha)} = i^{\frac{1}{2}\tau^*(\alpha)}\omega^{\tau^*(\alpha)} = -\omega^{\tau^*(\alpha)}$. Therefore, it follows from (10) that it is either $-\gamma^4 = -\omega^{\tau^*(\alpha)}$ or $-\gamma^4 = \omega^{\tau^*(\alpha)}$. The second equality is impossible, since $\omega^{\tau^*(\alpha)}$ is a fourth power of a Gaussian integer, while -1 is not.

Hence, in this case, $\pi^*(\alpha) = \alpha^{\frac{1}{2}\tau^*(\alpha)}$ as well.

3) $\alpha = \omega^2$. By proposition 6, $8\nmid\tau^*(\alpha)$, so the number $\frac{1}{4}\tau^*(\alpha)$ is odd and (11) implies that $\pi^*(\alpha) = -\gamma^4$. Relation (10) now takes the following form: $-\gamma^4 = \alpha^{\frac{1}{2}\tau^*(\alpha)}$ or $-\gamma^4 = -\alpha^{\frac{1}{2}\tau^*(\alpha)}$. By analogous reasoning, we now obtain that the first equality is impossible.

Thus, in this case, $\pi^*(\alpha) = -\alpha^{\frac{1}{2}\tau^*(\alpha)}$. □

Proposition 14 (properties of the π^* function)**.**

(a) $\pi^*(\alpha) = -1$ *if and only if* α *is a unit;*
(b) $\pi^*(\alpha) = \alpha^4$ *if and only if* α *is a Gaussian prime;*
(c) $\pi^*(\overline{\alpha}) = \overline{(\pi^*(\alpha))}$.

Доведення. Propositions (a) and (b) follow from the definition of the π^* function and theorem 7.

(c) If the divisors of α are $\delta_1, \delta_2, \ldots, \delta_{\tau^*(\alpha)}$, then by proposition 1, the divisors of $\overline{\alpha}$ are $\overline{\delta_1}, \overline{\delta_2}, ..., \overline{\delta_{\tau^*(\alpha)}}$. Hence,

$$\pi^*(\overline{\alpha}) = \prod_{j=1}^{\tau^*(\alpha)}\overline{\delta_j} = \overline{\prod_{j=1}^{\tau^*(\alpha)}\delta_j} = \overline{\pi^*(\alpha)}.$$ □

Proposition 15. $\pi^*(\alpha) = \pi^*(\beta)$ *if and only if* α *and* β *are associates.*

Доведення. Sufficiency follows from the fact that α and β have the same divisors.

Necessity. Suppose $\pi^*(\alpha) = \pi^*(\beta)$. Then, in particular, every prime divisor of α is a divisor of β, and vice versa. Hence, the canonical representations of α and β are of the form $\alpha = \mu\rho_1^{a_1}\rho_2^{a_2}\cdots\rho_k^{a_k}$ and $\beta = \nu\rho_1^{b_1}\rho_2^{b_2}\cdots\rho_k^{b_k}$, where all the exponents are positive. Since $\alpha^{\frac{1}{2}\tau^*(\alpha)} = \beta^{\frac{1}{2}\tau^*(\beta)}$ or $\alpha^{\frac{1}{2}\tau^*(\alpha)} = -\beta^{\frac{1}{2}\tau^*(\beta)}$, then

$$\alpha^{\frac{1}{2}\tau^*(\alpha)} = \mu^{\frac{1}{2}\tau^*(\alpha)}\rho_1^{\frac{1}{2}\tau^*(\alpha)a_1}\cdots\rho_k^{\frac{1}{2}\tau^*(\alpha)a_k} = \\ = \pm\beta^{\frac{1}{2}\tau^*(\beta)} = \pm\nu^{\frac{1}{2}\tau^*(\beta)}\rho_1^{\frac{1}{2}\tau^*(\beta)b_1}\cdots\rho_k^{\frac{1}{2}\tau^*(\beta)b_k}.$$

Thus, for each $j = \overline{1,k}$, the equality $a_j\tau^*(\alpha) = b_j\tau^*(\beta)$ holds. Hence, $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_k}{b_k} = \frac{\tau^*(\beta)}{\tau^*(\alpha)}$. If the inequality $\frac{\tau^*(\beta)}{\tau^*(\alpha)} > 1$ were true, then for each $j = \overline{1,k}$, the inequality $a_j > b_j$ would hold. But then, contrary to the initial condition, $\tau^*(\alpha) > \tau^*(\beta)$. In the same way, an assumption that $\frac{\tau^*(\beta)}{\tau^*(\alpha)} < 1$ leads to a contradiction. Thus, $a_1 = b_1, \ldots, a_k = b_k$, and $\alpha \sim \beta$. □

Proposition 16. *$\pi^*(\alpha) \in \mathbb{Z}$ if and only if $\alpha \sim \overline{\alpha}$.*

Доведення. By propositions 14.*c* and 15, we have:

$$\pi^*(\alpha) = \overline{\pi^*(\alpha)} \iff \pi^*(\alpha) = \pi^*(\overline{\alpha}) \iff \alpha \sim \overline{\alpha}. \square$$

Sums of products of divisors of a Gaussian integer

In this subsection, numbers $k, m, n, u, v, w, a, b, c, a_1, a_2, \ldots, a_k$ are positive integers, $\alpha \neq 0$, and $k \leqslant \tau^*(\alpha)$. We fix the divisors of α in a certain order $\delta_1, \delta_2, \ldots, \delta_{\tau^*(\alpha)}$, and introduce the following notation:

$$s^*_{k,m}(\alpha) = \sum_{1 \leqslant a_1 < a_2 < \ldots < a_k \leqslant \tau^*(\alpha)} \delta^m_{a_1} \delta^m_{a_2} \cdot \ldots \cdot \delta^m_{a_k};$$

$$p^*_{k,m}(\alpha) = \sum_{1 \leqslant a_1 < a_2 < \ldots < a_k \leqslant \tau^*(\alpha)} \frac{1}{\delta^m_{a_1} \delta^m_{a_2} \cdot \ldots \cdot \delta^m_{a_k}}.$$

Note that $s^*_{1,m}(\alpha) = \sigma^*_m(\alpha)$, $s^*_{\tau^*(\alpha),m}(\alpha) = (\pi^*(\alpha))^m$, $p^*_{\tau^*(\alpha),m}(\alpha) = \frac{1}{(\pi^*(\alpha))^m}$.

Proposition 17.

$$\sum_{\substack{1 \leqslant a,b \leqslant \tau^*(\alpha) \\ a \neq b}} \delta^u_a \delta^v_b = \sigma^*_u(\alpha)\sigma^*_v(\alpha) - \sigma^*_{u+v}(\alpha).$$

Доведення. This follows from the equalities below:

$$\sigma^*_u(\alpha)\sigma^*_v(\alpha) = \sum_{\delta|\alpha} \delta^u \cdot \sum_{\delta|\alpha} \delta^v = \sum_{\delta|\alpha} \delta^{u+v} + \sum_{\substack{1 \leqslant a,b \leqslant \tau^*(\alpha) \\ a \neq b}} \delta^u_a \delta^v_b = \sigma^*_{u+v}(\alpha) + \sum_{\substack{1 \leqslant a,b \leqslant \tau^*(\alpha) \\ a \neq b}} \delta^u_a \delta^v_b.$$

□

Corollary 5.

$$\sum_{1 \leqslant a,b \leqslant \tau^*(\alpha) a \neq b} \frac{1}{\delta^u_a \delta^v_b} = \frac{\sigma^*_u(\alpha)\sigma^*_v(\alpha) - \sigma^*_{u+v}(\alpha)}{\alpha^{u+v}}.$$

Propositions 17 and 9 imply

Corollary 6. $s^*_{2,m}(\alpha) = \frac{(\sigma^*_m(\alpha))^2 - \sigma^*_{2m}(\alpha)}{2} =$

$$= \begin{cases} 0 & \textit{if } m = 2n-1; \\ -\frac{\sigma^*_{2m}(\alpha)}{2} & \textit{if } m = 4n-2; \\ \frac{(\sigma^*_m(\alpha))^2 - \sigma^*_{2m}(\alpha)}{2} & \textit{if } m = 4n. \end{cases}$$

Proposition 18.

$$\sum_{\substack{1 \leqslant a,b,c \leqslant \tau^*(\alpha) \\ a \neq b, b \neq c, c \neq a}} \delta^u_a \delta^v_b \delta^w_c = \sigma^*_u(\alpha)\sigma^*_v(\alpha)\sigma^*_w(\alpha) - \sigma^*_{u+v}(\alpha)\sigma^*_w(\alpha) - \sigma^*_{u+w}(\alpha)\sigma^*_v(\alpha) - \sigma^*_{v+w}(\alpha)\sigma^*_u(\alpha) + 2\sigma^*_{u+v+w}(\alpha).$$

Доведення. This follows from proposition 17 and the chain of equalities:

$$\sigma^*_w(\alpha)\left(\sigma^*_u(\alpha)\sigma^*_v(\alpha) - \sigma^*_{u+v}(\alpha)\right) = \sum_{\delta|\alpha} \delta^w \cdot \sum_{\substack{1 \leqslant a,b \leqslant \tau^*(\alpha) \\ a \neq b}} \delta^u_a \delta^v_b = \sum_{\substack{1 \leqslant a,b \leqslant \tau^*(\alpha) \\ a \neq b}} \delta^{u+w}_a \delta^v_b + \sum_{\substack{1 \leqslant a,b \leqslant \tau^*(\alpha) \\ a \neq b}} \delta^u_a \delta^{v+w}_b + \sum_{\substack{1 \leqslant a,b,c \leqslant \tau^*(\alpha) \\ a \neq b, b \neq c, c \neq a}} \delta^u_a \delta^v_b \delta^w_c =$$
$$= \sigma^*_{u+w}(\alpha)\sigma^*_v(\alpha) - \sigma^*_{u+v+w}(\alpha) + \sigma^*_u(\alpha)\sigma^*_{v+w}(\alpha) - \sigma^*_{u+v+w}(\alpha) + \sum_{\substack{1 \leqslant a,b,c \leqslant \tau^*(\alpha) \\ a \neq b, b \neq c, c \neq a}} \delta^u_a \delta^v_b \delta^w_c.$$

□

Corollary 7.

$$\sum_{\substack{1 \leqslant a,b,c \leqslant \tau^*(\alpha) \\ a \neq b, b \neq c, c \neq a}} \frac{1}{\delta^u_a \delta^v_b \delta^w_c} = \frac{1}{\alpha^{u+v+w}}\big(\sigma^*_u(\alpha)\sigma^*_v(\alpha)\sigma^*_w(\alpha) - \sigma^*_{u+v}(\alpha)\sigma^*_w(\alpha) - \sigma^*_{u+w}(\alpha)\sigma^*_v(\alpha) - \sigma^*_{v+w}(\alpha)\sigma^*_u(\alpha) + 2\sigma^*_{u+v+w}(\alpha)\big).$$

Corollary 8.

$$\sum_{\substack{1 \leqslant a < b \leqslant \tau^*(\alpha) \\ 1 \leqslant c \leqslant \tau^*(\alpha) \\ c \neq a, c \neq b}} \delta^u_a \delta^u_b \delta^v_c = \frac{1}{2}\big((\sigma^*_u(\alpha))^2\sigma^*_v(\alpha) - 2\sigma^*_u(\alpha)\sigma^*_{u+v}(\alpha) - \sigma^*_{2u}(\alpha)\sigma^*_v(\alpha) + 2\sigma^*_{2u+v}(\alpha)\big).$$

Corollary 9.

$$\sum_{\substack{1 \leqslant a < b \leqslant \tau^*(\alpha) \\ 1 \leqslant c \leqslant \tau^*(\alpha) \\ c \neq a, c \neq b}} \frac{1}{\delta^u_a \delta^u_b \delta^v_c} = \frac{1}{2\alpha^{2u+v}}\big((\sigma^*_u(\alpha))^2\sigma^*_v(\alpha) - 2\sigma^*_u(\alpha)\sigma^*_{u+v}(\alpha) - \sigma^*_{2u}(\alpha)\sigma^*_v(\alpha) + 2\sigma^*_{2u+v}(\alpha)\big).$$

Propositions 18 and 9 imply

Corollary 10.

$$s^*_{3,m}(\alpha) = \frac{(\sigma^*_m(\alpha))^3 - 3\sigma^*_m(\alpha)\sigma^*_{2m}(\alpha) + 2\sigma^*_{3m}(\alpha)}{6} = \begin{cases} \frac{(\sigma^*_m(\alpha))^3 - 3\sigma^*_m(\alpha)\sigma^*_{2m}(\alpha) + 2\sigma^*_{3m}(\alpha)}{6} & \textit{if } m = 4n; \\ 0, & \textit{otherwise.} \end{cases}$$

Proposition 19.

(*a*) $p^*_{k,m}(\alpha)=\frac{s^*_{k,m}(\alpha)}{\alpha^{km}}$;
(*b*) *if* $\tau^*(\alpha)>k, k\in\mathbb{N}$, *then*

$$s^*_{k,m}(\alpha)=\frac{(\alpha)^{km}s^*_{\tau^*(\alpha)-k,m}(\alpha)}{(\pi(\alpha))^m};$$

(*c*) *if* $\tau^*(\alpha)>k, k\in\mathbb{N}$, *then*

$$p^*_{k,m}(\alpha)=\frac{(\alpha)^{km}p^*_{\tau^*(\alpha)-k,m}(\alpha)}{(\pi(\alpha))^m}.$$

Доведення. (*a*) If δ is a divisor of α, so is $\frac{\alpha}{\delta}$. Hence,

$$\frac{s^*_{k,m}(\alpha)}{\alpha^{km}}=\sum_{1\leqslant a_1<a_2<\ldots<a_k\leqslant\tau^*(\alpha)}\frac{\delta^m_{a_1}\delta^m_{a_2}\cdot\ldots\cdot\delta^m_{a_k}}{\alpha^{km}}=$$
$$=\sum_{1\leqslant a_1<a_2<\ldots<a_k\leqslant\tau^*(\alpha)}\frac{1}{\delta^m_{a_1}\delta^m_{a_2}\cdots\delta^m_{a_k}}=p^*_{k,m}(\alpha).$$

(*b*) This follows from (*a*) and the chain of equalities:

$$\frac{(\pi(\alpha))^m s^*_{k,m}(\alpha)}{\alpha^{km}}=$$
$$=\sum_{1\leqslant a_1<a_2<\ldots<a_k\leqslant\tau^*(\alpha)}\frac{\delta^m_1\delta^m_2\cdots\delta^m_{\tau^*(\alpha)}}{\delta^m_{a_1}\delta^m_{a_2}\cdots\delta^m_{a_k}}=$$
$$=\sum_{1\leqslant a_1<a_2<\ldots<a_{\tau^*(\alpha)-k}\leqslant\tau^*(\alpha)}\delta^m_{a_1}\delta^m_{a_2}\cdots\delta^m_{a_{\tau^*(\alpha)-k}}=$$
$$=s^*_{\tau^*(\alpha)-k,m}(\alpha).$$

Proposition (*c*) follows from (*a*) and (*b*). □

Proposition 20 (the recurrence formula for $s^*_{k,m}$)**.**

$$s^*_{k+1,m}(\alpha)=\frac{1}{k+1}\left((-1)^k\sigma^*_{(k+1)m}(\alpha)+\sum_{l=0}^{k-1}(-1)^l s^*_{k-l,m}(\alpha)\sigma^*_{(l+1)m}(\alpha)\right). \quad (12)$$

Доведення. Let $b_1,b_2,\ldots,b_k$ be distinct positive integers, where $1\leqslant b_1,b_2,\ldots,b_k\leqslant\tau^*(\alpha)$. By $q^*_{k,c,m}(\alpha)$, where $c\in\mathbb{N}\setminus\{1\}$, we denote the following sum:

$$q^*_{k,c,m}(\alpha)=\sum_{\substack{1\leqslant b_2<b_3<\ldots<b_k\leqslant\tau^*(\alpha)\\ 1\leqslant b_1\leqslant\tau^*(\alpha)\\ b_1\neq b_2,b_1\neq b_3,\ldots,b_1\neq b_k}}\delta^{cm}_{b_1}\delta^m_{b_2}\delta^m_{b_3}\cdots\delta^m_{b_k}.$$

Note that $q^*_{1,c,m}(\alpha)=\sigma^*_{cm}(\alpha)$. Then we have:

$$s^*_{k,m}(\alpha)\sigma^*_m(\alpha)=$$
$$=\left(\sum_{1\leqslant a_1<a_2<\ldots<a_k\leqslant\tau^*(\alpha)}\delta^m_{a_1}\delta^m_{a_2}\cdots\delta^m_{a_k}\right)\times$$
$$\times\left(\sum_{\delta|\alpha}\delta^m\right)=(k+1)s^*_{k+1,m}(\alpha)+q^*_{k,2,m}(\alpha).$$

$$s^*_{k,m}(\alpha)\sigma^*_{cm}(\alpha)=\left(\sum_{\delta|\alpha}\delta^{cm}\right)\times$$
$$\times\left(\sum_{1\leqslant a_1<a_2<\ldots<a_k\leqslant\tau^*(\alpha)}\delta^m_{a_1}\delta^m_{a_2}\cdots\delta^m_{a_k}\right)=$$
$$=\sum_{\substack{1\leqslant b_2<b_3<\ldots<b_k\leqslant\tau^*(\alpha)\\ 1\leqslant b_1\leqslant\tau^*(\alpha)\\ b_1\neq b_2,b_1\neq b_3,\ldots,b_1\neq b_k}}\delta^{(c+1)m}_{b_1}\delta^m_{b_2}\delta^m_{b_3}\cdots\delta^m_{b_k}+$$
$$+\sum_{\substack{1\leqslant b_2<b_3<\ldots<b_{k+1}\leqslant\tau^*(\alpha)\\ 1\leqslant b_1\leqslant\tau^*(\alpha)\\ b_1\neq b_2,b_1\neq b_3,\ldots,b_1\neq b_{k+1}}}\delta^{cm}_{b_1}\delta^m_{b_2}\delta^m_{b_3}\cdots\delta^m_{b_{k+1}}=$$
$$=q^*_{k,c+1,m}(\alpha)+q^*_{k+1,c,m}(\alpha).$$

From this, we obtain the following equalities:

$$s^*_{k,m}(\alpha)\sigma^*_m(\alpha)=(k+1)s^*_{k+1,m}(\alpha)+q^*_{k,2,m}(\alpha);$$
$$s^*_{k-1,m}(\alpha)\sigma^*_{2m}(\alpha)=q^*_{k-1,3,m}(\alpha)+q^*_{k,2,m}(\alpha);$$
$$s^*_{k-2,m}(\alpha)\sigma^*_{3m}(\alpha)=q^*_{k-2,4,m}(\alpha)+q^*_{k-1,3,m}(\alpha);$$
$$\cdots$$
$$s^*_{1,m}(\alpha)\sigma^*_{km}(\alpha)=q^*_{1,k+1,m}(\alpha)+q^*_{2,k,m}(\alpha),$$

which imply:

$$s^*_{k,m}(\alpha)\sigma^*_m(\alpha)-s^*_{k-1,m}(\alpha)\sigma^*_{2m}(\alpha)+\cdots+$$
$$+(-1)^{k-1}s^*_{1,m}(\alpha)\sigma^*_{km}(\alpha)=(k+1)s^*_{k+1,m}(\alpha)+$$
$$+(-1)^{k-1}q^*_{1,k+1,m}(\alpha)=(k+1)s^*_{k+1,m}(\alpha)+$$
$$+(-1)^{k-1}\sigma^*_{(k+1)m}(\alpha).$$

Hence,

$$(k+1)s^*_{k+1,m}(\alpha)=s^*_{k,m}(\alpha)\sigma^*_m(\alpha)-$$
$$-s^*_{k-1,m}(\alpha)\sigma^*_{2m}(\alpha)+\cdots$$
$$\cdots+(-1)^{k-1}s^*_{1,m}(\alpha)\sigma^*_{km}(\alpha)+(-1)^k\sigma^*_{(k+1)m}(\alpha),$$

which immediately implies (12). □

Proposition 21 (the recurrence formula for $p^*_{k,m}$)**.**

$$p^*_{k+1,m}(\alpha) = \frac{1}{k+1}\left(\frac{(-1)^k\sigma^*_{(k+1)m}(\alpha)}{\alpha^{(k+1)m}} + \sum_{l=0}^{k-1}\frac{(-1)^l p^*_{k-l,m}(\alpha)\sigma^*_{(l+1)m}(\alpha)}{\alpha^{(l+1)m}}\right).$$

Доведення. Propositions 19.a and 20 imply:

$$p^*_{k+1,m}(\alpha) = \frac{s^*_{k+1,m}(\alpha)}{\alpha^{(k+1)m}} = \frac{1}{(k+1)\alpha^{(k+1)m}}\left((-1)^k\sigma^*_{(k+1)m}(\alpha) + \sum_{l=0}^{k-1}(-1)^l s^*_{k-l,m}(\alpha)\sigma^*_{(l+1)m}(\alpha)\right) =$$
$$= \frac{1}{(k+1)\alpha^{(k+1)m}}\left((-1)^k\sigma^*_{(k+1)m}(\alpha) + \sum_{l=0}^{k-1}(-1)^l\alpha^{(k-l)m}p^*_{k-l,m}(\alpha)\sigma^*_{(l+1)m}(\alpha)\right) =$$
$$= \frac{1}{k+1}\left(\frac{(-1)^k\sigma^*_{(k+1)m}(\alpha)}{\alpha^{(k+1)m}} + \sum_{l=0}^{k-1}\frac{(-1)^l p^*_{k-l,m}(\alpha)\sigma^*_{(l+1)m}(\alpha)}{\alpha^{(l+1)m}}\right).$$ □

The definitions of the $s^*_{k,m}$ and $p^*_{k,m}$ functions immediately imply

Proposition 22. *If the numbers α and β are associates, then*

$$s^*_{k,m}(\alpha) = s^*_{k,m}(\beta) \quad \text{and} \quad p^*_{k,m}(\alpha) = p^*_{k,m}(\beta).$$

Proposition 23. *For all α $s^*_{k,m}(\overline{\alpha}) = \overline{s^*_{k,m}(\alpha)}$ and $p^*_{k,m}(\overline{\alpha}) = \overline{p^*_{k,m}(\alpha)}$.*

Доведення. This follows from the fact that if the divisors of α are $\delta_1, \delta_2, \ldots, \delta_{\tau^*(\alpha)}$, then the divisors of $\overline{\alpha}$ are $\overline{\delta_1}$, $\overline{\delta_2}$, ..., $\overline{\delta_{\tau^*(\alpha)}}$. □

Proposition 24. *If α and $\overline{\alpha}$ are associates, then $s^*_{k,m}(\alpha) \in \mathbb{Z}$ and $p^*_{k,m}(\alpha) \in \mathbb{Q}$.*

The proof is analogous to the proofs of propositions 10.g and 16.

Proposition 25. *If m is odd k is not divisible by 4, then $s^*_{k,m}(\alpha) = 0$ and $p^*_{k,m}(\alpha) = 0$*

Доведення. Let m be odd. Consider the following polynomial:

$$P(x) = \prod_{\delta|\alpha}(x - \delta^m).$$

By Vieta's theorem,

$$P(x) = x^{\tau^*(\alpha)} + \sum_{j=1}^{\tau^*(\alpha)}(-1)^j s^*_{j,m}(\alpha)x^{\tau^*(\alpha)-j}.$$

Since the set of divisors of α breaks down into quadruples:

$$(\delta_1, -\delta_1, i\delta_1, -i\delta_1), (\delta_2, -\delta_2, i\delta_2, -i\delta_2), \ \ldots, (\delta_k, -\delta_k, i\delta_k, -i\delta_k),$$

where $k = \frac{1}{4}\tau^*(\alpha)$, then the polynomial $P(x)$ can be rewritten as follows:

$$P(x) = \prod_{j=1}^{\frac{1}{4}\tau^*(\alpha)}((x - \delta_j^m)(x - (-\delta_j)^m)\times \times(x - (i\delta_j)^m)(x - (-i\delta_j)^m)).$$

The odd parity of m implies that

$$(x - \delta_j^m)(x - (-\delta_j)^m)(x - (i\delta_j)^m)(x - (-i\delta_j)^m) = x^4 - \delta_j^{4m}.$$

Hence,

$$P(x) = x^{\tau^*(\alpha)} + \sum_{j=1}^{\tau^*(\alpha)}(-1)^j s^*_{j,m}(\alpha)x^{\tau^*(\alpha)-j} = \prod_{j=1}^{\frac{1}{4}\tau^*(\alpha)}(x^4 - \delta_j^{4m}).$$

Thus, in the polynomial $P(x)$, coefficients of the terms whose exponent of x is not divisible by 4, are equal to 0. Hence, if $4 \nmid k$, then $s^*_{k,m}(\alpha) = 0$. By proposition 19.a, $p^*_{k,m}(\alpha) = 0$ as well. □

Proposition 26. *If $m \equiv 2 \pmod 4$ and k is odd, then $s^*_{k,m}(\alpha) = 0$ and $p^*_{k,m}(\alpha) = 0$.*

The proof of this proposition is analogous to the previous one.

Proposition 27. *Suppose $\beta \mid \alpha$. If $\delta_1, \delta_2, ..., \delta_{\tau(\frac{\alpha}{\beta})}$ are all the divisors of α that are divisible by β, then*

$$(a) \sum_{1 \leqslant a_1 < a_2 < ... < a_k \leqslant \tau^*(\frac{\alpha}{\beta})} \delta_{a_1}^m \delta_{a_2}^m \cdots \delta_{a_k}^m = \beta^{km} s_{k,m}^* \left(\frac{\alpha}{\beta}\right);$$

$$(b) \sum_{1 \leqslant a_1 < a_2 < ... < a_k \leqslant \tau^*(\frac{\alpha}{\beta})} \frac{1}{\delta_{a_1}^m \delta_{a_2}^m \cdots \delta_{a_k}^m} = \frac{p_{k,m}^*(\frac{\alpha}{\beta})}{\beta^{km}}.$$

The proof follows from the fact that if $\alpha = \beta\gamma$, then all the divisors of α that are divisible by β are of the form $\beta\delta$, where $\delta \mid \gamma$. □

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