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Superposing tidal force fields

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Abstract

The motion of the Earth relative to the Sun's inertial frame of reference, is a superposition of the motion of the Earth around the common center of mass of the Moon and the Earth and the motion of that center of mass around the Sun. However, you get the resulting tidal force field due to the simultaneous existence of the Moon and the Sun by superposing a *Moon- induced* tidal force field and a *Sun-induced* tidal force field. Where, the Moon-induced tidal force field is calculated as if the Sun were not existing, and the Sun-induced tidal force field is calculated as if the Moon were not existing. The situation may lead to confusion. Thus, we give a formal conceptual account of the state of affairs, and argue that the case is useful when introducing undergraduates to inertial forces and understanding Newtonian mechanics.

Keywords: tidal force fields, inertial forces, concepts of Newtonian mechanics

1. Introduction

In most introductory universities, physics textbooks tidal force fields are either not or extremely briefly mentioned [1–5], left as a problem for students to solve [6], or explained briefly and qualitatively [7]. The textbooks do not treat accelerating frames of reference quantitatively. They do not explain the simple basis of tidal force fields. Nevertheless, there are many pedagogical papers meeting the interest in the topic [8–13]. The common interests in, e.g., ocean tides and astrophysical phenomena [14, 15] invite to be obliged. An introduction conceptually convincing is, e.g., in an inertial frame of reference to analyze the stresses in a rod in an inhomogeneous gravitational field [8, 13]. However, in general, avoiding the formal comprehension of how to operate in accelerated frames of reference generates often confusion. In [9], the author writes: 'Specifically, it is the erroneous use of centrifugal force……that is the principle cause of confusion.' In [10], the author, using centrifugal forces, concludes that superposing the tidal force fields due to the Moon and due to the Sun is *approximately* correct. Since the correct mathematical treatment of tidal force fields, departing from an inertial frame of reference, technically only involve rather simple mathematics, we believe it most fruitful to introduce the topic to undergraduates in an exact formal manner as done in more advanced textbooks [16, 17]. Then, a discussion of tidal force fields is useful as a conceptual stepping-stone to the fictitious/inertial centrifugal and Coriolis forces in spinning frames of reference, derived by more demanding mathematics. I agree with J R Taylor [16] writing: 'One can take the view that the inertial force is a 'fictitious' force, introduced merely to preserve the form of Newton's second law. Nevertheless, for an observer in an accelerating frame, it is entirely real.' Thus, in what follows, we will use the term *inertial* force. According to E I Butikov (EIB) [18] 'a review of textbooks and related literature shows that the most important aspects of the origin and properties of tides are often treated inaccurately or even erroneously. Much of the confusion over generating tides is related to the roles of the orbital motion of the Moon and Earth about their common center of mass and the Earth's axial rotation.' The motion of the Earth relative to the Sun inertial frame of reference is a superposition of the motion of the Earth around the common center of mass of the Moon and the Earth and the motion of that center of mass around the Sun. However, you get the resulting tidal force field due to the simultaneous existence of the Moon and the Sun by superposing a *Moon-induced* tidal force field and a *Sun-induced* tidal force field. Where, the Moon-induced tidal force field is calculated as if the Sun weren´t existing, and the Sun-induced tidal force field is calculated as if the Moon weren´t existing. This situation may lead to confusion. Personally, it confused me, that the tidal force field from the Moon is analog to the tidal force field from the Sun. The point of departure, when deriving the Sun-induced tidal force field, is Newton's second law in the assumed inertial frame of reference with origin in the Sun. Whereas the point of departure, when deriving the Moon-induced tidal force field, is Newton's second law in a frame of reference with origin in the center of mass of the Moon and the Earth, obviously not an inertial frame of reference.

In what follows, I give a formal conceptual account of the state of affairs. I hope that it removes both mine and other kinds of confusion.

In section 2, we calculate the Sun-induced tidal force field in the non-spinning geocentric frame of reference. As usual, we assume the Sun and the Earth to form an isolated system, not taking the existence of the Moon into account. Then, departing from an assumed inertial frame of reference with origin at the center of the Sun, the tidal force field is the sum of the gravitational field from the Sun and the inertial force field due to the then assumed acceleration of the Earth caused by the gravitational field from the Sun. This is in contrast to reality, where the Sun is accelerating the center of mass of the Earth/Moon system, not the center of the Earth.

In section 3, we calculate the Moon-induced tidal force field in the non-spinning geocentric frame of reference. As usual, we assume that the Moon and the Earth together is an isolated system, not taking the existence of the Sun into account. Furthermore, in contrast to reality, we assume the frame of reference, with origin in the common center of mass of the Moon and the Earth and axes parallel to the axes of the Sun inertial frame of reference, an inertial frame of reference. Then, the tidal force field is the sum of the gravitational field from the Moon and the inertial force field due to the then assumed acceleration of the Earth caused by the gravitational field from the Moon.

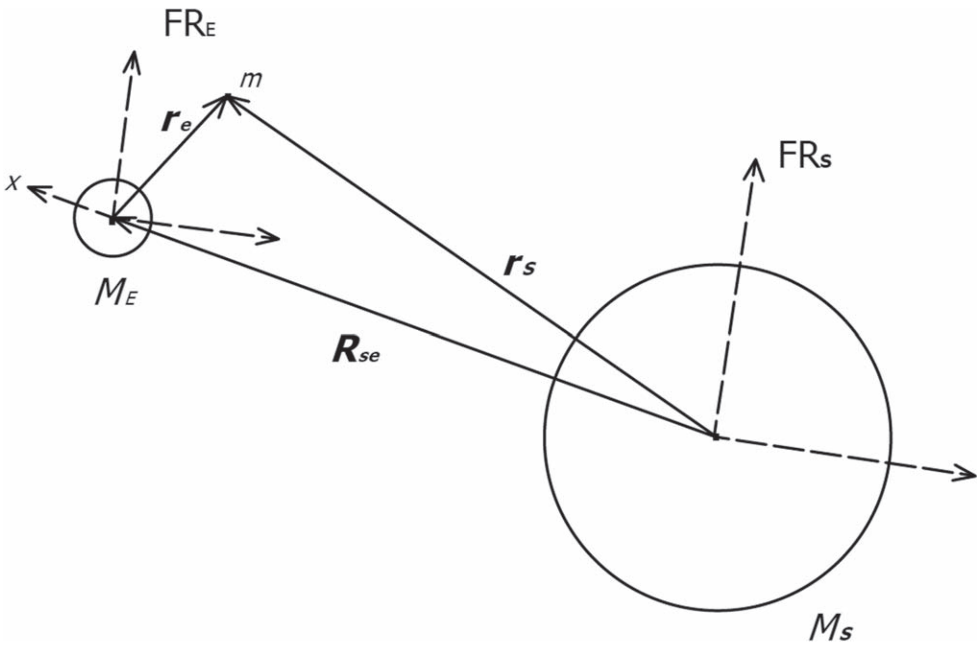

Figure 1. Two mutually non-spinning frames of reference in a system consisting of the Earth (mass M_E) and the Sun (mass M_S) only: $\mathrm{FR_S}$, with connected position vectors $\boldsymbol{r_s}$, has zero-point in the center of the Sun. $\mathrm{FR_E}$, with connected position vectors $\boldsymbol{r_e}$, has an origin in the center of the Earth. $\boldsymbol{R_{se}}$ is the position vector of the center of the Earth in $\mathrm{FR_S}$. m is a small test mass for which the different equations of motion in the different frames of reference may be derived. The directed x-axis along the line between the Sun and the Earth has origin at the center of the Earth.

In section 4, we calculate the tidal force field due to the Sun in the frame of reference with non-spinning axes and origin in the center of mass of the Moon/Earth system. It is this tidal force field, that e.g. in reality determines the accuracy of Newton´s calculation of the sidereal revolution time of the Moon, assuming the Moon and the Earth to form an isolated system.

In section 5, we depart from the Sun-induced tidal force field in the frame of reference with origin at the center of mass of the Moon/Earth system. We then calculate the tidal force field simultaneously caused by the Moon and the Sun, in the non-spinning geocentric frame of reference, as the result of two consecutive changes of frame of reference. This cumbersome calculation gives the same result as obtained by superposing the two tidal force fields induced by the Sun and the Moon, respectively, in section 2 and section 3. Finally, we explain why the principle of superposition is valid for tidal force fields in general.

In section 6, we discuss the educational relevance of the exercise here presented.

2. Sun-induced tidal force field in a non-spinning geocentric frame of reference

Figure 1 illustrates the essential reasoning behind the Sun-induced tidal force field. We consider a system where only gravitational forces from the Sun (with mass M_S) and the Earth (with mass M_E) matter. Moreover, we consider the frame of reference with axes pointing at distant stars and origin in the center of the Sun, $\mathrm{FR_S}$, an inertial frame of reference. Due to the much larger mass of the Sun than the mass of the Earth, the common center of mass of the

Earth and the Sun is very close to the center of the Sun. Thus, the Earth in FR_S is orbiting around the Sun. Since FR_S is an inertial frame of reference, Newton´s second law, describing the motion of a small test mass (m) in FR_S, becomes

$$md^2\boldsymbol{r_s}/dt^2 = m\boldsymbol{g_s}(r_s) + m\boldsymbol{g_e}(r_s), \tag{1}$$

$$\text{or } d^2\boldsymbol{r_s}/dt^2 = \boldsymbol{g_s}(\boldsymbol{r_s}) + \boldsymbol{g_e}(\boldsymbol{r_s}). \tag{2}$$

Here $\boldsymbol{r_s}$ is the position vector of the small mass in FR_S, and $\boldsymbol{g_s}(\boldsymbol{r_s})$ and $\boldsymbol{g_e}(\boldsymbol{r_s})$ are the gravitational field strengths due to the Sun and the Earth, respectively, at the position $\boldsymbol{r_s}$.

The Sun-induced tidal force field appears when we want to describe the motion of m in FR_E, a frame of reference with non-rotating axes relative to FR_S, but with origin in the orbiting center of the Earth. Since, the origin of FR_E is accelerated relative to FR_S, we have to adjust Newton´s second law.

From figure 1 we have $\boldsymbol{r_e} = \boldsymbol{r_s} - \boldsymbol{R_{se}}$, $\boldsymbol{r_e}$ being the position vector of m in FR_E and $\boldsymbol{R_{se}}$ being the position vector of the Earth in FR_S. Differentiating twice we get $d^2\boldsymbol{r_e}/dt^2 = d^2\boldsymbol{r_s}/dt^2 — d^2\boldsymbol{R_{se}}/dt^2$, and obtain after inserting equation (2)

$$d^2\boldsymbol{r_e}/dt^2 = \boldsymbol{g_s}(\boldsymbol{r_e}) + \boldsymbol{g_e}(\boldsymbol{r_e}) - d^2\boldsymbol{R_{se}}/dt^2. \tag{3}$$

This is the adjusted expression of Newton´s second law describing how motion and force fields correlates in FR_E. $\boldsymbol{g_s}(\boldsymbol{r_e})$ and $\boldsymbol{g_e}(\boldsymbol{r_e})$ are the gravitational field strengths at the position of m. Thus, they are equal to $\boldsymbol{g_s}(\boldsymbol{r_s})$ and $\boldsymbol{g_e}(\boldsymbol{r_s})$, respectively, in equation (2). We have just replaced their arguments, because we are referring to a different frame of reference. Due to the acceleration of the center of mass of FR_E, the adjusted Newton´s second law include the extra inertial force field—$d^2\boldsymbol{R_{se}}/dt^2$ compared to Newton´s second law in the inertial system.

Inserting $\boldsymbol{R_{se}}$ for $\boldsymbol{r_s}$ in equation (2) and skipping the term $\boldsymbol{g_e}(\boldsymbol{R_{se}})$ (the Earth is not interacting gravitational with its own center of mass) $d^2\boldsymbol{R_{se}}/dt^2$ is seen to be equal to $\boldsymbol{g_s}(\boldsymbol{R_{se}})$, the gravitational field strength from the Sun at the center of the Earth. In FR_E we denote it $\boldsymbol{g_s}(\boldsymbol{0_e})$, $\boldsymbol{0_e}$ being the origin of FR_E. Thus, Newton´s second law in FR_E is

$$d^2\boldsymbol{r_e}/dt^2 = \boldsymbol{g_e}(\boldsymbol{r_e}) + [\boldsymbol{g_s}(\boldsymbol{r_e}) - \boldsymbol{g_s}(\boldsymbol{0_e})]. \tag{4}$$

The term in the square brackets is 'the Sun-induced tidal force field'. Near the Earth compared to the distance to the Sun it is a small correction term relative to $\boldsymbol{g_e}$. It is much smaller than the gravitational field from the Sun itself, since the Sun, besides its gravitational field near the Earth, also causes an inertial force field due to its acceleration of the Earth.

It is easiest to evaluate the size of the tidal force field along the directed x-axis in figure 1. Reckoning $x = 0$ at the center of the Earth, and the tide field positive in the x direction, we get

$$\begin{aligned} g_s(x) - g_s(0) &= -GM_S/(R_{se} + x)^2 - (-GM_S/{R_{se}}^2) \\ &= GM_S/{R_{se}}^2(1 - 1/(1 + x/R_{se})^2) \\ &\approx GM_S/{R_{se}}^2(2x/R_{se}), \end{aligned} \tag{5}$$

where G is the gravitational constant. Thus, the tide force field is zero at the center of the Earth ($x = 0$) and pointing away from the Earth on both sides of the Earth (both for x positive and negative). At the surface of the Earth ($|x| = 6.4 \times 10^6$ m) the Sun-induced tide force field is a factor $2|x|/R_{se} = 2\ (6.4 \times 10^6\,\text{m})/(1.5 \times 10^{11}\,\text{m}) = 8.5 \times 10^{-5}$ smaller than the gravitational field from the Sun.

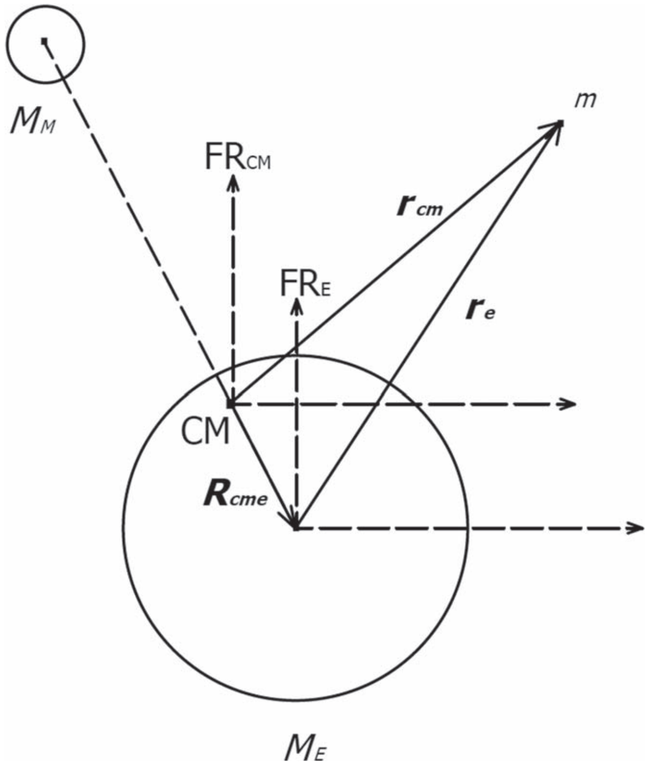

Figure 2. Two mutually non-spinning frames of reference in a system consisting of the Moon (mass M_M) and the Earth (mass M_E) only: FR_{CM}, with connected position vectors $\boldsymbol{r_{cm}}$, has origin in the common center of mass, CM, of the Moon and the Earth. FR_E, with connected position vectors $\boldsymbol{r_e}$, has origin in the center of the Earth. $\boldsymbol{R_{cme}}$ is the position vector of the center of the Earth in FR_{CM}. m is a small test mass for which the different equations of motion in the different frames of reference may be derived.

3. Moon-induced tidal force field in a non-spinning geocentric frame of reference

Figure 2 illustrates the essential reasoning behind the 'Moon-induced tidal force field'. We consider a system where only gravitational forces from the Moon (with mass M_M) and the Earth (with mass M_E) matter. In particular, we ignore the gravitational forces from the Sun. Thus, we assume the frame of reference with axes pointing at distant stars and origin in the common center of mass of the Moon and the Earth, FR_{CM}, an inertial frame of reference. Therefore, Newton´s second law describing the motion of a small test mass (m), in analogy with equation (2), in FR_{CM} becomes

$$d^2\boldsymbol{r_{cm}}/dt^2 = \boldsymbol{g_m}(\boldsymbol{r_{cm}}) + \boldsymbol{g_e}(\boldsymbol{r_{cm}}). \tag{6}$$

Here $\boldsymbol{r_{cm}}$ is the position vector of the small test mass in FR_{CM}, and $\boldsymbol{g_m}(\boldsymbol{r_{cm}})$ and $\boldsymbol{g_e}(\boldsymbol{r_{cm}})$ are the gravitational field strengths due to the Moon and the Earth, respectively, at the position $\boldsymbol{r_{cm}}$.

The Moon-induced tidal force field appears when we, with the assumption made, want to describe the motion of m in the non-spinning geocentric frame of reference, FR_E. Since the gravitational interaction between the Moon and the Earth makes them move around their common center of mass, FR_E is an accelerated frame of reference relative to FR_{CM}. Therefore, we have to adjust the expression of Newton´s second law.

From figure 2 we have $\boldsymbol{r_e} = \boldsymbol{r_{cm}}—\boldsymbol{R_{cme}}$, $\boldsymbol{r_e}$ being the position vector of m in FR_E and $\boldsymbol{R_{cme}}$ being the position vector of the Earth in $FR_{CM.}$ Thus, in analogy with equation (3) we get

$$d^2\boldsymbol{r_e}/dt^2 = \boldsymbol{g_m}(\boldsymbol{r_e}) + \boldsymbol{g_e}(\boldsymbol{r_e})—d^2\boldsymbol{R_{cme}}/dt^2, \tag{7}$$

as the adjusted expression of Newton´s second law, describing how motion and force fields correlate in FR_E. $\boldsymbol{g_m}(\boldsymbol{r_e})$ and $\boldsymbol{g_e}(\boldsymbol{r_e})$ are the gravitational field strengths at the position of m. Thus, they are equal to $\boldsymbol{g_m}(\boldsymbol{r_{cm}})$ and $\boldsymbol{g_e}(\boldsymbol{r_{cm}})$ respectively in equation (6). We have just replaced their argument because we are referring to a different frame of reference. Due to the acceleration of the center of the Earth in FR_{CM}, the adjusted Newton´s second law in FR_E includes the extra inertial force field—$d^2\boldsymbol{R_{cme}}/dt^2$ compared to Newton´s second law in FR_{CM}.

Inserting $\boldsymbol{R_{cme}}$ for $\boldsymbol{r_{cm}}$ in equation (6) and skipping the term $\boldsymbol{g_e}(\boldsymbol{R_{cme}})$ (the Earth is not interacting gravitationally with its own center of mass) $d^2\boldsymbol{R_{cme}}/dt^2$ is seen to be equal to $\boldsymbol{g_m}(\boldsymbol{R_{cme}})$, the gravitational field strength from the Moon at the center of the Earth. In FR_E we denote it $\boldsymbol{g_m}(\boldsymbol{0_e})$. Thus, in analogy with equation (4), we can rewrite Newton´s second law in FR_E as

$$d^2\boldsymbol{r_e}/dt^2 = \boldsymbol{g_e}(\boldsymbol{r_e}) + [\boldsymbol{g_m}(\boldsymbol{r_e}) - \boldsymbol{g_m}(\boldsymbol{0_e})]. \tag{8}$$

The term in the square brackets is the 'Moon-induced tidal force field'. The Moon-induced tidal force field and the derivation of it is completely analogous to the Sun-induced tidal force field and its derivation. Figure 1, illustrating the Sun-induced tidal field, and figure 2, illustrating the Moon-induced tidal field, does not look alike. However, apart from the figures, representing two different geometries, we have made the texts of section 2 and section 3 identical in order to make the analogy clear.

The size of the Moon-induced tidal force field along the direction towards the Moon, in analogy with what we found in equation (5), is $GM_M/R_{me}^2\ (2x/R_{me})$. Here x is the distance from the center of the Earth and R_{me} is the distance between the Moon and the Earth. Thus, at the surface of the Earth ($|x| = 6.4 \times 10^6$ m) the Moon-induced tide force field is a factor $2|x|/R_{me} = 2\ (6.4 \times 10^6\ \text{m})/(3.8 \times 10^8\ \text{m}) = 3.3 \times 10^{-2}$ smaller than the gravitational field from the Moon.

Inserting numbers in the expressions for the size of the Sun-induced tide force field and the Moon-induced tide force field at the surface of the Earth gives $5.0 \times 10^{-7}\ \text{m sec}^{-2}$ and $11.0 \times 10^{-7}\ \text{m sec}^{-2}$ respectively. Their similar magnitude makes it necessary to take both simultaneously into account when dealing with oceanic tides.

4. Sun-induced tidal force field in a non-spinning frame of reference with origin in the Moon/Earth center of mass

In section 2, we derived the Sun-induced tidal force field in FR_E as if the center of the Earth is orbiting around the Sun. Since it is a more real assumption, that it is the common center of mass of the Earth and the Moon, which is orbiting around the Sun, we will derive the Sun-induced force field in FR_{CM}. FR_{CM}, instead of FR_E, is the correct frame of reference for e.g. analyzing correction terms due to the Sun for bodies traveling between the Earth and the Moon.

Let us correct the formulas in section 2, taking the existence of the Moon into account, as illustrated in figure 3. In equation (2) we add $\boldsymbol{g_m}(\boldsymbol{r_s})$ on the right-hand side in order to get Newton´s second law in FR_S. Then, we have to change equation (3) to

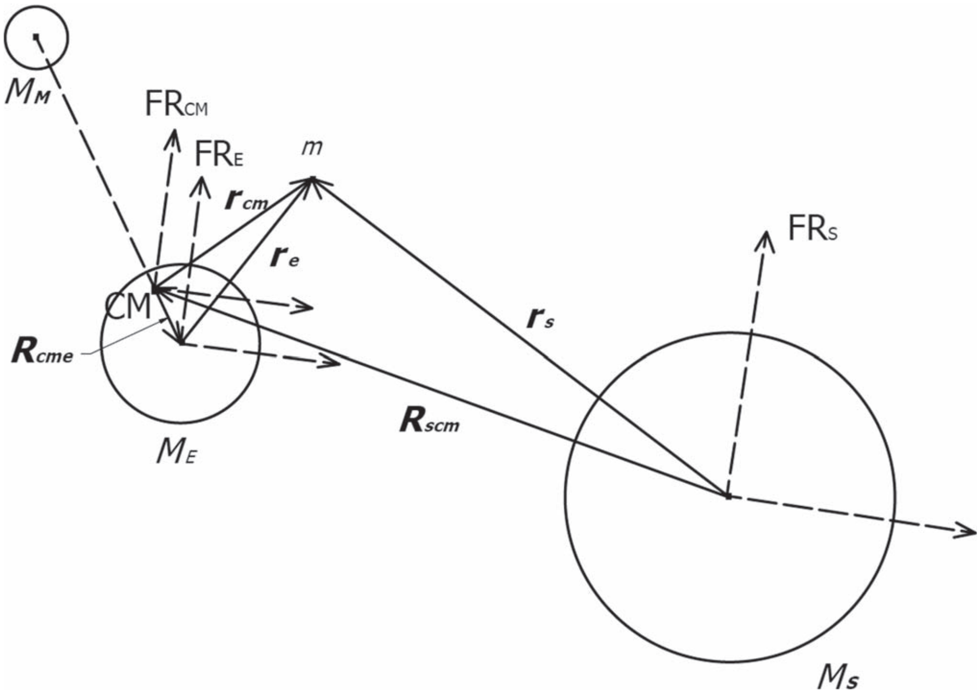

Figure 3. Three mutually non-spinning frames of reference in a system consisting of the Moon (mass M_M), the Earth (mass M_E), and the Sun (mass M_S): FR$_S$, with connected position vectors $\boldsymbol{r_s}$, has origin in the center of the Sun. FR$_{CM}$, with connected position vectors $\boldsymbol{r_{cm}}$, has origin in the common center of mass, CM, of the Moon and the Earth. FR$_E$, with connected position vectors $\boldsymbol{r_e}$, has origin in the center of the Earth. $\boldsymbol{R_{scm}}$ is the position of the center of mass of the Moon/Earth system in FR$_S$. $\boldsymbol{R_{cme}}$ is the position vector of the center of the Earth in FR$_{CM}$. m is a small test mass for which the different equations of motion in the different frames of reference may be derived.

$$d^2\boldsymbol{r_{cm}}/dt^2 = \boldsymbol{g_s}(\boldsymbol{r_{cm}}) + \boldsymbol{g_e}(\boldsymbol{r_{cm}}) + \boldsymbol{g_m}(\boldsymbol{r_{cm}}) - d^2\boldsymbol{R_{scm}}/dt^2 \tag{9}$$

in order to obtain Newton´s second law in FR$_{CM}$. Here the position vector $\boldsymbol{r_{cm}}$ now refers to FR$_{CM}$ instead of FR$_{E.}$ In addition, the inertial force field - $d^2\boldsymbol{R_{scm}}/dt^2$ now refers to the acceleration in FR$_S$ of the common center of mass of the Moon and the Sun instead of the acceleration of the center of the Earth in FR$_S$.

In section 2, we argued the acceleration of $\boldsymbol{R_{se}}$ to be equal to the strength of the gravitational field from the Sun, $\boldsymbol{g_s}(\boldsymbol{R_{se}})$, at the center of the Earth. This is exact if we assume the mass of the Earth is spherical and symmetrically distributed. Dealing with the mass distribution of the Moon and the Earth the analog assumption seems unlikely. However, the relative deviation of $\boldsymbol{g_s}(\boldsymbol{R_{scm}})$ from $d^2\boldsymbol{R_{scm}}/dt^2$ is only of the order of magnitude 10^{-6}. Thus, for most practical purposes we can replace $d^2\boldsymbol{R_{scm}}dt^2$ with $\boldsymbol{g_s}(\boldsymbol{0}_{\ cm})$ in equation (9), $\boldsymbol{0}_{\ cm}$ being the origin of FR$_{CM}$. Reorganizing equation (9) we then get the equation of motion for a body travelling in FR$_{CM}$ between the Moon and the Earth as

$$d^2\boldsymbol{r_{cm}}/dt^2 = \boldsymbol{g_e}(\boldsymbol{r_{cm}}) + \boldsymbol{g_m}(\boldsymbol{r_{cm}}) + [\boldsymbol{g_s}(\boldsymbol{r_{cm}}) - \boldsymbol{g_s}(\boldsymbol{0_{cm}})]. \tag{10}$$

Here the term in the square brackets is the Sun-induced tidal force field in FR$_{CM.}$

It is instructive to use equation (10) to calculate the sidereal revolution time of the Moon. On the one side, inserting numbers in Newton´s law of gravitation, we find the gravitational field from the Earth at the Moon to be 2,7 $\times$ 10^{-3} m $\sec^{-2}$, and the gravitational field from the Sun on the Moon to be 5,9 $\times$ 10^{-3} m/se c^2 . Thus, the Sun attracts the Moon with more than twice the attraction of the Moon from the Earth. On the other hand, we learned that Newton related the fall of an apple with the period of revolution of the Moon, treating the Moon/Earth as an isolated system. How can that be?

We take $\boldsymbol{r_{cm}}$ in equation (10) as the position vector of the Moon in FR_{CM} and cancel $\boldsymbol{g_m}(\boldsymbol{r_{cm}})$, since the Moon does not interact gravitationally with its own center of mass. Then, in analogy with equation (5), we find the Sun-induced tidal force field on the Moon, to be a factor $2r_{cm}/R_{scm} \approx 2(3{,}8 \times 10^8\ \mathrm{m})/(1{,}5 \times 10^{11}\ \mathrm{m}) = 5{,}1 \times 10^{-3}$ smaller than the gravitational field from the Sun. Thus, the influence of the Sun on Newton´s sidereal revolution time calculation is a tidal force field only of order of magnitude 1% of the gravitational field from the Earth at the Moon, not twice its magnitude. Thus, the calculation of the sidereal revolution time of the Moon, assuming the Moon/Earth system as an isolated system, is a good approximation.

The relative difference between the sidereal period of revolution of the Moon in FR_{CM} and in FR_E, assuming it as an inertial frame of reference, is about ½%. Thus, the error introduced because of this change of origin of the frame of reference is of the same order of magnitude as the error made by ignoring the tidal field from the Sun. This could be one of the reasons why, as cited in the introduction, 'much of the confusion over generating tides is related to the roles of the orbital motion of the Moon and the Earth about their common center of mass…'. Conceptually, FR_{CM} is a more correct frame of reference when calculating the sidereal period of revolution of the Moon than FR_E.

5. Superposing the tidal force field due to the Sun with the tidal force field due to the Moon

Let us find the force field in the non-spinning geocentric frame of reference taking the simultaneous existence of the Moon and the Sun into account, as illustrated in figure 3, by two consecutive changes of frame of reference. First from FR_S to FR_{CM}, second from FR_{CM} to FR_E.

The first change led us to equation (10) as the equation of motion in FR_{CM}.

Secondly, we insert $\boldsymbol{r_{cm}} = \boldsymbol{r_e} + \boldsymbol{R_{cme}}$ in equation (10) and get

$$d^2\boldsymbol{r_e}/dt^2 = \boldsymbol{g_e}(\boldsymbol{r_e}) + \boldsymbol{g_m}(\boldsymbol{r_e}) + [\boldsymbol{g_s}(\boldsymbol{r_e}) - \boldsymbol{g_s}(\boldsymbol{0_{cm}})] - d^2\boldsymbol{R_{cme}}/dt^2 \tag{11}$$

According to equation (10) for the particular case of $\boldsymbol{r_{cm}}$, $\boldsymbol{R_{cme}}$, we get

$$d^2\boldsymbol{R_{cme}}/dt^2 = \boldsymbol{g_m}(\boldsymbol{0_e}) + [\boldsymbol{g_s}(\boldsymbol{0_e}) - \boldsymbol{g_s}(\boldsymbol{0_{cm}})] \tag{12}$$

skipping the term $\boldsymbol{g_e}(\boldsymbol{0_e})$ (the Earth is not interacting with its own center of mass), and since $\boldsymbol{R_{cme}}$ in FR_{CM} is the same point in space as the origin $\boldsymbol{0_e}$ of FR_E.

Combining equations (11) and (12) the term $\boldsymbol{g_s}(\boldsymbol{0}_{cm})$ cancels out. Thus, after rearrangement, Newton's second law in FR_E, under the influence of the Moon the Sun and the Earth, is found to be

$$d^2\boldsymbol{r_e}/dt^2 = \boldsymbol{g_e}(\boldsymbol{r_e}) + [\boldsymbol{g_m}(\boldsymbol{r_e}) - \boldsymbol{g_m}(\boldsymbol{0_e})] + [\boldsymbol{g_s}(\boldsymbol{r_e}) - \boldsymbol{g_s}(\boldsymbol{0_e})] \tag{13}$$

In section 2 we derived the Sun-induced tidal force field on the Earth ignoring the existence of the Sun. In section 3 we derived the Moon-induced tidal force field on the Earth ignoring the existence of the Sun. The derivation of equation (13), taking the simultaneous existence of the Moon and the Sun into account, justifies the superposition of the two induced tidal force fields. Luckily. A different conclusion would have contradicted the usual treatment of oceanic tides.

In general, tidal force fields may be superposed, equation (13) being a special case.

Suppose, in an inertial frame of reference, that a given body is influenced by the gravitational force fields, $\boldsymbol{g_1}(\boldsymbol{r})$, $\boldsymbol{g_2}(\boldsymbol{r})$, …….$\boldsymbol{g_N}(\boldsymbol{r})$, from N other bodies. The acceleration of the body relative to the inertial frame of reference is the sum of the force fields at the center of the body (given the spherical symmetry of the body). Choosing the non-spinning frame of reference with origin in the center of the body, the contributing force fields due to the N bodies, in the transformed Newton´s second law in that frame of reference, is

$$[\boldsymbol{g_I}(\boldsymbol{r}_*) + \boldsymbol{g_2}(\boldsymbol{r}^*) + \cdots + \boldsymbol{g_N}(\boldsymbol{r}^*)] - [\boldsymbol{g_1}(\boldsymbol{0}) + \boldsymbol{g_2}(\boldsymbol{0}) + \cdots + \boldsymbol{g_N}(\boldsymbol{0})] \tag{14}$$

where the position vector pointing at the same point in space is shifted from $\boldsymbol{r}$ to $\boldsymbol{r}^*$, following the shift in frame of reference. The first bracket is the resulting gravitational field from the N bodies in the accelerated frame of reference. The second bracket is the inertial field due to this acceleration relative to the inertial frame of reference.

Rearranging (14) into

$$[\boldsymbol{g_1}(\boldsymbol{r}^*) - \boldsymbol{g_1}(\boldsymbol{0})] + [\boldsymbol{g_2}(\boldsymbol{r}^*) - \boldsymbol{g_2}(\boldsymbol{0})] + \cdots\cdots + [\boldsymbol{g_N}(\boldsymbol{r}^*) - \boldsymbol{g_N}(\boldsymbol{0})] \tag{15}$$

the Principle of Superposition for tidal force fields is justified. The resulting force field is the sum of partial tidal force fields, which would exist, if we leave the attracting bodies, one after the other, alone with the body attracted.

6. Educational relevance

For thousands of years people have observed ocean tides in many places on the Earth. They have observed spring tides at new Moon and full Moon, and have asked how the Moon and the Sun were causing the tides. Above I have explained the derivation and the superposition of the Moon-induced and Sun-induced tidal force fields. However, explaining the ocean tides around the globe is a much more demanding task, than explaining the force fields driving them. Relative to a frame of reference fixed to the Earth, the driving tidal force fields oscillate twice a day due to the rotation of the Earth about its axe. The oceans respond to this with forced oscillations with varying amplitudes and phases, depending on extensions, depths, and coastal geometry. The picture is very complex.

Nevertheless, understanding the tidal force fields meet the interest in tides partly, since it after all make us understand why high tide most often appears twice a day and spring tide at the new Moon and at full Moon. Meeting students‘ interest is of course of educational relevance. I have written sections 2, 3, and the general argument for superposing tidal force fields at the end of section 5, but not section 4 and the beginning of section 5, to meet that interest.

In all sections 2–5, I have tried to be explicit and formal in order to make all the sections usable as a training ground for understanding some basic concepts of Newtonian mechanics. It is my experience, when teaching Newtonian mechanics at bachelor level, that includes accelerated frames of reference and inertial forces deepen the student's understanding of motion as motion relative to something. The motion of m in figure 3 is determined by $d^2\boldsymbol{r_s}/dt^2 = \boldsymbol{g_s}(\boldsymbol{r_s}) + \boldsymbol{g_e}(\boldsymbol{r_s}) + \boldsymbol{g_m}(\boldsymbol{r_s})$ in $\mathrm{FR_S}$, by equation (10) in $\mathrm{FR_{CM}}$, and by equation (13) in $\mathrm{FR_E}$. Having the point of view, that real forces in principle may be measured by dynamometers and accelerations in contrast may be measured by clocks and rulers, inertial forces are real and Newton's second law an empirical law and not a mathematical identity or a definition of force. Teaching Newtonian mechanics, I find it most fruitful to take that point of view. The growing cress on a rotating turntable and a pendulum in an accelerated car cannot distinguish between the inertial force and the gravitational force. And much confusion in the minds of students has been created by naming g the acceleration in free fall, and not the gravitational field strength at the surface of the Earth. It is correct that the empirical Newton's second law states, that the acceleration a in the free fall is equal to g. However, that does not make them conceptually equal. We have the same problem when we name mv^2/r the centripetal force in a circular motion. Working with, e.g., the tidal force problems above may sharpen the student's understanding of the difference between equations expressing physical laws and equations expressing mathematical identities. In section 2, equation (2) is a physical law. In figure 1, $\boldsymbol{r_e} = \boldsymbol{r_s} - \boldsymbol{R_{se}}$, is seen to be a geometrical identity by definition. After differentiating twice and applying equation (2) (a physical law) for $\boldsymbol{r_s} = \boldsymbol{R_{se}}$ we reach equation (4), a physical law, that can be tested empirically. Along the road we have put $\boldsymbol{g}_s(\boldsymbol{r_s}) = \boldsymbol{g}_s(\boldsymbol{r_e})$, $\boldsymbol{g}_e(\boldsymbol{r_s}) = \boldsymbol{g}_e(\boldsymbol{r_e})$, and $\boldsymbol{g}_s(\boldsymbol{R_{se}}) = \boldsymbol{g}_s(\boldsymbol{0_e})$. $\boldsymbol{g}_s(\boldsymbol{r_s})$ and $\boldsymbol{g}_s(\boldsymbol{r_e})$ are of course different functions of $\boldsymbol{r_s}$ and $\boldsymbol{r_e}$ respectively. However, in physics we also think upon the gravitational field $\boldsymbol{g_s}$ as a function of the position in space, independent of a frame of reference. Thus, $\boldsymbol{g}_s(\boldsymbol{r_s})$ and $\boldsymbol{g}_s(\boldsymbol{r_e})$ are equal because $\boldsymbol{r_s}$ and $\boldsymbol{r_e}$ are pointing at the same position in space.

My point is that although mathematics has interwoven Newtonian mechanics, it also uses symbols and concepts of its own to be acquainted with. Sometimes partly in opposition to the conventional mathematical knowledge of the students. Then the above treatment of superposing tidal force fields may be useful as a training ground. Contrary to dealing with spinning frames of reference, its focus is not on mathematical technicalities, but rather on symbols and related concepts.

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Data availability statement

No new data were created or analysed in this study.

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