THE ASYMPTOTIC SAMUEL FUNCTION OF A FILTRATION

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A Dissertation presented to the Faculty of the Graduate School at

University of Missouri, Columbia

In Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

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MAY 2023

The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

THE ASYMPTOTIC SAMUEL FUNCTION OF A FILTRATION

presented by Smita Praharaj, a candidate for the degree of Doctor of Philosophy of Mathematics, and hereby certify that in their opinion it is worthy of acceptance.

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Ian Aberbach, Ph.D.

Petros Valettas, Ph.D.

Satish Nair, Ph.D.

To my parents, and my sister, my strength and my weakness!

ACKNOWLEDGEMENTS

I cannot thank my advisor, Professor Cutkosky, enough for his guidance and most of all his patience all throughout. My journey was far from linear but he was always understanding and supportive. I have learnt so much from his classes and our meetings. Be it the simplest or most complex of concepts, I have constantly been in awe of how meticulous and intuitive his explanations are. I am grateful to be his student.

I would like to express my deepest gratitude to my committee members.

I have always enjoyed talking to Professor Aberbach, be it regarding the comprehensive exams, or when I TAed my first abstract algebra class under him, or even when randomly seeing him on campus.

Probability and Computing with Professor Valettas is one of my most favorite classes, which is what encouraged me to do one of the comprehensive exams with him. The paper we chose was pretty recent which is why I was skeptical, but he was always encouraging and patient anytime I was stuck, and I had so much fun discussing with him.

Although the number of conversations with Professor Nair are limited, but each of them has lasted more than an hour. I have thoroughly enjoyed sharing ideas and learning about each other's research. He has always been very optimistic and motivating.

I will always be grateful to Professor Dana Weston for all the algebra classes as well as always being available to talk to. I would also like to thank Steven Goldschmidt for being a mentor, and a friend. I would like to Gwen Gwiplin and Yasuyo Knoll for being so prompt in their help and support.

I will forever be indebted to my family for never failing to put a smile on my face even in the worst of the days. Their support and encouragement always kept me going. They say friends are the family you choose. There are so many friends that I would raise a toast to: your camaraderie has been uplifting in ways unknown. I would like to thank Ashok Bagadiya, for being my companion in sickness and in health.

Finally, I would like to pat myself on the back. I did it!

Contents

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ABSTRACT

We extend the asymptotic Samuel function of an ideal to an arbitrary filtration of a Noetherian ring. We observe that although many properties that hold true in the ideal case are true for filtrations, there are many interesting differences as well. We study the notion of projective equivalence of filtrations, and consider the case of discrete valued filtrations, which have particularly nice properties.

Chapter 1

Introduction

Let R be a Noetherian ring. For an ideal I of R, the asymptotic Samuel function of I is defined as

$$\overline{\nu}_I(x) \coloneqq \lim_{m\to\infty} \frac{ord_I(x^m)}{m} \text{ for any } x \in R$$

where $ord_I(r) = \sup\{k \in \mathbb{N} \mid r \in I^k\}$ for $r \in R$. This was first defined by Samuel in [18]. Its basic properties and some beautiful theorems about it are proven in the articles [18], [12], [15], [17], [9], [10] and [11] and are surveyed in the book [19].

It is also shown in [19] Lemma 6.9.2 that this limit exists. Furthermore, Corollary 6.9.1 in [19] relates the asymptotic Samuel function of an ideal with the integral closure of the powers of the ideal, as mentioned in Lemma 3.2.

If $\mathcal{RV}(I) = \{v_1, \ldots, v_l\}$ is a set of Rees valuations of I ([15], [17], Section 10 [19]), then

$$\overline{\nu}_I(x) = \min\left\{\frac{v_1(x)}{v_1(I)}, \cdots, \frac{v_r(x)}{v_r(I)}\right\} \;\; \forall \;\; x \in R$$

This result is proven in [15] and after Lemma 10.1.5 in [19] (stated in Lemma 3.6). Furthermore, the Rees valuations are uniquely determined by the asymptotic Samuel function, up to equivalence of valuations. This is shown in Theorem 10.1.6 in [19] (stated in Theorem 3.7). This also proves that the range of the asymptotic Samuel function of I is contained in $\mathbb{Q}_{\geq 0} \cup \{\infty\}$.

Ideals I and J of R are said to be projectively equivalent if there exists $\alpha \in \mathbb{R}_{>0}$ such that $\overline{v}_I = \alpha\, \overline{v}_J$. Corollary 11.9 (ii) [10] or Exercise 10.26 of [19] provides a characterization of projectively equivalent ideals in terms of integral closures (which we state in Proposition 3.11).

In this thesis, we extend this notion of asymptotic Samuel function to arbitrary filtrations of R. Let $\mathcal{I} = \{I_m\}_{m\in\mathbb{N}}$ be a filtration of ideals in R, that is, $I_0 = R, I_n$ is an ideal in R, $I_{n+1} \subseteq I_n$ and $I_n \cdot I_m \subseteq I_{n+m}$, $\forall\ m, n \in \mathbb{N}$.

We show in Theorem 4.4 that for $x \in R$, the limit $\lim\limits_{m\to\infty} \dfrac{\nu_{\mathcal{I}}(x^m)}{m}$ exits, where $\nu_{\mathcal{I}}(r) = \max\{k \in \mathbb{N} \mid r \in I_k\}$. We define this function as the asymptotic Samuel function of the filtration $\mathcal{I}$, denoted by $\overline{\nu}_{\mathcal{I}}$. If $\mathcal{I} = \{I^m\}_{m\in\mathbb{N}}$ is the adic-filtration of powers of an ideal I then the asymptotic Samuel function $\overline{\nu}_{\mathcal{I}}$ of the filtration $\mathcal{I}$ is equal to the classical asymptotic Samuel function $\overline{\nu}_I$ of the ideal I.

The Rees algebra of a filtration $\mathcal{I} = \{I_m\}_{m\in\mathbb{N}}$ is the graded R-algebra

$$R[\mathcal{I}] = \sum_{m\in\mathbb{N}} I_m t^m \subseteq R[t],$$

where $R[t]$ is the polynomial ring in the variable t over R, which is viewed as a graded R-algebra where t has degree 1. Let $\overline{R[\mathcal{I}]} = \overline{\sum_{m\in\mathbb{N}} I_m t^m}$ be the integral closure of $R[\mathcal{I}]$ in $R[t]$.

If $\mathcal{I} = \{I^m\}_{m\in\mathbb{N}}$ is the adic-filtration of powers of an ideal I, then $R[\mathcal{I}] = \bigoplus_{m\in\mathbb{N}} I^m t^m$

is the usual Rees algebra of I, and $\overline{R[\mathcal{I}]} = \bigoplus_{m\in\mathbb{N}} \overline{I^m}t^m = \overline{R[I]}$, where $\overline{I^m}$ is the integral closure of the ideal I^m.

For a general filtration $\mathcal{I} = \{I_m\}_{m\in\mathbb{N}}$ of a Noetherian ring R, the integral closure of the Rees algebra $R[\mathcal{I}]$ is larger than the ring $\bigoplus_{m\in\mathbb{N}} \overline{I_m}t^m$. In fact, the integral closure of $R[\mathcal{I}]$ is

$$\overline{R[\mathcal{I}]} = \sum_{m\in\mathbb{N}} J_m t^m$$

where $J_m = \{f \in R \mid f^r \in \overline{I_{rm}} \text{ for some } r > 0\}$ and $\mathcal{IC}(\mathcal{I}) := \{J_m\}_{m\in\mathbb{N}}$ is a filtration of R. This is proven in Lemma 4.13.

Given a filtration $\mathcal{I} = \{I_m\}_{m\in\mathbb{N}}$ of R and $\alpha \in \mathbb{R}_{>0}$, define the twist of $\mathcal{I}$ by α to be the filtration

$$\mathcal{I}^{(\alpha)} = \{I_m^{(\alpha)}\}_{m\in\mathbb{N}} = \{I_{\lceil \alpha m\rceil}\}_{m\in\mathbb{N}}.$$

In Theorem 4.9 it is shown that if $\mathcal{I}$ is a filtration and $\alpha \in \mathbb{R}_{>0}$, then,

$$\overline{\nu}_{\mathcal{I}} = \alpha\overline{\nu}_{\mathcal{I}^{(\alpha)}}.$$

This is in contrast to the case of an ideal I in R, where range of $\overline{\nu}_I \subseteq \mathbb{Q}_{\geq 0} \cup \{\infty\}$.

The definition of projective equivalence for ideals extends naturally to filtrations. Filtrations $\mathcal{I}$ and $\mathcal{J}$ in a Noetherian ring R are said to be projectively equivalent if there exists $\alpha \in \mathbb{R}_{>0}$ such that $\overline{\nu}_{\mathcal{I}} = \alpha\overline{\nu}_{\mathcal{J}}$.

Suppose that I and J are ideals in a Noetherian ring R and $\mathcal{I} = \{I^n\}_{n\in\mathbb{N}}$, $\mathcal{J} = \{J^n\}_{n\in\mathbb{N}}$ are their associated adic-filtrations. We have that $\overline{\nu}_I = \overline{\nu}_{\mathcal{I}}$ and $\overline{\nu}_J = \overline{\nu}_{\mathcal{J}}$, so the ideals I and J are projectively equivalent if and only if the associated adic-filtrations $\mathcal{I}$ and $\mathcal{J}$ are projectively equivalent.

Theorem 4.9 shows that given any $\alpha \in \mathbb{R}_{>0}$, and a filtration $\mathcal{I} = \{I_m\}_{m\in\mathbb{N}}$ of R, the twist of $\mathcal{I}$ by α is projectively equivalent to $\mathcal{I}$ since $\overline{\nu}_{\mathcal{I}} = \alpha\overline{\nu}_{\mathcal{I}^{(\alpha)}}$. Thus, the conclusion of the rationality of α, as shown in Proposition 3.11 for projective equivalence of ideals, does not extend to filtrations.

We provide a necessary and sufficient condition for projective equivalence of filtrations in Theorem 4.9 (stated below).

Theorem 1.1. *Let $\mathcal{I} = \{I_m\}_{m\in\mathbb{N}}$ and $\mathcal{J} = \{J_m\}_{m\in\mathbb{N}}$ be filtrations in a Noetherian ring R. Then $\mathcal{I}$ and $\mathcal{J}$ are projectively equivalent if and only if $\exists\ \alpha,\ \beta \in \mathbb{R}_{>0}$ such that $\mathcal{IC}(\mathcal{I}^{(\alpha)}) = \mathcal{IC}(\mathcal{J}^{(\beta)})$, or equivalently, $\overline{R[\mathcal{I}^{(\alpha)}]} = \overline{R[\mathcal{J}^{(\beta)}]}$.*

We give an example in Example 4.20 of filtrations $\mathcal{I}$ and $\mathcal{J}$ which are projectively equivalent with $\overline{\nu}_{\mathcal{I}} = \overline{\nu}_{\mathcal{J}}$ but for no α or $\beta \in \mathbb{Q}$ do we have that $\overline{R[\mathcal{I}^{(\alpha)}]} = \overline{R[\mathcal{J}^{(\beta)}]}$. Thus the requirement of $\alpha, \beta \in \mathbb{R}_{>0}$ int he above Theorem cannot be weakened.

In the case that $\mathcal{I}$ and $\mathcal{J}$ are adic-filtrations of powers of ideals, we have by Proposition 3.11 that $\mathcal{I}$ and $\mathcal{J}$ are projectively equivalent if and only if $\overline{R[\mathcal{I}^{(m)}]} = \overline{R[\mathcal{J}^{(n)}]}$ for $m, n \in \mathbb{Z}_{>0}$ with $\overline{\nu}_{\mathcal{I}} = \dfrac{m}{n}\,\overline{\nu}_{\mathcal{J}}$. In this case, $\overline{R[\mathcal{I}^{(m)}]} = \overline{R[I^m]}$ and $\overline{R[\mathcal{J}^{(n)}]} = \overline{R[J^n]}$.

We show in Theorem 4.22 that given a filtration $\mathcal{I}$, there is a unique largest filtration $\mathcal{K}(\mathcal{I})$ such that $\mathcal{I}$ and $\mathcal{K}(\mathcal{I})$ have the same asymptotic Samuel function (also stated below).

Theorem 1.2. *(Theorem 4.22) For a filtration $\mathcal{I} = \{I_m\}_{m\in\mathbb{N}}$ of ideals in R, define*

$$K(\mathcal{I})_m := \{f \in R \mid \overline{\nu}_{\mathcal{I}}(f) \geq m\}\ \forall\ m \in \mathbb{N}.$$

Then $\mathcal{K}(\mathcal{I}) := \{K(\mathcal{I})_m\}_{m\in\mathbb{N}}$ *is a filtration of ideals in* R *and* $\mathcal{I} \subseteq \mathcal{K}(\mathcal{I})$*. Moreover,* $\overline{\nu}_{\mathcal{I}} = \overline{\nu}_{\mathcal{K}(\mathcal{I})}$ *and* $\mathcal{K}(\mathcal{I})$ *is the unique, largest filtration* $\mathcal{J}$ *such that* $\overline{\nu}_{\mathcal{I}} = \overline{\nu}_{\mathcal{J}}$*.*

If $\mathcal{I} = \{I^m\}_{m\in\mathbb{N}}$ is the adic-filtration of powers of an ideal, then $\mathcal{K}(\mathcal{I}) = \{\overline{I^m}\}_{m\in\mathbb{N}}$, the filtration of integral closures of powers of I (by Lemma 3.2).

In contrast, for a general filtration, it is possible for $\mathcal{K}(\mathcal{I})$ to be larger than the filtration $\mathcal{IC}(\mathcal{I})$, the integral closure of $\mathcal{I}$. Such an example is given in Example 4.23. By Lemma 4.25, the Rees algebra $R[\mathcal{K}(\mathcal{I})]$ is integrally closed. Thus for a filtration $\mathcal{I}$, we have inclusions of Rees algebras

$$R[\mathcal{I}] \subseteq R[\mathcal{IC}(\mathcal{I})] = \overline{R[\mathcal{I}]} \subseteq R[\mathcal{K}(\mathcal{I})] = \overline{R[\mathcal{K}(\mathcal{I})]} \tag{1.1}$$

where the two inclusions can be proper. In Theorem 4.24 we give another characterization of projective equivalence.

Theorem 1.3. *Suppose* $\mathcal{I}$ *and* $\mathcal{J}$ *are filtrations of a Noetherian ring* R*. Then* $\mathcal{I}$ *is projectively equivalent to* $\mathcal{J}$ *with* $\overline{\nu}_{\mathcal{I}} = \alpha\overline{\nu}_{\mathcal{J}}$ *if and only if* $\mathcal{K}(\mathcal{I}^{(\alpha)}) = \mathcal{K}(\mathcal{J})$*.*

In Section 5, we consider discrete valued filtrations (defined at the beginning of Section 5). We generalize some of the theory of Rees valuations of ideals (Section 10 [19]) to these filtrations.

This result is proven in [15] and after Lemma 10.1.5 in [19]. We prove the following Lemma, which generalizes Theorem 3.7 to discrete valued filtrations.

Lemma 1.4. *Let* $\mathcal{I} = \{I_m\}$ *where* $I_m = I(v_1)_{ma_1} \cap \cdots \cap I(v_s)_{ma_s}$ *be a discrete valued* *filtration of a Noetherian ring R. For* $f \in R \setminus \{0\}$,

$$\nu_{\mathcal{I}}(f) = \min\left\{\left\lfloor \frac{v_1(f)}{a_1} \right\rfloor, \cdots, \left\lfloor \frac{v_s(f)}{a_s} \right\rfloor\right\} \quad \textit{and} \quad \overline{\nu}_{\mathcal{I}}(f) = \min\left\{\frac{v_1(f)}{a_1}, \cdots, \frac{v_s(f)}{a_s}\right\}.$$

In Theorem 5.5, we generalize to discrete valued filtrations the proof of uniqueness of Rees valuations for ideals given in Theorem 10.1.6 [19]. We obtain the following Corollary.

Corollary 1.5. *(Corollary 5.7) Let* $\mathcal{I} = \{I_m\}_{m\in\mathbb{N}}$ *and* $\mathcal{J} = \{J_m\}_{m\in\mathbb{N}}$ *be discrete valued filtrations of a Noetherian ring R, where* $I_m = \bigcap_{i=1}^{s} I(v_i)_{a_i m}$ *and* $J_m = \bigcap_{i=1}^{r} I(v_i')_{a_i' m}$ $\forall\, m \in \mathbb{N}$ *are irredundant representations. If* $\overline{\nu}_{\mathcal{I}} = \overline{\nu}_{\mathcal{J}}$*, then* $r = s$ *and after reindexing,* $a_i = a_i'$ *and* $v_i = v_i'$.

If $\mathcal{I} = \{I_m\}_{m\in\mathbb{N}}$ where $I_m = \bigcap_{i=1}^{s} I(v_i)_{a_i m}$, then $\mathcal{I}^{[\alpha]}$ is the filtration $I_m^{[\alpha]} = \bigcap_{i=1}^{s} I(v_i)_{\alpha m a_i}$. This filtration is well defined (independent of representation of $\mathcal{I}$ as a discrete valued filtration).

Proposition 1.6. *(Proposition 5.8) Suppose that* $\mathcal{I}$ *is a discrete valued filtration of a Noetherian ring R and* $\alpha \in \mathbb{R}_{>0}$*. Then* $\mathcal{K}(\mathcal{I}^{(\alpha)}) = \mathcal{I}^{[\alpha]} = \mathcal{K}(\mathcal{I}^{[\alpha]})$.

In particular, the chain of inclusions of (1.1) are all equalities for discrete valued filtrations.

Theorem 1.7. *(Theorem 5.9) Let* $\mathcal{I} = \{I_m\}_{m\in\mathbb{N}}$ *and* $\mathcal{J} = \{J_m\}_{m\in\mathbb{N}}$ *be discrete valued filtrations of a Noetherian ring R and* $\alpha \in \mathbb{R}_{>0}$*. Then* $\overline{\nu}_{\mathcal{I}} = \alpha\, \overline{\nu}_{\mathcal{J}}$ *if and only if* $\mathcal{J} = \mathcal{I}^{[\alpha]}$.

Some of the results of this thesis appear in the paper [7] by Steven Dale Cutkosky and Smita Praharaj.

Chapter 2

Notation

Assume R is a commutative ring with identity.

$\mathbb{N}$	$\{0, 1, 2, \ldots\}$
$\mathbb{Z}_{>0}$	$\{1, 2, \ldots\}$
$\mathbb{R}$	All real numbers
$\mathbb{R}_{\geq 0}$	Non-negative real numbers
$\mathbb{R}_{>0}$	Positive real numbers
$\mathbb{Q}_{\geq 0}$	Non-negative rational numbers
$k^{\times}$	Nonzero elements of a field k
$\mathcal{RV}(I)$	Set of Rees valuations of the ideal I
$\overline{I}$	Integral closure of the ideal I
$\lfloor x \rfloor$	The largest integer less than or equal to x
$\lceil x \rceil$	The smallest integer which is greater than or equal to x

Chapter 3

The asymptotic Samuel function of an Ideal

Definition 3.1. For an ideal I of R, define a function $ord_I : R \to \mathbb{N} \cup \{\infty\}$ given by $f \mapsto sup \{m \mid f \in I^m\}$ for any $f \in R$. This is called the **order** of I.

Some observations regarding this function are as follows:

1. For any $x \in R \setminus I$, $ord_I(x) = 0$ and $ord_I(0) = \infty$.

2. For $f, g \in R$, $ord_I(f + g) \geq min \{ord_I(f), ord_I(g)\}$, and $ord_I(f \cdot g) \geq ord_I(f) + ord_I(g)$. So, this is not quite a valuation.

3. In fact, for $n \in \mathbb{N}$, $ord_I(f^n) \geq n \cdot ord_I(f)$ and this inequality could be strict. For example : Let $R = k[x, y]/(x^2 - y^3)$, where k is a field and x, y are variables over k and $\mathfrak{m} = (\bar{x}, \bar{y})$. Then $ord_{\mathfrak{m}}(\bar{x}) = 1$, but $ord_{\mathfrak{m}}(\bar{x}^2) = 3$.

The asymptotic Samuel function is a normalized version of the order function that gets around this situation in (3).

The following result has been proven in [19] Corollary 6.9.1.

Lemma 3.2. *Let R be a Noetherian ring, I an ideal of R, $x \in R\setminus\{0\}$, $c \in \mathbb{N}$. Then $x \in \overline{I^c}$ if and only if* $\limsup\limits_{m\to\infty} \dfrac{ord_I(x^m)}{m} \geq c$.

Here $\overline{I^c}$ denotes the integral closure of the ideal I^c in R. Furthermore, it is also shown in [19] Lemma 6.9.2 that

Lemma 3.3. *Let I be an ideal in a Noetherian ring R. For any $x \in R$,* $\lim\limits_{m\to\infty} \dfrac{ord_I(x^m)}{m}$ *exists.*

Definition 3.4. Let $\overline{v}_I : R \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be given by $\overline{v}_I(x) = \lim\limits_{m\to\infty} \dfrac{ord_I(x^m)}{m}$. This is called the **asymptotic Samuel function** of I.

Proposition 3.5. *For $f \in R$ and $n \in \mathbb{N}$, $\overline{v}_I(f^n) = n \cdot \overline{v}_I(f)$*

Proof. Since the limit exists, any subsequence also converges to the same limit. Thus,

$$\overline{v}_I(f) = \lim_{m\to\infty} \frac{ord_I(f^m)}{m} = \lim_{m\to\infty} \frac{ord_I(f^{nm})}{nm} = \frac{1}{n} \lim_{m\to\infty} \frac{ord_I((f^n)^m)}{m} = \frac{1}{n}\, \overline{v}_I(f^n)$$

Thus, $\overline{v}_I(f^n) = n\overline{v}_I(f)$. ∎

Rees published a series of papers ([14], [15], [16], [17]) which culminates in the proof of The Valuation Theorem (that relates the asymptotic Samuel function with the Rees valuations) for any ideal in a Noetherian ring. These results are also proven in Chapter 10, [19]. We mention these important results here (Lemma 3.6 - Lemma 3.9).

The following Lemma is proven in [19] (Lemma 10.1.5).

Lemma 3.6. *Let I be an ideal contained in no minimal prime of a Noetherian ring R. Let $\mathcal{RV}(I) = \{v_1, \ldots, v_l\}$ be a set of Rees valuations of I. Let $w : R \to \mathbb{R} \cup \{\infty\}$ be a function satisfying the following conditions:*

1. *For all $n \geq 1$, $\overline{I^n} = \{x \in R \mid w(x) \geq n\}$.*

2. *$w(x^n) = nw(x)$ for all $x \in R$ and $n \geq 1$.*

Then,

$$w(x) = \min\left\{\frac{v_1(x)}{v_1(I)}, \cdots, \frac{v_l(x)}{v_l(I)}\right\}.$$

Proof. Let $w' : R \to \mathbb{R} \cup \{\infty\}$ be given by $x \mapsto \min\limits_{1\leq i\leq l}\left\{\frac{v_i(x)}{v_i(I)}\right\}$. Observe that the function w' satisfies the two conditions above.

For $x \in R$, $x \in \overline{I^n}$ if and only if $v_i(x) \geq n\, v_i(I)\ \forall\ 1 \leq i \leq l$ if and only if $\frac{v_i(x)}{v_i(I)} \geq n\ \forall\ 1 \leq i \leq l$ if and only if $\min\limits_{1\leq i\leq l}\left\{\frac{v_i(x)}{v_i(I)}\right\} \geq n$, that is, $\omega'(x) \geq n$. Also, for $x \in R$ and $n \geq 1$,

$$w'(x^n) = \min_{1\leq i\leq l}\left\{\frac{v_i(x^n)}{v_i(I)}\right\} = \min_{1\leq i\leq l}\left\{\frac{nv_i(x)}{v_i(I)}\right\} = n \cdot \min_{1\leq i\leq l}\left\{\frac{v_i(x)}{v_i(I)}\right\} = n \cdot \omega'(x)$$

If $w \neq w'$, then $\exists\ x \in R$ such that $w(x) \neq w'(x)$.

If $w'(x) < w(x)$, then $\exists\ n \in \mathbb{N}$ such that $w'(x^n) \leq w(x^n) - 1$. If $w(x) < \infty$, set $k = \lfloor w(x^n) \rfloor$. If $w(x) = \infty$, we can set k to be an arbitrarily large integer. Since $w(x^n) \geq k$, by assumption $x^n \in \overline{I^k}$, but $w'(x^n) < w(x^n)$, that is, $w'(x^n) < k$, so, $x^n \notin \overline{I^k}$, by Definition of w'. This is a contradiction. Thus, $w'(x) \geq w(x)$.

Similarly, if $w(x) < w'(x)$. Then $\exists\ m \in \mathbb{N}$ such that $w(x^m) \leq w'(x^m) - 1$. If $w'(x) < \infty$, set $r = \lfloor w'(x^m) \rfloor$. If $w'(x) = \infty$, we can set r to be an arbitrarily large integer. Since $w'(x^m) \geq r$, by the Definition of w', $x^m \in \overline{I^r}$. But $w(x^m) <$

$w'(x^m) \implies w(x^m) < r$, so, $x^m \notin \overline{I^r}$, by assumption. This is a contradiction. Thus $w(x) \geq w'(x)$, proving that $w = w'$. ∎

The proof of the following result can be found in Theorem 10.1.6 in [19].

Theorem 3.7. *Let R be a ring and I be an ideal not contained in any minimal prime ideal of R. Let $v_1, \ldots, v_r$ be discrete valuations of rank 1 that are non-negative on R and each infinite on exactly one minimal prime ideal of R. Let $\omega : R \to \mathbb{Q}_{\geq 0} \cup \{\infty\}$ be defined by: for $x \in R$,*

$$\omega(x) = \min\left\{\frac{v_1(x)}{v_1(I)}, \cdots, \frac{v_r(x)}{v_r(I)}\right\} \tag{3.1}$$

If no v_i can be omitted from this expression, then the v_i are determined by the function ω up to equivalence of valuations.

Corollary 3.8. *For an ideal I of a Noetherian ring R, the Rees valuations of I are unique up to equivalence of valuations.*

Proof. We knows that Rees valuations exist for an ideal in a Noetherian ring. Say, $\mathcal{RV}(I) = \{v_1, \ldots, v_r\}$ and

$$w(x) = \min\left\{\frac{v_1(x)}{v_r(I)}, \cdots, \frac{v_r(x)}{v_r(I)}\right\} \forall\, x \in R \tag{3.2}$$

Since $\mathcal{RV}(I)$ is a minimal set of valuations defining $\overline{I^n} = \{x \in R \mid v(x^n) \geq nv(I) \;\forall\, v \in \mathcal{RV}(I)\} \;\forall\, n \in \mathbb{N}$, no v can be removed from 3.2. By Theorem 3.7, the Rees valuations are determined uniquely by the function w, up to equivalence of valuations.

∎

Lemma 3.9. *The range of $\overline{v}_I$ is contained in $\mathbb{Q}_{\geq 0} \cup \{\infty\}$.*

Proof. Lemma 3.2 and Proposition 3.5 shows that the asymptotic Samuel function, $\overline{v}_I$ satisfies the conditions of Lemma 3.6. Thus,

$$\overline{v}_I(x) = \min\left\{\frac{v_1(x)}{v_1(I)}, \cdots, \frac{v_r(x)}{v_r(I)}\right\} \; \forall \; x \in R \tag{3.3}$$

Since $\dfrac{v_i(x)}{v_i(I)} \in \mathbb{Q}_{\geq 0} \; \forall \; 1 \leq i \leq r$, the image of $\overline{v}_I$ is contained in $\mathbb{Q}_{\geq 0} \cup \{\infty\}$. ∎

Definition 3.10. Ideals I and J of a Noetherian ring R are said to be **projectively equivalent** if there exists $\alpha \in \mathbb{R}_{>0}$ such that $\overline{v}_I = \alpha \, \overline{v}_J$

Corollary 11.9 (ii) [10] and Exercise 10.26 of [19] provide a characterization of projectively equivalent ideals in terms of integral closures (which we state below).

Proposition 3.11. *Let I and J are ideals of a Noetherian ring R. Then I and J are projectively equivalent if and only if $\exists \; m, n \in \mathbb{Z}_{>0}$ such that $\overline{I^m} = \overline{J^n}$. If so, $\alpha = \dfrac{m}{n} \in \mathbb{Q}$ where $\overline{v}_I = \alpha \, \overline{v}_J$.*

Lemma 3.12. *Suppose the K is a field and $v : K^* \to \mathbb{Z}$ and $\omega : K^* \to \mathbb{Z}$ are discrete valuations. Then v is equivalent to ω if and only if $\exists \; c \in \mathbb{Q}_{>0}$ such that $v(a) = c \cdot \omega(a) \; \forall \; a \in K^*$.*

Proof. Since Γ_v and Γ_ω are subgroups of $\mathbb{Z}$, say, $\Gamma_v = m\mathbb{Z}$ and $\Gamma_\omega = n\mathbb{Z}$ for some $m, n \in \mathbb{Z}$.

Assume that v is equivalent to ω. Then $\exists$ an order-preserving isomorphism $\phi : \Gamma_v \to \Gamma_\omega$ such that $\phi(v(a)) = \omega(a) \; \forall \; a \in K^*$. If $\Gamma_v = \{0\}$, then $\Gamma_\omega = \{0\}$.

Without loss of generality, we can assume $m, n \in \mathbb{Z}_{>0}$. Since ϕ is an isomorphism between $m\mathbb{Z}$ and $n\mathbb{Z}$, $\phi(n) \in \{m, -m\}$ and ϕ being order-preserving $\implies \phi(m) = n$. This shows that for $a \in K^*$, if $v(a) = ml$ for some $l \in \mathbb{Z}$, then $\phi(v(a)) = \phi(ml) = nl \implies \omega(a) = nl$. In other words, $v(a) = \frac{m}{n}\,\omega(x)\ \forall\ x \in K^*$. Thus, $c = \dfrac{m}{n} \in \mathbb{Q}_{>0}$.

Now, suppose $\exists\ c \in \mathbb{Q}_{>0}$ such that $v(a) = c \cdot \omega(a)\ \forall\ a \in K^*$. Consider the map $\phi : \Gamma_\omega \to \Gamma_v$ given by $x \mapsto c \cdot x$. This is well-defined, since, any $x \in \Gamma_\omega$ is of the form $\omega(a)$ for some $a \in K^*$, and $\phi(x) = c \cdot x = c \cdot \omega(a) = v(a) \in \Gamma_v$. ϕ is an isomorphism, with the inverse map $\psi : \Gamma_v \to \Gamma_\omega$ given by $y \mapsto y/c$. ϕ is order-preserving since if $\omega(a) \leq \omega(b)$ in Γ_ω, then, $c \cdot \omega(a) \leq c \cdot \omega(b)$ as $c > 0$. This implies $v(a) \leq v(b)$. We have shown that ϕ is an order-preserving isomorphism between the value groups of ω and v proving that ω and v are equivalent valuations. ∎

Chapter 4

The asymptotic Samuel function of a Filtration

We extend the asymptotic Samuel function to any arbitrary filtration of R.

Let $\mathcal{I} = \{I_m\}_{m\in\mathbb{N}}$ be a filtration of ideals in R, that is, $I_0 = R$, I_n is an ideal in R, $I_n \supseteq I_{n+1}$ and $I_n \cdot I_m \subseteq I_{n+m}$, $\forall\ m, n \in \mathbb{N}$.

We say that a filtration $\mathcal{I} = \{I_m\}_{m\in\mathbb{N}}$ is a subset of a filtration $\mathcal{J} = \{J_m\}_{m\in\mathbb{N}}$ and write $\mathcal{I} \subseteq \mathcal{J}$ if $I_m \subseteq J_m\ \forall\ m \in \mathbb{N}$.

Example 4.1. Some examples of filtrations are as follows:

1. $\mathcal{I} = \{I^m\}_{m\in\mathbb{N}}$ for an ideal I of R.

2. $\mathcal{I} = \{\overline{I^m}\}_{m\in\mathbb{N}}$ for an ideal I of R.

3. For an arbitrary filtration $\mathcal{I}$ of R, $a \in \mathbb{Z}_{>0}$ we define a new filtration, called the **ath-truncated filtration**, $\mathcal{I}_a = \{I_{a,m}\}_{m\in\mathbb{N}}$ given by $I_{a,m} = I_m$ if $m \leq a$ and $I_{a,m} = \sum\limits_{\substack{i,j>0 \\ i+j=m}} I_{a,i}I_{a,j}$.

4. Given a filtration $\mathcal{I} = \{I_m\}_{m\in\mathbb{N}}$ of R and $\alpha \in \mathbb{R}_{>0}$, define a sequence of ideals $\mathcal{I}^{(\alpha)} = \{I_m^{(\alpha)}\}_{m\in\mathbb{N}} \coloneqq \{I_{\lceil \alpha m\rceil}\}_{m\in\mathbb{N}}$. This is also a filtration in R: For $m, n \in \mathbb{N}$, $\lceil \alpha n\rceil \leq \lceil \alpha(n+1)\rceil \implies I_{\lceil \alpha(n+1)\rceil} \subseteq I_{\lceil \alpha n\rceil}$ and $\lceil \alpha(n+m)\rceil \leq \lceil \alpha n\rceil + \lceil \alpha m\rceil \implies I_{\lceil \alpha n\rceil} \cdot I_{\lceil \alpha m\rceil} \subseteq I_{\lceil \alpha n\rceil + \lceil \alpha m\rceil} \subseteq I_{\lceil \alpha(n+m)\rceil}$. We call $\mathcal{I}^{(\alpha)}$ **the twist of $\mathcal{I}$ by α.**

Definition 4.2. For a filtration $\mathcal{I} = \{I_m\}_{m\in\mathbb{N}}$ in R, define a function $v_{\mathcal{I}} : R \to \mathbb{N} \cup \{\infty\}$ by $v_{\mathcal{I}}(f) \coloneqq \max\{m \mid f \in I_m\}$. We call this the **order** of $\mathcal{I}$.

Remark 4.3. For $x, y \in R$, $v_{\mathcal{I}}(xy) \geq v_{\mathcal{I}}(x) + v_{\mathcal{I}}(y)$ and $v_{\mathcal{I}}(x+y) \geq \min\{v_{\mathcal{I}}(x), v_{\mathcal{I}}(y)\}$. Observe that $v_{\mathcal{I}}(f) = \infty$ if and only if $f \in \bigcap\limits_{m\in\mathbb{N}} I_m$.

Theorem 4.4. *Let $\mathcal{I} = \{I_m\}_{m\in\mathbb{N}}$ be a filtration of ideals in a Noetherian ring R. For any $x \in R$, $\lim\limits_{n\to\infty} \frac{v_{\mathcal{I}}(x^n)}{n}$ exists.*

Proof. Let $x \in R$ and $u \coloneqq \limsup\limits_{n\to\infty} \frac{\nu_{\mathcal{I}}(x^n)}{n}$ (which could possibly be ∞).

If $u = 0$, then the limit exists since $0 \leq \liminf\limits_{n\to\infty} \frac{\nu_{\mathcal{I}}(x^n)}{n} \leq \limsup\limits_{n\to\infty} \frac{\nu_{\mathcal{I}}(x^n)}{n} = 0$.

Assume $u > 0$. Let $N \in \mathbb{R}_{>0}$ be such that $N < u$. We can choose $n_0 \in \mathbb{Z}_{>0}$ such that $\frac{\nu_{\mathcal{I}}(x^{n_0})}{n_0} > N$. Let n be any arbitrary positive integer. We have $n = qn_0 + r$ for some $q, r \in \mathbb{N}$ such that $0 \leq r < n_0$.

Using Remark 4.3 it follows that

$$\frac{\nu_{\mathcal{I}}(x^n)}{n} = \frac{\nu_{\mathcal{I}}(x^{qn_0+r})}{qn_0+r} \geq \frac{\nu_{\mathcal{I}}(x^{qn_0})}{qn_0+r} + \frac{\nu_{\mathcal{I}}(x^r)}{qn_0+r} \geq q\frac{\nu_{\mathcal{I}}(x^{n_0})}{qn_0+r} + \frac{\nu_{\mathcal{I}}(x^r)}{qn_0+r} \geq q\frac{\nu_{\mathcal{I}}(x^{n_0})}{qn_0+r}.$$

This implies $\frac{\nu_{\mathcal{I}}(x^n)}{n} \geq \frac{qn_0}{qn_0+r}\frac{\nu_{\mathcal{I}}(x^{n_0})}{n_0} \geq \frac{qn_0}{qn_0+r}N \geq \frac{qn_0}{qn_0+n_0}N = \frac{q}{q+1}N.$

Taking lim inf on both sides, we get $\liminf_{n\to\infty} \frac{\nu_{\mathcal{I}}(x^n)}{n} \geq \liminf_{n\to\infty} \frac{q}{q+1} N$.

Clearly, $\liminf_{n\to\infty} \frac{q}{q+1} \leq 1$. Since $r < n_0$, $\frac{q}{q+1} = \frac{n-r}{n+n_0-r} \geq \frac{n-n_0}{n}$ implying $\liminf_{n\to\infty} \frac{q}{q+1} \geq 1$. Thus $\liminf_{n\to\infty} \frac{q}{q+1} = 1$. This shows that $\liminf_{n\to\infty} \frac{\nu_{\mathcal{I}}(x^n)}{n} \geq N$ for any positive real number strictly smaller than u. Since N was arbitrarily chosen, $\liminf_{n\to\infty} \frac{\nu_{\mathcal{I}}(x^n)}{n} \geq \limsup_{n\to\infty} \frac{\nu_{\mathcal{I}}(x^n)}{n}$, proving that the limit exists. ∎

Definition 4.5. For a filtration $\mathcal{I} = \{I_m\}_{m\in\mathbb{N}}$ of ideals in R, we define the function $\overline{v}_{\mathcal{I}} : R \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ by $\overline{v}_{\mathcal{I}}(x) := \lim_{n\to\infty} \frac{v_{\mathcal{I}}(x^n)}{n}$ for $x \in R$.

The asymptotic Samuel function of an ideal I in a Noetherian ring R is defined to be $\overline{\nu}_I(x) = \lim_{n\to\infty} \frac{\mathrm{ord}_I(x^n)}{n}$ where $\mathrm{ord}_I(x) = \sup\{m \mid x \in I^m\}$ for $x \in R$. Then for any $x \in R$, $\overline{\nu}_I(x) = \overline{\nu}_{\mathcal{I}}(x)$, where $\mathcal{I}$ is the adic-filtration $\mathcal{I} = \{I^m\}_{m\in\mathbb{N}}$. This follows since $ord_I(x) = \nu_{\mathcal{I}}(x)$ for any $x \in R$.

Thus, $\overline{\nu}_{\mathcal{I}}$ extends the concept of the asymptotic Samuel function of an ideal to an arbitrary filtration of a Noetherian ring. We call $\overline{\nu}_{\mathcal{I}}$ the **asymptotic Samuel function of the filtration** $\mathcal{I}$.

Remark 4.6. Let $\mathcal{I} \subseteq \mathcal{J}$ be filtrations. Then $\overline{\nu}_{\mathcal{I}} \leq \overline{\nu}_{\mathcal{J}}$.

Proof. For $x \in R$, we have that $\nu_{\mathcal{I}}(x^i) \leq \nu_{\mathcal{J}}(x^i)\ \forall\ i \in \mathbb{N}$ so that $\overline{\nu}_{\mathcal{I}}(x) \leq \overline{\nu}_{\mathcal{J}}(x)$. ∎

Example 4.7. In a Noetherian local ring $(R, \mathfrak{m}_R)$, consider the filtration $\mathcal{I} = \{I_m\}_{m\in\mathbb{N}}$ given by $I_0 = R$ and $I_m = \mathfrak{m}_R$ for $m > 0$. In this case, $\overline{v}_{\mathcal{I}}(a) = \infty$ if $a \in \mathfrak{m}_R$ and $\overline{v}_{\mathcal{I}}(a) = 0$ if $a \notin \mathfrak{m}_R$.

Remark 4.8. The range of $\overline{\nu}_{\mathcal{I}}$ may not be contained in $\mathbb{Q}_{\geq 0} \cup \{\infty\}$. This follows from

Theorem 4.9 below. Observe that this is different from the case when $\mathcal{I} = \{I^m\}_{m\in\mathbb{N}}$ for an ideal I of R, in which case we do have $\overline{\nu}_{\mathcal{I}} = \overline{\nu}_I$ and then the range of $\overline{\nu}_{\mathcal{I}}$ is contained in $\mathbb{Q}_{\geq 0} \cup \{\infty\}$ (as shown after Lemma 10.1.5 [19]).

Theorem 4.9. *Let $\mathcal{I} = \{I_m\}_{m\in\mathbb{N}}$ be a filtration in R and $\alpha \in \mathbb{R}_{>0}$. Then, $\overline{v}_{\mathcal{I}} = \alpha\,\overline{v}_{\mathcal{I}^{(\alpha)}}$, where $\mathcal{I}^{(\alpha)} = \{I_m^{(\alpha)}\}_{m\in\mathbb{N}} = \{I_{\lceil \alpha m\rceil}\}_{m\in\mathbb{N}}$.*

Proof. For $x \in R$ and $i \in \mathbb{N}$, $x^i \in I_{\lceil \alpha\nu_{\mathcal{I}^{(\alpha)}}(x^i)\rceil}$, so, $\nu_{\mathcal{I}}(x^i) \geq \lceil \alpha\nu_{\mathcal{I}^{(\alpha)}}(x^i)\rceil$ which gives $\overline{\nu}_{\mathcal{I}}(x) = \lim\limits_{i\to\infty} \dfrac{\nu_{\mathcal{I}}(x^i)}{i} \geq \lim\limits_{i\to\infty} \dfrac{\lceil \alpha\nu_{\mathcal{I}^{(\alpha)}}(x^i)\rceil}{i}$. Since $\alpha\nu_{\mathcal{I}^{(\alpha)}}(x^i) \leq \lceil \alpha\nu_{\mathcal{I}^{(\alpha)}}(x^i)\rceil \leq \alpha\nu_{\mathcal{I}^{(\alpha)}}(x^i) + 1$,

$$\lim_{i\to\infty} \frac{\alpha\nu_{\mathcal{I}^{(\alpha)}}(x^i)}{i} \leq \lim_{i\to\infty} \frac{\lceil \alpha\nu_{\mathcal{I}^{(\alpha)}}(x^i)\rceil}{i} \leq \lim_{i\to\infty} \frac{\alpha\nu_{\mathcal{I}^{(\alpha)}}(x^i) + 1}{i}$$

This implies $\lim\limits_{i\to\infty} \dfrac{\lceil \alpha\nu_{\mathcal{I}^{(\alpha)}}(x^i)\rceil}{i} = \alpha\,\overline{\nu}_{\mathcal{I}^{(\alpha)}}(x)$. Thus, $\overline{\nu}_{\mathcal{I}}(x) \geq \alpha\,\overline{\nu}_{\mathcal{I}^{(\alpha)}}(x)$.

Note that if $x \in I_k$ for some $k \in \mathbb{N}$, then $x \in I_{\lceil \lfloor \frac{k}{\alpha} \rfloor \alpha\rceil}$. It follows that $\nu_{\mathcal{I}^{(\alpha)}}(x) \geq \left\lfloor \dfrac{\nu_{\mathcal{I}}(x)}{\alpha} \right\rfloor$. Thus, $\forall\ i \in \mathbb{Z}_{>0}$, $\dfrac{\nu_{\mathcal{I}^{(\alpha)}}(x^i)}{i} \geq \dfrac{\left\lfloor \frac{\nu_{\mathcal{I}}(x^i)}{\alpha} \right\rfloor}{i}$ which implies $\lim\limits_{i\to\infty} \dfrac{\nu_{\mathcal{I}^{(\alpha)}}(x^i)}{i} \geq \lim\limits_{i\to\infty} \dfrac{\left\lfloor \frac{\nu_{\mathcal{I}}(x^i)}{\alpha} \right\rfloor}{i}$. Since $\left\lfloor \dfrac{\nu_{\mathcal{I}}(x^i)}{\alpha} \right\rfloor > \dfrac{\nu_{\mathcal{I}}(x^i)}{\alpha} - 1$, $\lim\limits_{i\to\infty} \dfrac{\left\lfloor \frac{\nu_{\mathcal{I}}(x^i)}{\alpha} \right\rfloor}{i} \geq \lim\limits_{i\to\infty} \dfrac{\frac{\nu_{\mathcal{I}}(x^i)}{\alpha} - 1}{i} = \dfrac{\overline{\nu}_{\mathcal{I}}(x)}{\alpha}$.

This shows $\overline{\nu}_{\mathcal{I}^{(\alpha)}}(x) \geq \dfrac{\overline{\nu}_{\mathcal{I}}(x)}{\alpha}$, thus proving the result. ∎

Proposition 4.10. *Let $\mathcal{I}$ be a filtration of R. For $f, g \in R$,*

1. $\overline{\nu}_{\mathcal{I}}(f^n) = n\,\overline{\nu}_{\mathcal{I}}(f)\ \forall\ n \in \mathbb{Z}_{>0}$.

2. $\overline{\nu}_{\mathcal{I}}(f+g) \geq \min\{\overline{\nu}_{\mathcal{I}}(f), \overline{\nu}_{\mathcal{I}}(g)\}$.

Proof. Since the limit defining $\overline{\nu}_{\mathcal{I}}$ exists, any subsequence also converges to the same limit. Thus,

$$\overline{\nu}_{\mathcal{I}}(f) = \lim_{m\to\infty} \frac{\nu_{\mathcal{I}}(f^m)}{m} = \lim_{m\to\infty} \frac{\nu_{\mathcal{I}}(f^{nm})}{nm} = \frac{1}{n}\lim_{m\to\infty} \frac{\nu_{\mathcal{I}}((f^n)^m)}{m} = \frac{1}{n}\,\overline{\nu}_{\mathcal{I}}(f^n),\ \forall\, n \in \mathbb{Z}_{>0}.$$

This proves (1).

To prove (2), let $f, g \in R$ be such that $\overline{\nu}_{\mathcal{I}}(f) \geq \overline{\nu}_{\mathcal{I}}(g)$. For $\epsilon > 0$, $\exists\, m_0 \in \mathbb{Z}_{>0}$ such that $\forall\, m \geq m_0$, $\dfrac{\nu_{\mathcal{I}}(f^m)}{m} \geq \overline{\nu}_{\mathcal{I}}(g) - \epsilon$ and $\dfrac{\nu_{\mathcal{I}}(g^m)}{m} \geq \overline{\nu}_{\mathcal{I}}(g) - \epsilon$. For all $m, k \in \mathbb{Z}_{>0}$, $\nu_{\mathcal{I}}((f+g)^{mk}) \geq \min\limits_{i+j=mk}\{\nu_{\mathcal{I}}(f^i g^j)\}$, using Remark 4.3. Since $i + j = mk$, $mk \geq m\left\lfloor \dfrac{i}{m} \right\rfloor + m\left\lfloor \dfrac{j}{m} \right\rfloor \geq mk - 2m$. Thus by Remark 4.3,

$$\nu_{\mathcal{I}}(f^i g^j) \geq \nu_{\mathcal{I}}(f^i) + \nu_{\mathcal{I}}(g^j) \geq \nu_{\mathcal{I}}(f^{m\lfloor \frac{i}{m} \rfloor}) + \nu_{\mathcal{I}}(g^{m\lfloor \frac{j}{m} \rfloor}) \geq \left\lfloor \frac{i}{m} \right\rfloor \nu_{\mathcal{I}}(f^m) + \left\lfloor \frac{j}{m} \right\rfloor \nu_{\mathcal{I}}(g^m).$$

For $m \geq m_0$, $\nu_{\mathcal{I}}(f^i g^j) \geq \left\lfloor \dfrac{i}{m} \right\rfloor m(\overline{\nu}_{\mathcal{I}}(g) - \epsilon) + \left\lfloor \dfrac{j}{m} \right\rfloor m(\overline{\nu}_{\mathcal{I}}(g) - \epsilon) \geq (mk - 2m)(\overline{\nu}_{\mathcal{I}}(g) - \epsilon)$.

Thus, for $k >> 0$,

$$\frac{\nu_{\mathcal{I}}((f+g)^{mk})}{mk} \geq \frac{mk(\overline{\nu}_{\mathcal{I}}(g) - \epsilon) - 2m(\overline{\nu}_{\mathcal{I}}(g) - \epsilon)}{mk} = \overline{\nu}_{\mathcal{I}}(g) - \epsilon - \frac{2}{k}(\overline{\nu}_{\mathcal{I}}(g) - \epsilon).$$

Taking limits as $k \to \infty$, we get $\overline{\nu}_{\mathcal{I}}(f+g) \geq \overline{\nu}_{\mathcal{I}}(g) - \epsilon$. Since ϵ is arbitrary, $\overline{\nu}_{\mathcal{I}}(f+g) \geq \overline{\nu}_{\mathcal{I}}(g) = \min\{\overline{\nu}_{\mathcal{I}}(f), \overline{\nu}_{\mathcal{I}}(g)\}$. This completes the proof. ∎

Lemma 4.11. *Let R be a Noetherian ring and I be an ideal of R. For any $x \in R$, $\overline{v}_{\mathcal{I}}(x) = \overline{v}_{\overline{\mathcal{I}}}(x)$ where $\mathcal{I} = \{I^m\}_{m\in\mathbb{N}}$ and $\overline{\mathcal{I}} = \{\overline{I^m}\}_{m\in\mathbb{N}}$*

Proof. For $x \in R$ and $i \in \mathbb{N}$, $x^i \in \overline{I^{\nu_{\overline{\mathcal{I}}}(x^i)}}$ which gives $\overline{\nu}_{\mathcal{I}}(x^i) \geq \nu_{\overline{\mathcal{I}}}(x^i)$, by Lemma 3.2. By Proposition 4.10(a), $\overline{\nu}_{\mathcal{I}}(x) \geq \dfrac{\nu_{\overline{\mathcal{I}}}(x^i)}{i}$ $\forall\ i \in \mathbb{Z}_{>0}$. This implies $\overline{\nu}_{\mathcal{I}}(x) \geq \lim\limits_{i\to\infty} \dfrac{\nu_{\overline{\mathcal{I}}}(x^i)}{i} = \overline{\nu}_{\overline{\mathcal{I}}}(x)$.

Since $\mathcal{I} \subseteq \overline{\mathcal{I}}$, by Remark 4.6, $\overline{\nu}_{\mathcal{I}} = \overline{\nu}_{\overline{\mathcal{I}}}$. ∎

We can extend the concept to integral closure of arbitrary filtrations of a Noetherian ring.

Definition 4.12. The **Rees algebra of a filtration** $\mathcal{I} = \{I_m\}_{m\in\mathbb{N}}$ is the graded R-algebra $R[\mathcal{I}] = \sum_{m\in\mathbb{N}} I_m t^m \subseteq R[t]$, where $R[t]$ is the polynomial ring in variable t over R, which is viewed as a graded R-algebra where t has degree 1.

Let $\overline{R[\mathcal{I}]} = \overline{\sum_{m\in\mathbb{N}} I_m t^m}$ be the integral closure of $R[\mathcal{I}]$ in $R[t]$.

In [5] Lemma 5.5, there is a characterization of $\overline{R[\mathcal{I}]}$ when $(R, \mathfrak{m}_R)$ is a local ring and $\mathcal{I}$ is an $\mathfrak{m}_R$-filtration. The proof extends to the case where $\mathcal{I}$ is an arbitrary filtration of a Noetherian ring R.

Lemma 4.13. *Let $\mathcal{I} = \{I_m\}_{m\in\mathbb{N}}$ be a filtration in R. Then $\overline{R[\mathcal{I}]} = \sum_{m\in\mathbb{N}} J_m t^m$ where $J_m = \{f \in R \mid f^r \in \overline{I_{rm}}$ for some $r > 0\}$ and $\mathcal{IC}(\mathcal{I}) = \{J_m\}_{m\in\mathbb{N}}$ is a filtration in R.*

Proof. Using [19] Theorem 2.3.2, the ring $\overline{R[\mathcal{I}]}$ is graded. So, we can write $\overline{R[\mathcal{I}]} = \sum_{m\in\mathbb{N}} J_m t^m$ where J_m is the graded component of degree m. In order to show that J_m is as described in the statement of the Theorem, it suffices to show the required for homogeneous elements. Thus, for $f \in R$ and $n \in \mathbb{Z}_{>0}$ we want to show that ft^n is integral over $R[\mathcal{I}]$ if and only if $f^r \in \overline{I_{rn}}$ for some $r > 0$.

Now, ft^n is integral over $R[\mathcal{I}]$ if and only if there is a homogeneous relation (as below) for some $d > 0$ and $a_i \in I_{ni}$.

$$(ft^n)^d + a_1 t^n (ft^n)^{d-1} + \cdots + a_i t^{ni} (ft^n)^{d-i} + \cdots + a_{d-1} t^{n(d-1)} (ft^n) + a_d t^{nd} = 0$$

Suppose $f^r \in \overline{I_{rn}}$ for some $r > 0$. Then, there exists an equation of integral dependence over I_{rn} of degree, say d, given by,

$$(f^r)^d + a_1(f^r)^{d-1} + \cdots + a_i(f^r)^{d-i} + \cdots + a_{d-1}(f^r) + a_d = 0$$

where $a_i \in (I_{rn})^i \subseteq I_{rni}\ \forall\ 1 \leq i \leq d$. Multiplying the above equation by t^{rnd} we get:

$$[(ft^n)^r]^d + a_1 t^{rn}[(ft^n)^r]^{d-1} + \cdots + a_i t^{rni}[(ft^n)^r]^{d-i} + \cdots + a_{d-1}t^{rn(d-1)}[(ft^n)^r] + a_d t^{rnd} = 0$$

This shows that ft^n is integral over $R[\mathcal{I}]$.

Now assume that ft^n is integral over $R[\mathcal{I}]$. We will prove the needful, first, by assuming that $R[\mathcal{I}]$ is Noetherian and then, by reducing the case of an arbitrary filtration to the Noetherian case.

Assume $R[\mathcal{I}]$ is Noetherian. By [3] Proposition 3, §1.3, Chapter III, $\exists\ r > 0$ such that $I_{ri} = I_r^i\ \forall\ i \in \mathbb{Z}_{>0}$. By assumption, $(ft^n)^r$ is integral over $R[\mathcal{I}]$. So, we have a homogeneous relation (as below) for some $d > 0$ with $a_i \in I_{rni} = I_r^{ni} = I_{rn}^i$.

$$[(ft^n)^r]^d + a_1 t^{rn}[(ft^n)^r]^{d-1} + \cdots + a_i t^{rni}[(ft^n)^r]^{d-i} + \cdots + a_{d-1}t^{rn(d-1)}[(ft^n)^r] + a_d t^{rnd} = 0$$

$$\text{which implies } (f^r)^d + a_1(f^r)^{d-1} + \cdots + a_i(f^r)^{d-i} + \cdots + a_{d-1}(f^r) + a_d = 0$$

where $a_i \in I_{rn}^i\ \forall\ 1 \leq i \leq d$, proving $f^r \in \overline{I_{rn}}$.

Assume ft^n is integral over $R[\mathcal{I}]$ ($R[\mathcal{I}]$ need not be Noetherian). Say, ft^n satisfies an integral equation of degree d over $R[\mathcal{I}]$, given by

$$(ft^n)^d + A_1(ft^n)^{d-1} + \cdots + A_i(ft^n)^{d-i} + \cdots + A_{d-1}(ft^n) + A_d = 0$$

where $A_i \in R[\mathcal{I}]\ \forall\ 1 \leq i \leq d$. Thus $\exists\ a \in \mathbb{Z}_{>0}$ such that ft^n is integral over $R[\mathcal{I}_a]$, where $\mathcal{I}_a$ is the a^{th}-truncated filtration of $\mathcal{I}$. Observe that $R[\mathcal{I}_a]$ is Noetherian. Thus, by above, $f^r \in \overline{I_{a,rn}}$ for some $r \in \mathbb{Z}_{>0}$ but $\overline{I_{a,rn}} \subseteq \overline{I_{rn}} \implies f^r \in \overline{I_{rn}}$.

Clearly, $J_0 = R$. For $m \in \mathbb{N}$, if $f \in J_{m+1}$ then $f^r \in \overline{I_{r(m+1)}}$ for some $r > 0$. Since $\mathcal{I}$ is filtration, $I_{r(m+1)} \subseteq I_{rm}$, which implies $\overline{I_{r(m+1)}} \subseteq \overline{I_{rm}}$. Thus, $f^r \in \overline{I_{rm}}$, that is., $f \in J_m$. This shows that $J_{m+1} \subseteq J_m \ \forall\ m \in \mathbb{N}$.

Now suppose $f \in J_n$ and $g \in J_m$ for some $m, n \in \mathbb{N}$. Then $f^r \in \overline{I_{rm}}$ and $g^k \in \overline{I_{kn}}$ for some positive integers r and k. Observe that

$$(f^r)^k \in \left(\overline{I_{rm}}\right)^k \subseteq \overline{I_{rmk}} \text{ and } (g^k)^r \in \left(\overline{I_{kn}}\right)^r \subseteq \overline{I_{rkn}}$$

Thus, $(fg)^{rk} \in \overline{I_{rmk}} \cdot \overline{I_{rkn}} \subseteq \overline{I_{rk(m+n)}}$, that is., $fg \in J_{m+n}$. This proves that $\{J_m\}_{m \in \mathbb{N}}$ is a filtration. ∎

Definition 4.14. We will call the filtration $\mathcal{IC}(\mathcal{I})$ (in Lemma 4.13) the **integral closure of the filtration** $\mathcal{I}$.

Observe that if $\mathcal{I} = \{I^m\}_{m \in \mathbb{N}}$ for some ideal I of R, then $\mathcal{J} = \{\overline{I^m}\}_{m \in \mathbb{N}}$. In this particular case, we have already shown in Lemma 4.11 that $\overline{v}_{\mathcal{I}} = \overline{v}_{\mathcal{J}}$. In fact, this is true for any arbitrary filtration as well.

Theorem 4.15. *Let $\mathcal{I} = \{I_m\}_{m \in \mathbb{N}}$ be a filtration of R. Then $\overline{\nu}_{\mathcal{I}} = \overline{\nu}_{\mathcal{IC}(\mathcal{I})}$.*

Proof. Let $\mathcal{IC}(\mathcal{I}) = \{J_m\}_{m \in \mathbb{N}}$ (as in Lemma 4.13). Since $\mathcal{I} \subseteq \mathcal{IC}(\mathcal{I})$, $\overline{\nu}_{\mathcal{I}} \leq \overline{\nu}_{\mathcal{IC}(\mathcal{I})}$, by Remark 4.6.

Suppose $x \in J_m$ for some $m \in \mathbb{N}$, that is, $x^r \in \overline{I_{rm}}$ for some $r > 0$. The ideal I_{rm} is a reduction of $\overline{I_{rm}}$ by Corollary 1.2.5 [19]. By Remark 1.2.3 [19], $\exists$ $n \in \mathbb{Z}_{>0}$ such that $\forall\ k \geq n$, $x^{rk} \in (\overline{I_{rm}})^k \subset I^{k-n+1} \subseteq I_{rm(k-n+1)}$. This shows that

$\nu_{\mathcal{I}}(x^{rk}) \geq rm(k-n+1)$, which implies $\lim\limits_{k\to\infty} \frac{\nu_{\mathcal{I}}(x^{rk})}{rk} \geq \lim\limits_{k\to\infty} \frac{rm(k-n+1)}{rk} = m$. Thus, if $x \in J_m$, $\overline{\nu}_{\mathcal{I}}(x) \geq m$.

For $i \in \mathbb{N}$, since $x^i \in J_{\nu_{\mathcal{IC}(\mathcal{I})}(x^i)}, \overline{\nu}_{\mathcal{I}}(x^i) \geq \nu_{\mathcal{IC}(\mathcal{I})}(x^i)$. By Proposition 4.10, $\overline{\nu}_{\mathcal{I}}(x) \geq \frac{\nu_{\mathcal{IC}(\mathcal{I})}(x^i)}{i}$ $\forall\ i \in \mathbb{Z}_{>0}$. Thus, $\overline{\nu}_{\mathcal{I}}(x) \geq \lim\limits_{i\to\infty} \frac{\nu_{\mathcal{IC}(\mathcal{I})}(x^i)}{i} = \overline{\nu}_{\mathcal{IC}(\mathcal{I})}(x)$. This proves $\overline{\nu}_{\mathcal{I}}(x) = \overline{\nu}_{\mathcal{IC}(\mathcal{I})}(x)\ \forall\ x \in R$. ∎

Corollary 4.16. *Let $\mathcal{I} = \{I_m\}_{m\in\mathbb{N}}$ be a filtration in R and $\overline{\mathcal{I}} = \{\overline{I_m}\}_{m\in\mathbb{N}}$. Then $\overline{\nu}_{\mathcal{I}} = \overline{\nu}_{\overline{\mathcal{I}}}$.*

Proof. This follows from Theorem 4.15 and Remark 4.6 since $I_m \subseteq \overline{I_m} \subseteq J_m$ $\forall$ $m \in \mathbb{N}$, where $\mathcal{IC}(\mathcal{I}) = \{J_m\}_{m\in\mathbb{N}}$ is the integral closure of the filtration $\mathcal{I}$. ∎

Definition 4.17. We define filtrations $\mathcal{I}$ and $\mathcal{J}$ in a Noetherian ring R to be **projectively equivalent** if there exists $\alpha \in \mathbb{R}_{>0}$ such that $\overline{v}_{\mathcal{I}} = \alpha\, \overline{v}_{\mathcal{J}}$

This generalizes the classical definition of projective equivalence of ideals. Proposition 3.11 in the introduction gives the beautiful classical theorem characterizing projectively equivalent ideals. Proposition 3.11 is generalized to filtrations in Theorem 4.18.

Suppose that I and J are ideals in a Noetherian ring R and $\mathcal{I} = \{I^m\}_{m\in\mathbb{N}}$ and $\mathcal{J} = \{J^m\}_{m\in\mathbb{N}}$ are their associated adic-filtrations. We have that $\overline{\nu}_I = \overline{\nu}_{\mathcal{I}}$ and $\overline{\nu}_J = \overline{\nu}_{\mathcal{J}}$, so the ideals I and J are projectively equivalent if and only if the associated adic-filtrations $\mathcal{I}$ and $\mathcal{J}$ are projectively equivalent.

Theorem 4.9 shows that given any $\alpha \in \mathbb{R}_{>0}$, there are projectively equivalent filtrations $\mathcal{I}$ and $\mathcal{J}$ in a ring with $\overline{\nu}_{\mathcal{I}} = \alpha\overline{\nu}_{\mathcal{J}}$. Thus, the conclusion of the rationality of α (for projectively equivalent ideals commented after Proposition 3.11) does not extend to filtrations.

We provide the following necessary and sufficient condition for projective equivalence of filtrations.

Theorem 4.18. *Let $\mathcal{I}=\{I_m\}_{m\in\mathbb{N}}$ and $\mathcal{J}=\{J_m\}_{m\in\mathbb{N}}$ be filtrations in a Noetherian ring R. Then $\mathcal{I}$ and $\mathcal{J}$ are projectively equivalent if and only if $\exists\ \alpha,\ \beta\in\mathbb{R}_{>0}$ such that $\mathcal{IC}(\mathcal{I}^{(\alpha)})=\mathcal{IC}(\mathcal{J}^{(\beta)})$, or equivalently, $\overline{R[\mathcal{I}^{(\alpha)}]}=\overline{R[\mathcal{J}^{(\beta)}]}$.*

Proof. Suppose $\exists\ \alpha,\beta\in\mathbb{R}_{>0}$ such that $\mathcal{IC}(\mathcal{I}^{(\alpha)})=\mathcal{IC}(\mathcal{J}^{(\beta)})$. By Theorems 4.9 and 4.15, $\overline{\nu}_{\mathcal{I}}=\alpha\overline{\nu}_{\mathcal{I}^{(\alpha)}}=\alpha\overline{\nu}_{\mathcal{IC}(\mathcal{I}^{(\alpha)})}=\alpha\overline{\nu}_{\mathcal{IC}(\mathcal{J}^{(\beta)})}=\alpha\overline{\nu}_{\mathcal{J}^{(\beta)}}=\alpha\dfrac{\overline{\nu}_{\mathcal{J}}}{\beta}$. This shows that $\mathcal{I}$ and $\mathcal{J}$ are projectively equivalent. Assume $\mathcal{I}$ and $\mathcal{J}$ are projectively equivalent, that is, $\exists\ \gamma\in\mathbb{R}_{>0}$ such that $\overline{\nu}_{\mathcal{I}}=\gamma\overline{\nu}_{\mathcal{J}}$. Choose $\alpha,\ \beta\in\mathbb{R}_{>0}\setminus\mathbb{Q}$ such that $\dfrac{\alpha}{\beta}=\gamma$, or, $\alpha=\beta\gamma$. We show that $\mathcal{IC}(\mathcal{I}^{(\beta\gamma)})=\mathcal{IC}(\mathcal{J}^{(\beta)})$. Using Lemma 4.13, $\mathcal{IC}(\mathcal{I}^{(\beta\gamma)})=\{K_m\}_{m\in\mathbb{N}}$ where

$$K_m=\{f\in R\mid f^r\in\overline{I_{rm}^{(\beta\gamma)}}=\overline{I_{\lceil\beta\gamma rm\rceil}}\text{ for some }r>0\}$$

and $\mathcal{IC}(\mathcal{J}^{(\beta)})=\{L_m\}_{m\in\mathbb{N}}$ where

$$L_m=\{f\in R\mid f^t\in\overline{J_{tm}^{(\beta)}}=\overline{J_{\lceil\beta tm\rceil}}\text{ for some }t>0\}.$$

Recall the filtrations $\overline{\mathcal{I}}=\{\overline{I_m}\}_{m\in\mathbb{N}}$ and $\overline{\mathcal{J}}=\{\overline{J_m}\}_{m\in\mathbb{N}}$ defined in Corollary 4.16. Let $x\in K_m$, that is, $x^r\in\overline{I_{\lceil\beta\gamma rm\rceil}}$ for some $r>0$. Then $\forall\ i\in\mathbb{N}$, $x^{ri}\in(\overline{I_{\lceil\beta\gamma rm\rceil}})^i\subseteq\overline{I_{\lceil\beta\gamma rm\rceil i}}$, which implies $\nu_{\overline{\mathcal{I}}}(x^{ri})\geq\lceil\beta\gamma rm\rceil i$. This gives $\lim\limits_{i\to\infty}\dfrac{\nu_{\overline{\mathcal{I}}}(x^{ri})}{ri}\geq$

$\lim\limits_{i\to\infty} \dfrac{\lceil \beta\gamma rm\rceil i}{ri}$, that is, $\overline{\nu}_{\mathcal{I}}(x) \geq \dfrac{\lceil \beta\gamma rm\rceil}{r}$. By the assumption, $\overline{\nu}_{\mathcal{J}}(x) \geq \dfrac{\lceil \beta\gamma rm\rceil}{r\gamma}$, that is, $\lim\limits_{i\to\infty} \dfrac{\nu_{\mathcal{J}}(x^i)}{i} \geq \dfrac{\lceil \beta\gamma rm\rceil}{r\gamma}$.

Suppose $\lim\limits_{i\to\infty} \dfrac{\nu_{\mathcal{J}}(x^i)}{i} = \dfrac{\lceil \beta\gamma rm\rceil}{r\gamma}$. Then, given $\epsilon > 0$, $\exists\, n_0 = n_0(\epsilon) \in \mathbb{Z}_{>0}$ such that $-\epsilon < \dfrac{\nu_{\mathcal{J}}(x^i)}{i} - \dfrac{\lceil \beta\gamma rm\rceil}{r\gamma} < \epsilon \;\forall\, i \geq n_0$. For $\epsilon = \dfrac{\lceil \beta\gamma rm\rceil}{r\gamma} - \beta m > 0$, let $i_0 = n_0(\epsilon)$. We have that ϵ is positive since $\beta \notin \mathbb{Q}$. So, $-\epsilon = \beta m - \dfrac{\lceil \beta\gamma rm\rceil}{r\gamma} < \dfrac{\nu_{\mathcal{J}}(x^{i_0})}{i_0} - \dfrac{\lceil \beta\gamma rm\rceil}{r\gamma}$ implying $\nu_{\mathcal{J}}(x^{i_0}) > \beta m i_0$, or that, $\nu_{\mathcal{J}}(x^{i_0}) \geq \lceil \beta i_0 m\rceil$. This shows that $x^{i_0} \in \overline{J_{\lceil \beta i_0 m\rceil}}$, that is, $x \in L_m$.

If $\lim\limits_{i\to\infty} \dfrac{\nu_{\mathcal{J}}(x^i)}{i} > \dfrac{\lceil \beta\gamma rm\rceil}{r\gamma}$, $\exists\, j_0 \in \mathbb{Z}_{>0}$ such that $\dfrac{\nu_{\mathcal{J}}(x^{j_0})}{j_0} > \dfrac{\lceil \beta\gamma rm\rceil}{r\gamma}$. This shows that $\nu_{\mathcal{J}}(x^{j_0}) > \dfrac{j_0\lceil \beta\gamma rm\rceil}{r\gamma} \geq \dfrac{j_0\beta\gamma rm}{r\gamma}$ implying that $\nu_{\mathcal{J}}(x^{j_0}) \geq \lceil j_0\beta m\rceil$, so, $x^{j_0} \in \overline{J_{\lceil j_0\beta m\rceil}}$, or that, $x \in L_m$. Thus, we have shown that $K_m \subseteq L_m \;\forall\, m \in \mathbb{N}$. Similarly, we can show that $L_m \subseteq K_m \;\forall\, m \in \mathbb{N}$, proving that $\mathcal{IC}(\mathcal{I}^{(\beta\gamma)}) = \mathcal{IC}(\mathcal{J}^{(\beta)})$.

∎

Remark 4.19. In the proof above, the condition of α and β being real numbers cannot be weakened, as shown in the following example, where $\mathcal{I}$ and $\mathcal{J}$ are projectively equivalent filtrations with $\overline{\nu}_{\mathcal{I}} = \overline{\nu}_{\mathcal{J}}$ but for no α or $\beta \in \mathbb{Q}_{>0}$ do we have that $\overline{R[\mathcal{I}^{(\alpha)}]} = \overline{R[\mathcal{J}^{(\beta)}]}$. This follows from (4.3) below, since $\overline{R[\mathcal{I}^{(\alpha)}]} = \overline{R[\mathcal{J}^{(\beta)}]}$ implies $\alpha = \beta$ by Theorems 4.15 and 4.9.

Example 4.20. Let $R = k[[x]]$, a power series ring in the variable x over a field k. For $f \in R$, let $ord_{k[[x]]}(f) = \min\{r \mid f_r \neq 0\}$ where $f = \sum\limits_{m=0}^{\infty} f_m x^m$ with $f_m \in k$.

Fix $c \in \mathbb{Z}_{>0}$. Let $\mathcal{I} = \{I_m\}_{m\in\mathbb{N}}$ be given by $I_0 = R$ and let $I_m = (x^{m+c})$ for $m \in \mathbb{Z}_{>0}$ and $\mathcal{J} = \{J_m\}_{m\in\mathbb{N}}$ be given by $J_0 = R$ and $J_m = (x^m)$ for $m \in \mathbb{Z}_{>0}$. Both

$\mathcal{I}$ and $\mathcal{J}$ are filtrations in R and $\mathcal{I} \subseteq \mathcal{J}$.

Let $f \in R$ with $ord_{k[[x]]}(f) = c_0$. For $n \in \mathbb{Z}_{>0}, \nu_{\mathcal{J}}(f^n) = nc_0$ and $\nu_{\mathcal{I}}(f^n) = nc_0 - c$ (if $nc_0 > c$) and $= 0$ (if $nc_0 \leq c$).

$$\overline{\nu}_{\mathcal{I}}(f) = \lim_{n\to\infty} \frac{\nu_{\mathcal{I}}(f^n)}{n} = \lim_{n\to\infty} \frac{nc_0 - c}{n} = c_0 \text{ and } \overline{\nu}_{\mathcal{J}}(f) = \lim_{n\to\infty} \frac{\nu_{\mathcal{J}}(f^n)}{n} = \lim_{n\to\infty} \frac{nc_0}{n} = c_0$$

Thus $\mathcal{I}$ and $\mathcal{J}$ are projectively equivalent with $\overline{\nu}_{\mathcal{I}} = \overline{\nu}_{\mathcal{J}}$.

Observe that $\forall\ \alpha \in \mathbb{R}_{>0}$, $\overline{R[\mathcal{I}^{(\alpha)}]} \subseteq \overline{R[\mathcal{J}^{(\alpha)}]}$ in $R[t]$. We will show that

$$R[\mathcal{J}^{(\alpha)}] \text{ is integrally closed in } R[t]\ \forall\ \alpha \in \mathbb{R}_{>0}. \tag{4.1}$$

$$R[\mathcal{J}^{(\alpha)}] \subseteq \overline{R[\mathcal{I}^{(\alpha)}]} \text{ when } \alpha \in \mathbb{R}_{>0} \setminus \mathbb{Q}, \text{ proving that } \overline{R[\mathcal{I}^{(\alpha)}]} = \overline{R[\mathcal{J}^{(\alpha)}]}. \tag{4.2}$$

$$\overline{R[\mathcal{I}^{(\alpha)}]} \subsetneq R[\mathcal{J}^{(\alpha)}] \text{ when } \alpha \in \mathbb{Q}. \tag{4.3}$$

To prove (4.1), it is enough to show it for homogeneous elements in $R[t]$. Let $ft^n \in R[t]$ be integral over $R[\mathcal{J}^{(\alpha)}]$ with $ord_{k[[x]]}(f) = c_0$, and $n \in \mathbb{N}$. If $n = 0$ or $c_0 \geq \lceil \alpha n \rceil$, then $ft^n \in R[\mathcal{J}^{(\alpha)}]$. If $c_0 < \lceil \alpha n \rceil$, since ft^n is integral over $R[\mathcal{J}^{(\alpha)}]$, we have the following homogeneous equation for some $d \in \mathbb{Z}_{>0}$.

$$(ft^n)^d + a_1 t^n (ft^n)^{d-1} + \cdots + a_i t^{ni} (ft^n)^{d-i} + \cdots + a_{d-1} t^{n(d-1)} (ft^n) + a_d t^{nd} = 0$$

where $a_i \in J_{ni}^{(\alpha)} = (x^{\lceil \alpha ni \rceil})\ \forall\ 1 \leq i \leq d$.

In particular, the coefficient of $t^{nd} = 0$, that is,

$$f^d + a_1 f^{d-1} + \cdots + a_i f^{d-i} + \cdots + a_{d-1} f + a_d = 0.$$

Since $c_0 < \lceil \alpha n \rceil$, $c_0 < \alpha n$. However $ord_{k[[x]]}(f^d) = dc_0$ but

$$ord_{k[[x]]}(a_i f^{d-i}) \geq \lceil \alpha n i \rceil + (d-i)c_0 \geq \alpha n i + dc_0 - ic_0 > dc_0.$$

Therefore, the above equation is not possible. Thus, if ft^n is integral over $R[\mathcal{J}^{(\alpha)}]$, then $ft^n \in R[\mathcal{J}^{(\alpha)}]$, proving that $R[\mathcal{J}^{(\alpha)}]$ is integrally closed.

To prove (4.2), consider a homogeneous element in $R[\mathcal{J}^{(\alpha)}]$, say, ft^n where $ord_{k[[x]]}(f) = c_0 \geq \lceil \alpha n \rceil$, and $n \in \mathbb{N}$, which is integral over $R[\mathcal{I}^{(\alpha)}]$. If $c_0 \geq \lceil \alpha n \rceil + c$, then $ft^n \in R[\mathcal{I}^{(\alpha)}]$. If $c_0 < \lceil \alpha n \rceil + c$, since $c_0 \geq \lceil \alpha n \rceil > \alpha n$, we can let $d \in \mathbb{Z}_{>0}$ be such that $d \geq \dfrac{c}{c_0 - \alpha n}$. Then ft^n satisfies the following integral equation over $R[\mathcal{I}^{(\alpha)}]$:

$$(ft^n)^d + a_d t^{nd} = 0$$

where $a_d = -f^d$. By our choice of d, $dc_0 \geq \alpha n d + c$ which implies $dc_0 \geq \lceil \alpha n d \rceil + c$. So, $ord_{k[[x]]}(f^d) \geq \lceil \alpha n d \rceil + c$, which shows $a_d \in I_{\lceil \alpha n d \rceil} = I_{nd}^{(\alpha)}$.

For (4.3), say $\alpha = \dfrac{p}{q}$ for some $p, q \in \mathbb{Z}_{>0}$. Then $x^p t^q \in R[\mathcal{J}^{(\alpha)}]$ but $\notin \overline{R[\mathcal{I}^{(\alpha)}]}$. If $x^p t^q$ were integral over $R[\mathcal{I}^{(\alpha)}]$, then for some $d \in \mathbb{Z}_{>0}$ we would have that

$$(x^p)^d + a_1 (x^p)^{d-1} + \ldots + a_i (x^p)^{d-i} + \ldots + a_{d-1}(x^p) + a_d = 0$$

where $ord_{k[[x]]}(a_i) \geq \lceil \alpha q i \rceil + c = pi + c$. But this is not possible since $ord_{k[[x]]}(a_i (x^p)^{d-i}) \geq pi + c + p(d-i) > pd \; \forall \; 1 \leq i \leq d$.

Remark 4.21. The preceding example shows that given a filtration $\mathcal{I} = \{I_n\}_{n \in \mathbb{N}}$ in a Noetherian ring R, whose integral closure is $\mathcal{J} = \{J_n\}_{n \in \mathbb{N}}$ (say), $J_n \subsetneq \{x \in$

$R \mid \overline{v}_{\mathcal{I}}(x) \geq n\}$. This is contrast to the case when $\mathcal{I} = \{I^n\}_{n\in\mathbb{N}}$, as shown by Lemma 3.2, where $\overline{I^n} = \{x \in R \mid \overline{v}_I(x) \geq n\}$.

Theorem 4.22. *For a filtration $\mathcal{I} = \{I_m\}_{m\in\mathbb{N}}$ of ideals in R, define*

$$K(\mathcal{I})_m \coloneqq \{f \in R \mid \overline{\nu}_{\mathcal{I}}(f) \geq m\} \ \forall\, m \in \mathbb{N}.$$

Then $\mathcal{K}(\mathcal{I}) \coloneqq \{K(\mathcal{I})_m\}_{m\in\mathbb{N}}$ is a filtration of ideals in R and $\mathcal{I} \subseteq \mathcal{K}(\mathcal{I})$. Moreover, $\overline{\nu}_{\mathcal{I}} = \overline{\nu}_{\mathcal{K}(\mathcal{I})}$ and $\mathcal{K}(\mathcal{I})$ is the unique, largest filtration $\mathcal{J}$ such that $\overline{\nu}_{\mathcal{I}} = \overline{\nu}_{\mathcal{J}}$.

Proof. Denote $\mathcal{K}(\mathcal{I})$ by $\mathcal{K}$ and $K(\mathcal{I})_m$ by K_m $\forall\, m \in \mathbb{N}$. If $f \in I_n$ for some $n \in \mathbb{N}$, then $f^i \in (I_n)^i \subseteq I_{ni}$ implying $\nu_{\mathcal{I}}(f^i) \geq ni$ $\forall\, i \in \mathbb{N}$, which gives, $\overline{\nu}_{\mathcal{I}}(f) \geq n$. Thus, $I_n \subseteq K_n$ $\forall\, n \in \mathbb{N}$.

Suppose $f, g \in K_n$ for some $n \in \mathbb{N}$, that is, $\overline{\nu}_{\mathcal{I}}(f) \geq n$ and $\overline{\nu}_{\mathcal{I}}(g) \geq n$. For $\epsilon > 0$, $\exists\, m_0 \in \mathbb{Z}_{>0}$ such that $\forall\, m \geq m_0$, $\dfrac{\nu_{\mathcal{I}}(f^m)}{m} \geq n - \epsilon$ and $\dfrac{\nu_{\mathcal{I}}(g^m)}{m} \geq n - \epsilon$. Note that $\forall\, m, k \in \mathbb{Z}_{>0}$, $\nu_{\mathcal{I}}((f+g)^{mk}) \geq \min\limits_{i+j=mk}\{\nu_{\mathcal{I}}(f^i g^j)\}$, using Remark 4.3. Since $i + j = mk, mk \geq m\left\lfloor \dfrac{i}{m} \right\rfloor + m\left\lfloor \dfrac{j}{m} \right\rfloor \geq mk - 2m$. Using Remark 4.3 again,

$$\nu_{\mathcal{I}}(f^i g^j) \geq \nu_{\mathcal{I}}(f^i) + \nu_{\mathcal{I}}(g^j) \geq \nu_{\mathcal{I}}(f^{m\lfloor \frac{i}{m} \rfloor}) + \nu_{\mathcal{I}}(g^{m\lfloor \frac{j}{m} \rfloor}) \geq \left\lfloor \frac{i}{m} \right\rfloor \nu_{\mathcal{I}}(f^m) + \left\lfloor \frac{j}{m} \right\rfloor \nu_{\mathcal{I}}(g^m).$$

For $m \geq m_0$, $\nu_{\mathcal{I}}(f^i g^j) \geq \left\lfloor \dfrac{i}{m} \right\rfloor m(n-\epsilon) + \left\lfloor \dfrac{j}{m} \right\rfloor m(n-\epsilon) \geq (mk - 2m)(n - \epsilon)$.

Thus, for $k >> 0$,

$$\frac{\nu_{\mathcal{I}}((f+g)^{mk})}{mk} \geq \frac{mk(n-\epsilon) - 2m(n-\epsilon)}{mk} = n - \epsilon - \frac{2}{k}(n - \epsilon).$$

Taking limits as $k \to \infty$, we get $\overline{\nu}_{\mathcal{I}}(f+g) \geq n - \epsilon$. Since ϵ is arbitrary, $\overline{\nu}_{\mathcal{I}}(f+g) \geq n$. This proves that $f + g \in K_n$ $\forall\, f, g \in K_n$.

For $r \in R$ and $f \in K_n$, using Remark 4.3

$$\overline{\nu}_{\mathcal{I}}(rf) = \lim_{i\to\infty} \frac{\nu_{\mathcal{I}}((rf)^i)}{i} \geq \lim_{i\to\infty} \frac{\nu_{\mathcal{I}}(r^i)}{i} + \lim_{i\to\infty} \frac{\nu_{\mathcal{I}}(f^i)}{i} = \overline{\nu}_{\mathcal{I}}(r) + \overline{\nu}_{\mathcal{I}}(f) \geq n.$$

This shows that $rf \in K_n$, thus proving that K_n is an ideal in R.

Clearly $K_0 = R$. If $f \in K_{n+1}$ for some $n \in \mathbb{N}$, that is, $\overline{\nu}_{\mathcal{I}}(f) \geq n+1 > n$, then $f \in K_n$ proving that $K_{n+1} \subseteq K_n \ \forall \ n \in \mathbb{N}$.

Suppose $f \in K_m$ and $g \in K_n$ for some $m, n \in \mathbb{N}$. Then by Remark 4.3

$$\overline{\nu}_{\mathcal{I}}(fg) = \lim_{i\to\infty} \frac{\nu_{\mathcal{I}}((fg)^i)}{i} \geq \lim_{i\to\infty} \frac{\nu_{\mathcal{I}}(f^i)}{i} + \lim_{i\to\infty} \frac{\nu_{\mathcal{I}}(g^i)}{i} \geq n + m.$$

Thus, $K_n K_m \subseteq K_{n+m}$. This proves $\mathcal{K}$ is a filtration of ideals in R.

For $f \in R$ and $i \in \mathbb{N}$, it follows from the definition that $\overline{\nu}_{\mathcal{I}}(f^i) \geq \nu_{\mathcal{K}}(f^i)$ since $f^i \in K_{\nu_{\mathcal{K}}(f^i)}$. Using Proposition 4.10, $\overline{\nu}_{\mathcal{I}}(f) \geq \dfrac{\nu_{\mathcal{K}}(f^i)}{i} \ \forall \ i \in \mathbb{Z}_{>0}$. Thus, $\overline{\nu}_{\mathcal{I}}(f) \geq \overline{\nu}_{\mathcal{K}}(f) \ \forall \ f \in R$. Since $\mathcal{I} \subseteq \mathcal{K}$, $\overline{\nu}_{\mathcal{I}} \leq \overline{\nu}_{\mathcal{K}}$ by Remark 4.6. This proves $\overline{\nu}_{\mathcal{I}} = \overline{\nu}_{\mathcal{K}}$. For any filtration $\mathcal{L}$ of R, let $\mathcal{K}(\mathcal{L}) = \{K(\mathcal{L})_m\}_{m\in\mathbb{N}}$ where $K(\mathcal{L})_m = \{f \in R \mid \overline{\nu}_{\mathcal{L}}(f) \geq m\}$. If $\mathcal{J}$ is a filtration such that $\overline{\nu}_{\mathcal{I}} = \overline{\nu}_{\mathcal{J}}$, then $\mathcal{J} \subseteq \mathcal{K}(\mathcal{J}) = \mathcal{K}(\mathcal{I})$. This shows that every filtration $\mathcal{J}$ such that $\overline{\nu}_{\mathcal{I}} = \overline{\nu}_{\mathcal{J}}$ is contained in $\mathcal{K}(\mathcal{I})$, and we have shown earlier that $\overline{\nu}_{\mathcal{K}(\mathcal{I})} = \overline{\nu}_{\mathcal{I}}$, proving that $\mathcal{K}(\mathcal{I})$ is the unique, largest filtration $\mathcal{J}$ such that $\overline{\nu}_{\mathcal{I}} = \overline{\nu}_{\mathcal{J}}$. ∎

Example 4.23. In general, the integral closure $\mathcal{IC}(\mathcal{I})$ (so that $R[\mathcal{IC}(\mathcal{I})] = \overline{R[\mathcal{I}]}$) of a filtration $\mathcal{I}$ is strictly smaller than $\mathcal{K}(\mathcal{I})$ (or equivalently $\overline{R[\mathcal{I}]}$ is strictly smaller than $R[\mathcal{K}(\mathcal{I})]$). For adic-filtrations, however, $\mathcal{IC}(\mathcal{I}) = \mathcal{K}(\mathcal{I})$, by Lemma 3.2, where $\mathcal{I} = \{I^m\}_{m\in\mathbb{N}}$ for an ideal I of R.

Proof. We consider filtrations $\mathcal{I} = \{I_m\}_{m\in\mathbb{N}}$ where $I_0 = R$ and $I_m = (x^{m+c})$ for some fixed $c \in \mathbb{Z}_{>0}$ and $\mathcal{J} = \{J_m\}_{m\in\mathbb{N}}$ where $J_0 = R$ and $J_m = (x^m)$ for $m \in \mathbb{Z}_{>0}$ in $R = k[[x]]$, a power series ring in the variable x over a field k. We showed in Example 4.20 that $\overline{\nu}_{\mathcal{I}} = \overline{\nu}_{\mathcal{J}}$. Thus $\mathcal{K}(\mathcal{I}) = \mathcal{K}(\mathcal{J})$. By a direct calculation, $\mathcal{K}(\mathcal{J}) = \mathcal{J}$. Thus $\mathcal{K}(\mathcal{I}) = \mathcal{J}$. However we have shown in Example 4.20 that the integral closure $\overline{R[\mathcal{I}]}$ of the filtration $\mathcal{I}$ is a proper subset of $\overline{R[\mathcal{J}]} = R[\mathcal{J}]$. ∎

Theorem 4.24. *Suppose $\mathcal{I}$ and $\mathcal{J}$ are filtrations of a Noetherian ring R. Then $\mathcal{I}$ is projectively equivalent to $\mathcal{J}$ with $\overline{\nu}_{\mathcal{I}} = \alpha\overline{\nu}_{\mathcal{J}}$ if and only if $\mathcal{K}(\mathcal{I}^{(\alpha)}) = \mathcal{K}(\mathcal{J})$.*

Proof. $\overline{\nu}_{\mathcal{I}} = \alpha\overline{\nu}_{\mathcal{J}}$ if and only if $\overline{\nu}_{\mathcal{I}^{(\alpha)}} = \overline{\nu}_{\mathcal{J}}$ (by Theorem 4.9), which holds if and only if $\mathcal{K}(\mathcal{I}^{(\alpha)}) = \mathcal{K}(\mathcal{J})$ by Theorem 4.22. ∎

Lemma 4.25. *For a filtration $\mathcal{I}$ and the corresponding filtration $\mathcal{K}(\mathcal{I})$ (as defined in Theorem 4.22) in a Noetherian ring R, the Rees algebra $R[\mathcal{K}(\mathcal{I})]$ is integrally closed in $R[t]$.*

Proof. It suffices to prove the result for homogeneous elements in $R[t]$. Let ft^n be a homogeneous element in $R[t]$ that is integral over $R[\mathcal{K}(\mathcal{I})]$. Suppose ft^n satisfies the following homogeneous equation of degree $d > 0$:

$$(ft^n)^d + a_1t^n(ft^n)^{d-1} + \cdots + a_it^{ni}(ft^n)^{d-i} + \cdots + a_{d-1}t^{n(d-1)}(ft^n) + a_dt^{nd} = 0$$

where $a_i \in K(\mathcal{I})_{ni}$, that is, $\overline{\nu}_{\mathcal{I}}(a_i) \geq ni\ \forall\ 1 \leq i \leq d$. That gives

$$f^d + a_1f^{d-1} + \cdots + a_if^{d-i} + \cdots + a_{d-1}f + b_d = 0.$$

If $\overline{\nu}_{\mathcal{I}}(f) < n$, then the above equation is not possible since $\overline{\nu}_{\mathcal{I}}(f^d) = d\,\overline{\nu}_{\mathcal{I}}(f)$ but $\overline{\nu}_{\mathcal{I}}(a_i f^{d-i}) > d\overline{\nu}_{\mathcal{I}}(f)$ $\forall\ 1 \leq i \leq d$ since $\overline{\nu}_{\mathcal{I}}(a_i f^{d-i}) \geq \overline{\nu}_{\mathcal{I}}(a_i) + (d-i)\,\overline{\nu}_{\mathcal{I}}(f) > ni + d\overline{\nu}_{\mathcal{I}}(f) - ni = \overline{\nu}_{\mathcal{I}}(f)$. If $\overline{\nu}_{\mathcal{I}}(f) \geq n$, then $ft^n \in R[\mathcal{K}(\mathcal{I})]$, thus, proving that $R[\mathcal{K}(\mathcal{I})]$ is integrally closed in $R[t]$. ∎

Chapter 5

Discrete valued filtrations

Let R be a Noetherian ring. Let P be a minimal prime ideal of R and let v be a valuation of the quotient field $\kappa(P)$ of R/P which is nonnegative on R/P. Suppose Γ_v is the value group of v and $\mathcal{O}_v$ is the valuation ring of v with maximal ideal $\mathfrak{m}_v$. Note that $R/P \subseteq \mathcal{O}_v$. Let $\pi : R \to R/P$ be the natural surjection. We define a map $\tilde{v} : R \to \Gamma_v \cup \{\infty\}$ by

$$\tilde{v}(r) = \begin{cases} v(\pi(r)) & \text{if } r \notin P \\ \infty & \text{if } r \in P \end{cases}$$

We extend the order on Γ_v to $\Gamma_v \cup \{\infty\}$ by requiring that ∞ has order larger than all elements of Γ_v and $\infty + \infty = g + \infty = \infty \ \ \forall \ \ g \in \Gamma_v$.

$\tilde{v}$ gives a well-defined map that satisfies the following properties:

$$\tilde{v}(r \cdot s) = \tilde{v}(r) + \tilde{v}(s) \qquad \tilde{v}(r + s) \geq \min\{\tilde{v}(r), \tilde{v}(s)\} \qquad \tilde{v}^{-1}(\infty) = P$$

We will call $\tilde{v}$ a **valuation** on R. By abuse of notation, we will denote $\tilde{v}$ by v.

If v is discrete valuation of rank 1, we say that v is a **discrete valuation** of R. Through this, we can naturally identify Γ_v with $\mathbb{Z}$, by identifying the element of Γ_v with least positive value with $1 \in \mathbb{Z}$.

We define two valuations v and ω of R to be **equivalent** if $v^{-1}(\infty) = \omega^{-1}(\infty)$ ($= P$, say) and the valuations v and ω on $\kappa(P)$ are equivalent. In particular, since we have identified the value groups with $\mathbb{Z}$, if v and ω are rank 1 discrete valuations, using Lemma 3.12, v and ω are equivalent if and only if they are equal, that is, $v = \omega$.

Suppose that v is a discrete valuation of R. For $m \in \mathbb{N}$, define **valuation ideals**

$$I(v)_m = \{f \in R \mid v(f) \geq m\} = \pi^{-1}\left(\mathfrak{m}_v^m \cap R/P\right).$$

An **integral discrete valued filtration** of R is a filtration $\mathcal{I} = \{I_m\}_{m\in\mathbb{N}}$ such that there exist discrete valuations $v_1, \ldots, v_s$ of R and $a_1, \ldots, a_s \in \mathbb{Z}_{>0}$ such that for all $m \in \mathbb{N}$,

$$I_m = I(v_1)_{ma_1} \cap \cdots \cap I(v_s)_{ma_s}.$$

$\mathcal{I}$ is called an **$\mathbb{R}$-discrete valued filtration** if $a_1, \ldots, a_s \in \mathbb{R}_{>0}$ and $\mathcal{I}$ is called a **$\mathbb{Q}$-discrete valued filtration** if $a_1, \ldots, a_s \in \mathbb{Q}_{>0}$. If $a_i \in \mathbb{R}_{>0}$, then

$$I(v_i)_{ma_i} := \{f \in R \mid v_i(f) \geq ma_i\} = I(v_i)_{\lceil ma_i \rceil}.$$

We also call an $\mathbb{R}$-discrete valued filtration a **discrete valued filtration**. If the discrete valuations v_i are divisorial valuations of $\kappa(P_i)$, where P_i are minimal primes of R, then $\mathcal{I}$ is called a **divisorial filtration** of R.

Definition 5.1. Let $\mathcal{I} = \{I_m\}_{m\in\mathbb{N}}$ be a discrete valued filtration, which is represented as

$$I_m = I(v_1)_{ma_1} \cap \cdots \cap I(v_s)_{ma_s} \ \forall\ m \in \mathbb{N}. \tag{5.1}$$

If for each $i \in \{1, \dots, s\}$, the representation (5.1) of I_m is not valid for some m when the term $I(\nu_i)_{a_i m}$ is removed from I_m then the representation of (5.1) is said to be **irredundant**.

Lemma 5.2. *Let $\mathcal{I} = \{I_m\}_{m \in \mathbb{N}}$ where $I_m = I(v_1)_{ma_1} \cap \cdots \cap I(v_s)_{ma_s}$ be a discrete valued filtration of a Noetherian ring R. For $f \in R \setminus \{0\}$,*

$$\nu_{\mathcal{I}}(f) = \min\left\{\left\lfloor \frac{v_1(f)}{a_1} \right\rfloor, \cdots, \left\lfloor \frac{v_s(f)}{a_s} \right\rfloor\right\} \quad \textit{and} \quad \overline{\nu}_{\mathcal{I}}(f) = \min\left\{\frac{v_1(f)}{a_1}, \cdots, \frac{v_s(f)}{a_s}\right\}.$$

Proof. Let $\phi_{\mathcal{I}}(f) := \min\limits_{1 \le i \le s}\left\{\left\lfloor \frac{v_i(f)}{a_i} \right\rfloor\right\}$ and $\overline{\phi}_{\mathcal{I}}(f) := \min\limits_{1 \le i \le s}\left\{\frac{v_i(f)}{a_i}\right\}$. Let $f \in R \setminus \{0\}$.

Since $f \in I_{\nu_{\mathcal{I}}(f)} = \bigcap\limits_{i=1}^{s} I(v_i)_{a_i \nu_{\mathcal{I}}(f)}$, $v_i(f) \ge a_i \nu_{\mathcal{I}}(f)$, which implies, $\nu_{\mathcal{I}}(f) \le \left\lfloor \frac{v_i(f)}{a_i} \right\rfloor$ $\forall\ 1 \le i \le s$. This shows that $\nu_{\mathcal{I}}(f) \le \phi_{\mathcal{I}}(f)$. For each $i \in \{1, \dots, s\}$, $\phi_{\mathcal{I}}(f) \le \left\lfloor \frac{v_i(f)}{a_i} \right\rfloor$, which implies $\phi_{\mathcal{I}}(f) \le \frac{v_i(f)}{a_i}$, that is, $v_i(f) \ge a_i \phi_{\mathcal{I}}(f)$. Thus, $f \in \bigcap\limits_{i=1}^{s} I(v_i)_{a_i \phi_{\mathcal{I}}(f)} = I_{\phi_{\mathcal{I}}(f)}$. This implies $\nu_{\mathcal{I}}(f) \ge \phi_{\mathcal{I}}(f)$, proving $\nu_{\mathcal{I}}(f) = \phi_{\mathcal{I}}(f)$. Now

$$\overline{\nu}_{\mathcal{I}}(f) = \lim_{n \to \infty} \frac{\nu_{\mathcal{I}}(f^n)}{n} = \lim_{n \to \infty} \frac{\min\limits_{1 \le i \le s}\left\{\left\lfloor \frac{v_i(f^n)}{a_i} \right\rfloor\right\}}{n} = \lim_{n \to \infty} \frac{\min\limits_{1 \le i \le s}\left\{\left\lfloor \frac{n v_i(f)}{a_i} \right\rfloor\right\}}{n}$$

Since $x - 1 < \lfloor x \rfloor \le x$ for any $x \in \mathbb{R}$, $\forall\ n \in \mathbb{Z}_{>0}$ we have that

$$\frac{\min\limits_{1 \le i \le s}\left\{\frac{n v_i(f)}{a_i} - 1\right\}}{n} \le \frac{\min\limits_{1 \le i \le s}\left\{\left\lfloor \frac{n v_i(f)}{a_i} \right\rfloor\right\}}{n} \le \frac{\min\limits_{1 \le i \le s}\left\{\frac{n v_i(f)}{a_i}\right\}}{n}$$

Taking limits as $n \to \infty$, we get, $\overline{\nu}_{\mathcal{I}}(f) = \overline{\phi}_{\mathcal{I}}(f)$. ∎

Corollary 5.3. *Let $\mathcal{I}$ be a discrete valued filtration of a Noetherian ring R. Then $\mathcal{K}(\mathcal{I}) = \mathcal{I}$.*

Proof. Represent $\mathcal{I} = \{I_m\}_{m\in\mathbb{N}}$ by $I_m = \bigcap_{i=1}^{s} I(v_i)_{ma_i}$. By Lemma 5.2, for any nonzero $f \in R$, $\overline{\nu}_{\mathcal{I}}(f) \geq m$ if and only if $\dfrac{v_i(f)}{a_i} \geq m$, or, $v_i(f) \geq a_i m \ \forall \ 1 \leq i \leq s$. This is equivalent to $f \in I(v_i)_{a_i m} \ \forall \ 1 \leq i \leq s$, or that, $f \in I_m$. ∎

Lemma 5.4. *If $\tilde{v}$ and $\tilde{v}'$ are discrete valuations of R and $a, b \in \mathbb{R}_{>0}$ are such that $\dfrac{\tilde{v}}{a} = \dfrac{\tilde{v}'}{b}$ (as functions of R), then $\tilde{v} = \tilde{v}'$ and $a = b$.*

Proof. Since $b\tilde{v} = a\tilde{v}'$, $\tilde{v}^{-1}(\infty) = \tilde{v}'^{-1}(\infty)$ is a common minimal prime P of R. Thus, $\tilde{v}$ and $\tilde{v}'$ are induced by discrete valuations v and v' on $\kappa(P)$. Let $\pi : R \to R/P$ be the natural surjection. Suppose $\alpha \in \kappa(P)$ is nonzero, that is, $\alpha = \dfrac{\pi(f)}{\pi(g)}$ for some $f, g \in R \setminus P$. Then,

$$v(\alpha) = v(\pi(f)) - v(\pi(g)) = \tilde{v}(f) - \tilde{v}(g) = \frac{a}{b}(\tilde{v}'(f) - \tilde{v}'(g))$$
$$= \frac{a}{b}(v'(\pi(f)) - v'(\pi(g))) = \frac{a}{b}v'(\alpha)$$

This shows $bv = av'$ as function of $\kappa(P)^{\times}$. Since the value groups of v and v' are $\mathbb{Z}$, $\exists\ x, y \in \kappa(P)^{\times}$ such that $v(x) = 1$ and $v'(y) = 1$. Since $bv = av'$, $b = av'(x)$ and $bv(y) = a$. This implies $a|b$ and $b|a$. Thus, $a = b$ and hence, $\tilde{v} = \tilde{v}'$.

∎

Theorem 5.5. *Let $v_1, \ldots, v_s$ be discrete valuations of a Noetherian ring R. Let $a_1, \ldots, a_s \in \mathbb{R}_{>0}$, and define $\omega : R \setminus \{0\} \to \mathbb{R}_{\geq 0}$ by*

$$\omega(f) = \min\left\{\frac{v_1(f)}{a_1}, \cdots, \frac{v_s(f)}{a_s}\right\} \tag{5.2}$$

for $f \in R$. If no $\dfrac{v_i}{a_i}$ can be omitted from this expression, then the v_i and a_i are uniquely determined by the function ω, up to reindexing of the $\dfrac{v_i}{a_i}$.

Proof. We will say that the set $\{v_1, \ldots, v_s\}$ is irredundant if no $\dfrac{v_i}{a_i}$ can be removed from 5.2.

If $s = 1$, the assertion follows from Lemma 5.4.

Let $s > 1$. We define $S \subseteq R$ to be ω-consistent if for any $m \in \mathbb{N}$ and $f_1, \ldots, f_m \in S$,

$$\omega(f_1 \cdots f_m) = \sum_{i=1}^{m} \omega(f_i)$$

For $f \in R$, let $S_f = \{f^m \mid m \in \mathbb{N}\}$. Then S_f is ω-consistent as we now show.

Observe that for any $f \in R$ and $m \in \mathbb{N}$,

$$\omega(f^m) = \min_{1 \le i \le s} \left\{ \frac{v_i(f^m)}{a_i} \right\} = \min_{1 \le i \le s} \left\{ \frac{m v_i(f)}{a_i} \right\} = m \cdot \min_{1 \le i \le s} \left\{ \frac{v_i(f)}{a_i} \right\} = m \cdot \omega(f)$$

For $f^{t_1}, \ldots, f^{t_m} \in S_f, \omega(f^{t_1} f^{t_2} \cdots f^{t_m}) = \omega(f^{t_1 + \cdots + t_m}) = (t_1 + \cdots + t_m) \cdot \omega(f) = \sum_{i=1}^{m} t_i \cdot \omega(f) = \sum_{i=1}^{m} \omega(f^{t_i})$, proving that S_f is ω-consistent.

Let $\mathfrak{F}$ be the set of all ω-consistent subsets of R. Clearly, $\mathfrak{F} \neq \emptyset$ since it contains the sets S_f for any $f \in R$. Partially order $\mathfrak{F}$ by inclusion. Let $\{I_\lambda\}_{\lambda \in \Lambda}$ be a chain of ω-consistent subsets of R, then $I = \bigcup_{\lambda \in \Lambda} I_\lambda$ is an upper bound for this chain. For $m \in \mathbb{N}$, if we take $f_1, f_2, \ldots, f_m \in I$, $\exists\ \alpha \in \Lambda$ such that $f_1, \ldots, f_m \in I_\alpha$ because $\{I_\lambda\}_{\lambda \in \Lambda}$ is a chain. Since I_α is ω-consistent, $\omega(f_1 \cdots f_m) = \sum_{i=1}^{m} \omega(f_i)$. This shows that I is ω-consistent. Since every chain in $\mathfrak{F}$ has an upper bound in $\mathfrak{F}$, by Zorn's Lemma, $\mathfrak{F}$ has at least one maximal element.

We will provide an explicit description of all the maximal ω-consistent subsets of R.

For each $1 \leq i \leq s$, define the sets $\mathcal{S}_i := \left\{ f \in R \;\middle|\; \omega(f) = \frac{v_i(f)}{a_i} \right\}$.

Since $v_i(1) = 0 \;\forall\; 1 \leq i \leq s$, $\omega(1) = 0$. Thus, each $\mathcal{S}_i \neq \emptyset$ since $1 \in \mathcal{S}_i$. Observe that $\mathcal{S}_i$ is a multiplicatively closed subset of R. Take $f, g \in \mathcal{S}_i$.

$\omega(f) = \frac{v_i(f)}{a_i}$, that is, $\frac{v_i(f)}{a_i} \leq \frac{v_j(f)}{a_j}$ and $\omega(g) = \frac{v_i(g)}{a_i}$, that is, $\frac{v_i(g)}{a_i} \leq \frac{v_j(g)}{a_j} \;\forall\; 1 \leq j \leq s$

This gives $\frac{v_i(f)}{a_i} + \frac{v_i(g)}{a_i} \leq \frac{v_j(f)}{a_j} + \frac{v_j(g)}{a_j} \;\forall\; 1 \leq j \leq s$. Since v_i are valuations, we have $\frac{v_i(fg)}{a_i} \leq \frac{v_j(fg)}{a_j}$. In other words, $\omega(fg) = \frac{v_i(fg)}{a_i}$. This shows $fg \in \mathcal{S}_i$.

For $m \in \mathbb{N}$, take $f_1, \ldots, f_m \in \mathcal{S}_i$, then, $f_1 \cdots f_m \in \mathcal{S}_i$, so that $\omega(f_1 \cdots f_m) = \frac{v_i(f_1 \cdots f_m)}{a_i} = \sum_{j=1}^{m} \frac{v_i(f_j)}{a_i} = \sum_{j=1}^{m} \omega(f_j)$, proving that $\mathcal{S}_i$ is ω-consistent.

By the irredundancy condition on the $\frac{v_i}{a_i}$, we have the following remark.

Remark 5.5.1. For $i \neq j$, $\mathcal{S}_i \nsubseteq \bigcup_{j \neq i} \mathcal{S}_j$.

Since each $\mathcal{S}_i$ is a ω-consistent subset, it is contained in some maximal ω-consistent subset of R. We show that any maximal ω-consistent subset $\mathcal{S}$ of R equals to one of the $\mathcal{S}_i$, then by Remark 5.5.1 it follows that $\{\mathcal{S}_i \mid 1 \leq i \leq s\}$ are the distinct maximal ω-consistent subsets of R.

Remark 5.5.2. If $\omega(f) = \infty$ for some $f \in R$, then $v_i(f) = \infty \;\forall\; 1 \leq i \leq s$. Thus $f \in \mathcal{S}_i \;\forall\; 1 \leq i \leq s$.

Suppose $\mathcal{S}$ is a maximal ω-consistent subset of R, and $\mathcal{S} \neq \mathcal{S}_i$ for any $1 \leq i \leq s$. Then $\exists\; g_i \in \mathcal{S} \setminus \mathcal{S}_i \;\forall\; 1 \leq i \leq s$. Since $g_i \notin \mathcal{S}_i$, $\omega(g_i) < \infty$ (by Remark 5.5.2) and $\omega(g_i) < \frac{v_i(g_i)}{a_i} \;\forall\; 1 \leq i \leq s$. Let $g = g_1 \ldots g_s$. Since $\mathcal{S}$ is ω-consistent, $\omega(g) = \sum_{i=1}^{m} \omega(g_i)$, but $\sum_{i=1}^{m} \omega(g_i) < \sum_{i=1}^{m} \frac{v_j(g_i)}{a_j} \;\forall\; 1 \leq j \leq s$ since $\omega(g_i) \leq \frac{v_k(g_i)}{a_k} \;\forall\; k \neq i$ and

$\omega(g_i) < \dfrac{v_i(g_i)}{a_i}$ (when $k = i$). So, we get that $\omega(g) = \sum_{i=1}^{m} \omega(g_i) < \sum_{i=1}^{m} \dfrac{v_j(g_i)}{a_j} = \dfrac{v_j(g)}{a_j}$ $\forall\ 1 \leq j \leq s$. But that implies $\omega(g) < \min_{1\leq j\leq s} \left\{ \dfrac{v_j(g)}{a_j} \right\}$, contradicting the definition of ω. Thus, every maximal ω-consistent set $\mathcal{S}$ equals to one of the sets $\{\mathcal{S}_i \mid 1 \leq i \leq s\}$. Hence, $\{\mathcal{S}_i \mid 1 \leq i \leq s\}$ are all the maximal ω-consistent subsets of R. This completes the proof that $\{\mathcal{S}_i \mid 1 \leq i \leq s\}$ are all the maximal ω-consistent subsets of R.

In order to prove the Theorem, we will recover the valuations v_i and the numbers $a_i \in \mathbb{R}_{>0}$ from the function ω. Since each $\dfrac{v_i}{a_i}$ gives a distinct maximal set $\mathcal{S}_i$ and $\{v_i \mid 1 \leq i \leq s\}$ is irredundant, the number of v_i (and the number of a_i) equals the number of distinct maximal ω-consistent subsets of R.

Let $c \in R$ be such that $v_i(c) < \infty$. By Remark 5.5.1, $\exists\ x_i \in \mathcal{S}_i \setminus \bigcup_{j\neq i} \mathcal{S}_j$. By the choice of x_i, $\forall\ j \neq i$, $\dfrac{v_j(x_i)}{a_j} > \dfrac{v_i(x_i)}{a_i}$. For a sufficiently large positive integer d, and $\forall\ j \neq i$ we have that

$$\left(\frac{v_j(x_i)}{a_j} - \frac{v_i(x_i)}{a_i} \right) d > \frac{v_i(c)}{a_i}$$

This gives $\dfrac{v_j(x_i^d)}{a_j} > \dfrac{v_i(cx_i^d)}{a_i}$, which in turn implies $\dfrac{v_i(cx_i^d)}{a_i} < \dfrac{v_j(x_i^d)}{a_j} + \dfrac{v_j(c)}{a_j} = \dfrac{v_j(cx_i^d)}{a_j}$. The last inequality follows since the valuations are non-negative on R. This shows that $\omega(cx_i^d) = \dfrac{v_i(cx_i^d)}{a_i}$. In other words, $cx_i^d \in \mathcal{S}_i$.

What we have just shown is the following remark.

Remark 5.5.3. For $c \in R$ such that $v_i(c) < \infty$ and $x_i \in \mathcal{S}_i \setminus \bigcup_{j\neq i} \mathcal{S}_j$, $\exists$ a positive integer d such that $cx_i^d \in \mathcal{S}_i$. Moreover, if $cx_i^d \in \mathcal{S}_i$, then $cx_i^n \in \mathcal{S}_i\ \forall\ n \geq d$.

Let $S = \{a \in R \mid a \notin$ any minimal prime ideal of $R\}$. S is a multiplicatively closed set and for $a \in S$, $v_j(a) < \infty \ \forall\ 1 \leq j \leq s$ since each v_j is infinite only on some minimal prime ideal of R. In particular, $v_i(a) < \infty \ \forall\ a \in S$.

Remark 5.5.4. The construction in Remark 5.5.3 applies to every element in S.

Consider the quotient ring $K = S^{-1}R$. For each $1 \leq i \leq s$, we define a function $u_i : K \to \mathbb{Q} \cup \{\infty\}$ as follows:

Let $\alpha = f/g \in K$. Since $g \in S$, by Remark 5.5.3, $\exists$ a positive integer e such that $gx_i^e \in \mathcal{S}_i$. Now, if for some large positive integer d, $fx_i^d \in \mathcal{S}_i$, then by Remark 5.5.3, we can find a sufficiently large integer n such that $fx_i^n, gx_i^n \in \mathcal{S}_i$ and in that case we define $u_i\left(\dfrac{f}{g}\right) := \omega(fx_i^n) - \omega(gx_i^n)$

Otherwise, if $fx_i^d \notin \mathcal{S}_i$ for all positive integers d, then we define $u_i\left(\dfrac{f}{g}\right) := \infty$.

Remark 5.5.5. If $fx_i^d \in \mathcal{S}_i$ for some $d > 0$, $u_i\left(\dfrac{f}{g}\right) = \dfrac{v_i(f) - v_i(g)}{a_i}$ since $\omega(fx_i^n) = \dfrac{v_i(fx_i^n)}{a_i} = \dfrac{v_i(f) + v_i(x_i^n)}{a_i}$ and $\omega(gx_i^n) = \dfrac{v_i(gx_i^n)}{a_i} = \dfrac{v_i(g) + v_i(x_i^n)}{a_i}$. Since $g \in S$, $v_i(g) < \infty$, so, $u_i\left(\dfrac{f}{g}\right) = \infty$ if and only if $v_i(f) = \infty$.

Remark 5.5.6. If $fx_i^d \notin \mathcal{S}_i$ for all positive integers d, then $\omega(f) < \infty = v_i(a)$.

The remark follows because if $\omega(f) = \infty$, by Remark 5.5.2, $f \in \mathcal{S}_i$ which implies $fx_i^d \in \mathcal{S}_i$ for every positive integer d since $\mathcal{S}_i$ is multiplicatively closed, but that contradicts our assumption. Hence, $\omega(f) < \infty$. If $v_i(f) < \infty$, by Remark 5.5.3, we can find a large positive integer d such that $fx_i^d \in \mathcal{S}_i$, contradicting our assumption again. Thus, $v_i(f) = \infty$.

We need to show that u_i is well-defined, that is, it does not depend on the choice of n, x_i, f, or g. It follows from Remark 5.5.5 that u_i does not depend on n.

By the definition of u_i and Remarks 5.5.5 and 5.5.6, $u_i\left(\frac{f}{g}\right) = \infty$ if and only if $v_i(f) = \infty$. So $u_i\left(\frac{f}{g}\right)$ is independent of the choice of $x_i \in \mathcal{S}_i \setminus \bigcup_{j\neq i} \mathcal{S}_j$ if $v_i(f) = \infty$.

Suppose $v_i(f) < \infty$. If $x_i \in \mathcal{S}_i \setminus \bigcup_{j\neq i} \mathcal{S}_j$, then $fx_i^d \in \mathcal{S}_i$ for a large positive integer d by Remark 5.5.3, and thus $u_i\left(\frac{f}{g}\right) = \frac{v_i(f) - v_i(g)}{a_i}$, which is independent of the choice of $x_i \in \mathcal{S}_i \setminus \bigcup_{j\neq i} \mathcal{S}_j$. Thus, u_i is independent of the choice of x_i.

To prove that u_i doesn't depend on our choice of f and g, first we will show the following:

$$u_i\left(\frac{f}{g}\right) = u_i\left(\frac{cf}{cg}\right) \quad \forall \;\; c \in S$$

Let $c \in S$, then, by Remark 5.5.4, $\exists$ a positive integer e such that $cx_i^e \in \mathcal{S}_i$.

Suppose $\exists \; d' > 0$ such that $fx_i^{d'} \in \mathcal{S}_i$. By Remark 5.5.3 and 5.5.4, $fx_i^d, gx_i^d, \in \mathcal{S}_i$ for some $d > 0$, and thus, by Remark 5.5.5, $u_i\left(\frac{f}{g}\right) = \frac{v_i(f) - v_i(g)}{a_i}$. Since $fx_i^d, gx_i^d, cx_i^e \in \mathcal{S}_i$ (which is ω-consistent), $fx_i^d \cdot cx_i^e = fcx_i^{d+e}, gx_i^d \cdot cx_i^e = gcx_i^{d+e} \in \mathcal{S}_i$. Thus, we have

$$u_i\left(\frac{cf}{cg}\right) = \omega(fcx_i^{d+e}) - \omega(gcx_i^{d+e}) = \frac{v_i(fcx_i^{d+e})}{a_i} - \frac{v_i(gcx_i^{d+e})}{a_i} = \frac{v_i(f) - v_i(g)}{a_i}$$

This proves $u_i\left(\frac{cf}{cg}\right) = u_i\left(\frac{f}{g}\right)$, by Remark 5.5.5.

If $fx_i^d \notin \mathcal{S}_i$ for any positive integer d then $u_i\left(\frac{f}{g}\right) = \infty$. We show that $fcx_i^d \notin \mathcal{S}_i$ for any positive integer d, proving that $u_i\left(\frac{cf}{cg}\right) = \infty$.

Since $cg \in S$, by Remark 5.5.4, $\exists$ a positive integer d such that $cgx_i^d \in \mathcal{S}_i$. If for some positive integer e, $fcx_i^e \in \mathcal{S}_i$, then $\omega(fcx_i^e) = \dfrac{v_i(fcx_i^e)}{a_i} = \dfrac{v_i(f) + v_i(cx_i^e)}{a_i} = \infty$ (since $v_i(f) = \infty$, by Remark 5.5.6) this implies $v_j(fcx_i^e) = \infty \; \forall \; 1 \leq j \leq s$. Since $c \in S$, $v_j(c) < \infty \; \forall \; 1 \leq j \leq s$. Thus, $v_j(fcx_i^e) = v_j(fx_i^e) + v_j(c) = \infty$. This means $v_j(fx_i^e) = \infty \; \forall \; 1 \leq j \leq s$, or that, $\omega(fx_i^e) = \infty$ which implies $fx_i^e \in \mathcal{S}_i$ (by Remark 5.5.2), contradicting our assumption. Hence, $u_i\left(\dfrac{f}{g}\right) = u_i\left(\dfrac{cf}{cg}\right) \; \forall \; c \in S$.

Suppose $\dfrac{f}{g} = \dfrac{f'}{g'}$ in K. Then $\exists \; c \in S$ such that $c(fg' - gf') = 0$.

Since $c, g, g' \in S$ and S is multiplicatively closed, $cg, cg' \in S$. Thus, we get

$$u_i\left(\frac{f'}{g'}\right) = u_i\left(\frac{cgf'}{cgg'}\right) = u_i\left(\frac{cfg'}{cgg'}\right) = u_i\left(\frac{f}{g}\right).$$

This proves that u_i is well-defined. Since the sets $\mathcal{S}_i$ are determined only by ω and each u_i is determined by the set $\mathcal{S}_i$ and the function ω, we have that the functions u_i are determined only by ω. So, we have a well-defined function $u_i : K = S^{-1}R \to \mathbb{Q} \cup \{\infty\}$ given as follows:

$$\begin{aligned} u_i : \quad S^{-1}R \quad &\longrightarrow \quad \mathbb{Q} \cup \{\infty\} \\ \frac{f}{g} \quad &\mapsto \begin{cases} \omega(fx_i^d) - \omega(gx_i^d) & \text{if } fx_i^d \in \mathcal{S}_i \text{ for some } d > 0 \\ \infty & \text{if } fx_i^d \notin \mathcal{S}_i \;\; \forall \;\; d > 0 \end{cases} \end{aligned}$$

Remark 5.5.7. The functions u_i and $\dfrac{v_i}{a_i}$ agree on R.

The proof of the remark is as follows:

For $f \in R$, if $fx_i^d \in \mathcal{S}_i$ for some positive integer d, then

$$u_i\left(\frac{f}{1}\right) = \omega(fx_i^d) - \omega(x_i^d) = \frac{v_i(fx_i^d)}{a_i} - \frac{v_i(x_i^d)}{a_i} = \frac{v_i(f)}{a_i}.$$

If $fx_i^d \notin \mathcal{S}_i$ $\forall$ positive integers d, then $u_i\left(\frac{f}{1}\right) = \infty$ which gives $v_i(f) = \infty$ (by Remark 5.5.6) and thus $\frac{v_i(f)}{a_i} = \infty$.

It follows from Remark 5.5.7 that u_i satisfies the following properties for every $f, g \in R$:

$$u_i(fg) = u_i(f) + u_i(g) \quad \text{and} \quad u_i(f+g) \geq \min\{u_i(f), u_i(g)\}$$

Let P_i be the prime ideal $\{x \in R \mid u_i(x) = \infty\}$ of R. By Remark 5.5.7, $\{f \in R \mid v_i(x) = \infty\} = P_i$ and u_i induces a function on R/P_i which is equal to $\frac{v_i}{a_i}$ on R/P_i. Thus, P_i is a minimal prime of R and u_i induces a valuation on the quotient field of R/P_i which is equivalent to v_i. By abuse of notation, we will denote this valuation by u_i. By Remark 5.5.7, $u_i = \frac{v_i}{a_i}$. By Lemma 5.4, v_i and a_i are uniquely determined by u_i. Since the u_i are uniquely determined by the function ω, we have that the v_i and a_i are uniquely determined by the function ω. ∎

Corollary 5.6. *Let $\mathcal{I} = \{I_m\}_{m\in\mathbb{N}}$ be a discrete valued filtration of a Noetherian ring R, where $I_m = I(v_1)_{a_1 m} \cap \cdots \cap I(v_s)_{a_s m}$ $\forall$ $m \in \mathbb{N}$ is an irredundant representation. Then the valuations v_i and $a_i \in \mathbb{R}_{>0}$ are uniquely determined.*

Proof. Since $I_m = \{f \in R \mid v_i(f) \geq a_i m \text{ for } 1 \leq i \leq s\}$ and no v_i can be removed from this expression, by Lemma 5.2, no $\frac{v_i}{a_i}$ can be removed from the expression $\overline{\nu}_{\mathcal{I}}(f) = \min\limits_{1\leq i\leq s}\left\{\frac{v_i(f)}{a_i}\right\}$. Therefore, from Theorem 5.5 we have that v_i and $a_i \in \mathbb{R}_{>0}$ are uniquely determined. ∎

Corollary 5.7. *Let $\mathcal{I} = \{I_m\}_{m\in\mathbb{N}}$ and $\mathcal{J} = \{J_m\}_{m\in\mathbb{N}}$ be discrete valued filtrations of a Noetherian ring R, where $I_m = \bigcap_{i=1}^{s} I(v_i)_{a_i m}$ and $J_m = \bigcap_{i=1}^{r} I(v_i')_{a_i' m}$ $\forall\ m \in \mathbb{N}$ are irredundant representations. If $\overline{\nu}_{\mathcal{I}} = \overline{\nu}_{\mathcal{J}}$, then $r = s$ and after reindexing, $a_i = a_i'$ and $v_i = v_i'$.*

Proof. From Lemma 5.2 we have that $\min_{1\le i\le s}\left\{\dfrac{v_i(f)}{a_i}\right\} = \min_{1\le i\le r}\left\{\dfrac{v_i'(f)}{a_i'}\right\}$ $\forall\ f \in R$. The Corollary now follows from Theorem 5.5. ∎

Suppose that $\mathcal{I} = \{I_m\}_{m\in\mathbb{N}}$ is a discrete valued filtration where $I_m = \bigcap_{i=1}^{s} I(v_i)_{ma_i}$ and $\alpha \in \mathbb{R}_{>0}$. Then we have the explicit description of $\mathcal{I}^{(\alpha)}$ as

$$\mathcal{I}^{(\alpha)} = \{I_m^{(\alpha)}\}_{m\in\mathbb{N}} = \{I_{\lceil \alpha m\rceil}\}_{m\in\mathbb{N}} \text{ where } I_{\lceil \alpha m\rceil} = \bigcap_{i=1}^{s} I(v_i)_{\lceil \alpha m\rceil a_i}\ \forall\ m \in \mathbb{N}.$$

We define a new filtration

$$\mathcal{I}^{[\alpha]} = \{I_m^{[\alpha]}\}_{m\in\mathbb{N}} = \{I_{\alpha m}\}_{m\in\mathbb{N}} \text{ where } I_{\alpha m} = \bigcap_{i=1}^{s} I(v_i)_{\alpha m a_i}\ \forall\ m \in \mathbb{N}.$$

Observe that $\mathcal{I}^{(\alpha)}$ is, in general, not a discrete valued filtration, but $\mathcal{I}^{[\alpha]}$ is.

The filtration $\mathcal{I}^{[\alpha]}$ is well defined; that is, it is independent of (possibly redundant) representation $I_m = \bigcap_{i=1}^{s} I(v_i)_{a_i m}$ $\forall\ m \in \mathbb{N}$. To prove this, we first show that

$$I_m = \bigcap_{i=1}^{s} I(v_i)_{a_i m}\ \forall\ m \in \mathbb{N} \tag{5.3}$$

is an irredundant representation of $\mathcal{I}$ if and only if

$$I_m^{[\alpha]} = \bigcap_{i=1}^{s} I(v_i)_{\alpha a_i m}\ \forall\ m \in \mathbb{N} \tag{5.4}$$

is an irredundant representation of $\mathcal{I}^{[\alpha]}$. This follows since (5.3) is irredundant if and only if no $\frac{v_i}{a_i}$ can be eliminated from the function

$$\omega(f) = \min\left\{\frac{v_1(f)}{a_1}, \ldots, \frac{v_s(f)}{a_s}\right\}$$

which holds if and only if no $\frac{\nu_i}{\alpha a_i}$ can be eliminated from the function

$$\omega_\alpha(f) = \min\left\{\frac{v_1(f)}{\alpha a_1}, \ldots, \frac{v_s(f)}{\alpha a_s}\right\}$$

which is equivalent to (5.4) being irredundant. Now by Corollary 5.7, the valuations ν_i and $a_i \in \mathbb{R}_{>0}$ giving irredundant representations of $\mathcal{I}$ are uniquely determined and the valuations ν_i and $a_i\alpha \in \mathbb{R}_{>0}$ giving irredundant representations of $\mathcal{I}^{[\alpha]}$ are uniquely determined. Thus the filtration $\mathcal{I}^{[\alpha]}$ is independent of choice of representation of $\mathcal{I}$.

Proposition 5.8. *Suppose that $\mathcal{I}$ is a discrete valued filtration of a Noetherian ring R and $\alpha \in \mathbb{R}_{>0}$. Then $\mathcal{K}(\mathcal{I}^{(\alpha)}) = \mathcal{I}^{[\alpha]} = \mathcal{K}(\mathcal{I}^{[\alpha]})$.*

Proof. Since $\mathcal{I}^{[\alpha]}$ is a discrete valued filtration of R, by Corollary 5.3, $\mathcal{I}^{[\alpha]} = \mathcal{K}(\mathcal{I}^{[\alpha]})$. Now, $\mathcal{K}(\mathcal{I}^{(\alpha)}) = \{K(\mathcal{I}^{(\alpha)})_m\}_{m\in\mathbb{N}}$, where $K(\mathcal{I}^{(\alpha)})_m = \{x \in R \mid \overline{\nu}_{\mathcal{I}^{(\alpha)}}(x) \geq m\}$. For $x \in R$, $\overline{\nu}_{\mathcal{I}^{(\alpha)}}(x) \geq m$ if and only if $\overline{\nu}_{\mathcal{I}}(x) \geq \alpha m$ (by Theorem 4.9) if and only if $x \in I_m^{[\alpha]}$ (by Corollary 5.3). Thus, $\mathcal{K}(\mathcal{I}^{(\alpha)}) = \mathcal{I}^{[\alpha]}$. ∎

Theorem 5.9. *Let $\mathcal{I} = \{I_m\}_{m\in\mathbb{N}}$ and $\mathcal{J} = \{J_m\}_{m\in\mathbb{N}}$ be discrete valued filtrations of a Noetherian ring R and $\alpha \in \mathbb{R}_{>0}$. Then $\overline{\nu}_{\mathcal{I}} = \alpha\overline{\nu}_{\mathcal{J}}$ if and only if $\mathcal{J} = \mathcal{I}^{[\alpha]}$.*

Proof. Theorem 4.9 implies that $\overline{\nu}_{\mathcal{I}} = \alpha\overline{\nu}_{\mathcal{I}^{(\alpha)}}$. Thus $\overline{\nu}_{\mathcal{I}} = \alpha\overline{\nu}_{\mathcal{J}}$ if and only if $\overline{\nu}_{\mathcal{I}^{(\alpha)}} = \overline{\nu}_{\mathcal{J}}$. This holds if and only if $\mathcal{K}(\mathcal{I}^{(\alpha)}) = \mathcal{K}(\mathcal{J})$, by Theorem 4.22. Since $\mathcal{J}$

is a discrete valued filtration, by Corollary 5.3, $\mathcal{K}(\mathcal{J}) = \mathcal{J}$ and by Corollary 5.8, $\mathcal{K}(\mathcal{I}^{(\alpha)}) = \mathcal{I}^{[\alpha]}$. ∎

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