ON RELATIVISTIC POINT PARTICLES WITH CURVATURE-DEPENDENT ACTIONS*

By H. Arodź, A. Sitarz and P. Węgrzyn

Institute of Physics, Jagellonian University, Cracow**

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Systematic analysis of classical trajectories of a point-like analogue of the smooth string is presented. It is shown that the point-like analogue contains tachyons on the classical level.

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1. Introduction

Theories defined by Lagrangians with higher order derivatives have already been considered many times. Let us mention classical models of spinning particles [1], some versions of the general relativity [2], the regularized Yang-Mills theory [3], and some supersymmetric σ-models of Kähler type [4]. Recently, a new theory of this kind has appeared — the smooth string [5]. The Euclidean version of this string is relevant for the theory of random surfaces, while Minkowski space-time version is expected to find an application in the low-energy QCD. Also, it has been shown that the ordinary magnetic vortex line can be regarded as the physical realization of the classical smooth string [6]. The Lagrangian for the smooth string contains a term with the second order derivatives which represents the extrinsic curvature of the world-sheet of the string.

The smooth string is a bosonic string. It has an extremely complicated dynamics. Very little is known about it even on the classical level, due to the fact that equations of motion for the smooth string cannot be linearized by the appropriate choice of parametrization of the world-sheet, in contradistinction to the case of Nambu-Goto string. For this reason, a point-like analogue of the rigid string has been considered [7–12] with the hope that the study of this simpler model will help to understand the dynamics of the smooth string. The Lagrangian for the point-like analogue contains a power of the first curvature of the world-line of the particle with respect to the time.

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** Address: Instytut Fizyki UJ, Reymonta 4, 30-059 Kraków, Poland.

In the present paper we investigate two such point models in Minkowski space-time with the Lagrangian linear or quadratic in the first curvature of the world-line. The former case has been considered in papers [9–11]. We add to the results obtained in those papers the observation that in this model all trajectories have their first curvature constant. Due to that fact, equations of motion are essentially linear, and we can explicitly find all trajectories of the particle.

The model with the Lagrangian quadratic in the first curvature is much more difficult to analyze, because equations of motion cannot be linearized. In this case the first curvature is not a constant of the motion, in general. However, just this model is more interesting, because its equations of motion can be obtained from the equations of motion of the smooth string by neglecting internal degrees of freedom of the string [14]. Thus, this model may be called the point-like analogue of the smooth string. In the present paper we analyze the mass spectrum of the point-like analogue, the mass squared defined as the square of the energy-momentum four-vector. We carry out a systematic analysis of possible types of trajectories of the point-like analogue, and we find explicit formulae for the trajectories with the first curvature constant. We also find a rather interesting reformulation of the model in terms of the ordinary, relativistic point particle moving in an external electromagnetic field. This reformulation enables us to find an example of the trajectory with the first curvature variable.

Our main finding is that the point-like analogue of the smooth string is tachyonic already on the classical level. This result strongly suggests that the classical smooth string itself is also tachyonic. Let us recall that Nambu-Goto string is not tachyonic on the classical level — it becomes tachyonic only when quantized in the standard manner. If the smooth string is tachyonic, it will probably mean that it cannot be accepted as a satisfactory model. Perhaps its Lagrangian should be changed by adding some new terms. Such terms will likely contain the third order derivatives. Suggestions of this kind have already been expressed in papers [6, 15] with some other justification. Our investigation of the smooth string we will present in a forthcoming paper [14].

The present paper is organized as follows. In Section 2 we present the equations of motion and the integrals of motion for the point-like analogue of the smooth string. In Section 3 we study the model with the Lagrangian linear in the first curvature. In Section 4 we find all trajectories with the first curvature constant for the other model, i.e. the one with the Lagrangian quadratic in the first curvature. In Section 5 we use the angular momentum integral of motion in order to reduce the original equations of motion with the fourth order derivatives to the ordinary Newton equations for the relativistic particle in an external electromagnetic field. With the help of this reformulation of the problem we can find an explicit tachyonic solution with the first curvature variable. Section 6 contains conclusions and remarks. In particular, we point out that the reformulation presented in Section 5 suggests a new physical interpretation of the model with the higher order derivatives.

2. *The equations of motion*

The first curvature κ_1 of a curve $\vec{r}(s)$ in the Euclidean R^n space is defined as [13]

$$\kappa_1 = \left|\frac{d^2\vec{r}}{ds^2}\right|. \tag{1}$$

Here s is the length parameter along the curve, i.e. $\left|\frac{d\vec{r}}{ds}\right| = 1$. Passing to a general parametrization of the curve by the parameter τ, we obtain from (1)

$$\kappa_1^2 = \frac{\ddot{\vec{r}}^2\dot{\vec{r}}^2 - (\ddot{\vec{r}}\dot{\vec{r}})^2}{(\dot{\vec{r}}^2)^3}, \tag{2}$$

where the dots denote derivatives with respect to τ. As the Lorentz invariant counterpart of κ_1^2 we shall take a generalization of κ_1^2 to pseudo-Euclidean space-time, namely

$$k = \frac{(\ddot{x}\dot{x})^2 - \ddot{x}^2\dot{x}^2}{(\dot{x}^2)^3}, \tag{3}$$

where $x = (x^\mu)$, $\dot{x} = \left(\frac{dx^\mu}{d\tau}\right)$, $\dot{x}^2 = \dot{x}^0\dot{x}^0 - \dot{x}^i\dot{x}^i$, etc. In the proper-time gauge for time-like curves we have

$$\dot{x}^2 = 1. \tag{4}$$

Then $\dot{x}\ddot{x} = 0$ and

$$k = -\ddot{x}^2 \geqslant 0.$$

Free relativistic point particle is described by the Lagrangian

$$\mathscr{L} = -m\sqrt{\dot{x}^2}, \tag{5}$$

where the constant $m > 0$ is the mass of the particle. Motivated by the Lagrangian of the rigid string given in [5] we extend Lagrangian (5) by adding a term with k, [7],

$$\mathscr{L} = -\sqrt{\dot{x}^2}\,(m + \alpha K(k)), \tag{6}$$

where $K(k)$ is a function of k. In the following we shall consider the cases $K = \sqrt{k}$ and $K = k$.

Lagrangian (6) contains the second order derivatives $\ddot{x}$. The corresponding Euler-Lagrange equations have the form [16]

$$\frac{d^2}{d\tau^2}\left(\frac{\partial\mathscr{L}}{\partial\ddot{x}_\mu}\right) - \frac{d}{d\tau}\left(\frac{\partial\mathscr{L}}{\partial\dot{x}_\mu}\right) + \frac{\partial\mathscr{L}}{\partial x_\mu} = 0. \tag{7}$$

Because Lagrangian (6) does not depend on x, the last term on the l.h.s. of Eq. (7) vanishes, and equation (7) can be written in the form

$$\frac{d}{d\tau} P^\mu = 0, \tag{8}$$

where

$$P^\mu = \frac{d}{d\tau}\left(\frac{\partial \mathscr{L}}{\partial \ddot{x}_\mu}\right) - \frac{\partial \mathscr{L}}{\partial \dot{x}_\mu} \tag{9}$$

is the energy-momentum four-vector of the particle. Formula (9) can be obtained also from Noether theorem. Equation (8) expresses the conservation of the energy-momentum of the particle during the motion.

From Lagrangian (6) in the gauge (5) we obtain

$$P_\mu = (m+\alpha K(k)-4\alpha k K'(k))\dot{x}_\mu + 2\alpha K''(k)\dot{k}\dddot{x}_\mu + 2\alpha K'(k)\dddot{x}_\mu, \tag{10}$$

where $K' \equiv \dfrac{dK}{dk}$.

Let us multiply the both sides of formula (10) by $\dot{x}_\mu$. Using the formula $\dot{x}\dddot{x} = -\ddot{x}^2 = k$, which follows from (4), we obtain the following useful relation

$$P_\mu \dot{x}^\mu = m - 2\alpha K'(k)k + \alpha K(k). \tag{11}$$

Another useful formula is obtained by multiplying the both sides of formula (10) by P^μ and using formula (11) and its derivatives

$$P^\mu P_\mu = (m+\alpha K - 4\alpha k K')(m+\alpha K - 2\alpha k K') - 2\alpha^2 \frac{d^2}{d\tau^2}(kK'^2). \tag{12}$$

Thus, $P^2 \equiv P_\mu P^\mu$ is a function of k, $\dot{k}$ and $\ddot{k}$ only. By definition, P^2 is the mass squared of the particle. For the ordinary particle $K = 0$ and $P^2 = m^2$.

Lagrangian (6) is invariant under Lorentz transformations. Noether theorem gives the following formula for the angular momentum of the particle,

$$M_{\mu\nu} = \frac{\partial \mathscr{L}}{\partial \ddot{x}^\mu}\dot{x}_\nu - \frac{\partial \mathscr{L}}{\partial \ddot{x}^\nu}\dot{x}_\mu + \dot{x}_\mu P_\nu - \dot{x}_\nu P_\mu, \tag{13}$$

where P_μ is given by formula (9). $M_{\mu\nu}$ is integral of the motion of the particle. For Lagrangian (6) in the gauge (4) we have

$$M_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu + 2\alpha K'(k)(\ddot{x}_\mu \dot{x}_\nu - \ddot{x}_\nu \dot{x}_\mu), \tag{14}$$

where P_μ is given by formula (10).

Interesting information about the trajectory is provided by Pauli-Lubanski pseudovector

$$W_\lambda = -\tfrac{1}{4}\varepsilon_{\lambda\mu\nu\varrho} M^{\mu\nu} P^\varrho,$$

which is the integral of the motion because it is the function of P^{ϱ} and $M^{\mu\nu}$. Using formulae (10), (14) we obtain

$$W_{\lambda} = 2\alpha(K')^2 \varepsilon_{\lambda\mu\nu\varrho}\dot{x}^{\mu}\ddot{x}^{\nu}\dddot{x}^{\varrho}. \tag{15}$$

It is easy to check that the square of the four-vector $T_{\lambda} = \varepsilon_{\lambda\mu\nu\varrho}\dot{x}^{\mu}\ddot{x}^{\nu}\ddddot{x}^{\varrho}$ is related to the second curvature κ_2 of the world-line, [13],

$$T_{\lambda}T^{\lambda} = k^2\kappa_2^2.$$

In the next Section we shall prove that for $K = \sqrt{k}$, k is constant during the motion. Therefore, in this case all trajectories have the second curvature constant too. For $K(k) = k$ k does not have to be constant during the motion, in general.

3. *Classical trajectories in the case* $K = \sqrt{k}$

In this case we can find all trajectories explicitly. For $\alpha < 0$ we find the usual straight-line trajectories and no other trajectories. For $\alpha > 0$ we find several types of trajectories depending on initial data. In particular, there exist tachyonic trajectories, i.e. the ones with $P^2 < 0$. The presented below list of trajectories is complete, i.e. no other trajectories exist.

Let us start from the observation that for $K = \sqrt{k}$ formula (11) gives

$$P^{\mu}\dot{x}_{\mu} = m. \tag{16}$$

Thus, $P^{\mu}\dot{x}_{\mu}$ is constant during the motion. Formula (12) gives

$$P^{\mu}P_{\mu} = m^2 - m\alpha\sqrt{k}. \tag{17}$$

Because P_{μ} is constant during the motion, the first curvature k is constant too.

Let us first consider the case $k = 0$. Then, $k = -\ddot{x}^2 = 0$ means that $\ddot{x}$ is a light-like vector or $\ddot{x}_{\mu} = 0$. However, in the gauge (4) $\dot{x}\ddot{x} = 0$, where $\dot{x}$ is a time-like vector. This relation cannot be satisfied by a non-zero light-like vector $\ddot{x}$. Thus, $\ddot{x}_{\mu} = 0$, i.e.

$$x^{\mu}(\tau) = \frac{P^{\mu}}{m}\tau + x^{\mu}(0). \tag{18}$$

This is the usual straight-line trajectory of a free particle.

Now we shall pass to the case $k > 0$. It is easy to prove that for $\alpha < 0$ trajectories with $k > 0$ do not exist. From formula (17) we see that for $\alpha < 0$

$$P^2 \geqslant m^2. \tag{19}$$

On the other hand, calculating P^2 directly, by taking square of the r.h.s. of formula (10), we obtain

$$P^2 = m^2 + \frac{\alpha^2}{k}\dddot{x}^2 - \alpha^2 k. \tag{20}$$

Here we have used the relation

$$\dot{x}\dddot{x} = -\ddot{x}^2, \tag{21}$$

which follows from the gauge condition (4). $\ddot{x}$ has to be a space-like vector or zero. This follows from the condition

$$P\ddot{x} = 0,$$

which is a consequence of constraint (16), and from the fact that P is a time-like vector. Therefore, formula (20) implies that $P^2 \leqslant m^2$. This is compatible with inequality (19) only when $P^2 = m^2$. Then, formula (17) implies that $k = 0$.

Thus, for $k > 0$ the case $\alpha < 0$ is excluded. For $\alpha > 0$ we shall distinguish the following three subcases:

(a) $P^2 > 0$,

(b) $P^2 = 0$,

(c) $P^2 < 0$.

Because k is constant during the motion, formula (10) can be integrated. This yields the following equation for $x_\mu(\tau)$:

$$\frac{\alpha}{\sqrt{k}}\ddot{x}_\mu + \frac{P^2}{m}x_\mu = P_\mu\tau + C_\mu, \tag{22}$$

where (C_μ) is a constant four-vector. Here we have used formula (17). In the cases (a), (c) we can substitute in (22)

$$x_\mu(\tau) = \frac{m(P_\mu\tau + C_\mu)}{P^2} + z_\mu(\tau). \tag{23}$$

This leads to the harmonic oscillator-type equation for z_μ

$$\frac{\alpha}{\sqrt{k}}\ddot{z}_\mu + \frac{P^2}{m}z_\mu = 0. \tag{24}$$

In the case (a) the general solution of Eq. (24) has the form

$$z_\mu(\tau) = A_\mu \cos\omega\tau + B_\mu \sin\omega\tau, \tag{25}$$

where

$$\omega^2 = \frac{P^2\sqrt{k}}{\alpha m} = \frac{P^2(m^2-P^2)}{\alpha^2 m^2} > 0. \tag{26}$$

(It follows from formula (17) that for $\alpha > 0$ and $k < 0$ we have $P^2 < m^2$.) Solutions (23), (25) have to obey the following conditions for all τ:

$$\dot{x}^2 = 1, \quad \ddot{x}^2 = -k = -\left(\frac{m^2-P^2}{\alpha m}\right)^2, \quad \dot{x}_0 > 0. \tag{27}$$

These conditions are satisfied if and only if

$$A_\mu B^\mu = 0, \quad P_\mu A^\mu = P_\mu B^\mu = 0,$$

$$A_\mu A^\mu = B_\mu B^\mu = -\frac{\alpha^2 m^2}{(P^2)^2}, \quad P_0 > 0. \tag{28}$$

Thus, we have obtained the seven parameter family of solutions. For this solutions $0 < P^2 < m^2$.

In the case (c) we obtain from (24)

$$z_\mu(\tau) = D_\mu \exp(\varpi\tau) + E_\mu \exp(-\varpi\tau), \tag{29}$$

where

$$\varpi^2 = \frac{|P^2|(m^2 - P^2)}{\alpha^2 m^2}. \tag{30}$$

Again, we have to obey conditions (27). They are satisfied if and only if

$$D^2 = E^2 = 0, \quad DE = -\frac{\alpha^2 m^2}{2(P^2)^2},$$

$$EP = DP = 0, \quad D_0 > 0, \quad E_0 < 0. \tag{31}$$

Conditions (31) define the seven-parameter family of tachyonic solutions ($P^2 < 0$). There is no lower bound on the values of P^2.

In the case (b) Eq. (22) gives

$$x_\mu(\tau) = \frac{m}{6\alpha^2} P_\mu \tau^3 + \frac{m}{2\alpha^2} C_\mu \tau^2 + F_\mu \tau + G_\mu. \tag{32}$$

Conditions (27) are satisfied if and only if

$$P^2 = 0, \quad PC = 0, \quad CF = 0, \quad C^2 = -\alpha^2, \quad F^2 = 1, \quad PF = m,$$

$$mC_0^2 < 2\alpha^2 P_0 F_0. \tag{33}$$

We obtain the six-parameter family of solutions.

This ends our discussion of trajectories for Lagrangian (6) with $K = \sqrt{k}$. The presented list of trajectories is complete.

4. *The case $K(k) = k$ — the trajectories with the first curvature constant*

For $K(k) = k$ the first curvature k is not constant in τ, in general. In this Section we shall present the particular trajectories for which the first curvature is constant. Such trajectories are relatively easy to find. Trajectories with k variable will be studied in the next Section.

For $K(k) = k$ formulae (7), (10)–(12) in the gauge (4) give

$$2\alpha[x_\mu^{(4)} - (\dot{x}x^{(4)})\dot{x}_\mu] + (m+3\alpha\ddot{x}^2)\ddot{x}_\mu = 0, \tag{34}$$

where

$$x_\mu^{(4)} = \frac{d^4}{d\tau^4}\, x_\mu,$$

$$P_\mu = (m+3\alpha\ddot{x}^2)\dot{x}_\mu + 2\alpha\dddot{x}_\mu. \tag{35}$$

$$P^\mu \dot{x}_\mu = m+\alpha\ddot{x}^2 \equiv m-\alpha k, \tag{36}$$

and

$$P_\mu P^\mu = m^2 - 4\alpha mk + 3\alpha^2 k^2 - 2\alpha^2 \ddot{k}, \tag{37}$$

where $k \equiv -\ddot{x}^2$.

It follows from formula (37) that in general k is not constant during the motion. Formula (37) can be regarded as Newton-type equation:

$$2\alpha^2\ddot{k} = 3\alpha^2 k^2 - 4\alpha mk + m^2 - P^2. \tag{38}$$

The corresponding potential $V(k)$ has the form

$$V(k) = -\alpha^2 k^3 + 2\alpha mk^2 + (P^2 - m^2)k. \tag{39}$$

For our purpose we need solutions of Eq. (38) such that $k(\tau)$ is non-negative for all τ.

Let us examine the potential $V(k)$. We shall distinguish the following three cases:

(*i*) $P^2 \geqslant m^2$,

(*ii*) $m^2 > P^2 \geqslant -\frac{1}{3}m^2$,

(*iii*) $-\frac{1}{3}m^2 > P^2$.

The potential $V(k)$ has the shape presented in Fig. 1. We have pictured the cases $\alpha > 0$, $\alpha < 0$ separately. Trajectories with k constant correspond to the local extreme points of the potential $V(k)$.

Let us begin from the case (*i*). For both signs of α we have one local maximum of the potential in the region $k > 0$. It is located at

$$k_0 = \frac{2m}{3\alpha} + \frac{\sqrt{m^2+3P^2}}{3|\alpha|}. \tag{40}$$

Inserting $\ddot{x}^2 = -k_0$ in formula (35) we obtain a linear equation for $x_\mu(\tau)$ which is easy to solve. The solution has to obey the conditions

$$\dot{x}^2 = 1, \quad \dot{x}_0 > 0, \quad \ddot{x}^2 = -k_0. \tag{41}$$

It turns out that these conditions cannot be satisfied. Thus, there is no trajectory corresponding to $k = k_0$.

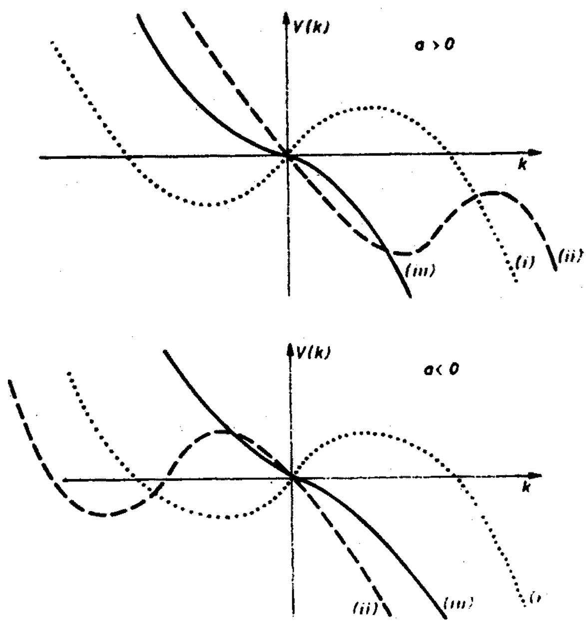

Fig. 1. The dotted line corresponds to the case (*i*), the dashed line corresponds to the case (*ii*), the continuous line corresponds to the case (*iii*)

For $P^2 = m^2$ we have another possibility:

$$k = 0. \tag{42}$$

Similarly as in Section 3, $k = 0$ implies the straight-line trajectory given by formula (18).

In the case (*iii*) no extreme points of $V(k)$ are present.

In the case (*ii*) we do not have any extreme points of $V(k)$ for $\alpha < 0$, and we have the maximum and the minimum of $V(k)$ for $\alpha > 0$. They are located at

$$k_{1,2} = \frac{2m \pm \sqrt{m^2+3P^2}}{3\alpha}. \tag{43}$$

The plus sign corresponds to the maximum of $V(k)$. Substituting $\ddot{x}^2 = -k_1$ in relation (35) we obtain the following equation for $x_\mu(\tau)$

$$2\alpha \dddot{x}_\mu - (m + \sqrt{m^2+3P^2})\dot{x}_\mu = P_\mu, \tag{44}$$

from which it follows that

$$\ddot{x}_\mu - \omega^2 x_\mu = (2\alpha)^{-1}(P_\mu \tau + C_\mu), \tag{45}$$

where

$$\omega^2 = (2\alpha)^{-1}(m + \sqrt{m^2+3P^2}).$$

Solutions of Eq. (45) have the following form

$$x_\mu(\tau) = -\frac{P_\mu\tau + C_\mu}{2\alpha\omega^2} + A_\mu \exp(\omega\tau) + B_\mu \exp(-\omega\tau). \tag{46}$$

This solution has to obey conditions (41) with k_0 replaced by k_1. This leads to the conditions

$$A^2 = B^2 = 0, \quad AP = BP = 0, \quad AB = -\tfrac{1}{2}k_1\omega^{-4},$$
$$A_0 < 0, \quad B_0 < 0. \tag{47}$$

These conditions cannot be satisfied if $P = (P_\mu)$ is a time-like or light-like vector. For $P^2 < 0$ they can be satisfied. Therefore, solution (46) exists only for

$$0 > P^2 \geqslant -\tfrac{1}{3}m^2.$$

It is a tachyonic solution.

Now let us investigate the last possibility

$$k_2 = \frac{2m - \sqrt{m^2+3P^2}}{3\alpha}, \quad \alpha > 0.$$

For $k = k_2$ the potential $V(k)$ has the local minimum, see Fig. 1. Substituting $\dddot{x}^2 = -k_1$ in relation (35) we obtain the following equation for $x_\mu(\tau)$

$$2\alpha\dddot{x}_\mu + (\sqrt{m^2+3P^2} - m)\dot{x}_\mu = P_\mu,$$

from which it follows that

$$2\alpha\ddot{x}_\mu + (\sqrt{m^2+3P^2} - m)x_\mu = P_\mu\tau + C_\mu. \tag{48}$$

Solutions of Eq. (48) are listed below. For $m^2 > P^2 > 0$:

$$x_\mu(\tau) = \frac{P_\mu\tau + C_\mu}{2\alpha\omega^2} + A_\mu\cos(\omega\tau) + B_\mu\sin(\omega\tau),$$

where

$$\omega^2 = (2\alpha)^{-1}(\sqrt{m^2+3P^2} - m),$$

and

$$A^2 = B^2 = -k_2\omega^{-4}, \quad AB = PA = PB = 0, \quad P_0 > 0. \tag{49}$$

Conditions (49) follow from the requirements

$$\dot{x}^2 = 1, \quad \dot{x}_0 > 0, \quad \ddot{x}^2 = -k_2.$$

For $P^2 = 0$:

$$x_\mu(\tau) = \frac{P_\mu}{12\alpha}\tau^3 + \frac{1}{4\alpha} C_\mu \tau^2 + D_\mu \tau + E_\mu, \tag{50}$$

where

$$PC = DC = 0, \quad D^2 = 1, \quad C^2 = -\tfrac{4}{3}m\alpha, \quad DP = \tfrac{4}{3}m\alpha,$$
$$C_0^2 < 4\alpha D_0 P_0, \quad P_0 > 0.$$

For $-\frac{1}{3}m^2 \leqslant P^2 < 0$:

$$x_\mu(\tau) = -\frac{P_\mu \tau + C_\mu}{m - \sqrt{m^2 + 3P^2}} + F_\mu \exp(\varpi\tau) + G_\mu \exp(-\varpi\tau),$$

where

$$\varpi^2 = (2\alpha)^{-1}(m - \sqrt{m^2+3P^2}),$$

and

$$F^2 = G^2 = 0, \quad PF = PG = 0, \quad FG = -\tfrac{1}{2}k_2\omega^{-4},$$
$$F_0 > 0, \quad G_0 < 0.$$

Let us summarize results presented in this Section. With the help of the auxiliary equation (38) we have found all trajectories with the first curvature constant. For $\alpha > 0$ these trajectories correspond either to the local minimum or to the local maximum of the curve (*ii*) on Fig. 1. In the former case P^2 can be arbitrary in the range $m^2 > P^2 \geqslant -\frac{1}{3}m^2$. In the latter case $-\frac{1}{3}m^2 \leqslant P^2 < 0$. Thus, the tachyonic trajectories are present. Of course, we also have the trivial straight-line trajectory (18) with $P^2 = m^2$. For $\alpha < 0$ we do not find any trajectory with the first curvature constant, except for the trivial trajectory (18). Nevertheless, we shall prove in the next Section that for $\alpha < 0$ tachyonic trajectories do exist. They have the first curvature variable.

5. *The case $K(k) = k$ — the trajectories with the first curvature variable*

The main difficulty lies in the fact that equation (34) cannot be replaced by a linear one in this case. Nevertheless, we can have some insight into the set of trajectories with k variable. We shall exploit the fact that P_μ and $M_{\mu\nu}$ are constant during the motion. We shall prove that each solution of Eq. (34) can be regarded as a trajectory of an ordinary point particle in a particular external electromagnetic field. This particle obeys the ordinary Newton equation with second order derivatives and Lorentz force. This is the main result of this Section. The external electromagnetic field is rather complicated. In spite of this we can find an explicit example of a tachyonic trajectory for $\alpha < 0$.

We see from Fig. 1 that if we take sufficiently low value of the total energy $\alpha^2\dot{k}^2 + V(k)$ corresponding to the equation (38), we can have a trajectory with τ-dependent positive

$k(\tau)$. It is clear that for this type of trajectories $k(\tau) \to +\infty$ when τ increases. Therefore, for sufficiently large τ we can ignore the last two terms in the potential $V(k)$, i.e.

$$V(k) \approx -\alpha^2 k^3.$$

Then, it is easy to find the approximate solution of Eq. (38) valid for large k:

$$k(\tau) = \frac{4}{(\tau_0 - \tau)^2}. \tag{51}$$

We see that $k(\tau)$, and therefore also $\ddot{x}_\mu(\tau)$ becomes infinite for some finite $\tau = \tau_0$. However this does not imply that we should reject this kind of trajectories without further investigation. The reason is that τ is not the time but only the proper time. The time variable is $t \equiv x_0(\tau)$. It may happen that $x_0(\tau) \to \infty$ for $\tau \to \tau_0$ in such a manner that $\vec{x}(t) = (x^i(t))$ is finite for all finite t. Then we would have a perfectly regular trajectory $\vec{x}(t)$. In other words, the fact that $x_\mu(\tau)$ becomes infinite for finite $\tau = \tau_0$ would only mean that the particle moves to the infinity with velocity so close to the velocity of light that the proper-time length of the trajectory is finite. We shall show that indeed, precisely this happens.

It is clear that Eq. (38) is not a good starting point for the study of trajectories in the present case. Even for the approximate form of $k(\tau)$ given by (51) it is rather difficult task to find solutions of Eq. (35) for $x_\mu(\tau)$ such that the condition $\dot{x}^2 = 1$ is satisfied. We have seen in Section 3 and 4 that it is not a trivial matter to obey that condition — some candidates for solutions have been eliminated because they are not compatible with it.

Much better approach is to exploit the fact that the angular momentum $M_{\mu\nu}$ given by formula (14) is the integral of the motion. For $K(k) = k$ formula (14) gives

$$M_{\mu\nu} = P_\nu x_\mu - P_\mu x_\nu + 2\alpha(\ddot{x}_\mu \dot{x}_\nu - \ddot{x}_\mu \dot{x}_\nu). \tag{52}$$

Multiplying the both sides of formula (52) by $\dot{x}^\nu$ and using the gauge condition $\dot{x}^2 = 1$ we obtain the following equation

$$2\alpha \ddot{x}_\mu = M_{\mu\nu}\dot{x}^\nu + P_\mu(x\dot{x}) - x_\mu(P\dot{x}). \tag{53}$$

Equation (53) can be interpreted as Newton equation of motion for ordinary relativistic point particle interacting with x_μ-dependent external electromagnetic field. Namely, Eq. (53) can be written in the form

$$2\alpha \ddot{x}_\mu = F_{\mu\nu}\dot{x}^\nu, \tag{54}$$

where

$$F_{\mu\nu} = M_{\mu\nu} + P_\mu x_\nu - P_\mu x_\nu \tag{55}$$

is the electromagnetic field strength tensor. For $F_{\mu\nu}$ one can find the corresponding gauge potential $A_\mu(x)$:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{56}$$

where

$$A_\mu(x) = -\tfrac{1}{2} M_{\mu\varrho}x^\varrho + (P_\varrho x^\varrho)x_\mu. \tag{57}$$

Equation (54) can be obtained from the following Lagrangian

$$\mathscr{L}^{(1)} = 2\alpha\sqrt{\dot{x}^2} + A_\mu \dot{x}^\mu. \tag{58}$$

$F_{\mu\nu}$ given by (55) is invariant under space-time translations parallel to P_μ

$$x'_\mu = x_\mu + aP_\mu, \tag{59}$$

where a is a parameter. The gauge potential A_μ is invariant under this transformation up to the gauge transformation

$$A'_\mu(x) = A_\mu(x) + \partial_\mu \chi, \tag{60}$$

where

$$\chi = \tfrac{1}{2}\, a(Px)^2 + (a^2P^2P_\mu - \tfrac{1}{2}\, aM_{\mu\varrho}P^\varrho)x^\mu + \tfrac{1}{2}\, aP^2x^2. \tag{61}$$

Lagrangian $\mathscr{L}^{(1)}$ changes by the proper-time derivative of χ, i.e.

$$\mathscr{L}^{(1)}(x') = \mathscr{L}^{(1)}(x) + \frac{d\chi}{d\tau}. \tag{62}$$

Noether theorem gives the following integral of the motion corresponding to the symmetry (59):

$$I_1 = 2\alpha\dot{x}_\mu P^\mu + \tfrac{1}{2}(Px)^2 - P^\mu M_{\mu\nu}x^\nu - \tfrac{1}{2}P^2x^2. \tag{63}$$

I_1 will play important role in our considerations. Equation (54) has the very nice feature that it automatically guarantees that

$$\dot{x}\ddot{x} = 0, \text{ i.e. } \dot{x}^2 = \text{constant} \equiv c_0. \tag{64}$$

Therefore, if we can prove that the original equation of motion (34) is equivalent to Eq. (54), we may forget about the troublesome step of checking whether candidates for the solution obey the gauge condition $\dot{x}^2 = 1$. The constant c_0 in (64) does not matter because it can always be set to 1 just by assuming that $\dot{x}^2 = 1$ for the initial data for equation (54).

Let us consider the problem of equivalence of Eq. (34) to Eq. (54). Equation (34) is equivalent to Eq. (35) just by trivial integration. The fact that $M_{\mu\nu}$ given by formula (52) is constant during the motion of the particle follows from formula (35) for P_μ — the τ-derivative of the r.h.s. of formula (52) vanishes if formula (35) holds. Because Eq. (54) has been deduced from formula (52), we conclude that Eq. (54) follows from formula (35). Thus in order to establish the equivalence it is sufficient to prove vice versa, i.e. that formula (35) can be derived from Eq. (54). Differentiation of Eq. (54) with respect to τ gives

$$2\alpha\dddot{x}_\mu = F_{\mu\nu}\ddot{x}^\nu + P_\mu - (P\dot{x})\dot{x}_\mu, \tag{65}$$

where $F_{\mu\nu}$ is given by formula (55). Eliminating $M_{\mu\nu}$ with the help of formula (52) we obtain

$$2\alpha \dddot{x}_\mu = -2\alpha \ddot{x}^2 \dot{x}_\mu - (P_\nu \dot{x}^\nu)\dot{x}_\mu + P_\mu. \tag{66}$$

In this step we have assumed that formula (52) holds independently of Eq. (54). We will return to this problem later. Equation (66) will become Eq. (35) if we additionaly require that

$$P_\nu \dot{x}^\nu = m + \alpha \ddot{x}^2. \tag{67}$$

This formula coincides with formula (36). However, formula (67) has to be regarded as an additional assumption, because formula (36) was derived from Eq. (35) which we are trying to obtain from Eq. (35).

Thus, we can obtain Eq. (35) if relations (52), (67) are valid. Let us recall that equation (54) follows from relation (52).

Instead of relation (67) we will use another equivalent relation. In order to obtain it, we write formula (52) in the form

$$2\alpha(\ddot{x}_\mu \dot{x}_\nu - \ddot{x}_\nu \dot{x}_\mu) = M_{\mu\nu} + P_\mu x_\nu - x_\mu P_\nu$$

and we take square of it. This gives the following formula

$$8\alpha \ddot{x}^2 = M_{\mu\nu}M^{\mu\nu} + 4M_{\mu\nu}P^\mu x^\nu + 2P^2x^2 - 2(Px)^2. \tag{68}$$

This formula is used to eliminate $\ddot{x}^2$ from formula (67). We obtain

$$2\alpha P\dot{x} = 2\alpha m + \tfrac{1}{4} M_{\mu\nu}M^{\mu\nu} + M_{\mu\nu}P^\mu x^\nu + \tfrac{1}{2} P^2 x^2 - \tfrac{1}{2}(Px)^2. \tag{69}$$

Relation (69) has rather complicated form — the fact that it has to be satisfied in addition to Eq. (54) would render Eq. (54) practically useless. Luckily, this is not the case. Let us compare relation (69) with the integral of motion (63). We see that they coincide if we take

$$I_1 = 2\alpha m + \tfrac{1}{4} M_{\mu\nu}M^{\mu\nu}.$$

Thus, it is sufficient to assume relation (69) only for the initial data for Eq. (54).

It is clear from considerations presented above that relations (52), (69) are equivalent to Eq. (35). Part of the content of relation (52) has the form of Eq. (54). Let us now analyse the full content of relation (52). Taking $\mu = i$, $\nu = 0$ or $\mu = i$, $\nu = k$ $(i, k = 1, 2, 3)$, we obtain from (52) the following relations

$$2\alpha \ddot{x}_i \dot{x}_0 = 2\alpha \ddot{x}_0 \dot{x}_i + M_{i0} + P_i x_0 - P_0 x_i, \tag{70}$$

$$2\alpha(\ddot{x}_i \dot{x}_k - \ddot{x}_k \dot{x}_i) = M_{ik} + P_i x_k - P_k x_i. \tag{71}$$

Relation (70) can be used to eliminate $\ddot{x}_i$, $\ddot{x}_k$ from formula (71). The result has the form

$$M_{ik} + P_i x_k - P_k x_i - \frac{1}{\dot{x}_0}\{\dot{x}_k M_{i0} - \dot{x}_i M_{k0} + x_0(P_i \dot{x}_k - P_k \dot{x}_i) + P_0(\dot{x}_i x_k - \dot{x}_k x_i)\} = 0. \tag{72}$$

Formulae (72), (70) imply formula (71). Formula (72) does not contain second derivatives. Recalling our experience with relation (69) we may hope that it is sufficient to satisfy relation (72) only for the initial data. Indeed, this is the case. It is easy to check that the τ-derivative of the l.h.s. of relation (72) vanishes identically if Eq. (70) is satisfied. Thus, relation (52) is equivalent to relations (70), (72), and it is sufficient that relation (72) is fulfilled at a single instant of the proper-time τ, e.g. at the instant at which the initial data are specified.

Relation (70) has the form of an equation of motion. We would like to clarify its relation to Eq. (54). Because relations (70) and (72) are equivalent to relation (52), and equation (54) follows from relation (52), we conclude that relation (70) together with condition (72) imposed on the initial data imply Eq. (54). One can also obtain this result by a direct calculation. The first step is to eliminate $\ddot{x}_0$ from Eq. (70) with the help of formula $\dot{x}\ddot{x} = \dot{x}_0\ddot{x}_0 - \dot{x}_i\ddot{x}_i = 0$. Next, we extract from (70) $\ddot{x}^i$ using the fact that the matrix $\delta^k_i + \dfrac{\dot{x}^k\dot{x}_i}{\dot{x}_0^2}$ has as the inverse matrix $\delta^p_k - \dot{x}^p\dot{x}_k$. Finally, we eliminate some terms using relation (72). As the result we obtain the spatial part (i.e. $\mu = 1, 2, 3$) of Eq. (54). The $\mu = 0$ component follows from the spatial part because $\ddot{x}_0 = \dot{x}_0^{-1}(\dot{x}_i\ddot{x}_i)$.

Let us summarize our considerations. We have found that the fourth-order equation (34) can be integrated twice with the help of momentum and angular momentum integrals of motion. The resulting second-order equation has the form (54). It is equivalent to Eq. (34) provided that the initial data obey the two constraints (69), (72).

We find it rather enlightening to write Eq. (54) in the old-fashioned way, with the ordinary time $t = x^0(\tau)$ as the independent variable and $\vec{x}(t) \equiv (x^i(t))$ as the trajectory. The ordinary velocity is

$$\vec{v} = \frac{d\vec{x}}{dt} = \left(\frac{\dot{x}^i}{\dot{x}_0}\right),$$

where, as usual, the dot denotes derivatives with respect to τ. It is easy to check that $\dot{x}_0 = \gamma$, where $\gamma = (1-\vec{v}^2)^{-1/2}$ is the Lorentz factor. Equation (54) is equivalent to the following equation

$$2\alpha\frac{d(\gamma\vec{v})}{dt} = \vec{E} + \vec{v}\times\vec{B}, \tag{73}$$

where

$$\vec{E} = (E^k), \quad E^k \equiv F_{0k}, \quad \vec{B} = (B^s), \quad B^s \equiv -\tfrac{1}{2}\,\varepsilon_{iks}F_{ik},$$
$$F_{ik} = -\varepsilon_{iks}B^s,$$

and the cross denotes the vector product. Let us recall that we use the metric with the signature $(+, -, -, -)$, thus $E^k = -E_k$, etc.

We obtain from formula (55) that

$$\vec{E} = \vec{N} + t\vec{P} - P_0\vec{x}, \tag{74}$$
$$\vec{B} = -\vec{M} + \vec{x}\times\vec{P}. \tag{75}$$

where

$$\vec{N} = (M_{0k}), \quad \vec{M} = (M^s), \quad M^s = \tfrac{1}{2}\varepsilon_{iks}M_{ik}.$$

Conditions (69), (72) can be written as, respectively,

$$2\alpha m+\tfrac{1}{2}(\vec{M}^2-\vec{N}^2) = 2\alpha\gamma(P_0-\vec{P}\vec{v})+\vec{P}\vec{N}t-P_0\vec{x}\vec{N}$$
$$+(\vec{x}\times\vec{P})\vec{M}+\tfrac{1}{2}(P_0t-\vec{P}\vec{x})^2-\tfrac{1}{2}P^2(t^2-\vec{x}^2), \tag{76}$$

$$\vec{M} = \vec{x}\times\vec{P}-t\vec{v}\times\vec{P}-\vec{v}\times\vec{N}-P_0\vec{x}\times\vec{v}. \tag{77}$$

The expressions on the r.h.s. of formulae (76), (77) are the first integrals for the equation of motion (73). Therefore, it is sufficient to assume that these conditions are satisfied at a single instant of time.

It is easy to check that at the points lying on the trajectory of the particle $\vec{B} = \vec{x}\times\vec{E}$ Let us also note that one can use formula (77) in order to eliminate $\vec{M}$ from formula (76) — then we obtain

$$2\alpha m = \tfrac{1}{2}(1-\vec{v}^2)\vec{E}^2+\tfrac{1}{2}(\vec{v}\vec{E})^2+2\alpha\gamma(P_0-\vec{P}\vec{v}. \tag{78}$$

Equation (73) and condition (78) are relatively simple. We can find an explicit tachyonic solution of them. For such solution $P^2 < 0$. Therefore, using a Lorentz transformation one can have $P_0 = 0$. Let us also assume that $\vec{N} = 0$. Then $\vec{E} = t\vec{P}$, and condition (78) gives for $t = 0$

$$m = -\gamma(t=0)\vec{P}\vec{v}(t=0). \tag{79}$$

Equation (73) takes the form

$$2\alpha\frac{d(\gamma\vec{v})}{dt} = t\vec{P}(1-\vec{v}^2)+t(\vec{P}\vec{v})\vec{v}. \tag{80}$$

This form of the equation of motion suggests the following Ansatz:

$$\vec{v}(t) = v(t)\frac{\vec{P}}{|\vec{P}|}, \tag{81}$$

where $|\vec{P}| = \sqrt{\vec{P}^2}$, and $-1 < v(t) < 1$ in order that $0 \leqslant \vec{v}^2 < 1$.

Condition (79) gives

$$v(0) = -\frac{m}{|\vec{P}|}\left(1+\frac{m^2}{\vec{P}^2}\right)^{-1/2}. \tag{82}$$

Equation (80) reduces to

$$2\alpha\frac{d}{dt}\left(\frac{v}{\sqrt{1-v^2}}\right) = t|\vec{P}|. \tag{83}$$

This equation is easily solved by the substitution $v = \sin\varphi$. The result is

$$v(t) = \frac{a(t)}{\sqrt{1-a^2(t)}}, \tag{84}$$

where

$$a(t) = -\frac{m}{|\vec{P}|} + \frac{|\vec{P}|}{4\alpha}t^2. \tag{85}$$

From formulae (81), (84) one can recover $\dot{x}_\mu$, $\ddot{x}_\mu$, $\dddot{x}_\mu$:

$$\dot{x}_0 = \gamma = \sqrt{1+a^2}, \quad \dot{x}^i = \dot{x}_0 v^i = a(t)\frac{P^i}{|\vec{P}|},$$

$$\ddot{x}_0 = a\frac{da}{dt}, \quad \ddot{x}^i = \sqrt{1+a^2}\frac{da}{dt}\frac{P^i}{|\vec{P}|},$$

$$\dddot{x}_0 = \sqrt{1+a^2}\frac{d}{dt}\left(a\frac{da}{dt}\right), \quad \dddot{x}^i = \sqrt{1+a^2}\frac{d}{dt}\left(\frac{da}{dt}\sqrt{1+a^2}\right)\frac{P^i}{|\vec{P}|}: \tag{86}$$

Substituting these derivatives in Eq. (35) we check that indeed, we have obtained the solution. In order to find the trajectory $\vec{x}(t)$ we integrate the equation

$$\frac{d\vec{x}}{dt} = v(t)\frac{P^i}{|\vec{P}|}.$$

It follows that

$$\vec{x}(t) = f(t)\frac{\vec{P}}{|\vec{P}|} + \vec{x}(0), \tag{87}$$

where

$$f(t) = \int_0^t \frac{a(t')}{\sqrt{1+a^2(t')}}dt'.$$

For large t the function $f(t)$ behaves like $\operatorname{sign}(\alpha)t$. Therefore, for $t \to \infty$

$$\vec{x}(t) \approx \operatorname{sign}(\alpha)t\frac{\vec{P}}{|\vec{P}|}. \tag{88}$$

For $\alpha < 0$ the velocity $\vec{v}(t)$ is antiparallel to the momentum $\vec{P}$.

Solution (87) is characterized by the following values of the momentum and of the angular momentum

$$P_0 = 0, \quad P^i \neq 0, \quad M_{\mu\nu} = 0. \tag{89}$$

Let us recall that the momentum is given by formula (35) — it is not proportional to $\dot{x}$. Trajectory (87) is a straight line in the ordinary 3-space. The four-dimensional world-line is not a straight-line, of course. The particle self-accelerates because $|\vec{v}| \to 1$ as $t \to +\infty$. From (89) we see that $P^2 < 0$, i.e. the solution is tachyonic. It has the first curvature variable, $k = -\ddot{x}^2 = \left(\frac{da}{dt}\right)^2$. The proper-time length of the trajectory is finite,

$$\tau_\infty = \int_0^\infty \gamma^{-1} dt = \int_0^\infty (1+a^2)^{-1/2} dt < \infty,$$

in accordance with the remark following formula (51).

6. *Conclusions and remarks*

a) We have shown that Lagrangians (6) with $K(k) = \sqrt{k}$ and $\alpha > 0$, or $K(k) = k$, with the both signs of α, lead to tachyonic trajectories. These trajectories lie inside the upper light-cone ($\dot{x}^2 \equiv 1$, $\dot{x}_0 > 0$). Thus, the particle moves with velocity smaller than the velocity of light. However, the momentum of the particle does not have to be parallel to the velocity — it can point at a space-like direction. For $K(k) = \sqrt{k}$ and $\alpha < 0$ we notice rather interesting fact that there do not exist any non-trivial solutions, apart from the straight world-lines in space-time. In this case the particle behaves like the usual free particle, in spite of the presence of higher derivatives in its Lagrangian.

b) The presented in Section 5 reformulation of Eq. (34) in terms of ordinary particle moving in the external electromagnetic field suggests the following new physical interpretation of the theory with higher order derivatives. In order to uniquely specify a solution of Eq. (34) we have to fix $x^i(0)$, $\dot{x}^i(0)$, $\ddot{x}^i(0)$, $\dddot{x}^i(0)$, $i = 1, 2, 3$, as the initial data. We omit the $\mu = 0$ components because in order to satisfy the $\mu = 0$ component of Eq. (34) it is sufficient to satisfy the $\mu = i$ components of that equation and the constraint $\dot{x}^2 = 1$ for all τ. All τ-derivatives of $x_0(\tau)$ can be computed from that constraint and its τ-derivatives: $\dot{x}\ddot{x} = 0, \ddot{x}^2 + \dot{x}\dddot{x} = 0$, etc. We have also to specify the initial value of x_0, i.e. $x_0(0)$ — this corresponds to the choice of the initial instant of time. Equivalently, we may fix $M_{\mu\nu}$, P_ϱ, $\vec{x}(t_0)$, $\vec{v}(t_0)$ at the time $t_0 = x_0(0)$. According to formula (55), fixing $M_{\mu\nu}$, P_ϱ is equivalent to fixing of the external electromagnetic field. Thus, those degrees of freedom of the object described by Eq. (34) which are non-standard, i.e. $\ddot{x}^i$, $\dddot{x}^i$, can be interpreted as belonging to the electromagnetic field. Thus, Eq. (34) with the higher order derivatives can be interpreted as describing the system composed of an ordinary point particle and of the 10-parameter family of external electromagnetic fields of the form (74), (75). This ordinary point particle is described by Newton equation (73) and its initial data are restricted by the four constraints (77), (78).

We have found this interpretation in the case of Lagrangian (6) with $K(k) = k$. We do not know whether a similar interpretation is possible for other models with higher order derivatives. The problem of physical interpretation of theories with higher order derivatives is rather important one. For example, a sensible physical interpretation might help to find the right approach to quantization of such theories. It is a well-known fact that theories with higher order derivatives quantized in the standard manner are full of severe problems. In our opinion, the primary reason for these problems is just the lack of the proper physical interpretation of such theories.

c) The above presented idea of the new physical interpretation of theories with higher order derivatives is a by-product of our considerations. Our main goal is to gain some knowledge about dynamics of the point-like limit of the smooth string. Therefore, we interpret the additional degrees of freedom of objects described by Eq. (34) as some remnants of internal dynamics of the smooth string. The fact that the model $K(k) = k$ is tachyonic suggests that the rigid string is tachyonic already on the classical level, in contradistinction to Nambu-Goto string. Derivation of the point-like limit and an analysis of the mass spectrum of the classical smooth string will be presented in the forthcoming paper [14].

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