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THE PROBLEM OF EFFECTIVENESS OF CLASSICAL DIMENSIONAL REDUCTION MECHANISM IN HOMOGENEOUS ARBITRARY-DIMENSIONAL COSMOLOGY*

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Thermodynamical functions are determined for the bosonic gas distribution in the external gravitational field described by multidimensional cosmological models having the structure FRW$\times B^D$, where B^D is any D-dimensional compact space with the scalar curvature $R^{(D)}$. Universal asymptotics of this function are found for the following situations: 1) at high temperatures, with $\beta^2 R \ll 1$ and $\beta m_0 \ll 1$, 2) at low temperatures, with $\beta^2 R \gg 1$ and $\beta m_0 \gg 1$, where R is the scale factor of the physical space, and β is the reciprocal of temperature. It is shown that if $R^{(3)}/R^{(D)} \gg 1$ (where $R^{(3)}$ and $R^{(D)}$ are curvature scalars of macro- and microspace, correspondingly), the Casimir energy is always negative. These results are applied to discuss the dimensional reduction generated by the classical Einstein equations with quantum corrections. The idea of the dynamical dimensional reduction is expressed, in terms of the dynamical system theory, as the problem of the existence of a single stable critical point representing a configuration with the static internal space. It is demonstrated that, in the low-temperature approximation, there is no effective mechanism of the dimensional reduction to a static internal space, if B^D is a group manifold (with the same scale factor in all internal directions). On the other hand, the effective mechanism of the dimensional reduction to the zero size does exist. The existence of such mechanism for the full class of multidimensional homogeneous cosmologies with the hydrodynamic energy-momentum tensor is also discussed.

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Introduction

Multidimensional theories of the Universe, such as Kaluza–Klein theories, supergravity [1] or superstring theories [2], belong to the contemporary cosmological paradigm. It is a common feeling that they pave the way towards the unification of all physical forces, accompanied — as a necessary by-product — by the unification of physics and cosmology.

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Although all these theories present quite different pictures of the world, they share certain common features which completely change everything we were accustomed to when studying the very early Universe (among many other things also the problem of the chaotic behaviour in the asymptotic initial state, see for instance [4]). According to these theories, at the very early epochs of the world evolution the size of the physical space (macrospace) and that of the internal space (microspace) were comparable, and the internal space could substantially influence the dynamics of the Universe. The question of how the size of the microspace has been reduced to non-observable dimensions is known as the dimensional reduction problem. The idea of the purely dynamical dimensional reduction, i.e. the reduction done entirely by the Einstein dynamical equations, has been proposed by *Chodos and Detweiller* [5]. Many solutions of these equations are known [6] for which such a reduction occurs.

Main objections against the above approach are connected with the fact that the existence of solutions with the static microspace and expanding macrospace is a consequence of a special choice of the initial conditions, whereas a "correct" mechanism of the dimensional reduction should be independent of such a choice. Indeed, in the present work it will be shown that, within the class of multidimensional homogeneous world models being solutions to the Einstein field equations with the cosmological constant and hydrodynamic energy-momentum tensor, the set of those solutions which admit the static microspace is a "zero-measure" set in the space of all initial conditions for these cosmological models (Section 2, 3). On the other hand, the dimensional reduction took place at Planck-length scale and quantum effects were then important for cosmological evolution [7]. Therefore, conclusions drawn from purely classical equations of motion can be illusive. Quantum effects should probably be computed at finite temperatures if they are to be used within the cosmological context [8].

Main objections against the above approach are also connected with the fact of the non-existence of asymptotically stable configurations with the static microspace (Section 4).

Our analysis is based on the full clasification of homogeneous arbitrarily-dimensional cosmological models.

1. Einstein's equations for multidimensional homogeneous cosmological models

The assumption of homogeneity of the macrospace has an observational justification in cosmology. We assume, by analogy, that the D-dimensional internal space B is also a homogeneous but anisotropic space, i.e. it is a group manifold rather than only a coset manifold. In other words, the total space-time has $(D+3)$-dimensional spatial sections being orbits of simply transitive isometry groups.

First, we classify all $(D+3)$-dimensional isometry groups. (In Ref. [9] the classification of 10-dimensional groups has been performed). In the spirit of the Bianchi classification we enumerate all the relevant $(D+3)$-dimensional Lie algebras. The total space-time is a trivial principal bundle $P(M, G_D)$ with $P = M \times G_D$, where M is external physical space-time (of the Bianchi type), $G_D = B$ is the structure group. The bundle space P is a metric product space in the sense that: $g_P = g_M \otimes g_B$. The isometry group for $P = M_3$

TABLE I

Simple compact Lie algebra	Dimensions
SL (N, C)	$(N+1)(N-1)$
SO $(2N, \mathrm{C})$	$N(2N-1)$
SO $(2N+1, \mathrm{C})$	$N(2N+1)$
SP (N, C)	$N(2N+1)$
special algebra G_2	14
E_4	52
E_6	78
E_7	133
E_8	248

TABLE II

B_7	M_3 (the Bianchi type)									
	I	II	VI_0	VII_0	VIII	IX	V	IV	VI_h	VII_h
A	0	3	5	5	6	6	3	5	6	6
B	6	9	11	11	12	12	9	11	12	12
C	12	15	17	17	18	18	15	17	18	18

$A = \bigoplus_{i=1}^{7} L_1^i$, $B = \bigoplus_{i=1}^{4} L_1^i \oplus \mathrm{B(IX)}$, $C = L_1 \oplus \mathrm{B(IX)} \oplus \mathrm{B(IX)}$, $L_3(\mathrm{IX})$ — 3 dim algebra $\mathrm{B(IX)}$, $L_3(\mathrm{IX}) \simeq \mathrm{SO}(3)$ algebra, L_1 — 1 dim abelian algebra.

$\times G_D$ is a direct product $G_{D+3} = G_3 \otimes G_D$ of the standard Bianchi isometry group G_3 and G_D. Therefore for its Lie algebra $\mathscr{L}_{D+3}$, one has $\mathscr{L}_{D+3} = L_3 \oplus L_{D+3}$; the problem of classifying all the $\mathscr{L}_{D+3}$ is thereby reduced to enumerating all relevant Lie algebras $\mathscr{L}_D$.

We assume, for physical reasons, that G_D is a compact Riemannian space, and consequently we classify all distinct real D-dimensional compact Lie algebras.

By using the theorem on decomposing compact Lie algebras into the sum of simple Lie algebras and the centre, we are able to generalize the classification given by Demiański et al. (in Ref. [9]).

The classification of compact simple D-dimensional real Lie algebras is known (for example [10]). Table I contains all simple real forms of the Lie algebras which can be used as algebras of the isometry group of a D-dimensional compact internal space. By forming direct sums of simple compact Lie algebras and of Abelian ones we obtain all possible algebras generating algebras of isometry groups of the internal space B^D (the total dimension must be D because the group of isometry is acting simply transitively on B^D [11]. Since these Lie algebras have the structure of direct sums, one can easily determine the set of equivalence classes of the structure constants [12]. The case $D = 7$, for example, is shown in Table II. The types: VI_h, VII_h, VIII, and IX with the microspace of the form $\mathrm{B(IX)} \times \mathrm{B(IX)} \times S^1$, have the highest dimensions, i.e. any open subset in the space of the initial data must have a non-empty intersection with the one of the above distinguished four types.

From Table II, one can also see that the world models with the microspace of the torus type (which are most often analysed in the literature) form a "zero-measure" set (in the Collins-Hawking sense) within the class of all multidimensional homogeneous world models (for details see [12]).

One can make use of the tetrad formalism [13] and take into consideration the metric of the form

$$ds^2 = dt^2 - g_{ij}(t)e^i_a(x)e^j_b(x)dx^a dx^b, \tag{1}$$

where g_{ij} are functions of the cosmological time t only, and $e^i_a(x)$, $i, j = 1, 2, 3, \ldots, D+3$ are the basis vectors. We will assume the Einstein equations in the form

$$R_{\mu\nu} = T_{\mu\nu} - \frac{1}{D+2} g_{\mu\nu} T - \frac{2}{D+2} \Lambda, \tag{2}$$

where

$$T_{\mu\nu} = (\varrho + p)u_\mu u_\nu - p g_{\mu\nu} \tag{3}$$

is the hydrodynamic energy-momentum tensor, ϱ the energy density, p pressure, and Λ the $(D+3)$-dimensional cosmological constant; $\alpha, \beta, \mu, \nu = 0, 1, \ldots, D+3$. The components of the Ricci tensor for the metric (1) are

$$R^0_0 = -\tfrac{1}{2}\dot{\kappa}^\alpha_\alpha - \tfrac{1}{4}\kappa^\beta_\alpha \kappa^\alpha_\beta, \tag{4}$$

$$R^0_i = -\tfrac{1}{2}\kappa^j_k(C^k_{ij} - \delta^k_i C^l_{lj}), \tag{5a}$$

$$R^j_i = -P^j_i - \frac{1}{2\sqrt{|g|}}\frac{d}{dt}(\sqrt{|g|}\,\kappa^j_i), \tag{5b}$$

$$\kappa^j_k = \dot{g}_{ki}g^{ij}, \qquad |g| = |\det(g_{ij})|,$$

$$P_{ij} = -\Gamma^k_{il}\Gamma^l_{jk} - C^l_{lk}\Gamma^k_{ij},$$

$$\Gamma^k_{ij} = \tfrac{1}{2}(C^k_{ij} + C^m_{il}g_{mj}g^{kl} + C^m_{jl}g_{mi}g^{kl}),$$

where P_{ij} is the Ricci curvature tensor expressed in terms of the structure constans; the dot denotes differentiation with respect to the cosmological time t.

For a compact Lie algebra $\mathscr{L}$ the structure constants C^l_{rs} may be represented with the help of a third-order totally antisymmetric covariant tensor. Indeed, if we use the metric tensor g_{tl} in $\mathscr{L}$ for lowering indices of contravariant tensors, then the tensor $C_{rsl} = C^t_{rs}g_{tl}$*, by virtue of equations $g_{ls} = C^i_{lk}C^k_s$ (Cartan metric tensor), may be written in the form:

$$C_{rsl} = C^t_{sm}C^n_{rt}C^m_{ln} + C^t_{mr}C^n_{ts}C^m_{nl}.$$

The last expresion is invariant under the cyclic permutation of the indices and is skew symmetric in r and s, hence the tensor C_{rsl} is totally antisymmetric. On the other hand, for a compact Lie algebra $\mathscr{L}$, the Cartan metric tensor may be assumed to be in the form

$g_{tl} = \delta_{tl}$; hence $C_{rsl} = C^l_{rs}$, i.e. the structure constants C^l_{rs} and the components C_{rsl} of tensor $*$ coincide [10].

The above properties enable us to construct the Ricci curvature tensor of a constant time-space. For the metric (1) we can write the components of the Ricci tensor as

$$P^i_i = \tfrac{1}{2} \sum_{j,k} (C_{ijk})^2 (A_i A_j A_k)^{-2} \{A_i^4 - (A_j^2 - A_k^2)^2\}, \tag{6}$$

where C_{ijk} are the structure constants of the Lie algebra of the respective isometry group, and

$$g_{ij}(t) = \mathrm{diag}\,(A_1^2, \ldots, A_{D+3}^2), \tag{7}$$

where A_i ($i = 1, \ldots, D+3$) are the scale factors of the macro- ($i = 1, 2, 3$) and micro- ($i = 4, \ldots, D+3$) space, respectively.

By generalizing the standard reduction procedure of a metric to the diagonal form, for the classical Bianchi types expressed in terms of the group of inner automorphism preserving commutation relations (see e.g. [11]), one can show that for multidimensional homogeneous world models with the type A macrospace metrics (together with their first derivatives) can always be reduced to the diagonal form (7) at any time instant t_0. The Einstein equations transfer this property to any other time instant. In such a case the components $(0, i)$ of the Einstein equations are identically satisfied. For the considered world models, (having the macrospace of type B) the metric can be reduced to the form, having one non-diagonal component $g_{12}(t)$, with the help of the group of inner automorphisms. For such a case, the components $(0, i)$ of the Einstein equations give the additional condition $R^0_3 = 0$. In the exceptional case, when the macrospace is of B(V) type, the metric can be reduced to the diagonal form with the additional constraint $R^0_3 = 0$. In the following, we shall assume the diagonal form of the metric (7).

Dimension 11 is distinguished by a realistic supersymmetric version of Kaluza–Klein theories based on the gauge group $SU(3) \times SU(2) \times U(1)$. In this case, $N = 1+3+7$ (N is space-time dimension) and the Einstein equations (2) assume the following form:

$$\sum_{i=1}^{10} \frac{\ddot{A}_i}{A_i} = -\frac{8\varrho + 10p}{9} + \tfrac{2}{9}\Lambda,$$

$$\left(\frac{\dot{A}_i}{A_i}\right)^{\cdot} + \frac{\dot{A}_i}{A_i}\left(\sum_{i=1}^{3} \frac{\dot{A}_i}{A_i} + \sum_{j=4}^{10} \frac{\dot{a}_j}{a_j}\right) + P^i_i = \frac{\varrho - p}{9} + \tfrac{2}{9}\Lambda, \quad (i = 1, 2, 3),$$

$$\left(\frac{\dot{a}_j}{a_j}\right)^{\cdot} + \frac{\dot{a}_j}{a_j}\left(\sum_{i=1}^{3} \frac{\dot{A}_i}{A_i} + \sum_{j=4}^{10} \frac{\dot{a}_j}{a_j}\right) + Q^j_j = \frac{\varrho - p}{9} + \tfrac{2}{9}\Lambda, \quad (j = 4, 5, 6),$$

$$\left(\frac{\dot{a}_k}{a_k}\right)^{\cdot} + \frac{\dot{a}_k}{a_k}\left(\sum_{i=1}^{3} \frac{\dot{A}_i}{A_i} + \sum_{j=4}^{10} \frac{\dot{a}_j}{a_j}\right) + R^k_k = \frac{\varrho - p}{9} + \tfrac{2}{9}\Lambda, \quad (k = 7, 8, 9),$$

$$\left(\frac{\dot{a}_{10}}{a_{10}}\right)^{\cdot} + \frac{\dot{a}_{10}}{a_{10}}\left(\sum_{i=1}^{3}\frac{\dot{A}_i}{A_i} + \sum_{j=4}^{10}\frac{\dot{a}_j}{a_j}\right) = \frac{\varrho - p}{9} + \tfrac{2}{9}\Lambda,$$

$$-2R_3^0 = a(\kappa_1^1 + \kappa_2^2 - 2\kappa_3^3) + n_1\kappa_1^2 - n_2\kappa_2^1 = 0, \tag{8}$$

where n_1, n_2 are eigenvalues of the symmetric matrix n^{ab} such that we can write $n^{ab} = n^{(a)}\,\delta^{ab}$ for the standard decomposition of the structure constants $C^a_{bc} = \varepsilon_{abc}n^{ad} + \delta^a_b a_a + \delta^a_b a_c$ (for details, see [12]), and they refer to the macrospace. P^i_i, Q^j_j, R^k_k are the Ricci tensors of the macro- and microspace, respectively, (Q^j_j, R^k_k are either always null, for B(I) microspace sector or have the B(IX) form), $A_j = a_j$ ($j = 4, \ldots, 10$).

From equation (8), one can see that the solutions with the static microspace $\dot{A}_4 = \ldots = \dot{A}_{10} = 0$, in the case of $(1+3+7)$ dimension are admissible only if $Q^j_j = R^k_k = 0$ and $\Lambda = 0$, $p = \varrho$ (massless scalar field). If $p = \varrho = 0$ a transition to the vacuum case takes place. We can formulate:

Conclusion 1. Within the class of homogeneous 11-dimensional world models the only general solutions with the static microspace, are the following ones: (Bianchi type) $\times\, T^7$. They form the "zero-measure" set within the considered class of models (Table II).

The conditions stated above are, of course, the necessary but not sufficient ones for the existence of solutions with a static microspace. In the work by Demiański et al. [13], the case $\mathrm{B(V)} \times \mathrm{T}^7$, admitting asymptotic solutions with the static microspace, was investigated. It will be demonstrated that, within the class of $\mathrm{B(V)} \times \mathrm{T}^D$ type solutions, the set of models, which admit the static microspace is of a non-zero measure.

Conclusion 1 is also valid in the case when the dimension of space-time is: 5, 6, 8, or 9 (from Table II, we can see that the S^1 sector is present in this case). In this case, the only solutions having static microspace are the following ones: (Bianchi type) $\times\, \mathrm{T}^D$.

If space-time dimension is 7 or 10 we can formulate the following conclusion:

Conclusion 2. Within the class of homogeneous 7 or 10-dimensional world models, the only solutions having the static microspace are the solutions: (Bianchi type) $\times\, \mathrm{S}^3$, (Bianchi type) $\times\, \mathrm{S}^3 \times \mathrm{S}^3$, where S^3 is a maximally symmetric space. These models are either sourceless or with an energy-momentum tensor describing a massless scalar field ($p = \varrho$).

From Eq. (8), one can see that, in general, solutions with the static microspace, $\dot{a}_4 = \dot{a}_5 = \dot{a}_6 = \dot{a}_7 = \dot{a}_8 = \dot{a}_9 = 0$, are admisible (for the case $1+3+6$ dimension only if $Q^j_j = R^k_k = \frac{2\Lambda}{D+2}\,\delta^k_k\left(R^k_k = \frac{D-1}{a^2}\,\delta^k_k\right)$ and $p = \varrho$). In the next Section it will be demonstrated that, within the class of (Bianchi type) $\times\, \mathrm{S}^3 \times \mathrm{S}^3$ solutions, those models which admit an asymptotically static microspace form a "zero-measure" set in the full class of solutions.

Now, we shall investigate the existence of solutions with a static microspace in the general case: (Bianchi type) $\times\, B^D$. In this case, the dynamics is described by equations (4) and (5) with curvature tensor (6). Although system of Eqs (4) and (5), with the curvature tensor (6), does not behave chaotically near the singularity [11, 14], the type of solutions in general depends on the space symmetry and it is hard to give a general solution. However for our purposes, it is sufficient to investigate the system in the physically motivated approximation: $a_4, \ldots, a_{D+3} = a$. In such a case the solutions with the static misrospace are admitted if $p = \varrho$, and one has

$$\sum_{\substack{(i,j,k)\\ \text{no sum over } i}} (C_{ijk})^2 = \beta \quad \text{for all } i = 4, \ldots, D+3.$$

The radius of the static microspace is equal to $a = \sqrt{\dfrac{\beta(D+2)}{2\Lambda}}$. System of equations (4) and (5) with curvature tensor (6) is then reduced to the following form

$$\sum_{i=1}^{3} \frac{\ddot{A}_i}{A_i} + D\frac{\ddot{a}}{a} = \frac{2\Lambda}{D+2},$$

$$\left(\frac{\dot{A}_i}{A_i}\right)^{\cdot} + \frac{\dot{A}_i}{A_i}\left(\sum_{i=1}^{3} \frac{\dot{A}_i}{A_i} + D\frac{\dot{a}}{a}\right) + P^i_i = \frac{2\Lambda}{D+2},$$

$$\left(\frac{\dot{a}}{a}\right)^{\cdot} + \frac{\dot{a}}{a}\left(\sum_{i=1}^{3} \frac{\dot{A}_i}{A_i} + D\frac{\dot{a}}{a}\right) + \frac{\beta}{a^2} = \frac{2\Lambda}{D+2}. \tag{9}$$

Although equations (9) have been, for simplicity, written for the vacuum case ($p = \varrho = 0$), our conclusions remain valid that only a "zero-measure" set of trajectories, being solutions to the system (9) with a hydrodynamical energy-momentum tensor, leads to a static internal space. If $\beta = D-1$, we obtain the previously discussed case.

2. Asymptotic stability of classical solutions with the static microspace

2.1. The method of the dynamical system stability

First of all, equations describing a cosmological model should be reduced to the form of a dynamical system: $\dot{x}_i = f_i(x_1, \ldots, x_n)$ in such a way that solutions with a static microspace (with some other property of interest) should be critical points of the system, i.e. all $f_i(x_1, \ldots, x_n) = 0$, $(i = 1, \ldots, n)$, and $(\overset{\bullet}{x}_1, \ldots, \overset{\bullet}{x}_n) = \overset{\bullet}{P}$ should be a critical point. As it is well known, such points represent asymptotic states of system [16]. If a critical point $\overset{\bullet}{x} = (x_1, \ldots, x_n)$ is non degenerate, i.e. if at this point all real parts of the eigenvalues ($\mathrm{Re}\,\lambda_i$) of the linearization matrix $A^i_j = \left.\dfrac{\partial f_i}{\partial x_j}\right|_{x=\overset{\bullet}{x}}$ do not vanish, then there exists a one-to-one continuous mapping of a neighbourhood of this point which transforms trajectories of the original system into a linearized one. In this sense, the qualitative behaviour of the original system is equivalent to the behaviour of its linearized part. If $(\xi^1_i), \ldots, (\xi^n_i)$ are eigenvectors of the linearization matrix A^i_j, the solution of the linearized system has, in general, the following form:

$$x_i(t)-\overset{\bullet}{x}_i = \operatorname{Re} \sum_{k=1} C_k \xi_i^k e^{\lambda_k t},$$

where C_k are constants. A non-degenerate critical point is called the attracting point if, for all eigenvalues, $\operatorname{Re} \lambda_i < 0$. In this case, all trajectories from the neighbourhood of this point go to this point if $t \to \infty$. A non-degenerate critical point is said to be repulsing point if, for all eigenvalues, $\operatorname{Re} \lambda_i > 0$.

In this case, all trajectories from the neighbourhood of the point $\overset{\bullet}{x}$ go to it if $t \to -\infty$. A non-degenerate critical point is said to be non-stable saddle point if the dynamical system has, at $\overset{\bullet}{x}$, negative eigenvalues $\alpha_1 \leqslant \ldots \leqslant \alpha_d < 0$, $\operatorname{Re} \lambda_i = \alpha_i$ and $n-d$ eigenvalues with positive real parts.

When investigating stability of solutions with a static microspace, the following theorem proves to be of special interest.

If $\overset{\bullet}{x}$ is a non-degenerate critical point and if the dynamical system has, at $\overset{\bullet}{x}$, d eigenvalues $\lambda_1, \ldots, \lambda_d$ with negative real parts $\alpha_1 \leqslant \ldots \leqslant \alpha_d < 0$, then there exist (locally) an invariant d-dimensional manifold W^d_{atr}, on which all trajectories of the system go to $\overset{\bullet}{x}$ as $t \to \infty$. A manifold M is said to be an invariant manifold of the system if every trajectory passing through a non-degenerated point of M lies entirely in M (for $-\infty < t < +\infty$). For every such solution there exists the asymptotic

$$\lim_{t\to\infty} t^{-1} \ln \left[\left(\sum_{j=1}^{n} (x_j(t)-\overset{\bullet}{x}_j)^2\right)^{1/2}\right] = \alpha_i \tag{10}$$

for a certain i. Analogously, if at certain point $\overset{\bullet}{x}$ the system has k eigenvalues with positive real parts, then there exists an invariant k-dimensional manifold W^k_{rep}, on which all trajectories emanate from $\overset{\bullet}{x}$ [16].

From the last theorem it follows that, for a saddle point, there are two invariant manifolds W^d_{atr} and W^{n-d}_{rep} containing this point and filled with trajectories (separatrices) going to, and emanating from this point. These manifolds are said to be stable and non--stable manifolds, respectively. All other trajectories (not contained in W^d_{atr} or in W^{n-d}_{rep}) do not meet the critical point in question.

2.2. The stability of solutions with the static microspace

2.2.1. Stability of solutions with the static microspace within the class of $B(V)\times T^D$ models

By using the last equation of (8), one obtains: $A_1(t) = A(t)$; $A_2(t) = A(t)\cdot S(t)$; $A_3(t) = a\cdot A(t)\cdot S^{-1}(t)$ (for the B(V) model all n_i are zero), where $S(t)$ is an unknown function of t.

Equations (8), for sourceless case or for a massless scalar field, can be reduced to the form of the dynamical system:

$$\dot{H} = -3H^2 - H(h_4 + \ldots + h_{D+3}) + \frac{2}{R^2},$$

$$\dot{\phi} = -\phi(3H + h_4 + \ldots + h_{D+3}),$$

$$\dot{h}_4 = -h_4(3H+h_4+ \dots +h_{D+3}),$$
$$\vdots$$
$$\dot{h}_{D+3} = -h_{D+3}(3H+h_4+ \dots +h_{D+3}), \tag{11}$$

where $H = \dot{A}/A$, $\phi = \dot{S}/S$, $h_j = \dot{a}_j/a_j$, $j = 4, \dots, D+3$.

By introducing the new variables $U = H \cdot R$, $V = \phi \cdot R$, $y_j = h_j \cdot R$, the above set of equations can be given the form of the autonomous dynamical system:

$$U' = -2U^2 - Uy_4 - Uy_5 - \dots - Uy_{D+3} + 2,$$
$$V' = -2UV - Vy_4 - Vy_5 - \dots - Vy_{D+3},$$
$$y_4' = -2y_4U - y_4^2 - y_4y_5 - \dots - y_4y_{D+3},$$
$$\vdots$$
$$y_{D+3}' = -2y_{D+3}U - y_{D+3}y_4 - \dots - y_{D+3}^2, \tag{12}$$

where prime denotes differentiation with respect to the new time variable τ, $d\tau = dt/R$.

System (9) has, in a finite region, the only critical point $(\mathring{P}) = (\mathring{y}_4 = \dots = \mathring{y}_{D+3} = 0$, $\mathring{v} = 0$, $\mathring{u} = \pm 1)$. It represents the solution with the static microspace and the isotropic Friedman $(k = -1)$ macrospace. System (12) is determined on a region given by the (0, 0) component of the Einstein equations:

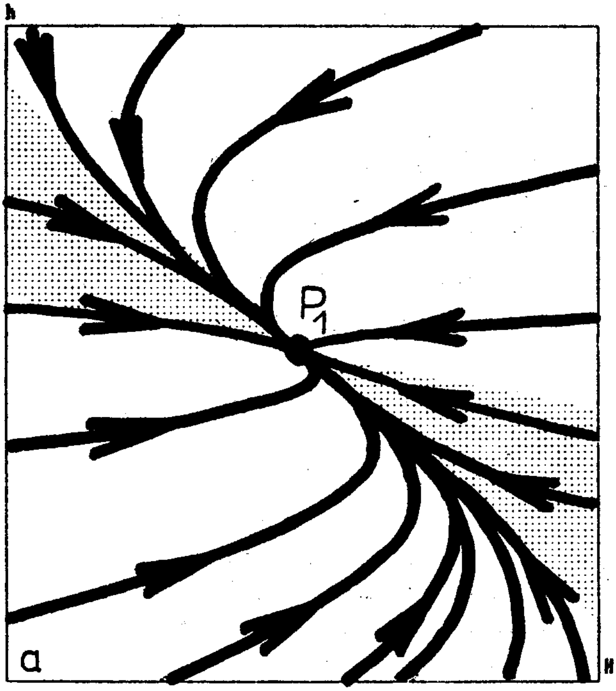
$$2\varrho A^2 = 6U^2 - 2V^2 - 6 + \Big(\sum_{j=4}^{D+3} y_j\Big)^2 - \sum_{j=4}^{D+3} y_j^2 \geq 0. \tag{13}$$

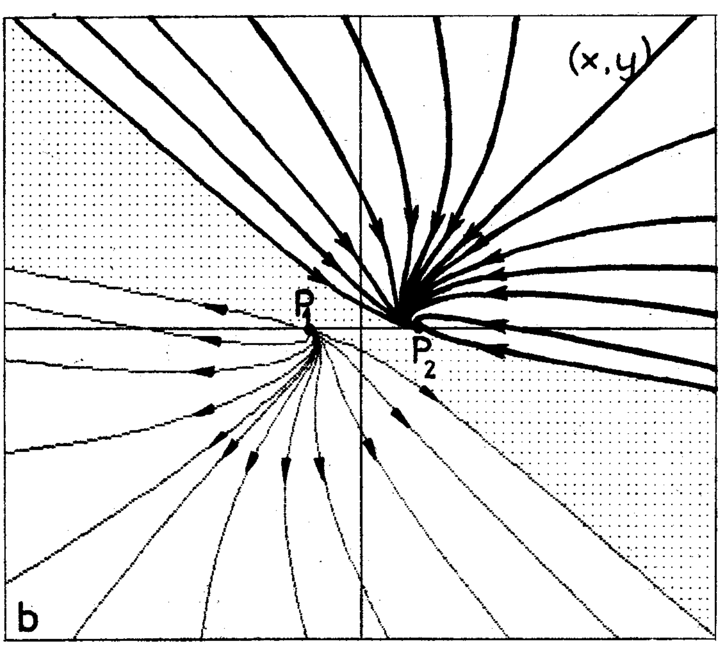
If $p = \varrho = 0$ (sourceless case), one can easily verify that the critical point, representing solutions with a static microspace and an isotropic macrospace with the Milne-like evolution: $R \propto t$, is situated on the boundary of "constraint condition" (13). The phase portrait, for the case of {FRW with $k = -1$} $\times T^7$ with the identical scale factors, is shown in Fig. 1. Now, we will demonstrate the stability of the critical point $\mathring{P}$. At this point, system (9) has real negative eigenvalues of the linearization matrix, i.e.

$$\lambda_1 = -4\mathring{u}, \quad \lambda_2 = -2\mathring{u}, \dots, \quad \lambda_{D+2} = -2\mathring{u}. \tag{14}$$

This means that $\mathring{P}$ is an attractive point (for $\mathring{u} > 0$). One can see that, in our case, there exists $(D+2)$-dimensional invariant manifold in the $(D+2)$-dimensional phase space. In other words, there exists a non-zero measure set of trajectories for which the solution with the static microspace is an attractor. It can be also shown that this property is characteristic for the macrospace with Ellis-Mac Callum metric [15, 16] and with $n_a^a = 0$.

Cosmologies resulting from the bosonic sector of the $N = 1$, $D = 10$ supergravity theory, or equivalently from the theory with the Chapline-Manton action and vanishing Yang-Mills field, have been studied in the literature [17]. If one assumes that fields F_{NPQ} vanish and dilaton field is homogeneous: $\phi = \phi(t)$ (see Gleiser, Stein-Schabes [17]) one obtains solutions with the static microspace, for $\{B(V) \times T^6\}$ models. This solution is asymptotically stable in the full class of solutions. Fig. 2 shows the phase portrait for the FRW space-time (with $k = -1$) $\times T^6$ (identical microspace scale factors) with the dilaton


h
P1
a
H


(x,y)
P1
P2
b

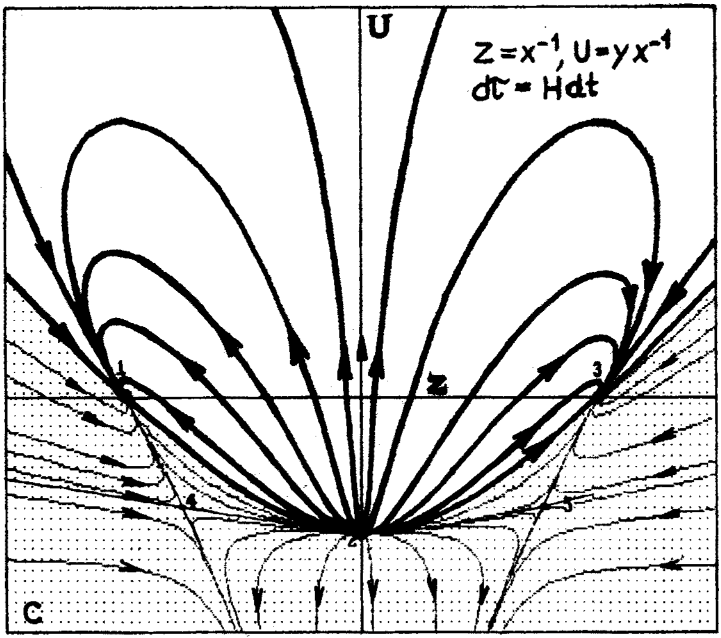


Fig. 1. The phase portrait for space-time FRW (with $k = -1) \times T^7$ with the identical scale factors of the microspace in the vacuum (a) and with radiation (b); $H = \dot{A}/A$, $h = \dot{a}/a$, $x = HA$, $y = hA$ are Hubble's functions in the macro- and microspace, respectively. The shadowed region is excluded by the constraint condition. The critical point $P_1(1, 0)$ representing the asymptotic state of vacuum solutions is an atracting point. The behaviour of trajectories of the dynamical system FRW (with $k = -1) \times T^7$ with radiation, in neighbourhood of the singularity, is represented in (c). To investigate the system at infinity, the projective coordinates turn out to be useful ($z = 1/x$, $u = y/x$). Typical behaviour of the system near the singularity is represented by the repulsing point P_1

field. Critical point P_1 represents the asymptotic state of the classical (sourceless) solutions. It is an attracting point because the macrospace is expanding. If the system is initialy in the neighbourhood of this critical point there is a non-zero measure set of trajectories for which it is an attractor ($\dim W^{\mathrm{Atr}} = D-2$).

2.2.2. Stability of solutions with the static microspace within the class of (Bianchi type) $\times B^D$ models

For simplicity, we assume that $A_1, A_2, A_3 \gg a$ so that the curvature of physical space can be neglected. However, the conclusion remains valid for the case when the physical space is a generalization of Bianchi V model to the model of Bianchi VI_h with the Ellis-Mac Callum metric and $n^a_a = 0$ [15]. In this case equation (9) can be reduced to the form of the autonomous dynamical system

$$x' = -\frac{3D}{D+2}x^2 + xy\,\frac{-D^2+5D+2}{D+2} + \frac{D(D-1)}{D+2}y^2 + \frac{D}{D+2},$$

$$y' = \frac{6}{D+2}x^2 - \frac{2(D-1)}{D+2}y^2 + \frac{3(D-2)xy}{D+2} - \frac{2}{D+2}, \tag{15}$$

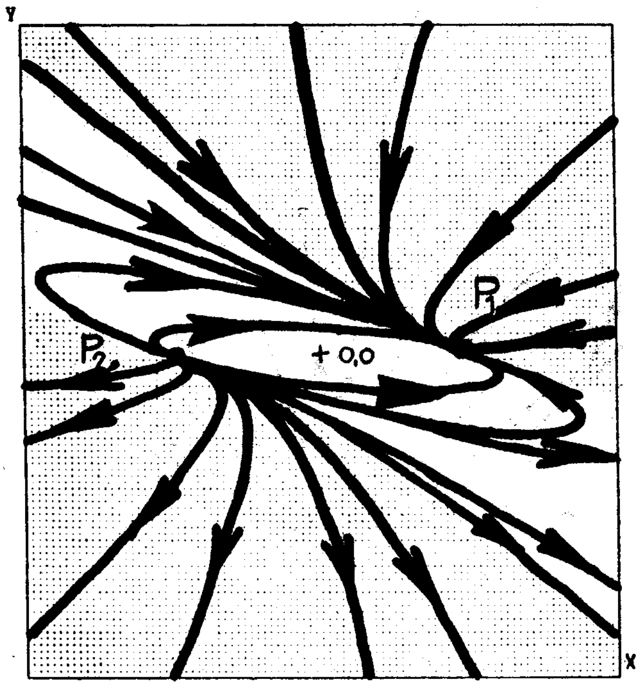


Fig. 2. The phase portrait for FRW space-time (with $k = -1) \times T^6$ (scale factors of the microspace are identical) with the dilaton field in $N = 1$, $d = 10$ theory of supergravity. The critical point $P_1(1, 0)$ represents the asymptotic state of the classical (sourceless) solutions. It is an atracting point $x = HA$, $y = hA$, where H and h are Hubble's functions of the macro- and microspace, respectively; A represents the scale factors factors of the macrospace

with the constraint condition

$$\Lambda a^2 = 3x^2 + 3Dxy + \frac{D(D-1)}{2} y^2 + \frac{D}{2} > 0, \tag{16}$$

where $x = \frac{\mathring{A}}{A} a$, $\dot{y} = a$, and prime denotes the differentiation with respect to the parameter τ: $d\tau = dt/a$; if it is assumed that the radius of the microspace is now equal to $\sqrt{\frac{D-1}{\beta}}\, a$. Dynamical system (15) has the critical point P: $x_0 = \pm\sqrt{3}/3 \left(H_0 = \pm\sqrt{\frac{2\Lambda}{3(D+2)}}\right)$, $y_0 = 0 \left(a = \sqrt{\frac{(D-1)(D+2)}{2\beta\Lambda}}\right)$, which represents the solution with the static microspace and inflationary phase on the macrospace. This critical point is a saddle point, i.e. if the system is in the neighbourhood of this point then only a zero-measure set of trajectories leads to P (dim $W^{\mathrm{Atr}} = 1$), see Fig. 3a.

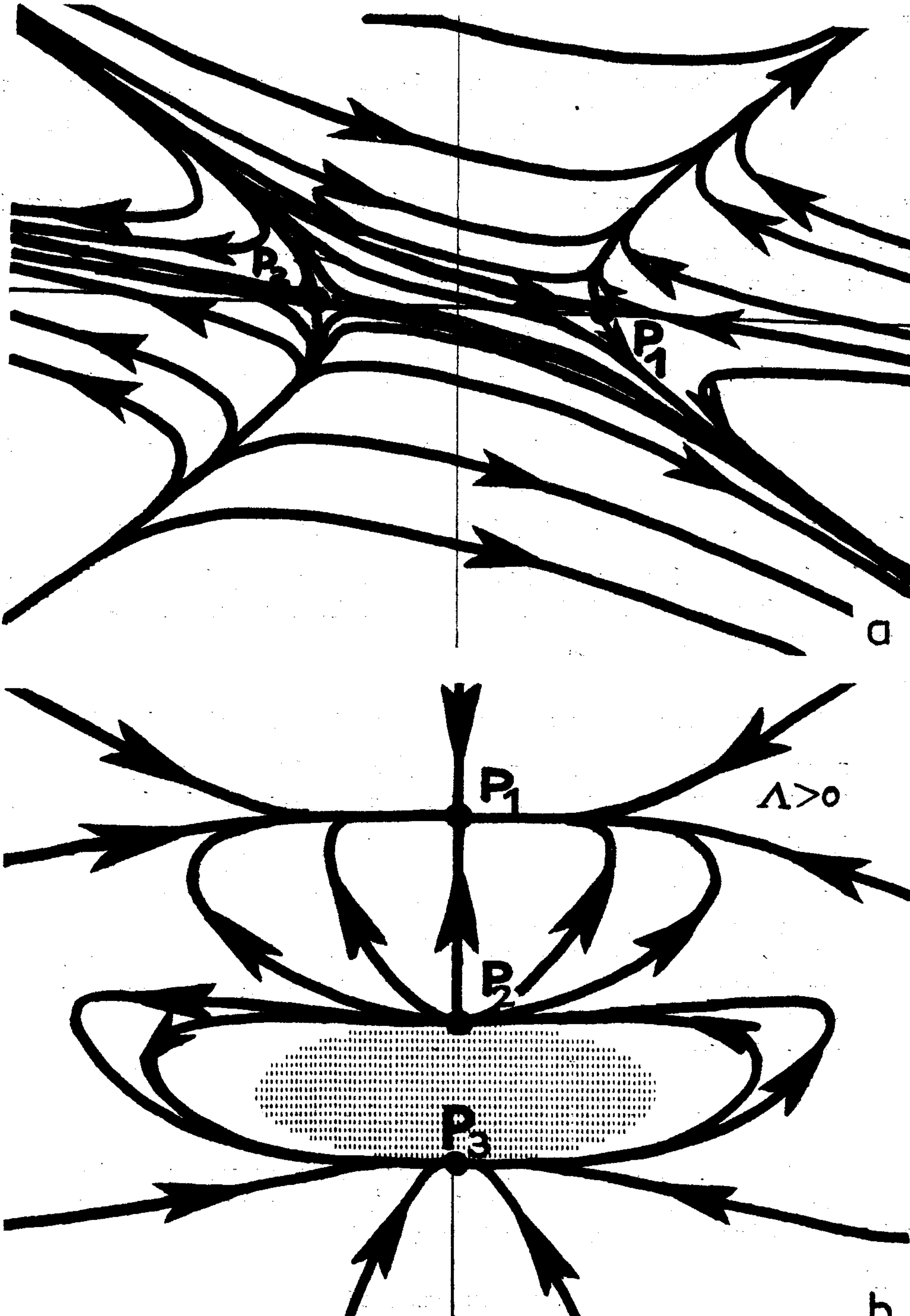


Fig. 3. The phase portrait of FRW space-time (with $k = 0) \times B^D$ (scale factors of the microspace are identical). The critical point P_1 (representing a solution with the static microspace) is a nonstable saddle point (a), (b) shows the behaviour of the system near the singularity in projective coordinates $z = 1/x$, $u = y/x$. The typical behaviour near the singularity is represented by the repulsing critical point P_2

In Fig. 3b the behaviour of the system near the singularity is shown. For this purpose, we introduce, as previously (Fig. 1c), the following projective coordinates (z, u): $\left(z = \frac{1}{x}, u = \frac{y}{x}\right)$, $\left(w = \frac{1}{y}, v = \frac{x}{y}\right)$. In these coordinates, the sphere S^1 is covered by two straight lines which correspond to points (x, y) at infinity; $z = 0$, $-\infty < u < \infty$; $w = 0$, $-\infty < v < \infty$ [16]. In the coordinates (z, u) system (15) takes the form

$$\frac{dz}{d\tau_1} = \frac{3D}{D+2} z - uz \frac{5D+2-D^2}{D+2} - \frac{D(D-1)}{D+2} u^2 z - \frac{D}{D+2} z^3,$$

$$\frac{du}{d\tau_1} = \frac{6}{D+2} - \frac{D(D-7)}{D+2} u^2 + \frac{6(D-1)}{D+2} u - \frac{2}{D+2} z^2 - \frac{D(D-1)}{D+2} u^3 \tag{17}$$

in the region $3+3Du+\frac{D(D-1)}{2}u^2+\frac{D}{2}z^2 \geqslant 0$, and where $d\tau_1 = z^{-1}d\tau$. From Fig. 3b we see, that the typical behaviour near the singularity (represented by the repulsing point P_2) corresponds to the situation in which the dimensions of the micro- and macrospace are not compatible (as Fig. 1c).

3. *Quantum effects in homogeneous multidimensional cosmological models*

Quantum vacuum energy can essentially influence both the cosmological evolution or the stability of compactified solutions [18]. Let us now consider the thermal bosonic gas in external gravitational field of multidimensional cosmological models.

For simplicity we shall assume one-loop quantum effects arsing from scalar particles contained in multidimensional cosmological models. We shall determine quantum distribution function in case when scale factors are slowly variable in time (quasistatic approximation). Let the background space have the $M^3 \times B^D$ structure, where M^3 is a maximally symmetric space, whereas B^D is an arbitrary D-dimensional compact space with the curvature scalar ${}^DR$.

The action for the system under consideration is

$$S = S_g + \tfrac{1}{2}\int d^N x \sqrt{-g}\,(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m_0^2 + \zeta R\phi^2), \tag{18}$$

where R is curvature scalar of the total space, ζ the coupling parameter, m_0 the effective mass. We shall quantize the scalar field only, whereas the gravitational field remains classical (for the review of problems connected with quantum field theory in a curved space-time see Birrel-Davies [19]).

We shall determine quantum distribution function by using the ζ-function regularization method

$$\ln Z^Q(\beta) = -\tfrac{1}{2}\left(\frac{d}{ds}\zeta(s)|_{s=0} + \ln \mu^2\zeta(0)\right), \tag{19}$$

where

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \operatorname{tr} \exp[-t\Delta].$$

The operator Δ for the metric $g_{M^3} \otimes g_{B^D}$, after the Wick rotation $t \to -i\tau$ assumes the form

$$\Delta = -\frac{\partial^2}{\partial \tau^2} + \Delta^{(3)}_{\mathrm{L.B.}} + \Delta^{(D)}_{\mathrm{L.B.}} + m^2, \tag{20}$$

where $\Delta^{(3)}_{\mathrm{L.B.}}$ is the Laplace–Beltrami operator on the sphere (pseudosphere) S^3 of a unit radius, $m^2 = m_0^2 + \zeta R$ is the effective mass, $\Delta^{(D)}_{\mathrm{L.B.}}$ is the Laplace–Beltrami operator on D-dimensional compact manifold

$$\Delta_{\mathrm{L.B.}} = -\frac{1}{\sqrt{^{(D)}g}} \partial_A(\sqrt{^{(D)}g}\, g^{AB} \partial_B), \tag{21}$$

where $^{(D)}g = \det {}^{(D)}g_{AB}$.

We shall discuss cases when M^3 is a sphere S^3 or a pseudosphere PS^3, or R^3 or $T^3 = S^1 \times S^1 \times S^1$ whereas B^D is D-dimensional compact space with the curvature scalar $^{(D)}R$ (not necessary homogeneous).

In order to determine the function $\zeta(s, \beta)$ we must know the spectrum and degeneracies of the operators $\Delta^{(3)}_{\mathrm{L.B.}}$ and $\Delta^{(D)}_{\mathrm{L.B.}}$. In the case when the physical space is a sphere S^3, the spectrum and degeneracy of operators $\Delta^{(3)}_{\mathrm{L.B.}}$ and $\Delta^{(D)}_{\mathrm{L.B.}}$ are equal to $l(l+2)$ and $(l+1)^2$ respectively, where $l = 0, 1, 2, \ldots$ (the respective eigenfunctions are spherical harmonics on S^3). In this case the function $\zeta(s, \beta)$ has the following form

$$\zeta(s, \beta) = \frac{1}{\Gamma(s)} \sum_{n=-\infty}^{\infty} \sum_{\lambda_D} d_{\lambda_D} (1+1)^2 \int_0^\infty dt t^{s-1} e^{-t\left[\left(\frac{2\pi}{\beta} n\right)^2 + \frac{l(l+2)}{R^2} + \lambda_D + m^2\right]}, \tag{22}$$

where

$$m^2 = m_0^2 + \zeta\left(\frac{6}{R^2} + {}^{(D)}R\right).$$

If the physical space is a flat $\mathbf{R}^3$ one, the spectrum of the operator $\Delta^{(3)}_{\mathrm{L.B.}}$ is $(k_1^2 + k_2^2 + k_3^2)$ with the degeneracy equal to 1; $k \in (-\infty, \infty)$ (respective eigenfunctions are plane waves in $\mathbf{R}^3$). The function $\zeta(s, \beta)$ takes the form

$$\zeta(s, \beta) = \frac{1}{\Gamma(s)} \sum_{n=-\infty}^{+\infty} \sum_{\lambda_D} d_{\lambda_D} \frac{V_3}{(2\pi)^3} \int d^3k \int_0^\infty dt t^{s-1} e^{-t\left[\left(\frac{2\pi n}{\beta}\right)^2 + (\vec{k})^2 + \lambda_D + m^2\right]}, \tag{23}$$

where V_3 is the volume of $\mathbf{R}^3$, $(\vec{k})^2 = k_1^2+k_2^2+k_3^2$, $m^2 = \zeta^{(D)}R+m_0^2$. By using the fact that $\int d^3ke^{-t\vec{k}^2} = \frac{\pi^{3/2}}{t^{3/2}}$ we can write (23) in the form

$$\zeta(s, \beta) = \frac{\pi^{3/2}V_3}{(2\pi)^3\Gamma(s)} \sum_{n=-\infty}^{+\infty} \sum_{\lambda_D} d_{\lambda_D} \int_0^\infty dt \cdot t^{s-\frac{5}{2}} e^{-t\left[\left(\frac{2\pi n}{\beta}\right)^2+\lambda_D+m^2\right]}. \tag{24a}$$

In the case when the physical space is a three-torus $T^3 = S^1 \times S^1 \times S^1$, the spectrum of the operator $\Delta^{(3)}_{\mathrm{L.B.}}$ is the following:

$$\left(\frac{2\pi}{R_1}\right)^2 n_1^2+\left(\frac{2\pi}{R_2}\right)^2 n_2^2+\left(\frac{2\pi}{R_3}\right)^2 n_3^2,$$

where $n_l = 0, \pm 1, \ldots$, R_i $(i = 1, 2, 3)$ are radii of respective spheres S^1. In this case the degeneracy is equal to 1 (the respective eigenfunctions are the product of three eigenfunctions of the operator $-\frac{d^2}{dx^2}$ on the sphere S^1). The function $\zeta(s, \beta)$ is equal to

$$\zeta(s, \beta) = \frac{1}{\Gamma(s)} \sum_{n=-\infty}^{n=+\infty} \sum_{n_1,n_2,n_3=-\infty}^{+\infty} \sum_{\lambda_D} d_{\lambda_D} \int_0^\infty dt t^{s-1} e^{-t\left[\left(\frac{2\pi}{\beta}\right)^2 n^2+\sum_i\left(\frac{2\pi}{R_i}n_i\right)^2+\lambda_D+m^2\right]}. \tag{24b}$$

When the physical space is a pseudosphere PS^3, we can use the identity

$$\ln Z^Q(\beta) = -\tfrac{1}{2}\operatorname{tr}\ln\left(\frac{\Delta}{\mu^2}\right) = -\tfrac{1}{2}\int_0^\beta d\tau \int_{PS^3} d^3x \int_{M^D} d^Dy$$

$$\times \sum_B |\langle x, y, \tau|B\rangle|^2 \langle B|\ln\frac{\Delta}{\mu^2}|B\rangle \sqrt{{}^3g}\sqrt{{}^Dg}, \tag{25}$$

where the set of eigenvalues $\{B\}$ contains "energy modes" $\left(\frac{2\pi}{\beta}\right)^2 n^2$, modes (k, J, M) corresponding to Laplace-Beltrami operators on PS^3 and modes λ_D corresponding to L.-B. operators on M_D.

Hence

$$\langle x, y, \tau|B\rangle = \frac{1}{\sqrt{\beta}}\exp\left[-i\left(\frac{2\pi}{\beta}\right)n\tau\right]\langle x|k, J, M\rangle\langle y|\lambda_D\rangle$$

and

$$\langle x|\Delta^3_{\mathrm{L.B.}}|k, J, M\rangle = (k^2+1)\langle x|K, J, M\rangle,$$

where $k \in [0, \infty]$, $J = 0, 1, \ldots, M \in [-J, J]$.

As a result of summation over J, M of the squared modulus of the function (25), we have:

$$\sum_{J,M} |\langle x|k, J, M\rangle|^2 = \frac{1}{2\pi^2}\left(\frac{k}{a}\right)^2 \frac{1}{a^2}$$

and

$$\sum_{\lambda_D} |\langle y|\lambda_D\rangle|^2 = \frac{d_{\lambda_D}}{\mathrm{Vol}\, M_D},$$

where Vol M_D is a volume of M_D; Vol $M_D = \int \sqrt{^{(D)}g}d^Dy$. Therefore we, obtain

$$\ln Z^Q(\beta) = \frac{\mathrm{Vol\,PS}^3}{4\pi^2 a^2}\sum_{n=-\infty}^{+\infty}\sum_{\lambda_D} d_{\lambda_D}\int_0^\infty \frac{dk}{a}k^2 \ln\left\{\frac{1}{\mu^2}\left(\frac{2\pi n}{\beta}\right)^2 + \frac{k^2+1}{a^2}+\lambda_D+m^2\right\}. \quad (26)$$

Because $\ln A = -\left.\frac{\partial A^{-\zeta}}{\partial s}\right|_{s=0}$, $\int_0^\infty dx x^2 e^{-r^2x} = \frac{\sqrt{\pi}}{4r^3}$, we obtain the following formula for $\zeta(s, \beta)$

$$\zeta(s, \beta) = \frac{\mathrm{Vol\,PS}^3}{8\pi^{3/2}\Gamma(s)}\sum_{n=-\infty}^{+\infty}\sum_{\lambda_D} d_{\lambda_D}\int_0^\infty dt t^{s-\frac{5}{2}} e^{-t\left[\left(\frac{2\pi}{\beta}\right)^2 n^2+\lambda_D+\frac{1}{a^2}+m^2\right]}, \quad (27)$$

where $m^2 = m_0^2+\zeta\left(^{(D)}R-\frac{6}{a^2}\right)$, a — radius of the microspace.

By comparing formulae (24) and (27) we see that formulae for the physical spaces $\mathbf{R}^3$ and PS^3 are similar. The negativness of the curvature of PS^3 is taken into account in $\zeta(s, \beta)$ through the effective mass m^2.

3.1. Universal high-temperature approximation

Our task will be to show the universal high-temperature ($\beta \to 0$) behaviour for cases of $\mathrm{S}^3\times B^D$, $\mathrm{PS}^3\times B^D$ and $\mathbf{R}^3\times B^D$, respectively.

The function $\zeta(s, \beta)$ for the thermal scalar bosonic gas in the space $\mathrm{S}^3\times B^D$ has the form

$$\zeta(s, \beta) = \frac{1}{\Gamma(s)}\sum_{n=-\infty}^{n=+\infty}\int_0^\infty dt\cdot t^{s-1}e^{-t\left[\left(\frac{n}{a}\right)^2+m^2\right]}\sum_{\lambda_3} d_{\lambda_3}e^{-t\lambda_3}\sum d_{\lambda_D}e^{-t\lambda_D}, \quad (28)$$

where $a = \dfrac{\beta}{2\pi}$, $m^2 = m_0^2+\zeta({}^{(3)}R+{}^{(D)}R)$. Using the fact that

$$\sum_{\lambda_3} d_{\lambda_3} e^{-t\lambda_3} = \operatorname{tr} e^{-t\Delta_{\text{L.B.}}^{(3)}} = \int_{M_3} d^3x \langle x|e^{-t\Delta_{\text{L.B.}}^{(3)}}|x\rangle,$$
$$\sum_{\lambda_D} d_{\lambda_D} e^{-t\lambda_D} = \operatorname{tr} e^{-t\Delta_{\text{L.B.}}^{(D)}} = \int_{M_D} d_y^D \langle y|e^{-t\Delta_{\text{L.B.}}^{(D)}}|y\rangle$$

and if the quantity $\left[\left(\dfrac{n}{a}\right)^2+m^2\right]$ tends to infinity, i.e. if $a \to 0$ ($\beta \to 0$) or if $m^2 \to \infty$ (i.e. ${}^{(3)}R+{}^{(D)}R \to \infty$), one can see the major contribution to the integral over t comes from a neighbourhood of $t = 0$.

For small values of t, the following asymptotic expansion are valid:

$$\operatorname{tr} e^{-t\Delta_{\text{L.B.}}^{(3)}} = \frac{1}{(4\pi t)^{3/2}} \int_{M_3} \sqrt{{}^{(3)}g} \sum_{l=0} A_l^{(3)} t^l d^3x, \tag{29}$$

$$\operatorname{tr} e^{-t\Delta_{\text{L.B.}}^{(D)}} = \frac{1}{(4\pi t)^{D/2}} \int_{M_D} \sqrt{{}^{(D)}g} \sum_{j=0} A_j^{(D)} t^j d^D y, \tag{30}$$

where $A_l^{(3)}$, $A_j^{(D)}$ are DeWitt–Schwinger coefficients [21, 22]. For the spaces considered by us, these coefficients do not depend on coordinates; they are constants dependent on scale factors. By using equations (29), (30) we obtain

$$\zeta(s,\beta) = \frac{\operatorname{Vol} M_3 \operatorname{Vol} B_0}{\Gamma(s)\,(4\pi)^{\frac{D+3}{2}}} \sum_{n=-\infty}^{+\infty} \sum_{l,j=0} A_l^{(3)} A_j^{(D)} \int_0^\infty dt \cdot t^{\left(s+l+j-1-\frac{D+3}{2}\right)} \cdot e^{-t\left[\left(\frac{n}{a}\right)^2+m^2\right]}, \tag{31}$$

where $\operatorname{Vol} M_3 = \int d^3x \sqrt{{}^{(3)}g}$, $\operatorname{Vol} B^D = \int d^D y \sqrt{{}^{(D)}g}$.

After some transformation of (31) we obtain the following formula

$$\zeta(s,\beta) = \frac{\operatorname{Vol} M_3 \operatorname{Vol} B_D \beta^{2s}}{\Gamma(s)\,(4\pi)^{(D+4)/2}\beta^{D+3}} \sum_{l,j=0} \sum_{n=-\infty}^{+\infty} (\beta^{2l} A_l^3)(\beta^{2j} A_j^D) \Bigg\{ (\beta m)^{D+4-2(s+l+j)}$$
$$\times \left[\Gamma\left(s+l+j-\frac{D+4}{2}\right) + 2 \sum_{n=-\infty}^{+\infty}{}' \frac{K_{\left(\frac{D+4}{2}-(s+l+j)\right)}(\beta m n)}{\left(\frac{\beta m n}{2}\right)^{\frac{D+4}{2}-(s+l+j)}} \right] \Bigg\}, \tag{32}$$

where K_ν is the third order Bessel function and prime means that we omit the term $n = 0$ in the sum.

The terms, for which $l+j < \dfrac{D+3}{2}$, give the leading contribution to the quantum distribution function for $\beta \to 0$. If $D+4$ is an odd number then $\Gamma\left(s+l+j-\dfrac{D+4}{2}\right)$ is non-singular at $s = 0$, for any l and j. Therefore $\zeta(0, \beta) = 0$, and

$$\ln Z^{Q}(\beta) = \tfrac{1}{2} \frac{\mathrm{Vol}\, M_3\ \mathrm{Vol}\, B^2}{(4\pi)^{(D+4)/2} \cdot \beta^{D+3}} \sum_{l,j,l+j<\frac{D+3}{2}} (\beta^{2l} A_l^{(3)})(\beta_D{}^{j} A_j^{(D)})$$

$$\times \left\{ (\beta m)^{D+4-2(l+j)} \left[\Gamma\left(l+j-\frac{D+4}{2}\right) + 2 \sum{}' \frac{K_{\left(\frac{D+4}{2}+l+j\right)}(\beta mn)}{\left(\frac{\beta mn}{2}\right)^{\frac{D+4}{2}-(l+j)}} \right] \right\}. \tag{33}$$

If $\beta^2 m^2 \to 0$, then

$$\zeta(s, \beta) = \frac{\mathrm{Vol}\, M_3 \cdot \mathrm{Vol}\, B_D \beta^{2s}}{\Gamma(s)\,(4\pi)^{(D+3)/2} \beta^{D+3}} \sum_{n=-\infty}^{+\infty} \sum_{\substack{l,j=0 \\ l+j<\frac{D+3}{2}}} (\beta^{2l} A_l^{(3)})$$

$$\times (\beta^{2j} A_j^{(D)}) \frac{\Gamma\left(s+l+j-\frac{D+3}{2}\right)}{(2\pi n)^{2\left(s+l+j-\frac{D+3}{2}\right)}}, \tag{34}$$

where we have omitted the contribution of the zero mode $n = 0$ to the sum. By using the identity: $\pi^{-\frac{x}{2}} \Gamma\left(\frac{x}{2}\right) \zeta_R(x) = \pi^{\frac{x-1}{2}} \Gamma\left(\frac{1-x}{2}\right) \zeta_R(1-x)$ we obtain the following high--temperature asymptotics of the function $\zeta(s, \beta)$:

$$\zeta(s, \beta) = \frac{2\,\mathrm{Vol}\, M_3 \cdot \mathrm{Vol}\, B_D \beta^{2s} \pi^s}{\pi^{\frac{D+4}{2}} \Gamma(s) \beta^{D+3}} \sum_{\substack{l,j=0 \\ l,j<\frac{D+3}{2}}} \left(\frac{\beta}{2}\right)^{2l} A_l^{(3)} \left(\frac{\beta}{2}\right)^{2j} A_j^{(D)}$$

$$\times \Gamma\left(\frac{D+4}{2} - s - (l+j)\right) \zeta_R(D+4-2s-2(l+j)). \tag{35}$$

If $l+j < \dfrac{D+3}{2}$, the Riemann function $\zeta_R(x)$ does not pass through the pole $x = 1$ and we can go to the limit $s \to 0$. Hence we obtain $\zeta(s, \beta) \xrightarrow[s\to 0]{} 0$, and

$$\ln Z^{Q}(\beta) \cong \frac{\mathrm{Vol}\, M_3 \cdot \mathrm{Vol}\, B_D}{\pi^{(D+4)/2} \beta^{D+3}} \sum_{\substack{l,j=0 \\ l+j<\frac{D+3}{2}}} \left(\left(\frac{\beta}{2}\right)^{2l} A_l^{(3)}\right)\left(\left(\frac{\beta}{2}\right)^{2j} A_j^{(D)}\right)$$

$$\times \Gamma\left(\frac{D+4}{2} - (l+j)\right) \zeta_R(D+4-2(l+j)). \tag{36}$$

From formula (36) we see that the leading term in the expansion of the function is

$$\ln Z^Q(\beta) \xrightarrow[\beta\to 0]{} \frac{\mathrm{Vol}\, M_3\ \mathrm{Vol}\, B_D}{\pi^{(D+4)/2}\beta^{D+3}}\,\Gamma\left(\frac{D+4}{2}\right)\zeta_{\mathrm{R}}(D+4). \tag{37}$$

As it is known, the distribution function for a scalar thermal radiation in a three--dimensional box of the volume V_3 is equal to $\ln Z^Q(\beta) = \pi^2 V_3/(90\beta^3)$. Because $\zeta_{\mathrm{R}}(4) = \dfrac{\pi^2}{90}$ and $\Gamma(2) = 1$, we see, from formula (37), that the high-temperature asymptotics of quantum distribution function of a thermal gas contained in the $(D+3)$-dimensional compact space, corresponding to an arbitrary cosmological model, is the same as the distribution function for a scalar thermal radiation contained in a $(D+3)$-dimensional box. This fact can be explained as follows: at very high temperatures particles do not "perceive" the geometry of the space in which they are placed, they behave as particles in a flat multidimensional box. It can also be shown that when $D+4$ is an even number (then $\zeta(0,\beta) \neq 0$), asymptotics (37) remains valid, i.e. the anomaly effects are negligible in high-temperature approximation, as compared with term (37).

In the case when the physical space is a pseudosphere PS^3, the function $\zeta(s,\beta)$, for the thermal gas of scalar bosons in the space $\mathrm{PS}^3\times \mathrm{B}^D$, is the following:

$$\zeta(s,\beta) = \frac{\mathrm{Vol}\,(\mathrm{PS}^3)}{(4\pi)^{3/2}\Gamma(s)}\sum_{n=-\infty}^{+\infty}\int_0^\infty dt\cdot t^{s-\frac{5}{2}}e^{-t\left[\frac{4\pi^2}{\beta^2}n^2+M^2\right]}\sum_{\lambda_D} d_{\lambda_D}e^{-t\lambda_D}, \tag{38}$$

where $M^2 = m_0^2+\zeta^{(D)}R+(1-6\zeta)/A^2$ and A is the scale factor on the pseudosphere. For the minimal coupling $\zeta = 0$, for massless particles, we analogously obtain the distribution function in the form

$$\ln Z^Q(\beta) = \frac{\mathrm{Vol.}(\mathrm{PS}^3)\cdot \mathrm{Vol}\, B^D}{2(4\pi)^{\frac{D+4}{2}}\beta^{D+3}}\sum_{\substack{j=0\\ j<\frac{D+3}{2}}}(\beta^{2j}A_j^{(D)})\left\{\left(\frac{\beta}{A}\right)^{D+4-2j}\right.$$

$$\left.\times\left[\Gamma\left(j-\frac{D+4}{2}\right)+2\sum_{n=-\infty}^{+\infty}\frac{K_{\left(\frac{D+4}{2}-j\right)}\left(\frac{\beta}{A}n\right)}{\left(\frac{\beta}{2A}n\right)^{\frac{D+4}{2}-j}}\right]\right\}. \tag{39a}$$

If $(\beta M)^2 \to 0$ ($M = 1/A$) then (39) takes the universal form (37). We can obtain quantum distribution function for the case $M_3 = \mathrm{R}^3$ from the one for PS^3 by substituting $M^2 \to \mathcal{M}^2 = m_0^2+\zeta^{(D)}R$ and $\mathrm{Vol}\,(\mathrm{PS}^3) \to V_3$, where V_3 is the volume of R^3.

3.2. Universal low-temperature approximation

We shall find the low-temperature ($\beta \to \infty$) approximation of quantum distribution function for massless scalar bosons contained in spaces $S^3 \times B^{(D)}$ and $PS^3 \times B^{)D)}$, respectively. After simple transformation the following formula is obtained for the function $\zeta(s, \beta)$ in the case of $S^3 \times B^{(D)}$

$$\zeta(s, \beta) = \frac{\beta}{2\pi^{1/2}\Gamma(s)} \int_0^\infty dt \cdot t^{s-\frac{3}{2}} e^{-t\mu^2} \sum_{p=0}^{\infty} p^2 e^{-t\left(\frac{p}{A}\right)^2} \sum_{\lambda_D} d_{\lambda_D} e^{-t\lambda_D} \tag{39b}$$

where A is the radius of S^3. If we now change the variables: $t/A^2 = z$, we obtain, from (39b), the following formula:

$$\zeta(s, \beta) = -\frac{\beta A^{2s}}{2\pi^{1/2}\Gamma(s)A} \frac{\partial}{\partial A} I(\lambda)|_{\lambda=1}, \tag{40}$$

where

$$I(\lambda) = \tfrac{1}{2} \sum_{p=0}^{\infty} \sum_{\lambda_D} d_{\lambda_D} \frac{\Gamma(s-\frac{3}{2})}{[(\sqrt{\lambda}\, p)^2 + A^2\lambda_D + (A\mu)^2]^{s-\frac{3}{2}}}.$$

Let us consider, for simplicity, the case of massless particles ($m_0 = 0$). In such a case, $A^2\mu^2 = \zeta A^{2(D)}R + (6\zeta - 1)$ and if $A^2\mu^2 \to \infty$ (which takes place when ${}^{(D)}RA^2 \to \infty$, i.e. when the scale factor on the macrospace is much greater then the scale factors on microspace), the major contribution to the integral over z comes from the neighbourhood of $z = 0$.

In this case, we can make use of the following asymptotic expansion:

$$\sum_{p=-\infty}^{+\infty} e^{-z(\sqrt{\lambda}p)^2} = \frac{\pi^{1/2}}{\lambda^{1/2} z^{1/2}} \{1 + O(e^{-\frac{1}{z}})\},$$

$$\sum_{\lambda_D} d_\lambda e^{-zA^2\lambda_D} = \frac{\mathrm{Vol}\, B^{(D)}}{(4\pi z A^2)^{D/2}} \sum_{j=0} A_j^{(D)} (zA^2)^j.$$

Then we obtain

$$\zeta(s, \beta) = \frac{\beta A^{2s}\pi^{1/2}\, \mathrm{Vol}\, B^{(D)}}{8\pi\Gamma(s)A^{D+1}(4\pi)^{D/2}} \sum_{j=0} A_j^{(D)} A^{2j} \frac{\Gamma\left(s+j-\dfrac{D+4}{2}\right)}{(A\mu)^{2\left(s+j-\frac{D+4}{2}\right)}}. \tag{41}$$

If $D+4$ is an odd number then $\zeta(0, \beta) = 0$, and we obtain

$$\ln z^Q(\beta) \cong \frac{\beta\, \mathrm{Vol}\, B^{(D)} \mathrm{Vol}\, S^3}{2(4\pi)^{\frac{D+4}{2}} A^{D+4}} \sum_{j=0} (A_j^{(D)} A^{2j}) \frac{\Gamma\left(j-\dfrac{D+4}{2}\right)}{(A\mu)^{2j-(D+4)}}. \tag{42}$$

In the case of $PS^3 \times B^D$, the function $\zeta(s, \beta)$ has the form

$$\zeta(s, \beta) = \frac{\beta \operatorname{Vol}(PS^3)}{16\pi^2 \Gamma(s)} \int_0^\infty dt \cdot t^{s-3} e^{-tM^2} \sum_{\lambda_D} e^{-t\lambda_D},$$

where $M^2 = m_0^2 + \zeta^{(D)} R + \dfrac{1-6\zeta}{A^2}$, A is the radius of pseudosphere.

By introducing the variable $z = t/A^2$, we obtain

$$\zeta(s, \beta) = \frac{\beta \operatorname{Vol}(PS^3) A^{2s-4}}{16\pi^2 \Gamma(s)} \int_0^\infty dz z^{s-3} e^{-zA^2M^2} \sum_{\lambda_D} d_{\lambda_D} e^{-zA^2\lambda_D}. \tag{43}$$

In analogy to the previous case, if $A^2M^2 \to \infty$, i.e. if $A^{2(D)}R \to \infty$ (for $m_0 = 0$), the major contribution to the integral in (43) comes from the neighbourhood of $z = 0$, and by using the asymptotic expansion

$$\sum_{\lambda_D} d_{\lambda_D} e^{-zA^2\lambda_D} = \frac{\operatorname{Vol} B^D}{(4\pi z A^2)^{D/2}} \sum_{j=0} (A^{2j} A_j^{(D)}) z^j$$

we obtain

$$\zeta(s, \beta) = \frac{\beta \operatorname{Vol}(PS^3) \operatorname{Vol} B^{(D)} A^{2s}}{2(4\pi)^{\frac{D+4}{2}} \cdot A^{D+4}} \sum_{j=0} (A^{2j} A_j^{(D)}) \frac{\Gamma\left(j - \frac{D+4}{2}\right)}{(A^2M^2)^{j-\frac{D+4}{2}}}. \tag{44}$$

If $D+4$ is an odd number we have $\zeta(0, \beta) = 0$ (the terms from anomaly are, in the general case of the low-temperature approximation, of a lower order as compared with the term $j = 0$ — for details see [23]). Finally, we obtain

$$\ln Z^Q(\beta) = \frac{\beta \operatorname{Vol}(PS^3) \operatorname{Vol} B^D}{2(4\pi)^{\frac{D+4}{2}} \cdot A^{D+4}} \sum_{j=0} (A^{2j} A_j^{(D)}) \frac{\Gamma\left(j - \frac{D+4}{2}\right)}{(A^2M^2)^{j-\frac{D+4}{2}}}. \tag{45}$$

Formulae (42) and (45) are of general character and are valid when the scale factors on the macrospace are much larger then scale factors on the microspace. By assuming that particles are massless, and that coupling is conformal $\left(\zeta = \dfrac{D+2}{4(D+3)}\right)$, we obtain

$$\mu^2 = \frac{D+2}{4(D+3)} {}^{(D)}R + \frac{1}{A^2}\left(\frac{3}{2} \frac{D+2}{D+3} - 1\right). \tag{46}$$

If the scale factor $A \to \infty$, we can neglect the second term in (46); in such a case, $\mu \simeq \left(\frac{D+2}{4(D+3)}\,{}^{(D)}R\right)^{1/2}$, and the quantum distribution function takes the form

$$\ln Z^{Q}(\beta) \cong \frac{\beta \,\mathrm{Vol}\, B^D \,\mathrm{Vol}\, S^3}{2(4\pi)^{\frac{D+4}{2}}} \sum_{j=0} A_j^{(D)} \frac{\Gamma\left(j-\frac{D+4}{2}\right)}{\left(\frac{(D+2)}{4(D+3)}\,{}^{(D)}R\right)^{j-\frac{D+4}{2}}}. \tag{47}$$

Analogously, quantum distribution function (45) is

$$\ln Z^{Q}(\beta) \cong \frac{\beta \,\mathrm{Vol}\, B^D \,\mathrm{Vol}\, PS^3}{2(4\pi)^{\frac{D+4}{2}}} \sum_{j=0} A_j^{(D)} \frac{\Gamma\left(j-\frac{D+4}{2}\right)}{\left(\frac{(D+2)}{4(D+3)}\,{}^{(D)}R\right)^{j-\frac{D+4}{2}}}. \tag{48}$$

The first term of the expansion ($A_0^{(D)} = 1$) is the leading one in (47) and (48), for the low--temperature approximation $A \gg a_4, \ldots, a_{D+3}$. In this approximation terms from anomaly (if $D+4$ is even) are of a lower order.

4. *Metric back-reaction on low-temperature quantum corrections*

The assumption of the quasi-static approximation enables us to determine thermodynamical characteristics of a bosonic gas in external gravitational fields multidimensional cosmological models. The internal energy can be computed from the formula

$$E = -\frac{\partial}{\partial \beta} \ln Z^{Q}(\beta) \tag{49}$$

and the energy-density is

$$\varrho = \frac{E}{\mathrm{Vol}\, M^3 \cdot \mathrm{Vol}\, B^D}. \tag{50}$$

Free energy and pressures on the macro- and microspace are given by the following formulae

$$F = -\frac{1}{\beta} \ln Z^{Q}(\beta), \tag{51}$$

$$p(M^3) = -\frac{1}{\mathrm{Vol}\,(B^D)} \frac{\partial F}{\partial(\mathrm{Vol}\, M^3)}, \tag{52}$$

$$p(B^D) = -\frac{1}{\mathrm{Vol}\, M^3} \frac{\partial F}{\partial(\mathrm{Vol}\, B^D)} = p'. \tag{53}$$

By using the general high-temperature asymptotic (37), from (49)–(53), one obtains

$$E \xrightarrow[\beta\to 0]{} \frac{(D+3)\,\mathrm{Vol}\,M_3\,\mathrm{Vol}\,B^D}{\pi^{(D+4)/2}\beta^{D+4}}\,\Gamma\left(\frac{D+4}{2}\right)\zeta_{\mathrm{R}}(D+4),$$

$$\varrho \xrightarrow[\beta\to 0]{} \frac{(D+3)}{\pi^{(D+4)/2}\beta^{D+4}}\,\Gamma\left(\frac{D+4}{2}\right)\zeta_{\mathrm{R}}(D+4),$$

$$p' = p = \frac{1}{D+3}\,\varrho. \tag{54}$$

The above relations show that high-temperature quantum effects are dynamically equivalent to effects of radiative matter, i.e. to a hydrodynamic energy-momentum tensor

$$T^\mu_\nu = \mathrm{diag}\,(\varrho, -p, \ldots, -p) \quad \text{with} \quad p = \frac{\varrho}{D+3}.$$

The problem of the existence of solutions with the static microspace and their stability has been discussed in Sections 1 and 2.

Now let us investigate the existence and the stability of solutions in the low-temperature approximation of quantum effects. For simplicity let us assume that we investigate the system in a physically justified approximation $a_4, \ldots, a_{D+3} \to a$. From the universal asymptotics (47) and (48) and for relations (49)–(53) we obtain

$$p(M^3) = p = -\varrho = \frac{\alpha}{a^{D+4}}; \quad p(B^D) = p' = \frac{4}{D}\,\varrho, \quad \alpha = \mathrm{const} > 0. \tag{55}$$

From the above, it follows that low-temperature quantum effects are dynamically equivalent to effects of the anisotropic energy-momentum tensor

$$T^\mu_\nu = \mathrm{diag}\,(\varrho, -p, -p, -p, -p', \ldots, -p').$$

The Einstein equations, with quantum corrections on $M^3 \times B^D$, take the form

$$3\frac{\ddot{A}}{A} + D\frac{\ddot{a}}{a} = -\frac{\varrho(D+1)+3p+Dp'}{D+2} + \frac{2\Lambda}{D+2},$$

$$\frac{2K}{A^2} + \left(\frac{\dot{A}}{A}\right)^{\cdot} + \frac{\dot{A}}{A}\left(3\frac{\dot{A}}{A} + D\frac{\dot{a}}{a}\right) = -\frac{(1-D)p+Dp'-\varrho}{D+2} + \frac{2\Lambda}{D+2}, \tag{56}$$

where p, p' and ϱ are given by (55), $K = 0, \pm 1$.

System of equations (56) can be reduced to the form of the autonomous dynamical system

$$\dot{H} = \frac{dH}{dt} = -H^2 + DH\cdot h + \frac{D(D-1)}{3}h^2 + \frac{2(1-D)\Lambda}{3(D+2)} - \tfrac{5}{3}\,\bar{\varrho}_0 u^{\frac{4+D}{2}},$$

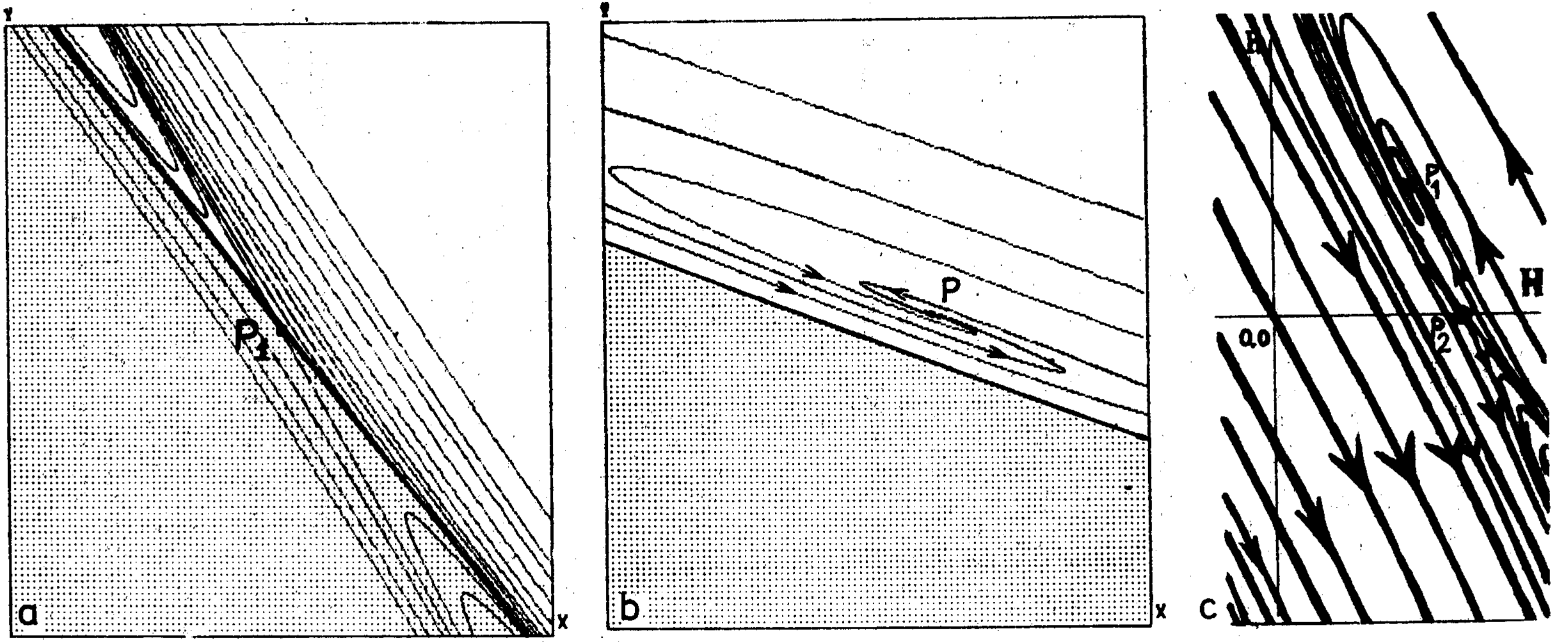


Fig. 4. The behaviour of trajectories of the dynamical system for FRW space-time (with $k = -1) \times T^7$ with quantum corrections in the low-temperature aproximation: (a) the behaviour of trajectories in the neighbourhood of the critical point P: $x_0 = +1$, $y_0 = 0$ representing the classical vacuum solution. ($\Lambda = 0$), (b) the behaviour of trajectories in the neighbourhood of the critical point P: $y_0 = \frac{2}{D+4} x_0$ $\left(1/r^{D+4} \propto 1/R^2\right.$e.g. curvature terms is proportional to the matter term, $\left. x_0 = \sqrt{\frac{(D+4)}{(D+8/3)(D+2)}}\right)$, (c) the behaviour of trajectories of the dynamical system for FRW space-time (with $k = 0) \times T^7$ in low-temperature approximation. $H = \dot{A}/A$, $h = \dot{a}/a$ — Hubble's functions of the macro- and microspace, respectively. The critical point $h_0 = 0$, $H_0 = \pm \sqrt{\frac{D+4}{6(D+2)}\Lambda}$ is a saddle point, the critical point $H_0 = h_0 = \pm \sqrt{\frac{2\Lambda}{(D+3)(D+2)}}$, representing the asymptotic state of the classical vacuum solutions, is an attracting knot

$$\dot{h} = \frac{dh}{dt} = -Dh^2 - 3H \cdot h - \beta u + \frac{2\Lambda}{D+2} + \frac{4}{D} \varrho_0 u^{\frac{4+D}{2}},$$

$$\dot{u} = \frac{du}{dt} = -2uh, \tag{57}$$

where $u = 1/a^2$, $\varrho = \varrho_0/a^{(D+4)}$, $\varrho_0 = \bar{\varrho}_0^{(D+4)}$, $H = \dot{A}/A$, $h = \dot{a}/a$.

System (57) is defined in the region

$$\frac{3K}{A^2} = \varrho_0 u^{\frac{4+D}{2}} + \Lambda - 3H^2 - \frac{D(D-1)}{2} h^2 - 3DH \cdot h - \frac{D}{2} \beta u \begin{cases} \geq 0, & K = 0, +1 \\ < 0, & K = -1 \end{cases} \tag{58}$$

We are interested in the stability of the solution with static microspace, i.e. in the behaviour of the system in the neighbourhood of the critical point $h_0 = 0$, $u = u_0$, $H = H_0$. In order to investigate the character of this critical point it is convenient to use the Routh-Hurwitz criterion [24] which gives us the necessary and sufficient condition for the existence of negative real parts of eigenvalues of the linearization matrix.

This criterion, when applied to our case, gives us the following stability condition for a solution with the static microspace and an inflationary phase in the macrospace:

$$\frac{2(4+D)}{D} \varrho_0 u_0^{\frac{D+2}{2}} > \beta. \tag{59}$$

Condition (59) should be understood in the following way: if system (57) admits the solution $H = H_0$, $h_0 = 0$, $u = u_0$, this solution is represented by an attractive critical point, provided condition (59) is satisfied. If, for example, $\beta = 0$ (the microspace is a hypertorus) condition (59) requires that Casimir energy be positive. Because it is not satisfied in the low-temperature approximation, dim $W^{\mathrm{Atr}} < 3$ and only a zero-measure set of trajectories leads to the solution which is interesting for us. Condition (59) is not satisfied in the general case of low temperatures which proves the asymptotic non-stability of solutions with a static microspace. If $K = 0$ and $\beta = 0$, i.e. for the case of FRW (with $K = 0) \times \mathrm{T}^7$ model, the behaviour of the system in the neighbourhood of the saddle point $h_0 = 0$, $H_0 = \pm\sqrt{\frac{D+4}{6(D+2)}}\Lambda$ is illustrated in Fig. 4c.

If $\Lambda = 0$, the solutions, for which quantum effects are negligible, are asymptotically admissible (for example FRW (with $K = -1) \times \mathrm{T}^D$), but they are nonstable (Fig. 4a). The solutions, for which curvature terms are proportional to the terms arising from quantum fluctuations are stable (Fig. 4b).

From first integral (58) one sees that quantum effects do not provide effective mechanism of dimensional reduction which could lead to a static microspace, but they can provide an effective reduction mechanism leading to the zero size.

For example, if $K = +1$ and $\Lambda = 0$, one sees, from (58), that the physical space expands if and only if the microspace contracts.

5. A simple mechanism of dimensional reduction with the help of low-temperature corrections for FRW × {small world of Ellis}

So far our conclusions have shown the noneffectiveness of mechanisms of dimensional reduction. Finally, we shall show the existence of a certain theoretical possibility of internal space compactification with the help of low-temperature quantum effects for the case when the internal space is a compact form of negative curvature. As it is well known, the Einstein theory of gravity determines the local structure of space-time, i.e. the metric, whereas the compactness of the internal space is its topological property [25]. The problem of topological classification of three-dimensional maximally symmetric spaces was investigated by Wolf [26], and in cosmological context by Ellis [27]. In the case of negative curvature the classification is not known, but for compact spaces one can use their volume for classification purposes (see Thurston [28]). The number of such spaces is infinite. We shall assume that the internal space is compact D-dimensional space of negative curvature PS^D/Γ (Γ is a discrete subgroup of the isometry group) whereas the physical space is an FRW model. If $\Gamma \neq I$ (I is the identity), its action lowers the dimension of the isometry group, and the space does not admit the full $\frac{D(D+1)}{2}$-dimensional isometry group any longer.

In this case, Einstein equations with quantum effects can be reduced to the form of the following three-dimensional dynamical system

$$\dot{H} = -H^2+DH\cdot h+\frac{D(D-1)}{3}h^2+\frac{D(D-1)}{3}u-\tfrac{5}{3}\varrho_0 u^{\frac{4+D}{2}},$$

$$\dot{h} = -3H\cdot h-Dh^2+(D-1)u+\frac{4}{D}\varrho_0 u^{\frac{4+D}{2}},$$

$$\dot{u} = -2uh \tag{60}$$

in the region

$$\frac{3K}{A^2} = \varrho-3H^2-3DH\cdot h-\frac{D(D-1)}{2}h^2-\frac{D(D-1)}{2a^2}\begin{cases}\geqslant 0 & \text{for} \quad K=0,+1\\ <0 & \quad K=-1,\end{cases}$$

where $H = \dot{A}/A$, $h = \dot{a}/a$ are Hubble functions on the macro- and microspace, respectively; $u = 1/a^2$, $\varrho = \varrho_0/a^{(D+4)}$ is the density of Casimir energy, $\varrho_0 < 0$.

System (60) has the critical point $u_0^{(D+2)/2} = \frac{D(D-1)}{4|\varrho_0|}$, $H_0 = \frac{3}{4}D(D-1)u_0$, $h_0 = 0$ which represents the state of the system with a static microspace and an inflationary phase on macrospace. Eigenvalues of linearization matrix (60) are negative at this point

$$\lambda_1 = -2H_0 < 0 \text{ when } H_0 > 0, \quad \lambda_2, \lambda_3 < 0$$

where λ_2, λ_3 satisfy the equation

$$\lambda^2+3H_0\lambda+2u_0\left[(D-1)+\frac{2(4+D)}{D}\varrho_0 u_0^{\frac{2+D}{2}}\right] = 0$$

if the following condition is satisfied:

$$\det A = (D-1)+\frac{2(4+D)}{D}\varrho_0 u_0^{\frac{2+D}{2}} \geqslant 0 \Leftrightarrow a_0 < \sqrt{\frac{4+D}{2}},$$

$$\operatorname{Tr} A < 0 \Leftrightarrow H_0 > 0. \tag{61}$$

If condition (61) is satisfied, $\dim W^{\mathrm{Atr}} = 3$, then a non-empty set of trajectories of the system leads to the critical point $(H_0, 0, h_0)$. This proves the asymptotic stability of solutions with a static microspace when the physical space expands.

6. *Conclusions*

In this paper we have investigated the effectiveness of the mechanism of the dimensional reduction leading to the static size of the internal space. We have discussed the classical Einstein equations and Einstein's equations with quantum corrections arising from massless scalar fields. The universal low-temperature asymptotic has been determined by using the additional assumption that the scale factors on the physical space are much larger than the scale factors on the microspace. The fact that the Casimir energy is negative turns out to be a basic property of this approximation. Consequently, there are no asymptotically stable configurations with the static microspace. The negative character of the Casimir energy has not been taken into account in many papers [29]. The asymptotically stable configurations with the static microspace are admissible by the classical Einstein equations only for the torus as a model of the internal space. However, such models form the zero-measure set in the space of all initial data for homogeneous multidimensional cosmological models. When investigating the stability of configurations with the static microspace, we have used the dynamical system methods and we have based our considerations on the full classification of arbitrarily dimensional cosmological models.

A criterion of the existence of an effective dimensional reduction mechanism can be formulated in terms of the dynamical system theory: there must exist exactly one critical point in the phase-space (in the physical region), for which $\dim W^{\mathrm{Atr}} = n$ (n is the dimension of the phase-space) and which represents the solution with a static microspace.

Finally, we have constructed a toy model FRW × {small world of Ellis} for which the mechanism of the dimensional reduction by low-temperature quantum effects leads to the configuration with the static microspace and the inflation phase on the macrospace (the problem of cosmological constant is here solved in a natural way).

The obtained results, concerning the existence of the effective dimensional reduction mechanism, are based on the assumption that the internal space is a group manifold (of Bianchi types generalized to higher dimensions). The assumptions of homogeneity and anisotropy seem to be reasonable from the cosmological point of view. Some classical 3-dimensional Bianchi types (e.g. B(IX) and B(VIII)) turn out to be, in a sense, "close" to "general" non-homogeneous world models. However the vanishing of the chaotic behaviour in higher dimensional generalized Bianchi types [11] suggests that these models are not generic. In such a case the Einstein equations for the multidimensional homogeneous cosmology would be not rich enough to describe the early cosmic evolution [11]. The problem of the generic character of the Einstein field equations for such a cosmology has been discussed in [30].

In the present work, we have investigated the influence of quantum effects, in the low--temperature approximation, on the existence of assymptotically stable configurations with the static internal space. If one takes into account field effects with the Freund-Rubin Ansatz, the effective reduction mechanism exists, provided that the internal space is non-isotropic and the cosmological constant different from zero. The existence of such a mechanism has been also demonstrated for the case of non-isotropic physical space of the type (Kantowski-Sachs) $\times T^D$ (see [31]).

One should also notice that, although quantum effects do not lead to asymptotically stable configuration with the static internal space, they do provide an effective mechanism of the reduction to the zero dimension. When the internal space is sufficiently small, one--loop approximation breaks down.

In investigating the back reaction we have used approximations of thermodynamical distribution functions. The assumption that the size of the physical space is much larger than that of the internal space leads to a negative Casimir energy. To solve the problem of the existence of the effective reduction mechanism one should first solve the problem of the back reaction on fully determined distribution functions. This could be done with the help of numerical methods. Let us stress this out that exact formulae of quantum distribution functions, found in the present work, are independent of the assumption that the internal space is a group manifold B^D (it is an arbitrary compact Riemann with the curvature scalar ${}^{(D)}R$).

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REFERENCES

[1] H. J. Duff, B. E. Nilsson, C. N. Pope, *Phys. Rep.* **130**, 1 (1986).

[2] Selected papers can be found in: J. Schwartz Ed., *Superstring* World Scientic, Singapore 1985.

[3] E. W. Kolb, Preprint, Fermilab-Pub-86, 138-A, 1986.

[4] A. Hosoya, L. G. Jensen, J. A. Stein-Schabes, *Nucl. Phys.* **B283**, 657 (1987); Y. Elskens, M. Henneaux, *Nucl. Phys.* 1987, to be published; Y. Elskens, M. Henneaux, *Class. Quantum Grav.* **4**, L161 (1987).

[5] A. Chodos, S. W. Detweiller, *Phys. Rev.* **D21**, 2167 (1980).

[6] For example: D. Sahdev, *Phys. Lett.* **B1558**, 1378 (1984); M. Demiański, Z. Gołda, M. Heller, M. Szydłowski, *Class. Quantum Grav.* **3**, 1199 (1986).

[7] P. Candelas, S. Wienberg, *Nucl. Phys.* **B237**, 397 (1984); S. Randjbar-Daemi, A. Salam, J. Strathdee, *Phys. Lett.* **B135**, 388 (1984); M. A. Rubin, B. D. Roth, *Nucl. Phys.* **B226**, 444 (1983); T. Appelquist, A. Chodos, *Phys. Rev. Lett.* **50**, 141 (1983); M. A. Rubin, B. D. Roth, *Phys. Lett.* **B127**, 55 (1983); T. Appelquist, A. Chodos, E. Mayers, *Phys. Lett.* **B127**, 51 (1983); D. Rohrlich, *Phys. Rev.* **D29**, 330 (1983); M. Szydłowski, J. Szczęsny, M. Biesiada, *Class. Quantum Grav.* **4**, 1731 (1987).

[8] T. Futumase, *Phys. Rev.* **D29**, 2783 (1984); M. Yoshimura, *Phys. Rev.* **D30**, 344 (1984); T. Koikawa, M. Yoshimura, *Phys. Lett.* **B155**, 137 (1985); D. Birmingham, S. Sen, *Ann. Phys.* **161**, 121 (1985); B. Allen, *Nucl. Phys.* **B226**, 228 (1983).

[9] M. Demiański, Z. Gołda, L. Sokołowski, M. Szydłowski, P. Turkowski, *J. Math. Phys.* **28**, 171 (1987).
[10] See e.g. A. O. Barut, R. Rączka, *Theory of Group Representation and Its Application*, Polish Scientific Publisher, Warszawa 1980.
[11] M. Szydłowski, *Phys. Lett.* **B195**, 31 (1987).
[12] C. B. Collins, S. W. Hawking, *Astrophys. J.* **180**, 317 (1973).
[13] M. Demiański, M. Heller, Z. Gołda, M. Szydłowski, *Class. Quantum Grav.* **3**, 1199 (1986).
[14] M. Szydłowski, M. Biesiada, J. Szczęsny, *Phys. Lett.* **B195**, 27 (1987).
[15] G. F. R. Ellis, M. A. H. MacCallum. *Comm. Math. Phys.* **12**, 108 (1969).
[16] O. I. Bogoyavlenskij, *Metody kachestvennoj teorii dinamicheskikh system*, Nauka, Moscow 1980.
[17] M. Gleiser, J. A. Stein-Schabs, Preprint Fermilab-Pub-86, 106-A; A. H. Chamseddine, *Nucl. Phys.* **B185**, 403 (1981); G. F. Chapline, N. S. Manton, *Phys. Lett.* **B120**, 105 (1983).
[18] O. Yasuda, *Phys. Lett.* **B133**, 180 (1983); K. Kikkawa et al., *Nucl. Phys.* **B260**, 429 (1985); C. S. Lim, *Phys. Rev.* **D31**, 2507 (1985); D. Bailin et al., *Nucl. Phys.* **B253**, 387 (1987); M. Gleiser et al., *Ann. Phys. (USA)* **160**, 299 (1985); J. S. Dowker, *Phys. Rev.* **D29**, 2773 (1984).
[19] N. D. Birrel, P. C. W. Davies, *Quantum Fields in Curred Spaces*, Cambridge University Press 1982.
[20] L. Parker, S. A. Fulling, *Phys. Rev.* **D9**, 341 (1974); H. Y. Lee, S. A. Sakai, Preprint TIT/HE9-89 (Jan. 1986).
[21] N. E. Hurt, *Geometric Quantization in Action*, D. Reidel Publishing Company 1983.
[22] I. Chavel, *Eigenvalues in Riemannian Geometry*, Academic Press, Inc. 1984.
[23] M. Szydłowski, J. Szczęsny, *Phys. Rev.* D (in press).
[24] J. Barrow, H. Sonoda, *Phys. Rep.*, Vol. 1, No 3 (1987).
[25] Ch. Misner, in *Relativity, Groups and Topology*, ed. DeWitt, New York-London 1964.
[26] J. Wolf, *Spaces and Constant Curvature*, McGraw-Hill, N. Y. 1967.
[27] G. F. R. Ellis, *Gen. Relativ. Gravitation* **2**, 7 (1971).
[28] B. Thurston, *The Geometry and Topology of 3-manifolds*, Princeton University Press 1978.
[29] F. C. Accetta, M. Gleiser, R. Holman, E. Kolb, *Nucl. Phys.* **B276**, 501 (1986).
[30] M. Szydłowski, G. Pajdosz, *Class. Quantum Grav.* 1989 (in press).
[31] M. Szydłowski et al., *Class. Quantum Grav.* **5**, 1097 (1988).