FACTORIZED RENORMALIZATION FORMULA*

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A factorized subtraction formula for R-operation is discussed. The notion of T-diagrams as a special class of Feynman diagrams, is introduced. For T-diagrams the factorized renormalization formula is shown to be reducible to Zimmermann's forest formula for R-operation.

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1. Introduction

The formalism of the R-operation for removing the ultraviolet divergences and extracting finite parts from divergent Feynman amplitudes was given and developed by Bogoliubov, Parasiuk, Hepp, Zimmermann and Zavialov [1–4, 10]. The renormalized integrand is obtained by performing all subtractions connected with divergent subdiagrams (i.e. renormalization parts).

Thus, the R-operation may be defined as the product of subtraction operators taken over all divergent subdiagrams. It is the factorized renormalization formula.

However, by reason of difficulties connected with overlapping divergent subdiagrams this product is usually decomposed to the well-known sum over Zimmermann's forests of divergent subdiagrams.

In this paper, using the κ-representation for subtraction operators similar to that given by Bergere and Zuber [5], we show that the factorized formula has well defined meaning. In contradistinction to Bergere and Zuber [5, 6], we do not define the factorized R-operation as the product taken over all possible subdiagrams, but simply as the product connected only with divergent subdiagrams.

In Sect. 3, we introduce the notion of T-diagrams. It is a very wide class of Feynman diagrams for which one can use a simplified form of κ-representation for subtraction operators. The factorized renormalization formula for T-diagrams is shown to reduce to Zimmermann's forest formula of R-operation in the scalar theories case. The equivalence

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of both renormalization formulas for any type of Feynman diagrams we hope to examine in future.

The factorized formula for the R-operation is a very handy tool to deal with ultraviolet divergences. From a purely computational point of view, ir is much easier to calculate the product of subtraction operators without performing the tedious decomposition into forests.

Moreover, the presented factorization may be considered also in the Smirnov and Cheryrkin formalism of the R-operation [7, 8], which is an extension of the BPHZ subtraction scheme for the case when both ultraviolet and infrared divergences are involved.

2. *Factorized renormalization formula and Zimmermann's renormalization formula*

Let us limit ourselves to the scalar theories case. In order to compare both renormalization methods we write the Feynman amplitude, associated with the connected diagram Γ, in the standard parametric integral representation (omitting a numerical factor) [4, 9, 10]

$$I_\Gamma = \delta^{(D)}\left(\sum_{n=1}^{N} k_n\right)\int_0^\infty \dots \int_0^\infty d\boldsymbol{\alpha} D^{-D/2}(\boldsymbol{\alpha}) \exp\left(i\,\frac{A(\boldsymbol{\alpha}, \boldsymbol{k})}{D(\boldsymbol{\alpha})}\right)$$

$$\times \exp\left(-i \sum_{l=1}^{L} \alpha_l (m_l^2 - i\varepsilon)\right) \tag{1}$$

where L — the number of internal lines of Γ, N — the number of vertices $\boldsymbol{k} = (k_1, \dots, k_N)$ denotes the set of external momenta, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_L)$, $d\boldsymbol{\alpha} = d\alpha_1 \dots d\alpha_L$. The functions appearing in (1) are defined as [see 4, 10]

$$D(\boldsymbol{\alpha}) = \sum_{T_1} \left(\prod_{l \notin T_1} \alpha_l\right) \tag{2}$$

where the sum runs over all trees in Γ,

$$A(\boldsymbol{\alpha}, \boldsymbol{k}) = \sum_{T_2} \left\{\left(\prod_{l \notin T_2} \alpha_l\right)\left(\sum_{i \in V_1} k_i\right)^2\right\}, \tag{3}$$

where the sum runs over all two-trees in Γ (each two-tree T naturally separates the vertices into two disjoint non-empty sets V_1 and V_2).

We use the standard definition of the index ω of the diagram (subdiagram)

$$\omega = D(L - N + 1) - 2L, \tag{4}$$

where D denotes the number of dimensions, and the following definitions:

A subdiagram is said to be full if any two vertices in this subdiagram are joined by all the lines which already joined them in the original diagram.

A subdiagram is said to be one-particle irreducible (API) if it is connected and each of its internal line belongs to at least one loop.

A subdiagram is said to be divergent if it is full, one-particle irreducible and its index is non-negative.

A family of all divergent subdiagrams included in the diagram Γ we denote by $\mathscr{R}_\Gamma$

$$\mathscr{R}_\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_R\}. \tag{5}$$

In particular, the original diagram Γ may belong to this family.

Two subdiagrams γ_1 and γ_2 are overlapping if $\gamma_1 \cap \gamma_2 \neq \emptyset$ and neither $\gamma_1 \subset \gamma_2$ nor $\gamma_2 \subset \gamma_1$.

A forest is a subfamily of non-overlapping divergent subdiagrams.

By the R-operation we mean a standard product of subtractions connected with all divergent subdiagrams $\gamma_1, \gamma_2, \ldots, \gamma_R$. The subtraction operation O_γ amounts to the subtraction from I_Γ of a Maclaurin polynomial with respect to external momenta of a degree equal to the index of γ. To realize it, we take the usual κ-representation [4, 9, 10] modified by Bergere and Zuber [5].

Let us denote

$$\phi = D^{-D/2}(\alpha) \exp\left(i \frac{A(\alpha, k)}{D(\alpha)}\right). \tag{6}$$

The parametrical function $\phi(\kappa)$ is obtained from ϕ by dilatation of all $\alpha_l \in \gamma_r$ by κ_r (for any $1 \leqslant r \leqslant R$) and multiplication by the factor $\prod_{r=1}^{R} \kappa_r^{\frac{1}{2}Dp_r+\Omega_r}$

$$\phi(\kappa) = \prod_{r=1}^{R} \kappa_r^{\frac{1}{2}Dp_r+\Omega_r} D_\kappa^{-D/2}(\alpha) \exp\left(i \frac{A_\kappa(\alpha, k)}{D_\kappa(\alpha)}\right), \tag{7}$$

where $p_r = l_r - n_r + 1$ denotes the number of loops in the subdiagram γ_r. The subtraction operation O_{γ_r} is now expressed in the standard way [9, 10]

$$O_{\gamma_r} = 1_{\gamma_r} - M_{\gamma_r}, \tag{8}$$

$$1_{\gamma_r} \equiv \psi(\kappa_r)|_{\kappa_r=1}, \tag{8a}$$

$$M_{\gamma_r} \equiv \sum_{n=0}^{\frac{1}{2}\omega_r+\Omega_r} \frac{1}{n!} \frac{\partial^n}{\partial \kappa_r^n} \psi(\kappa_r)\Big|_{\kappa_r=0}. \tag{8b}$$

ψ denotes ϕ or any expression of the form $O_{\gamma_{i_1}} \ldots O_{\gamma_{i_k}}\phi$. The number $\Omega_r \geqslant 0$ is large enough to make ψ and its derivatives singularity free for every point $\kappa_r \in [0, 1]$.

The formula (8) is in fact independent of the auxiliary parametr Ω_r, nevertheless, Ω_r cannot be fixed because its smallest possible value is always determined by the actual structure of the argument ψ. For some diagrams we can simplify the expressions (7) and (8) by setting $\Omega_r = 0$. But, in general, we need $\Omega_r \neq 0$. This problem is discussed in Sect. 3.

The finite part of the Feynman amplitude is obtained by applying R-operation. As it was said before, by the R-operation we mean the product of the subtraction operations connected with the family of divergent subdiagrams

$$R \equiv O_{\gamma_1} O_{\gamma_2} \ldots O_{\gamma_R} = (1-M_{\gamma_1})(1-M_{\gamma_2}) \ldots (1-M_{\gamma_R}). \tag{9}$$

For the sake of simplicity, we use somewhat simplified notation: $\mathbf{1}_{\gamma_r} \equiv \mathbf{1}$.

The well-known difficulties connected with the definition of the products $M_{\gamma_i} M_{\gamma_j}$ for overlapping subdiagrams γ_i and γ_j give rise to replacing formula (9) by the Zimmermann's forest formula

$$R = \mathbf{1} + \sum_{\mathscr{F} \subset \mathscr{R}_\Gamma} \{\prod_{r \in \mathscr{F}} (-M_{\gamma_r})\}, \tag{10}$$

where the sum runs over the set of all non-empty forests. The mentioned products do not occur in (10).

In this paper we prove that the formula (9) is correct if we use the modified κ-representation (8) for subtractions.

3. Renormalization of Feynman diagrams with tree families of divergent subdiagrams (T-diagrams)

Let us limit ourselves to some special sort of Feynman diagrams.

DEFINITION. A family of divergent subdiagrams $\mathscr{R}_\Gamma = \{\gamma_1, \ldots, \gamma_R\}$ is called a "tree family" if there exists a tree $T_\perp$ of the diagram Γ satisfying the following: for any r ($1 \leqslant r \leqslant R$) the intersection $T_\perp \cap \gamma_r$ is a tree of the subdiagram γ_r.

DEFINITION. A Feynman diagram Γ is called a T-diagram if its family of divergent subdiagrams is a tree family.

The topological structure of a tree family is described by the following theorem

THEOREM 1. The family $\mathscr{R}_\Gamma = \{\gamma_1, \ldots, \gamma_R\}$ is a tree family if and only if

$$\forall I_n \subset \{1, \ldots, R\} \quad \forall k \leqslant n,$$

$$C\Big(\bigcup_{i_1, \ldots, i_k \in I_n} \gamma_{i_1} \cap \ldots \cap \gamma_{i_k}\Big) \leqslant n-k+1, \tag{1}$$

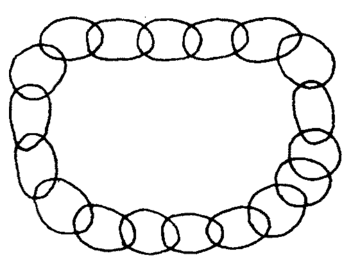
where I_n denotes a n-element subset of natural numbers, $C(\ldots)$ denotes the number of connected components for the given diagram.

Since we shall not make use of the foregoing theorem, we omit its complicated proof. It is interesting and useful for us to consider the particular cases of condition (1)

1°. $k = 2$. We obtain the following conditions

$$\forall I_n \subset \{1, \ldots, R\} : C\Big(\bigcup_{i,j \in I_n} \gamma_i \cap \gamma_j\Big) \leqslant n-1. \tag{2}$$

The above implies that there is no "diagram loop"



2°. $k = n$. In this case the condition (1) leads to

$$\forall \{i_1, \ldots, i_n\} \subset \{1, \ldots, R\}:$$
$$C(\gamma_{i_1} \cap \ldots \cap \gamma_{i_n}) \leqslant 1. \quad (3)$$

It means that all possible intersections of subdiagrams belonging to a tree family are empty of connected.

The important analytic properties of Feynman amplitudes connected with T-diagrams are stated in the next theorems.

THEOREM 2. The κ-parametrized integrand (2.7) and its derivatives with $\boldsymbol{\Omega}$ assumed to be equal to zero (i.e. $\Omega_1 = \Omega_2 = \ldots = \Omega_R = 0$) have no singularities for every point $\boldsymbol{\kappa} \in [0, 1]^R$ (i.e. $(\kappa_1, \ldots, \kappa_R) \in [0, 1] \times \ldots \times [0, 1]$) if and only if the Feynman amplitude is associated with a T-diagram.

COROLLARY. One may set $\boldsymbol{\Omega} = 0$ in κ-representation of subtraction operations (2.8) if and only if the Feynman amplitude is associated with a T-diagram.

THEOREM 3. If we work with the subtraction operators only within the framework of Zimmermann's forest formula for the R-operation (2.10), then for any Feynman diagram we are allowed to set $\boldsymbol{\Omega} = 0$ in the κ-representation.

The proofs are given in Appendix I.

As it was previously mentioned, the results do not depend on the particular choice of auxiliary parameters $\boldsymbol{\Omega}$, but sometimes, for the sake of mathematical correctness, it is not possible to choose $\boldsymbol{\Omega} = 0$. Moreover, for some products of subtraction operations, the choice of $\boldsymbol{\Omega}$ is dependent on the order of their factors.

We now begin to investigate the product of subtraction operators $M_{\gamma_1} M_{\gamma_2}$ associated with two overlapping subdiagrams of some T-diagram Γ. The first step is to introduce the (κ_1, κ_2)-parametrization.

$$\phi(\kappa_1, \kappa_2) = \frac{\kappa_1^{Dp_1/2} \kappa_2^{Dp_2/2}}{\boldsymbol{D}^{D/2}_{\kappa_1,\kappa_2}(\boldsymbol{\alpha})} \exp\left(i \frac{\boldsymbol{A}_{\kappa_1,\kappa_2}(\boldsymbol{\alpha}, \boldsymbol{k})}{\boldsymbol{D}_{\kappa_1,\kappa_2}(\boldsymbol{\alpha})}\right), \quad (4)$$

where:

$$\boldsymbol{D}_{\kappa_1,\kappa_2}(\boldsymbol{\alpha}) = \kappa_1^{p_1} \kappa_2^{p_2} \sum_{k,l \geqslant 0} D_{kl}(\boldsymbol{\alpha}) \kappa_1^k \kappa_2^l, \quad (4a)$$

$$\boldsymbol{A}_{\kappa_1,\kappa_2}(\boldsymbol{\alpha}, \boldsymbol{k}) = \kappa_1^{p_1} \kappa_2^{p_2} \sum_{k,l \geqslant 0} A_{kl}(\boldsymbol{\alpha}, \boldsymbol{k}) \kappa_1^k \kappa_2^l. \quad (4b)$$

The coefficients D_{kl} and A_{kl} are given by

$$D_{kl}(\boldsymbol{\alpha}) = \sum_{T_1: \left\{ \begin{smallmatrix} w_1(T_1) = k + p_1 \\ w_2(T_1) = l + p_2 \end{smallmatrix} \right\}} \left(\prod_{s \notin T_1} \alpha_s \right), \quad (5a)$$

$$A_{kl}(\boldsymbol{\alpha}, \boldsymbol{k}) = \sum_{T_2: \left\{ \begin{smallmatrix} w_1(T_2) = k + p_1 \\ w_2(T_2) = l + p_2 \end{smallmatrix} \right\}} \left\{ \left(\prod_{s \notin T_2} \alpha_s \right) \left(\sum_{i \in V_1} k_i \right)^2 \right\}, \quad (5b)$$

where $w_i(T_{1(2)})$ denotes the number of lines belonging to γ_i but not to given T_1 (T_2), and the sums are taken over all trees (two-trees) satisfying our conditions.

Using the fact that the intersection (in the sense of common lines and vertices) of subdiagrams belonging to a tree family is connected, one can easily show that

$$D_{00}(\alpha) \not\equiv 0. \tag{6}$$

From (4a, b) and (4) we have

$$\phi(\kappa_1, \kappa_2) = [\sum_{k,l\geqslant 0} D_{kl}(\alpha)\kappa_1^k\kappa_2^l]^{-D/2} \exp\left(i \frac{\sum_{k,l\geqslant 0} A_{kl}(\alpha, k)\kappa_1^k\kappa_2^l}{\sum_{k,l\geqslant 0} D_{kl}(\alpha)\kappa_1^k\kappa_2^l} \right). \tag{7}$$

The product of subtraction operators $M_{\gamma_1}M_{\gamma_2}$ is given by

$$M_{\gamma_1}M_{\gamma_2}\phi = \sum_{p=0}^{\frac{1}{2}\omega_1} \frac{1}{p!} \frac{\partial^p}{\partial\kappa_1^p} \sum_{q=0}^{\frac{1}{2}\omega_2} \frac{1}{q!} \frac{\partial^q}{\partial\kappa_1^q} \phi \Big|_{\kappa_1=\kappa_2=0} \tag{8}$$

so that

$$M_{\gamma_1}M_{\gamma_2}\phi = \sum_{i,j,s,t,f,g} \frac{C_{ijstfg}}{D_{00}^{D/2+B_{ijstfg}}} D_{i_1j_1} \ldots D_{i_fj_f}$$

$$\times A_{s_1t_1} \ldots A_{s_gt_g} \exp\left(i\frac{A_{00}}{D_{00}}\right), \tag{9}$$

where B_{ijstfg}, C_{ijstfg} are some coefficients.

Let us imagine that κ_1 and κ_2 are of dimension a, the cofficients D_{kl} and A_{kl} are of dimension a^{-k-l}, so that $D_{\kappa_1,\kappa_2}(\alpha)$, $A_{\kappa_1,\kappa_2}(\alpha, k)$ and $\phi(\kappa_1, \kappa_2)$ are dimensionless. Comparing (8) and (9) one can establish the following inequality

$$\sum_{k=1}^{f} i_k + \sum_{k=1}^{f} j_k + \sum_{l=1}^{g} s_l + \sum_{l=1}^{g} t_l \leqslant \tfrac{1}{2}(\omega_1 + \omega_2). \tag{10}$$

We now turn to carry out the "dimensional analysis" in another way. Let us assign dimension a to each coefficient D_{kl} and A_{kl}. In this case we obtain the following relation

$$B_{ijstfg} = f + g. \tag{11}$$

The formula (9) can be rewritten as

$$M_{\gamma_1}M_{\gamma_2}\phi = \sum_{i,j,s,j,f,g} \frac{C_{ijstfg}}{D_{00}^{D/2+f+g}} \prod_{k=1}^{f} D_{i_kj_k} \prod_{l=1}^{g} A_{s_lj_l} \times \exp\left(i\frac{A_{00}}{D_{00}}\right), \tag{12}$$

where the sum runs over all indices fulfilling (10).

For what follows, we need two other subdiagrams

$$\gamma_\cup = \gamma_1 \cup \gamma_2 \text{ and } \gamma_\cap = \gamma_1 \cap \gamma_2,$$

where $\gamma_\cup$ is the full subdiagram of Γ which contains all vertices belonging to γ_1 and γ_2. We denote by $l_{1\,2}$ the number of lines which belong to $\gamma_\cup$ but neither to γ_1 nor to γ_2. The subdiagram $\gamma_\cup$ is also full. Moreover, $\gamma_\cup$ is connected (because γ_1 and γ_2 overlap) and $\gamma_\cap$ is connected as well (because Γ is a T-diagram). We also need two parameters $\kappa_\cup$ and $\kappa_\cap$ defined for $\gamma_\cup$ and $\gamma_\cap$ in the natural way. Let us introduce them into $D_{kl}(\boldsymbol{\alpha})$ and $A_{kl}(\boldsymbol{\alpha}, \boldsymbol{k})$ (5ab)

LEMMA 1. The coefficients A_{kl} and D_{kl} in (4ab) after the $(\kappa_\cup, \kappa_\cap)$-parametrization become of the form

$$D_{kl}(\boldsymbol{\alpha}, \kappa_\cup, \kappa_\cap) = \kappa_\cup^{p_\cup}\kappa_\cap^{p_\cap} \sum_{k',l'\geqslant 0} \kappa_\cup^{k'}\kappa_\cap^{l'} \Delta_{klk'l'}(\boldsymbol{\alpha}), \tag{13a}$$

$$A_{kl}(\boldsymbol{\alpha}, \boldsymbol{k}, \kappa_\cup, \kappa_\cap) = \kappa_\cup^{p_\cup}\kappa_\cap^{p_\cap} \sum_{k',l'\geqslant 0} \kappa_\cup^{k'}\kappa_\cap^{l'} A_{klk'l'}(\boldsymbol{\alpha}, \boldsymbol{k}), \tag{13b}$$

and the following inequalities are valid

$$k'+l' \leqslant k+l. \tag{13c}$$

Proof. Let us consider a term of (5a) connected with a tree T_1

$$\prod_{s\notin T_1} \alpha_s. \tag{14}$$

This term after $(\kappa_\cup, \kappa_\cap)$-parametrization is multiplied by a factor $\kappa_\cup^{w_\cup(T_1)}\kappa_\cap^{w_\cap(T_1)}$, where

$$w_\cup(T_1) = k'+p_\cup \quad \text{and} \quad w_\cap(T_1) = l'+p_\cap. \tag{15}$$

Note the following relations

$$w_\cup(T_1) = w_1(T_1)+w_2(T_1)-w_\cap(T_1)+w_{1\,2}(T_1), \tag{16}$$

$$p_\cup = p_1+p_2-p_\cap+l_{12}, \tag{17}$$

where $w_{1\,2}(T_1)$ denotes the number of lines belonging to $\gamma_\cup$, but not to γ_1, γ_2, T_1.

Using (15), (16), (17) and the foregoing definitions of w_1 and w_2 and taking into account that $w_{1\,2}(T_1) \leqslant l_{1\,2}$ we obtain (13c).

The above lemma, together with inequality (10) (note that D_{00} and A_{00} are monomials with respect to $(\kappa_\cup, \kappa_\cap)$-parametrization), implies that the $(\kappa_\cup, \kappa_\cap)$-parametrized product $M_{\gamma_1}M_{\gamma_2}\phi$ may be viewed as the polynomial being of the form

$$(M_{\gamma_1}M_{\gamma_2}\phi)_{\kappa_\cup,\kappa_\cap} = \sum_{p,q} c_{pq}\kappa_\cup^p\kappa_\cap^q,$$

where

$$p+q \leqslant \tfrac{1}{2}(\omega_1+\omega_2). \tag{18}$$

To go further, we need the following relation (proof in Appendix II)

$$\omega_\cup+\omega_\cap-\omega_1-\omega_2 = (D-2)l_{12}. \tag{19}$$

As a simple consequence of (19) we have $\omega_{\cup}+\omega_{\cup} \geqslant 0$. We now consider the three possible cases

a. $\omega_{\cup} \geqslant 0, \omega_{\cap} < 0$
b. $\omega_{\cup} < 0, \omega_{\cap} \geqslant 0$
c. $\omega_{\cup} \geqslant 0, \omega_{\cap} \geqslant 0$

Case a

Since $\gamma_{\cup}$ is full, connected, one-particle irreducible and its index is non-negative, then $\gamma_{\cup}$ is a divergent subdiagram of Γ. It can be easily seen from (19) that

$$\tfrac{1}{2}\omega_{\cup} \geqslant \tfrac{1}{2}(\omega_1+\omega_2). \tag{20}$$

It means that the highest power with respect to $\kappa_{\cup}$ in the polynomial (14) is not greater than the range of the subtraction operation connected with $\gamma_{\cup}$. It leads to the identity

$$\boldsymbol{O}_{\gamma_{\cup}}\boldsymbol{M}_{\gamma_1}\boldsymbol{M}_{\gamma_2}\phi = 0 \tag{21}$$

Case b

In this case relation (19) implies

$$\tfrac{1}{2}\omega_{\cap} \geqslant \tfrac{1}{2}(\omega_1+\omega_2). \tag{22}$$

The subdiagram $\gamma_{\cap}$ is connected. It follows from the fact that γ_1 and γ_2 belong to the tree family. If $\gamma_{\cap}$ is one-particle irreducible, then it is a divergent subdiagram of Γ and the same reasoning as in the case a leads to the identity

$$\boldsymbol{O}_{\gamma_{\cap}}\boldsymbol{M}_{\gamma_1}\boldsymbol{M}_{\gamma_2}\phi = 0. \tag{23}$$

Suppose now that the subdiagram $\gamma_{\cap}$ is not one-particle irreducible, i.e. it contains lines which do not belong to any loops. Let us remove each such line. The remaining diagram is composed of r one-particle irreducible, full and mutually disjoint subdiagrams $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r$ Moreover, because the removed lines were not associated with loops

$$\omega_{\varepsilon_1}+\omega_{\varepsilon_2}+ \ldots +\omega_{\varepsilon_r} \geqslant \omega_{\cap} \geqslant 0. \tag{24}$$

We can number the subdiagrams with non-negative indices as $\varepsilon_1, \ldots, \varepsilon_s$ $(s \leqslant r)$. They are divergent subdiagrams of Γ. Considering subtractions associated with subdiagrams $\varepsilon_1, \ldots, \varepsilon_s$, in a quite analogous manner as before, we obtain the following expression

$$(\boldsymbol{M}_{\gamma_1}\boldsymbol{M}_{\gamma_2}\phi)_{\kappa_{\varepsilon_1},\ldots,\kappa_{\varepsilon_s}} = \sum_{q_1,\ldots,q_s} c_{q_1,\ldots,q_s}\kappa_{\varepsilon_1}^{q_1} \ldots \kappa_{\varepsilon_s}^{q_s}, \tag{25}$$

where

$$q_1+ \ldots +q_s \leqslant \tfrac{1}{2}(\omega_1+\omega_2).$$

Relations (22) and (24) lead to conclusion that at least one operation $\boldsymbol{O}_{\varepsilon}$ cancels the polynomial (25)

$$\boldsymbol{O}_{\varepsilon_1} \ldots \boldsymbol{O}_{\varepsilon_s}\boldsymbol{M}_{\gamma_1}\boldsymbol{M}_{\gamma_2}\phi = 0. \tag{26}$$

Case c

From (19) we have

$$\tfrac{1}{2}(\omega_{\cup}+\omega_{\cap}) \geqslant \tfrac{1}{2}(\omega_1+\omega_2). \tag{27}$$

The situation is similar as it was in Case *b*. If the subdiagram $\gamma_{\cap}$ is one-particle irreducible, we may prove that

$$\boldsymbol{O}_{\gamma_{\cup}}\boldsymbol{O}_{\gamma_{\cap}}\boldsymbol{M}_{\gamma_1}\boldsymbol{M}_{\gamma_2}\phi = 0. \tag{28}$$

More generally, if the subdiagram contains mutually disjoint and divergent subdiagrams ε_l we obtain

$$\boldsymbol{O}_{\gamma_{\cup}}\boldsymbol{O}_{\gamma_{\varepsilon_1}} \cdots \boldsymbol{O}_{\gamma_{\varepsilon_s}}\boldsymbol{M}_{\gamma_1}\boldsymbol{M}_{\gamma_2}\phi = 0. \tag{29}$$

The above considerations lead to a common conclusion. If a Feynman diagram Γ is a T-diagram, then for any two overlapping subdiagrams $\gamma_a, \gamma_b \in \mathscr{R}_\Gamma$, there exist a set of divergent subdiagrams $\gamma_{i_1}, \ldots, \gamma_{i_r} \in \mathscr{R}_\Gamma$ for which

$$\boldsymbol{O}_{\gamma_{i_1}} \cdots \boldsymbol{O}_{\gamma_{i_r}}\boldsymbol{M}_{\gamma_a}\boldsymbol{M}_{\gamma_b}\phi = 0, \tag{30}$$

where

$$\forall 1 \leqslant k \leqslant r\colon \gamma_{i_k} \subset \gamma_1 \cap \gamma_2 \quad \text{or} \quad \gamma_{i_k} = \gamma_1 \cup \gamma_2. \tag{30a}$$

It should be emphasized here that the relations (30), valid for each pair of divergent subdiagrams, do not directly ensure the reduction of the factorized formula (2.9) into the Zimmermann's one (2.10). If there is only one pair of overlapping subdiagrams, the reduction is trivially verified. But if we have more than one such pair, the reduction cannot be obtained by means of succesive application of the suitable relations (30), because we are not allowed to make use of them separately. However, both renormalization formulae (2.9) and (2.10) are equivalent, as the following Theorem proves.

THEOREM 4. For any Feynman T-diagram we are allowed to make use of the κ-representation of subtraction operations with fixed $\boldsymbol{\Omega} = 0$ and, in this case, factorized renormalization formula (2.9) is equivalent to Zimmermann's forest formula (2.10).

$$\boldsymbol{O}_{\gamma_1} \cdots \boldsymbol{O}_{\gamma_R}\phi = [1+ \sum_{\mathscr{F}\subset\mathscr{R}_\Gamma} \prod_{r\in\mathscr{F}} (-\boldsymbol{M}_{\gamma_r})] \tag{31}$$

Proof. Let us expand the left-hand side of (31)

$$\boldsymbol{O}_{\gamma_1} \cdots \boldsymbol{O}_{\gamma_R}\phi = [1-\sum_r \boldsymbol{M}_{\gamma_r}+\sum_{r,s}\boldsymbol{M}_{\gamma_r}\boldsymbol{M}_{\gamma_s}-\sum_{r,s,t}\boldsymbol{M}_{\gamma_r}\boldsymbol{M}_{\gamma_s}\boldsymbol{M}_{\gamma_t}+\ldots]\phi. \tag{32}$$

First, we consider two-element overlapping products of the expansion (32). We have to group together several terms of (32) in order to form expressions like (30), and remove them. It is important that we always construct a relation (30) taking into account all subdiagrams satisfying (30a). Then we turn to three-element products of (32) and repeat the above procedure, and so on. It is possible to meet a number of products $\boldsymbol{M}_{\gamma_a}\boldsymbol{M}_{\gamma_b}$ of overlapping subdiagrams in a given many-element product of the operators $\boldsymbol{M}_\gamma$, so we need some rule to choose the correct product, for which the construction (30) is to be built. Let us define the following partial ordering relation

$$(\gamma_1, \gamma_2) < (\gamma_3, \gamma_4) \equiv \begin{cases} \gamma_1 \cup \gamma_2 \subsetneq \gamma_3 \cup \gamma_4 & \text{or} \\ \gamma_1 \cup \gamma_2 = \gamma_3 \cup \gamma_4 & \text{and} \quad \gamma_1 \cap \gamma_2 \subsetneq \gamma_3 \cap \gamma_4. \end{cases}$$

We extend the foregoing relation to the linear order relation (in any way). Let the product we choose be the maximal one with respect to the linear order relation. Now we should prove that a systematic procedure based upon the above algorithm will actually enable us to remove from (32) all "overlapping" products, i.e. all products associated with families of subdiagrams not being forests.

First we introduce the notions and the terminology which will be used (all following notions are connected with divergent subdiagrams).

$\gamma_1 \leftrightarrow \gamma_2$ means that the subdiagrams γ_1 and γ_2 overlap.

$\{\gamma_1 \leftrightarrow \gamma_2\}$ designates the set of all divergent subdiagrams γ satisfying $\gamma = \gamma_1 \cup \gamma_2$ or $\gamma \subset \gamma_1 \cap \gamma_2$.

$(\gamma_1, \gamma_2) \to \gamma_3$ means that $\gamma_1 \leftrightarrow \gamma_2$ and $\gamma_3 \in \{\gamma_1 \leftrightarrow \gamma_2\}$.

$(\gamma_i, \gamma_j) = \max(\gamma_1, \gamma_2, \ldots, \gamma_N)$ means that the pair (γ_i, γ_j) is the maximal one among all pairs included in the set $\{\gamma_1, \ldots, \gamma_N\}$ with respect to the linear order relation.

Let $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, ... denote products of the operators M.

A product $\mathscr{A}$ is said to be a base product if it has been chosen to built the construction (30).

A product $\mathscr{B}$ is said to be an adjoint product if it has been used to built the construction (30) for some base product $\mathscr{A}$.

A product $\mathscr{C}$ is said to be an original product if it does not contain operator M_γ for which $\gamma \in \{\gamma_A \leftrightarrow \gamma_B\}$, where (γ_A, γ_B) is the maximal pair in this product.

Of course, looking at the given product we cannot say whether it is a base one or an adjoint one. It depends on some "organization" of the process of removing overlapping products.

$\mathscr{A} \to \mathscr{B}$ means that the product $\mathscr{A}$ is an original one and furthermore if it is a base product then the product $\mathscr{B}$ is an adjoint one used to built the construction (30) for $\mathscr{A}$.

A chain for the product $\mathscr{A}$ is the sequence of the products $\mathscr{B}_1, \ldots, \mathscr{B}_N$ for which

$$\mathscr{B}_N \to \mathscr{B}_{N-1} \to \ldots \to \mathscr{B}_2 \to \mathscr{B}_1 \to \mathscr{A}$$

and there exist no product $\mathscr{C}$ satisfying: $\mathscr{C} \to \mathscr{B}_N$.

The following lemma is a first step to prove that the provided algorithm for removing overlapping products is consistent.

LEMMA 2. If $\gamma_A \leftrightarrow \gamma_B$, $\gamma_1 \leftrightarrow \gamma_2$, $\gamma_1 \subset \gamma_A \cap \gamma_B$, $\gamma_A = \gamma_{A_1} \cup \ldots \cup \gamma_{A_N}$ and there exist γ_{A_i} such that $\gamma_{A_i} \leftrightarrow \gamma_B$, then

$$(\gamma_1, \gamma_2) \neq \max(\gamma_1, \gamma_2, \gamma_{A_1}, \ldots, \gamma_{A_N}, \gamma_B). \tag{33}$$

Proof. There are two cases to consider

1. $\gamma_2 \leftrightarrow \gamma_B$,
2. $\gamma_2 \subset \gamma_B$.

Case 1

We note that in virtue of the assumptions of the Lemma it is evident that

$$\gamma_1 \cup \gamma_2 \subset \gamma_B \cup \gamma_2, \tag{33a}$$

$$\gamma_1 \cap \gamma_2 \subset \gamma_B \cap \gamma_2. \tag{33b}$$

In order to prove the validity of (33) we need only to show that (33a) and (33b) cannot hold simultaneously. If they were then

$$\gamma_B = \gamma_B \cap (\gamma_1 \cup \gamma_2) = (\gamma_B \cap \gamma_1) \cup (\gamma_B \cap \gamma_2)$$

$$= \gamma_1 \cup (\gamma_1 \cap \gamma_2) \subset \gamma_1 \subset \gamma_A$$

and this would contradict overlapping γ_A and γ_B. Thus the above reasoning essentially gives a relation

$$(\gamma_1, \gamma_2) < (\gamma_B, \gamma_2).$$

Case 2

Consider γ_{A_i} such that $\gamma_{A_i} \leftrightarrow \gamma_B$.

Because we have $\gamma_1 \cup \gamma_2 \subset \gamma_B$ it guarantees that

$$\gamma_1 \cup \gamma_2 \subsetneq \gamma_B \cup \gamma_{A_i}$$

and accordingly $(\gamma_1, \gamma_2) < (\gamma_B, \gamma_{A_i})$.

Remark. It is easy to see that the statement remain valid in the special cases

1° $N = 1$,

2° $\gamma_1 = \gamma_{A_i}$ or $\gamma_2 = \gamma_{A_k}$.

COROLLARY. If $(\gamma_A, \gamma_B) = \max(\gamma_A, \gamma_B, \gamma_1, \gamma_2)$ and $(\gamma_1, \gamma_2) \to \gamma_A$ then $\gamma_A = \gamma_1 \cup \gamma_2$.
Proof. By utilizing the contradiction of the Lemma, we can exclude the case $\gamma_A \subset \gamma_1 \cap \gamma_2$.
LEMMA 3. For any product $\mathscr{A}$ there exist at most two different products $\mathscr{B}_i$ satisfying $\mathscr{B}_i \to \mathscr{A}$.
LEMMA 4. For any original product $\mathscr{A}$ there exist at most one product $\mathscr{B}$ satisfying $\mathscr{B} \to \mathscr{A}$.

Proof. Let (γ_A, γ_B) be the maximal pair in the product $\mathscr{A}$. If $\mathscr{A}$ is not an original product, it contains operators M_γ for which $\gamma \in \{\gamma_A \leftrightarrow \gamma_B\}$. We remove all such operators and using remaining operators we create the product $\mathscr{B}$. Clearly $\mathscr{B} \to \mathscr{A}$ and it is one of the products mentioned in Lemma 3. To establish both Lemmas it is sufficient to show that there exists at most one product $\mathscr{B}_1$, different from $\mathscr{B}$, such that $\mathscr{B}_1 \to \mathscr{A}$. Assume, to the contrary, that there exist two products $\mathscr{B}_1$ and $\mathscr{B}_2$ satisfying $\mathscr{B}_i \to \mathscr{A}$. Let (γ_1, γ_2) and (γ_3, γ_4) be the maximal pairs in the $\mathscr{B}_1$ and $\mathscr{B}_2$ respectively. There is no essential loss of generality in assuming that

$$(\gamma_1, \gamma_2) < (\gamma_3, \gamma_4) < (\gamma_A, \gamma_B).$$

We can limit ourselves to examine two following situations (the other can be obtained by reversing the roles of the subdiagrams)

1° $\gamma_3, \gamma_A \in \{\gamma_1 \leftrightarrow \gamma_2\}$ and $\gamma_A \in \{\gamma_3 \leftrightarrow \gamma_4\}$,

2° $\gamma_3, \gamma_A \in \{\gamma_1 \leftrightarrow \gamma_2\}$ and $\gamma_B \in \{\gamma_3 \leftrightarrow \gamma_4\}$.

Ad 1°. Applying the Corollary to Lemma 2 we obtain

$$\gamma_3 = \gamma_1 \cup \gamma_2 = \gamma_A = \gamma_3 \cup \gamma_4.$$

Accordingly $\gamma_4 \subset \gamma_3$ and it contradicts overlapping of γ_3 and γ_4.
Ad 2°. From the Corollary to Lemma 2 we have

$$\gamma_3 = \gamma_1 \cup \gamma_2 = \gamma_A \quad \text{and} \quad \gamma_B = \gamma_3 \cup \gamma_4.$$

Thus $\gamma_A \subset \gamma_B$, but this contradicts overlapping of γ_A and γ_B.

This completes the proof.

LEMMA 5. For any product $\mathscr{A}$ there are no more than two non-equivalent chains

$$\mathscr{B}_N \to \mathscr{B}_{N-1} \to \ldots \to \mathscr{B}_1 \to \mathscr{A},$$
$$\mathscr{C}_M \to \mathscr{C}_{M-1} \to \ldots \to \mathscr{C}_1 \to \mathscr{A}, \tag{34a}$$

where the products $\mathscr{B}_1, \ldots, \mathscr{B}_N, \mathscr{C}_1, \ldots, \mathscr{C}_M$ are original ones and the lengths of the chains are related by the condition

$$N = M+1 \quad (N \geqslant 2). \tag{34b}$$

Moreover, the products $\mathscr{B}_{k+1}$ and $\mathscr{C}_k$ have the same maximal pair $(\gamma_{a_k}, \gamma_{b_k})$.

The foregoing construction will be termed the *ladder for the product* $\mathscr{A}$. There are some exceptions to the general statement of this Lemma which will be termed as *degenerate ladders*, namely

1. $\mathscr{A}$,
2. $\mathscr{B}_N \to \ldots \to \mathscr{B}_1 \to \mathscr{A}$.

For a given product $\mathscr{A}$ there exists only one chain in the latter case and none in the former case. Moreover, if the product $\mathscr{A}$ is not an original one, then there is only one form of degenerate ladder

$$\mathscr{B} \to \mathscr{A}.$$

Proof. From Lemma 2 and Lemma 3 it follows that there exist at most two non-equivalent chains for the given product $\mathscr{A}$. Therefore, excluding the possibility of the existence of a degenerate ladder, the lemma just formulated above will be proved if we succeed in establishing the assertion (34b). We will prove it by induction. Let (γ_A, γ_B) be the maximal pair in the product $\mathscr{A}$. Let us assume that for some n there exist the following ladder

$$\mathscr{B}_{n+1} \to \mathscr{B}_n \to \ldots \to \mathscr{B}_1 \to \mathscr{A},$$
$$\mathscr{C}_n \to \mathscr{C}_{n-1} \to \ldots \to \mathscr{C}_1 \to \mathscr{A},$$

where for any $k \leqslant n$ the products $\mathscr{B}_{k+1}$ and $\mathscr{C}_k$ have the same maximal pair $(\gamma_{a_k}, \gamma_{b_k})$. Moreover

$$\mathscr{C}_k = \mathscr{B}_{k+1}\mathscr{D}_k, \tag{35}$$

where $\mathscr{D}_k$ is the product of all operators present in $\mathscr{C}_k$ which are associated with subdiagrams belonging to the set $\{\gamma_A \leftrightarrow \gamma_B)$. Obviously, if $\mathscr{C}_k$ contains no such operators then $\mathscr{D}_k = 1$ and $\mathscr{C}_k = \mathscr{B}_{k+1}$.

We will establish the following equivalence. There exists product $\mathscr{C}_{n+1}$ such that $\mathscr{C}_{n+1} \to \mathscr{C}_n$ if ond only if there exists the product $\mathscr{B}_{n+2}$ such that $\mathscr{B}_{n+2} \to \mathscr{B}_{n+1}$. Moreover, the products $\mathscr{B}_{n+2}$ and $\mathscr{C}_{n+1}$ have the same maximal pair and $\mathscr{C}_{n+1} = \mathscr{B}_{n+2}\mathscr{D}_{n+1}$.
($\Rightarrow$). Suppose that there exist the product $\mathscr{C}_{n+1}$ such that $\mathscr{C}_{n+1} \to \mathscr{C}_n$, and let $(\gamma_{a_{n+1}}, \gamma_{b_{n+1}})$ be the maximal pair in $\mathscr{C}_{n+1}$. From the Corollary to Lemma 2 up to a change of the roles between γ_{a_k} and γ_{b_k} we obtain

$$\gamma_A = \gamma_{a_1} \cup \gamma_{b_1},$$

$$\gamma_{a_k} = \gamma_{a_{k+1}} \cup \gamma_{b_{k+1}}, \text{ for any } k \leqslant n.$$

Thus

$$\gamma_A = \gamma_{b_1} \cup \gamma_{b_2} \cup \ldots \cup \gamma_{b_{n+1}} \cup \gamma_{a_{n+1}}. \tag{36}$$

We note that all operators associated with the subdiagrams which appear in the right-hand side of (36) are involved in the product $\mathscr{C}_{n+1}$. Then the following statement is true

$$(\gamma_{a_{n+1}}, \gamma_{b_{n+1}}) = \max\,(\gamma_{a_{n+1}}, \gamma_{b_1}, \ldots, \gamma_{b_{n+1}}, \gamma_B). \tag{37}$$

First we prove that

$$\gamma_{a_{n+1}}, \gamma_{b_{n+1}} \notin \{\gamma_A \leftrightarrow \gamma_B\}. \tag{38}$$

Assume, to the contrary, that $\gamma_{a_{n+1}} \in \{\gamma_A \leftrightarrow \gamma_B\}$. Because $(\gamma_{a_{n+1}}, \gamma_{b_{n+1}}) < (\gamma_A, \gamma_B)$ it implies that $\gamma_{a_{n+1}} \subset \gamma_A \cap \gamma_B$. Referring to relation (36) and applying Lemma 2 we would contradict relation (37). With this contradiction the condition (38) is established. Keeping in mind the conditions (38) and (35), we define $\mathscr{B}_{n+2} := \mathscr{C}_{n+1}/\mathscr{D}_{n+1}$ and conclude that $\mathscr{B}_{n+2} \to \mathscr{B}_{n+1}$.
($\Leftarrow$). Now suppose that there exists the product $\mathscr{B}_{n+2}$ with the maximal pair $(\gamma_{a_{n+1}}, \gamma_{b_{n+1}})$ satisfying $\mathscr{B}_{n+2} \to \mathscr{B}_{n+1}$. We construct the product $\mathscr{D}_{n+1}$ from the product by the removal of all operators associated with subdiagrams belonging to the set $\{\gamma_{a_{n+1}} \leftrightarrow \gamma_{b_{n+1}}\}$. Then we define $\mathscr{C}_{n+1} := \mathscr{B}_{n+2}\mathscr{D}_{n+1}$. To prove that $\mathscr{C}_{n+1} \to \mathscr{C}_n$ we should only show that the pair $(\gamma_{a_{n+1}}, \gamma_{b_{n+1}})$ is the maximal one in $\mathscr{C}_{n+1}$. Let us suppose that there exists a pair (γ_1, γ_2) in the product $\mathscr{C}_{n+1}$ such that $\gamma_1 \in \{\gamma_A \leftrightarrow \gamma_B\}$. Clearly $\gamma_1 \subset \gamma_A \cap \gamma_B$ and, because the relations (36) and (37) hold, under Lemma 2 we obtain

$$(\gamma_1, \gamma_2) < (\gamma_{a_{n+1}}, \gamma_{b_{n+1}}).$$

This completes the proof. In a similar way we prove the additional statement of Lemma 5 referring to degenerate ladders for non-original product.

LEMMA 6. Let us consider any product $\mathscr{A}$ with the non-degenerate ladder

$$\mathscr{B}_{N+1} \to \ldots \to \mathscr{B}_1 \to \mathscr{A},$$
$$\mathscr{C}_N \to \ldots \to \mathscr{C}_1 \to \mathscr{A}.$$

The products $\mathscr{B}_1$ and $\mathscr{C}_1$ are not simultaneously base ones nor simultaneously adjoint ones.

Proof: Because the products $\mathscr{B}_{N+1}$, $\mathscr{C}_N$ are base ones, then the products $\mathscr{B}_N$, $\mathscr{C}_{N-1}$ are adjoint ones, the products $\mathscr{B}_{N-1}$, $\mathscr{C}_{N-2}$ are base ones and so on.

Now, we are in a position to complete the proof of Theorem 4. We will establish by induction that the provided algorithm of the removal of overlapping products is consistent. Let us assume that the process of removing overlapping products has been accomplished in N stages, without fail. Now we deal with $(N+1)$-element products. Consider any $(N+1)$-element overlapping product $\mathscr{A}$.

1°. Is $\mathscr{A}$ an original product?

If $\mathscr{A}$ is not an original product and it has not been removed so far, then it has got an non--degenerate ladder. But from Lemma 6 we conlude that it should be removed. This leads to a contradiction.

2°. Do we still have all adjoint products necessary to built the construction (30) for $\mathscr{A}$?

Let us consider any adjoint product $\mathscr{B}$ such that $\mathscr{B} \to \mathscr{A}$. If $\mathscr{B}$ has been removed previously, then $\mathscr{B}$ has an non-degenerate ladder. Considering Lemma 6 we establish a contradiction by concluding that the product $\mathscr{A}$ should be removed previously.

3°. Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be $(N+1)$-element original products which have not been removed so far. Is there any product $\mathscr{B}$ being adjoint simultaneously to $\mathscr{A}_1$ and $\mathscr{A}_2 : \mathscr{A}_1 \to \mathscr{B}$ and $\mathscr{A}_2 \to \mathscr{B}$?

Because the product $\mathscr{B}$ has got an non-degenerate ladder, the assumption that $\mathscr{A}_1$ and $\mathscr{A}_2$ have not been removed leads to a contradiction.

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APPENDIX I

This Appendix is devoted to prove the theorems from Sect. 3 which show the analytic properties of Feynman amplitudes associated with T-diagrams. We intend to give only the proof of Theorem 2. The Corollary is a straightforward application of this theorem, the Theorem 3, in turn, can be proved analogously.

Suppose that there exists a tree T_1^* satisfying the assertion that for any r $(1 \leqslant r \leqslant R)$ the intersection $T_1 \cap \gamma_r$ is a tree of the subdiagram γ_r. Denoting by $w_r(T_1)$ the number of lines belonging to γ_r but not to T_1, we have

$$D_\kappa(\alpha) = \sum_{T_1} (\kappa_1^{w_1(T_1)} \ldots \kappa_R^{w_R(T_1)} \prod_{l \notin T_1} \alpha_l)$$
$$= \kappa_1^{p_1} \ldots \kappa_R^{p_R} [\prod_{l \notin T_1^*} \alpha_l + D_\kappa^*(\alpha)], \tag{1}$$

where

$$D^*_\kappa(\alpha) = \sum_{T_1 \neq T_1^*} (\prod_{r=1}^{R} \kappa_r^{w_r(T_1)-p_r} \prod_{l \notin T_1} \alpha_l).$$

Because $w_r(T_1) \geqslant p_r$, $D^*_\kappa(\alpha)$ is a polynomial with respect to κ. Similarly, it is possible to write

$$A_\kappa(\alpha, k) = \sum_{T_2} \{\prod_{r=1}^{R} \kappa_r^{w_r(T_2)} (\prod_{l \notin T_2} \alpha_l) (\sum_{i \in V_1} k_i)^2\} = \kappa_1^{p_1} \dots \kappa_R^{p_R} A^*_\kappa(\alpha, k), \tag{2}$$

where $A^*_\kappa(\alpha, k)$ is a polynomial of κ as well.
Substituting (1) and (2) to (2.7) we obtain

$$\phi(\kappa) = \frac{1}{[\prod_{l \in T_1^*} \alpha_l + D^*_\kappa(\alpha)]^{D/2}} \exp\left(i \frac{A^*_\kappa(\alpha, k)}{\prod_{l \notin T_1^*} \alpha_l + D^*_\kappa(\alpha)}\right).$$

It is apparent that $\phi(\kappa)$ and its derivatives have no singularities.

To prove necessity, suppose that the Feynman diagram is not a T-diagram. It implies that for any tree T_1 there is a subdiagram γ_r such that $w_r(T_l) > p_r$. Thus,

$$\left.\frac{D^*_\kappa(\alpha)}{\kappa_1^{p_1} \dots \kappa_R^{p_R}}\right|_{\kappa_1 = \dots = \kappa_R = 0} = 0$$

and ϕ has got a singularity at the point $\kappa = 0$.

This completes the proof.

APPENDIX II

We use the same notation as in Sect. 3. We intend to prove the general relation

$$\omega_\cup + \omega_\cap - \omega_1 - \omega_2 = (D-2)l_{12} + D(s-1),$$

where s is the number of connected components of the subdiagram $\gamma_\cap$. To obtain the above statement it suffices to combine the following relations

$$\omega_i = D(l_i - n_i + 1) - 2l_i,$$

where i stands for 1, 2 or $\cup$,

$$\omega_\cap = D(l_\cap - n_\cap + s) - 2l_\cap,$$

$$l_\cup = l_1 + l_2 - l_\cap + l_{12},$$

$$n_\cup = n_1 + n_2 - n_\cap.$$

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